# Time-dependent electromagnetic scattering from thin layers

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# Outline

Scattering from generalized impedance b.c.

Functional-analytic setting

Well-posedness via time-dependent boundary integral equations

BEM & CQ discretization

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# Picture: 3D-scattering from a torus

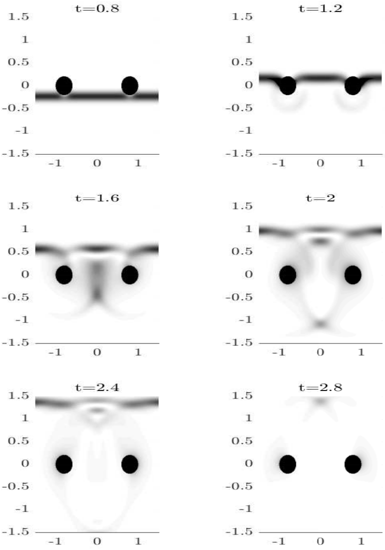


Figure: total electric field strength in the slice $x_2 = 0$ at different times

# Time-dependent scattering

Time-dependent Maxwell's equations in an exterior domain for the total electric field $E^{\text{tot}}(x,t)$ and total magnetic field $H^{\text{tot}}(x,t)$,

$$\begin{aligned} \varepsilon\,\partial_t E^{\text{tot}} - \operatorname{curl} H^{\text{tot}} &= 0 \\ \mu\,\partial_t H^{\text{tot}} + \operatorname{curl} E^{\text{tot}} &= 0 \end{aligned} \qquad \text{in the exterior domain } \Omega.$$

(We set $\varepsilon = 1$, $\mu = 1$ in the following.)

Given incident fields $(E^{\text{inc}}, H^{\text{inc}})$: solution to Maxwell's equations in $\mathbb{R}^3$, with support in $\Omega$ at $t = 0$

Compute scattered fields

$$E^{\text{scat}} = E^{\text{tot}} - E^{\text{inc}}, \qquad H^{\text{scat}} = H^{\text{tot}} - H^{\text{inc}}$$

on a time interval $0 \le t \le T$ at selected space points $x \in \Omega$.

# Scattering: boundary conditions

B.c. depend on material (often multiscale) $\to$ effective models

- Thin coating of depth $\delta \ll 1$ on perfectly conducting material

$$(E^{\mathrm{tot}} \times \nu) \times \nu = \delta\,(\partial_t - \partial_t^{-1} \nabla_\Gamma \mathrm{div}_\Gamma)(H^{\mathrm{tot}} \times \nu)$$

Engquist & Nédélec 1993

- Skin effect: strong absorption by a highly conducting material

$$(E^{\mathrm{tot}} \times \nu) \times \nu = \delta\,\partial_t^{1/2}(H^{\mathrm{tot}} \times \nu)$$

Haddar, Joly & Nguyen 2008

Both cases are examples of generalized impedance b.c.

$$\begin{aligned}(E^{\mathrm{tot}} \times \nu) \times \nu &= Z(\partial_t)(H^{\mathrm{tot}} \times \nu) \\ &= (\mathcal{L}^{-1} Z) * (H^{\mathrm{tot}} \times \nu)\end{aligned}$$

temporal convolution

# Computation of the scattered field

1. Solve a time-dependent boundary integral equation for the tangential traces $E \times \nu$, $H \times \nu$ on $\Gamma = \partial\Omega$, use boundary elements for space discretization and convolution quadrature for time discretization
2. Apply a representation formula to compute the scattered fields $(E, H)$ at arbitrary points $x \in \Omega$, $0 \le t \le T$. Use again convolution quadrature for time discretization.

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Well-proven strategy: L. 1994, Laliena & Sayas 2009, Sayas 2016, Banjai, L., Nick 2021 (acoustic)

Ballani, Banjai, Sauter & Veit 2013, Chan & Monk 2015, Kovács & L. 2017 (electromagnetic)

... but does it work here? And how?

# Challenges

- Derive a time-dependent boundary integral equation and prove its well-posedness in appropriate trace spaces
- Prove well-posedness of the time-dependent scattering problem in appropriate spaces

- Prove stability and convergence / error bounds of the discretization in space and time

# Outline

# Tangential trace and Green's formula

Tangential trace:

$$\gamma_T v = v|_\Gamma \times \nu \qquad \text{on } \Gamma, \qquad \text{for continuous } \; v : \overline{\Omega} \to \mathbb{C}^3$$

Green's formula:

$$\int_\Omega \big(u \cdot \operatorname{curl} v - \operatorname{curl} u \cdot v\big)\mathrm{d}x = [\gamma_T u, \gamma_T v]_\Gamma$$

with the skew-hermitian pairing

$$[\phi, \psi]_\Gamma = \int_\Gamma (\phi \times \nu) \cdot \psi \, \mathrm{d}\sigma.$$

# Trace space

Hilbert space $X_\Gamma$ such that

$$\gamma_T : H(\mathrm{curl}, \Omega) \to X_\Gamma \quad \text{is a surjective bounded linear map.}$$

$[\phi, \psi]_\Gamma$ is a nondegenerate sesquilinear form on $X_\Gamma \times X_\Gamma$.

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Alonso & Valli 1996, Buffa, Costabel & Sheen 2002

# Solution space $V_\Gamma \subset X_\Gamma$

Hilbert space $V_\Gamma$ dense in $X_\Gamma$

- $V_\Gamma = X_\Gamma \cap H(div_\Gamma, \Gamma)$ for thin coating b.c.
- $V_\Gamma = X_\Gamma \cap L^2(\Omega)^3$ for skin effect b.c.

$$\|\phi\|^2_{V_\Gamma} = \|\phi\|^2_{X_\Gamma} + |\phi|^2_{V_\Gamma}$$

# Impedance operator

$Z(s) : V_\Gamma \to V_\Gamma'$, Re $s > 0$, analytic family of polynomially bounded linear operators: for Re $s \geq \sigma > 0$,

$$\|Z(s)\|_{V_\Gamma{}' \leftarrow V_\Gamma} \leq M_\sigma \, |s|^\kappa$$

- $\kappa = 1$ for thin coating b.c.
- $\kappa = 1/2$ for skin effect b.c.

of positive type: for all $\phi \in V_\Gamma$ and Re $s \geq \sigma > 0$,

$$\mathrm{Re}\langle \phi, Z(s)\phi \rangle \geq c_\sigma \, \mathrm{Re}\, s \, \left| s^{-1}\phi \right|^2_{V_\Gamma}$$

# Temporal convolution

$$Z(\partial_t)g := (\mathcal{L}^{-1}Z) * g \qquad \text{for suff. regular } g : [0, T] \to V_\Gamma$$

Note the operational calculus: $Y(\partial_t)Z(\partial_t)g = (YZ)(\partial_t)g$

Space-time Hilbert space:

$H_0^r(0, T; V) = \{g|_{(0,T)} \, : \, g \in H^r(\mathbb{R}, V) \text{ with } g = 0 \text{ on } (-\infty, 0)\}$

The bound on $Z(s)$, $\mathrm{Re}\, s > 0$, implies a bound on $Z(\partial_t)$:

$$\|Z(\partial_t)\|_{H_0^r(0,T;V_\Gamma{}') \leftarrow H_0^{r+\kappa}(0,T;V_\Gamma)} \le e\, M_{1/T}$$

# Weak formulation of the impedance b.c.

The boundary condition relates the tangential traces of $E$ and $H$.

Determine the tangential traces

$$\gamma_T E \in L^2(0,T;X_\Gamma) \quad \text{and} \quad \gamma_T H \in H_0^\kappa(0,T;V_\Gamma)$$

such that for almost every $t \in (0,T)$, we have for all $\phi \in V_\Gamma$

$$[\phi, \gamma_T E]_\Gamma + \langle \phi, Z(\partial_t)\gamma_T H\rangle_\Gamma = \langle \phi, g^{\mathrm{inc}}\rangle_\Gamma .$$

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Wellposedness of Maxwell's equations in the exterior domain with these boundary conditions?

Stable and convergent numerical method?

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# Potential operators in the Laplace domain

## Fundamental solution

$$G(s,x) = \frac{e^{-s|x|}}{4\pi\,|x|}, \qquad \operatorname{Re} s > 0,\ x \in \mathbb{R}^3 \setminus \{0\}.$$

## Single layer potential operator $\mathcal{S}(s)$

$$\big(\mathcal{S}(s)\varphi\big)(x) = -s \int_\Gamma G(s,x-y)\varphi(y)\mathrm{d}y + s^{-1}\nabla \int_\Gamma G(s,x-y)\,\mathrm{div}_\Gamma\,\varphi(y)\mathrm{d}y$$

## Double layer potential operator $\mathcal{D}(s)$

$$\big(\mathcal{D}(s)\varphi\big)(x) = \mathrm{curl} \int_\Gamma G(s,x-y)\varphi(y)\mathrm{d}y.$$

defined for $x \in \mathbb{R}^3 \setminus \Gamma = \Omega^+ \dot\cup\, \Omega^-$.

# Potential operators in the Laplace domain: properties

Maxwell solutions:

$$s\,\mathcal{S}(s) - \mathrm{curl} \circ \mathcal{D}(s) = 0, \qquad s\,\mathcal{D}(s) + \mathrm{curl} \circ \mathcal{S}(s) = 0$$

Jump relations: with $[\gamma_T] = \gamma_T^+ - \gamma_T^-$,

$$[\gamma_T] \circ \mathcal{S}(s) = 0, \qquad [\gamma_T] \circ \mathcal{D}(s) = -\mathrm{Id}$$

Bounds of $\mathcal{S}(s), \mathcal{D}(s) : X_\Gamma \to H(\mathrm{curl}, \mathbb{R}^3 \setminus \Gamma)$:

$$\|\mathcal{S}(s)\|_{H(\mathrm{curl},\mathbb{R}^3\setminus\Gamma)\leftarrow X_\Gamma} \le C_\Gamma \frac{|s|^2 + 1}{\mathrm{Re}\, s}, \qquad \text{same bound for } \mathcal{D}(s),$$

where $C_\Gamma = \|\{\gamma_T\}\|_{X_\Gamma \leftarrow H(\mathrm{curl},\mathbb{R}^3\setminus\Gamma)}$ with $\{\gamma_T\} = \frac{1}{2}\left(\gamma_T^+ + \gamma_T^-\right)$

# Boundary operators and Calderón operator

Single and double layer boundary operators from $X_\Gamma$ to $X_\Gamma$:

$$V(s) = \{\gamma_T\} \circ \mathcal{S}(s), \qquad K(s) = \{\gamma_T\} \circ \mathcal{D}(s)$$

Calderón operator: $B(s) = \begin{pmatrix} -V(s) & K(s) \\ -K(s) & -V(s) \end{pmatrix}$

is such that it transforms jumps to averages:

$$B(s) \begin{pmatrix} [\gamma_T]\widehat{H} \\ -[\gamma_T]\widehat{E} \end{pmatrix} = \begin{pmatrix} \{\gamma_T\}\widehat{E} \\ \{\gamma_T\}\widehat{H} \end{pmatrix}.$$

Coercivity: for $\mathrm{Re}\, s > 0$ and for all $(\varphi, \psi) \in X_\Gamma \times X_\Gamma$,

$$\mathrm{Re} \left[ \begin{pmatrix} \varphi \\ \psi \end{pmatrix}, B(s) \begin{pmatrix} \varphi \\ \psi \end{pmatrix} \right]_\Gamma \geq \frac{1}{c_\Gamma^2} \frac{\mathrm{Re}\, s}{|s|^2 + 1} \left( \|\varphi\|_{X_\Gamma}^2 + \|\psi\|_{X_\Gamma}^2 \right),$$

where $c_\Gamma = \|[\gamma_T]\|_{X_\Gamma \leftarrow H(\mathrm{curl}, \mathbb{R}^3 \setminus \Gamma)}$.

# Modified Calderón operator

$$B_{\mathrm{imp}}(s) := B(s) + \begin{pmatrix} 0 & -\frac{1}{2} I \\ -\frac{1}{2} I & 0 \end{pmatrix}$$

is such that

$$B_{\mathrm{imp}}(s) \begin{pmatrix} \gamma_T \widehat{H} \\ -\gamma_T \widehat{E} \end{pmatrix} = \begin{pmatrix} \gamma_T \widehat{E} \\ 0 \end{pmatrix}$$

for solutions $(\widehat{E}, \widehat{H})$ in $\Omega^+$ of Maxwell's eqs. in the Laplace domain

On the right-hand side, we insert $\gamma_T \widehat{E}$ from the impedance boundary condition to obtain a boundary integral equation.

# Time-dependent boundary integral equation

To a given sufficiently regular function $g^{\text{inc}} : [0, T] \to V_\Gamma{}'$, find $(\varphi, \psi) : [0, T] \to V_\Gamma \times X_\Gamma$ such that for all $(\upsilon, \xi) \in V_\Gamma \times X_\Gamma$,

$$\left[ \begin{pmatrix} \upsilon \\ \xi \end{pmatrix}, B_{\text{imp}}(\partial_t) \begin{pmatrix} \varphi \\ \psi \end{pmatrix} \right]_\Gamma + \langle \upsilon, Z(\partial_t)\varphi \rangle = \langle \upsilon, g^{\text{inc}} \rangle.$$

Well-posedness. Let $r \geq 0$. For $g^{\text{inc}} \in H_0^{r+3}(0, T; V_\Gamma{}')$, there is a unique solution $(\varphi, \psi) \in H_0^{r+1}(0, T; V_\Gamma \times X_\Gamma)$, and

$$\left\| \begin{pmatrix} \varphi \\ \psi \end{pmatrix} \right\|_{H_0^{r+1}(0,T;V_\Gamma \times X_\Gamma)} \leq C_T \left\| g^{\text{inc}} \right\|_{H_0^{r+3}(0,T;V_\Gamma{}')}.$$

# Time-dependent representation formulas

$$E = -\mathcal{S}(\partial_t)\varphi + \mathcal{D}(\partial_t)\psi$$
$$H = -\mathcal{D}(\partial_t)\varphi - \mathcal{S}(\partial_t)\psi$$

It then follows that

$$(\varphi, \psi) = (\gamma_T H, -\gamma_T E).$$

# Well-posedness of the time-dep. scattering problem

Let $g^{\text{inc}} := \gamma_T E^{\text{inc}} \times \nu - Z(\partial_t)\gamma_T H^{\text{inc}} \in H_0^{r+3}(0, T; V_\Gamma{}'), \;\; r \geq 0.$

(a) The scattering problem has a unique solution

$$(E, H) \in H_0^r(0, T; H(\text{curl}, \Omega)^2) \cap H_0^{r+1}(0, T; (L^2(\Omega)^3)^2),$$

given by the representation formulas with the solution of the boundary integral equation $(\varphi, \psi) = (\gamma_T H, -\gamma_T E)$.

(b) The electromagnetic fields are bounded by

$$\|E\|_{H_0^r(0,T;H(\text{curl},\Omega))} + \|H\|_{H_0^r(0,T;H(\text{curl},\Omega))} \leq C_T \|g^{\text{inc}}\|_{H_0^{r+3}(0,T;V_\Gamma{}')},$$

and the same bound is valid for the $H_0^{r+1}(0, T; (L^2(\Omega)^3)^2)$ norms.

# Outline

# Space and time discretization

We implemented and analyzed

- Raviart–Thomas boundary elements of order $k \geq 0$ for the space discretization of the boundary integral equation
- Runge–Kutta convolution quadrature based on the Radau methods with $m \geq 2$ stages for the time discretization of the boundary integral equation and the representation formulas.

# Discretized BIE and representation formulas

Formally, the discretization of the weak formulation of the boundary integral equation reads: for all $(\upsilon_h, \xi_h) \in V_h \times X_h$,

$$\left[ \begin{pmatrix} \upsilon_h \\ \xi_h \end{pmatrix}, B_{\mathrm{imp}}(\partial_t^\tau) \begin{pmatrix} \varphi_h^\tau \\ \psi_h^\tau \end{pmatrix} \right]_\Gamma + \langle \upsilon_h, Z(\partial_t^\tau) \varphi_h^\tau \rangle = \langle \upsilon_h, g^{\mathrm{inc}} \rangle$$

with spatial meshsize $h$, time stepsize $\tau$.

From the approximate tangential traces $(\varphi_h^\tau, \psi_h^\tau)$, the electromagnetic fields at selected points $x \in \Omega$ of interest are computed by the discretized representation formulas

$$\begin{aligned} E_h^\tau &= -\mathcal{S}(\partial_t^\tau)\varphi_h^\tau + \mathcal{D}(\partial_t^\tau)\psi_h^\tau \\ H_h^\tau &= -\mathcal{D}(\partial_t^\tau)\varphi_h^\tau - \mathcal{S}(\partial_t^\tau)\psi_h^\tau. \end{aligned}$$

# Error bounds

Under sufficient regularity, for $n\tau \leq T$,

$$\left\| \begin{pmatrix} E_h^n - E(t_n) \\ H_h^n - H(t_n) \end{pmatrix} \right\|_{H(\mathrm{curl},\Omega)^2} \leq C_T \big( \tau^{m-1/2} + h^{k+1} \big).$$

Away from the boundary, there is the full order $2m-1$ in time:

$$\left\| \begin{pmatrix} E_h^n - E(t_n) \\ H_h^n - H(t_n) \end{pmatrix} \right\|_{\left(H(\mathrm{curl},\Omega_d)\cap C^1(\overline{\Omega}_d)^3\right)^2} \leq C_{d,T} \big( \tau^{2m-1} + h^{k+1} \big)$$

where $\Omega_d = \{x \in \Omega \,:\, \mathrm{dist}(x,\Gamma) > d\}$ with $d > 0$.

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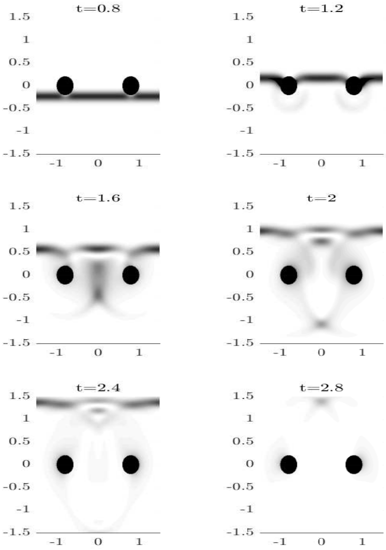
cf. Banjai, L., Melenk 2011 (acoustic scattering with Dirichlet b.c.)

# Picture once again: 3D-scattering from a torus




impedance operator $Z(\partial_t) = \delta \partial_t^{1/2}$ with $\delta = 0.1$ on the torus

# Condition numbers

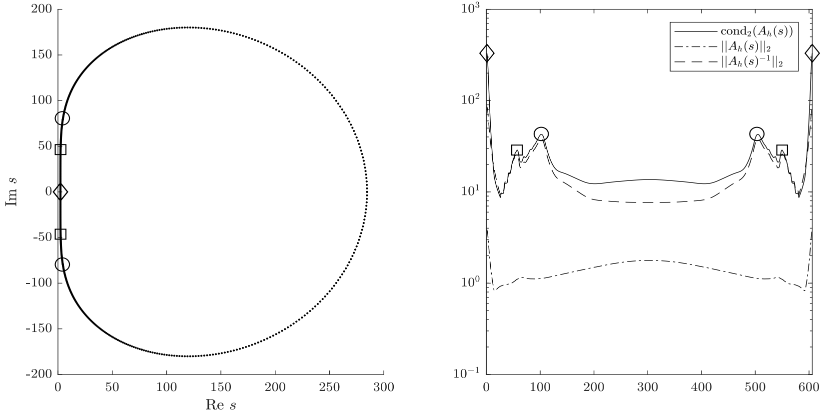



Figure: The left-hand side plot shows a plot of the occuring frequencies for the 3-stage Radau IIA method for $N = 100$ and $T = 4$. On the right-hand side, the condition numbers and the euclidean norms of the occuring matrices are shown, as they appear when following the integral contour on the left-hand side. The markers on both plots localize the corresponding spikes of the condition numbers on the integral contour.