Some results on boundary integral equation methods and their applications in numerics

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Outline

- A simple example to illustrate boundary integral equation (BIE) methods
- Regularization on the hypersingular boundary integral operators
  - Fast solutions for dynamic poroelasticity
  - Multigrid method via variable substitution for solving BIE with negative order
- Uniqueness of the reduced boundary value problem with integral operators
  - Accurate solutions for the fluid-solid transmission problem
  - *Accurate solutions for the exterior elastic wave problem*
  - A new FE-BIE method for the exterior problem

Boundary integral equation methods

- We assume that $\Omega \subset \mathbb{R}^3$ with Lipschitz boundary $\Gamma = \partial\Omega$, and consider: given a function ϕ, find u such that
$$-\triangle u = 0 \quad \text{in } \Omega$$
with the Dirichlet boundary condition $u = \phi$ on $\partial\Omega$
- (PDE→BIE) A boundary integral equation reads: find $\sigma \in H^{-1/2}(\Gamma)$, such that
$$V\sigma = (\frac{1}{2}I + K)\phi.$$
Here, V and K are the single and double layer boundary integral operators
$$V\sigma(x) = \frac{1}{4\pi}\int_\Gamma \frac{1}{|x-y|}\sigma(y)ds_y \qquad K\phi(x) = \frac{1}{4\pi}\int_\Gamma \frac{\partial}{\partial n_y}(\frac{1}{|x-y|})\phi(y)ds_y \quad x \in \Gamma$$
- (Variational equation) The weak formulation: find $\sigma \in H^{-1/2}(\Gamma)$ such that
$$a(\sigma,\psi) = \ell_\phi(\psi) \quad \text{for all } \psi \in H^{-1/2}(\Gamma),$$
where
$$a(\sigma,\psi) := \langle V\sigma, \psi\rangle \qquad \ell_\phi(\psi) = \langle(\frac{1}{2}I + K)\phi, \psi\rangle$$
- (Numerical schemes) Collocation methods, Moment methods, Galerkin methods, etc.

On BIE methods

- A good counterpart (surface or boundary methods) with respect to domain discretization methods (FDM, FEM, DG, spectral methods, etc.)
  - Analysis on uniqueness and existence, errors, convergence, singularity regularization, surface regularity etc.
- The breakthrough for the Galerkin boundary element methods for practical, three dimensional problems was achieved through (S.A. Sauter and C. Schwab, Springer 2010)
  - The development of fast algorithms to represent the non-local boundary integral operators (efficiency)- Fast multipole methods, Preconditioner etc.
  - The development of numerical methods for the approximation of integrals in order to determine the system matrix (accuracy) - Numerical quadrature, quadrature on manifold etc.

$$V\sigma(x) = \frac{1}{4\pi}\int_\Gamma \frac{1}{|x-y|}\sigma(y)ds_y \qquad K\phi(x) = \frac{1}{4\pi}\int_\Gamma \frac{\partial}{\partial n_y}(\frac{1}{|x-y|})\phi(y)ds_y \quad x \in \Gamma$$

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Understanding the hyper-singular boundary integral operator

Curse: The hypersingular boundary integral operator (BIO) is defined as

$$W\varphi(x) = -\lim_{z\to x\in\Gamma, z\notin\Gamma} n_x \cdot \nabla_z \int_\Gamma \frac{\partial E(z,y)}{\partial n_y}\varphi(y)ds_y. \tag{1}$$

For $\Gamma \in C^{2+\alpha}, \varphi \in C^{1+\alpha}, \alpha \in (0,1), x \in \Gamma$, it can be shown that

$$\begin{aligned} W\varphi(x) &= -\text{HFP}\int_\Gamma \frac{\partial^2 E(x,y)}{\partial n_x \partial n_y}\varphi(y)ds_y \\ &= \lim_{\epsilon\to 0}\left\{\int_{\Gamma_\epsilon} \frac{\partial^2 E(x,y)}{\partial n_x \partial n_y}\varphi(y)ds_y + \frac{\varphi(x)}{\pi\epsilon} - O(\epsilon^\alpha)\right\}, \end{aligned}$$

where $\Gamma_\epsilon = \{y \in \Gamma : |y - x| > \epsilon\}$.

Regularization of hyper-singular boundary integral operators

Lemma

Let $\Gamma \in C^2$ and let ϕ be a Hölder continuously differentiable function. Then the limit (1) exists uniformly with respect to all $x \in \Gamma$ and all ϕ with $\|\phi\|_{C^{1+\alpha}} \leq 1$ $(0 < \alpha < 1)$. Furthermore, the operator W can be expressed as a composition of tangential derivatives and the simple layer potential operator V:

$$W\phi(x) = -\frac{d}{ds_x} V \left(\frac{d\phi}{ds} \right) (x) \quad \text{for} \quad n=2$$

and

$$W\phi(x) = -(n_x \times \nabla_x) \cdot V(n_y \times \nabla_y \phi)(x) \quad \text{for} \quad n=3$$

Lemma 1.2.2, Applied Mathematical Sciences 164, Springer, Hsiao-Wendland 2008

Maue (1949), Günter (1953), Giroire-Nédélec (1978), Bonnemay (1979), Nédélec (1982), Han (1988,1994), Kupradze et al (1979), Schwab et al, (1992), Kohr (2006) etc.

Laplace equations, Stokes equations, Lamé equations, biharmonic equations etc.

Regularization of hyper-singular boundary integral operators (acoustic and electromagnetic wave equations)

Theorem

The hypersingular BIO W associated to the Helmholtz equation can be expressed as

$$W\phi(x) = -\frac{d}{ds_x} V\left(\frac{d\phi}{ds}\right)(x) - k^2 n_x^\top V(\phi n_y)(x).$$

in 2D, and

$$W\phi(x) = -(n_x \times \nabla_x) \cdot V(n_y \times \nabla\phi_y)(x) - k^2 n_x^\top V(\phi n_y)(x).$$

in 3D. Here, V is the single-layer BIO given by

$$V\sigma(x) \quad := \quad \int_\Gamma \gamma_k(x,y)\sigma(y)\, ds_y, \quad x \in \Gamma.$$

Theorem 3.4.2, Applied Mathematical Sciences 144, Springer, Jean-Claude Nédélec 2000

Elastic wave equations???

Hyper-singular BIO for elastic wave equations

The hypersingular BIO associated with the elastic wave equation reads

$$\begin{aligned} Wu(x) &= -T_x \int_\Gamma (T_y E(x,y))^\top u(y)\, ds_y, \quad x \in \Gamma. \\ (W\varphi(x) &= -\lim_{z\to x\in\Gamma, z\notin\Gamma} n_x \cdot \nabla_z \int_\Gamma \frac{\partial E(z,y)}{\partial n_y} \varphi(y) ds_y.) \end{aligned}$$

T is the traction operator

$$Tu := 2\mu\, \partial_\nu u + \lambda\, \nu\, \mathrm{div}\, u + \mu\nu \times \mathrm{curl}\ u.$$

$E(x,y)$ is the fundamental displacement tensor

$$E(x,y) = \frac{1}{\mu}(I + \frac{\nabla_x \nabla_x}{k_s^2})\gamma_{k_s}(x,y) - \frac{1}{\lambda + 2\mu}\frac{\nabla_x \nabla_x}{k_p^2}\gamma_{k_p}(x,y), \quad x \neq y.$$

Here, I denotes the identity matrix, and $\gamma_{k_t}(x,y)$ is the fundamental solution of the Helmholtz equation in $\mathbb{R}^d$

$$\gamma_{k_t}(x,y) = \begin{cases} \frac{i}{4} H_0^{(1)}(k_t|x-y|), & d = 2, \\ \frac{\exp(ik_t|x-y|)}{4\pi|x-y|}, & d = 3, \end{cases} \qquad x \neq y, \quad t = p, s,$$

where $H_0^{(1)}(\cdot)$ is the first kind Hankel function of order zero.

Hyper-singular BIO for elastic wave in 2D

Theorem (Yin-Hsiao-X. 2017, Bao-X.-Yin 2017)

The hyper-singular BIO W in two dimensions can be expressed as

$$\begin{aligned} Wu(x) \quad = \quad & \rho\omega^2 \int_\Gamma \left[\gamma_{k_s}(x,y)(n_x n_y^\top - n_x^\top n_y I - J_{n_x,n_y}) - \gamma_{k_p}(x,y) n_x n_y^\top\right] u(y) ds_y \\ & -4\mu^2 \int_\Gamma \frac{dE(x,y)}{ds_x}\frac{du(y)}{ds_y} ds_y + \frac{4\mu^2}{\lambda+2\mu}\int_\Gamma \frac{d\gamma_{k_p}(x,y)}{ds_x}\frac{du(y)}{ds_y} ds_y \\ & +2\mu \int_\Gamma n_x \nabla_x^\top R(x,y) A \frac{du(y)}{ds_y} ds_y + 2\mu \int_\Gamma A \frac{d}{ds_x}(\nabla_y R(x,y)) n_y^\top u(y) ds_y, \end{aligned}$$

where $R = \gamma_{k_s} - \gamma_{k_p}$ *and* $J_{n_x,n_y} = n_y n_x^\top - n_x n_y^\top$, $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

- 3D (thermo-)elastic wave (Bao-X.-Yin 2019); Porous medium elastic wave in 2D/3D (Zhang-X.-Yin 2021, Zhang-X.-Yin preprint)

Why hypersingular boundary integral operators?

Why we need such formulations of hypersingular BIO or hypersingular BIO

- Theoretical definition required
- There is rich physics in the hypersingularity (for instance: circuit physics)
- Use of BEM:
  - Computing Neumann boundary value problems with a double layer potential: Hsiao-Wendland, J. Math. Anal. Appl. 1977; Giroire-Nédélec Math. Comp. 1978; H. Han Numer. Math. 1994; Bao-X.-Yin JCP 2017, CMAME 2019
  - Solvability of the resulting boundary integral equations (Uniqueness, Lipschitz domain): Burton-Miller, Proc. Roy. Soc. London Ser. A, 1971; M. Costabel, SIAM Math. Anal. 1988; Yin-X.-Hsiao, SINUM 2017
  - Fast boundary integral equation methods (”Good” system, Preconditioning): Hsiao-X.-Zhang, JSC 2014; Zhang-X.-Yin SISC 2021, Zhang-X.-Yin Preprint
  - Coupling methods of boundary elements and finite elements (Symmetric): Johnson-Nédélec Math. Comp., 1980; M. Costabel, Boundary Element IX 1987; H. Han J. Comput. Math. 1990
  - Applications: crack/eigenvalue/optimal/inverse problems, layered/porous/thermo-effect medium, etc. Wendland-Stephan ARMA 1990, Bruno-X.-Yin, IJNME 2021, X.-Yin, Preprint,

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Dynamic poroelasticity equation

The solid displacements $u = (u_1, u_2)^\top$ and the pore pressure p satisfy (Biot-41,55,56)

$$\Delta^* u + (\rho - \beta\rho_f)\omega^2 u - (\alpha - \beta)\nabla p = 0$$
$$\Delta p + qp + i\omega\gamma\nabla\cdot u = 0$$

in Ω^c, together with the Neumann boundary condition

$$\widetilde{T}(\partial,\nu)U := \begin{bmatrix} T(\partial,\nu) & -\alpha\nu \\ -i\omega\beta\nu^\top & \frac{i\beta}{\omega\rho_f}\partial_\nu \end{bmatrix} U = F,$$

on Γ. Here, $U = (u^\top, p)^\top$, $\Omega^c = \mathbb{R}^2\backslash\overline{\Omega}$ with $\Omega \subset \mathbb{R}^2$ being a bounded domain with smooth boundary Γ, $T(\partial,\nu)$ is the traction operator defined as

$$T(\partial,\nu)u := 2\mu\partial_\nu u + \lambda\nu\nabla\cdot u + \mu\nu^\perp(\partial_2 u_1 - \partial_1 u_2), \quad \nu^\perp = (-\nu_2, \nu_1)^\top,$$

and Δ^* is the Lamé operator defined by

$$\Delta^* := \mu\Delta + (\lambda+\mu)\nabla\nabla\cdot .$$

Fundamental solutions of Biot model

Let $E(x,y)$ be the fundamental solution of the adjoint problem in $\mathbb{R}^2$ given by

$$E(x,y) = \begin{bmatrix} E_{11}(x,y) & E_{12}(x,y) \\ E_{21}^{\top}(x,y) & E_{22}(x,y) \end{bmatrix}, \quad x \neq y,$$

with

$$E_{11}(x,y) = \frac{1}{\mu}\gamma_{k_s} I + \frac{1}{(\rho - \beta\rho_f)\omega^2}\nabla_x\nabla_x^{\top}\left[\gamma_{k_s} - \frac{k_p^2 - k_2^2}{k_1^2 - k_2^2}\gamma_{k_1} + \frac{k_p^2 - k_1^2}{k_1^2 - k_2^2}\gamma_{k_2}\right],$$

$$E_{12}(x,y) = \frac{i\omega\gamma}{(\lambda + 2\mu)(k_1^2 - k_2^2)}\nabla_x[\gamma_{k_1} - \gamma_{k_2}],$$

$$E_{21}(x,y) = -\frac{\gamma}{(\lambda + 2\mu)(k_1^2 - k_2^2)}\nabla_x[\gamma_{k_1} - \gamma_{k_2}],$$

$$E_{22}(x,y) = \frac{i\rho_f\omega}{\beta(k_1^2 - k_2^2)}[(k_p^2 - k_1^2)\gamma_{k_1} - (k_p^2 - k_2^2)\gamma_{k_2}],$$

in which

$$\gamma_{k_t}(x,y) = \frac{i}{4}H_0^{(1)}(k_t|x-y|), \quad x \neq y, \quad t = s, p, 1, 2,$$

denotes the fundamental solution of the Helmholtz equation in $\mathbb{R}^2$ with wave number k_t.

Fundamental solutions of Biot model

Here, k_p and k_s, referred as the compressional and shear wave numbers, respectively, are given by

$$k_p := \omega\sqrt{\frac{\rho - \beta\rho_f}{\lambda + 2\mu}}, \quad k_s := \omega\sqrt{\frac{\rho - \beta\rho_f}{\mu}}.$$

The wave numbers k_1, k_2, satisfying $\mathrm{Im}(k_i) \geq 0, i = 1, 2$, are the roots of the characteristic system

$$k_1^2 + k_2^2 = q(1 + \epsilon) + k_p^2, \quad k_1^2 k_2^2 = q k_p^2, \quad \epsilon = \frac{i\omega\gamma(\alpha - \beta)}{q(\lambda + 2\mu)}.$$

Boundary integral representation

From the potential theory, the unknown function U can be represented as

$$U(x) = (D - i\eta S)(\varphi)(x), \qquad x \in \Omega^c, \quad Re(\eta) \neq 0,$$

where

$$S(\varphi)(x) := \int_\Gamma (E(x,y))^\top \varphi(y) ds_y,$$

$$D(\varphi)(x) := \int_\Gamma (\widetilde{T}^*(\partial_y, \nu_y) E(x,y))^\top \varphi(y) ds_y,$$

denote the single-layer and double-layer potentials, respectively. Here, $\widetilde{T}^*$ denotes the corresponding Neumann boundary operator of adjoint problem given by

$$\widetilde{T}^*(\partial, \nu) = \begin{bmatrix} T(\partial, \nu) & -i\omega\alpha\nu \\ -\beta\nu^\top & \frac{i\beta}{\omega\rho_f}\partial_\nu \end{bmatrix}.$$

Combined boundary integral equation

Operating with the traction operator on boundary integral representation, taking the limit as $x \to \Gamma$, and applying the boundary condition, we obtain the BIE on Γ

$$[i\eta(\frac{I}{2} - K') + N](\varphi(x)) = F \quad on \quad \Gamma,$$

where the strong-singular BIO K' and the hyper-singular BIO N are defined by

$$K'(\varphi(x)) = \widetilde{T}(\partial_x, \nu_x) \int_\Gamma (E^\top(x,y))^\top \varphi(y) ds_y,$$

$$N(\varphi(x)) = \widetilde{T}(\partial_x, \nu_x) \int_\Gamma (\widetilde{T}^*(\partial_y, \nu_y) E^\top(x,y))^\top \varphi(y) ds_y.$$

Operator spectrum

Theorem (Zhang-X.-Yin 2021)

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with smooth boundary Γ. Then $K'^2 - \begin{bmatrix} C^2_{\lambda,\mu} I & 0 \\ 0 & 0 \end{bmatrix}$ is compact where $C_{\lambda,\mu}$ is a constant that can be represented by Lamé parameters.

$$C_{\lambda,\mu} = \frac{\mu}{2(\lambda + 2\mu)} < \frac{1}{2}$$

In addition, the spectrum of K' consists of three sequences of eigenvalues which accumulate at 0, $C_{\lambda,\mu}$ and $-C_{\lambda,\mu}$ respectively.

Both theoretical and numerical proofs of this theorem can be given.

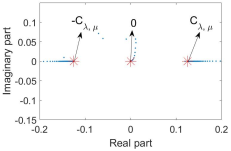

Figure: Eigenvalue distribution for the Integral operator K'.

Operator spectrum

In view of the Calderón relation

$$NS = -\frac{I}{4} + K'^2,$$

together with the inequalities $0 < C_{\lambda,\mu} < 1/2$, we conclude that the eigenvalues of the composite operator *NS* are bounded away from zero and infinity.

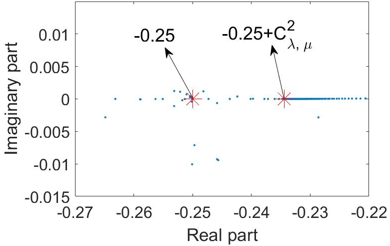

Figure: Eigenvalue distribution for the Integral operator *NS*.

Regularized integral equation

Relying on the studies of the spectra of various relevant integral operators, we propose the regularized combined field equations

$$U(x) = (DR - i\eta S)(\varphi)(x), \qquad x \in \Omega^c, \qquad Re(\sigma) \neq 0, \tag{2}$$

$$[i\eta(\frac{I}{2} - K') + NR](\varphi(x)) = F \quad on \quad \Gamma. \tag{3}$$

Note that for the elastic scattering problem, R usually takes the form $R = S$ (We could simply choose the single-layer potential corresponding to the static problem as R expressed as $R = S_0$).

Operator spectrum of regularized integral equation

Theorem (Zhang-X.-Yin 2021)

The spectrum of the regularized combined field integral operator on the left hand side of that equation consists of three non-empty sequences of eigenvalues which converge to $-1/4 + i\eta/2$, $-1/4 + C^2_{\lambda,\mu} + i\eta(1/2 + C_{\lambda,\mu})$ *and* $-1/4 + C^2_{\lambda,\mu} + i\eta(1/2 - C_{\lambda,\mu})$, *respectively.*

The spectrum of the corresponding regularized combined field operator is displayed in following Fig. Clearly the eigenvalues accumulate as prescribed by the theorem, and, in particular, they do not accumulate either at zero or infinity.

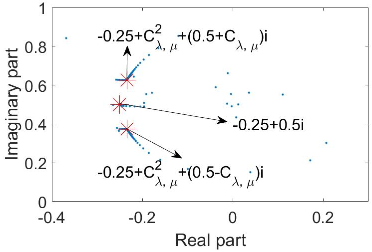

Figure: Eigenvalue distribution for the Integral operator $i\eta(\frac{I}{2} - K') + NR$.

Numerical Examples

Let $U^{\mathrm{exa}} = (u^{\mathrm{exa}}, p^{\mathrm{exa}})^\top$ be the exact solution of the poroelastic problem given by

$$u^{\mathrm{exa}}(x) = E_{21}(x,z), \qquad p^{\mathrm{exa}}(x) = E_{22}(x,z), \qquad x \in \Omega^c,$$

with $z = (0, 0.5)^\top \in \Omega$.

Table: Number of iterations and computing time (seconds) required for the problem of poroelastic scattering by a kite-shaped obstacle. GMRES tol: 10^{-5}. T_p: Precomputation time, N_{it}: Number of iterations, T_{it}^{tot}: Total computing time

ω	N	CBIE			RBIE		
		T_p	N_{it}	T_{it}^{tot}	T_p	N_{it}	T_{it}^{tot}
1	15	0.06	30	0.15	0.07	19	0.09
10	150	1.63	136	5.30	1.93	77	3.78
20	300	7.58	232	32.0	7.97	119	21.0
50	750	40.0	456	342.4	42.1	177	162.4

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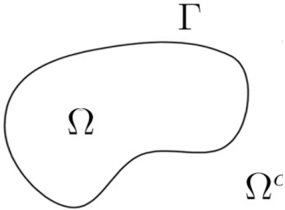
Application of Multigrid to BIE (Hsiao-X.-Zhang 2014)

- We consider the following exterior Dirichlet problem in 2D ($d = 2$) or 3D ($d = 3$):

$$\begin{aligned} -\Delta u &= 0 && \text{in } \Omega^c := \mathbf{R}^d \setminus (\Omega \cup \Gamma), \\ u &= f_0 && \text{on } \Gamma := \partial\Omega, \\ u &= O(1) \quad \text{or} \quad O(|\mathbf{x}|^{-1}) && \text{as } |\mathbf{x}| \to \infty \text{ if } d = 2 \text{ or } 3, \end{aligned}$$

  where $f_0 \in H^{1/2}(\Omega)$ is a given function. Here Ω is assumed to be a smooth domain.



Boundary integral equation

- Boundary integral equation: the boundary integral equation reads

$$V\sigma = -(\frac{1}{2}I - K)f_0 \qquad on \quad \Gamma.$$

Here, V and K denote the single-layer and double-layer potential

$$\begin{aligned} K(v)(\mathbf{x}) &= \int_\Gamma \frac{\partial}{\partial \mathbf{n_y}}\gamma(\mathbf{x},\mathbf{y})v(\mathbf{y})ds_\mathbf{y} \qquad \forall v \in H^{1/2}(\Gamma), \\ V(\sigma)(\mathbf{x}) &= \int_\Gamma \gamma(\mathbf{x},\mathbf{y})\sigma(\mathbf{y})ds_\mathbf{y} \qquad \forall \sigma \in H^{-1/2}(\Gamma), \end{aligned}$$

$\sigma = \frac{\partial u^+(\mathbf{y})}{\partial \mathbf{n_y}}$ is required to satisfy the condition $\int_\Gamma \sigma ds = 0$ in the case that $d = 2$, and

$$\gamma(\mathbf{xy}) = \begin{cases} -\log|\mathbf{x}-\mathbf{y}|/(2\pi) & \text{if } d = 2, \\ 1/(4\pi|\mathbf{x}-\mathbf{y}|) & \text{if } d = 3. \end{cases}$$

Variable substitutions

- Variational problem: Find $\sigma \in \mathcal{H}$ such that

$$\langle V\sigma, \chi \rangle = \langle f, \chi \rangle \quad \forall \chi \in \mathcal{H}, \tag{4}$$

where $f = -(\frac{1}{2}I - K)f_0$.

- Variable substitution for the unknown density function σ:

$$\sigma = W\,w \to \langle V\,W\,w, W\,v \rangle = \langle f, W\,v \rangle \quad \forall w, v \in H_0^{1/2}(\Gamma). \tag{5}$$

Here, W is a hypersingular boundary integral operator defined by

$$W\,w(\mathbf{x}) = -\frac{\partial}{\partial \mathbf{n_x}} \int_\Gamma \frac{\partial \gamma}{\partial \mathbf{n_y}}(\mathbf{x}, \mathbf{y}) w(y) ds_{\mathbf{y}}, \qquad \mathbf{x} \in \Gamma.$$

- The notation: Denote the space

$$\mathcal{H} = \begin{cases} H^{-1/2}(\Gamma) & \text{in} \quad 3D, \\ H_0^{-1/2}(\Gamma) = \{\chi \in H^{-1/2}(\Gamma) |\, \langle 1, \chi \rangle = 0\} & \text{in} \quad 2D, \end{cases}$$

and introduce the notation

$$\langle v, \chi \rangle = \int_\Gamma v\chi ds \qquad \forall v \in H^{1/2}(\Gamma),\ \forall \chi \in \mathcal{H}.$$

Equivalence

Theorem (Hsiao-X.-Zhang 2014)

For any Dirichlet data $f \in H^{1/2+\alpha}(\Gamma)$ *in (4), there is a unique solution* $\sigma \in H_0^{-1/2+\alpha}(\Gamma)$ *in (4) such that*

$$\|\sigma\|_{H^{-1/2+\alpha}(\Gamma)} \leq C \, \|f\|_{H^{1/2+\alpha}(\Gamma)} \, .$$

When we make variable substitution in (5), the original PDE solutions (4) are exactly the same as that obtained from the potential solution:

$$\langle V\, W\, w, W\, v\rangle = \langle f, W\, v\rangle \quad \forall w, v \in H_0^{1/2}.$$

That is

$$\sigma = W\, w.$$

Discrete equations

- Discretizing the equations by continuous boundary elements: Find $w_h \in H_{h_k} = \left\{ v \in H_0^{1/2}(\Gamma) \cap C^0(\Gamma) | \, v|_{E_{k,j}} \in P_1 \; \forall E_{k,j} \in \Gamma_k \right\}$, such that

$$\left\langle V\, W\, w_{h_k}, W\, v_{h_k} \right\rangle = \left\langle f, W\, v_{h_k} \right\rangle \qquad \forall v_{h_k} \in H_{h_k}. \tag{6}$$

- Represent linear systems for (6) by $A_k \overline{w}_{h_k} = \overline{g}_{h_k}$ where

$$(A_k)_{ij} = \left\langle V\, W\, \psi_j, W\, \psi_i \right\rangle,$$

$$(\overline{w}_{h_k})_j = w_j \qquad \left(\text{so that } w_{h_k} = \sum_{1 \le j \le N_k} w_j \psi_j, \right),$$

$$(g_{h_k})_j = \left\langle f, W\, \psi_j \right\rangle.$$

- The multigrid (MG) method: Given an initial w_0 approximating the solution w_{h_k} in (6), one k-th level multigrid iteration produces a new approximation w_{m+1}.

Convergence

Theorem (Hsiao-X.-Zhang 2014)

(Constant-rate convergence for MG) For any positive $\gamma < 1$, there exist constants m (large enough) and $p(> 1)$, both independent of k, such that

$$|||w_{h_k} - w_{m+1}|||_{k,1/2} \leq \gamma \,|||\, w_{h_k} - w_0 \,|||_{k,1/2} \,.$$

Here w_{h_k} and w_i are corresponding numerical solutions defined above.

Theorem (Hsiao-X.-Zhang 2014)

(Optimal order of computation) The multigrid solution $\tilde{w}_{h_k}$ ($\approx w_{h_k}$) of (6) approximates the true solution σ of original variational problem (4) at the optimal order:

$$\left\|\sigma - W\tilde{w}_{h_k}\right\|_{L^2(\Gamma)} \leq Ch^2 \left\|f_0\right\|_{H^1(\Gamma)} \,.$$

However, numerically, we founded that it has the order 3 of convergence.

Numerical results

We solve a 2D exterior Laplace equation where the exact solution is

$$u = \log \frac{(x-1)^2 + (y-1)^2}{(x-1)^2 + (y-1)^2},$$

and the Γ is the circle of radius $\sqrt{8}$ centered at the origin.

Table: The errors and the iteration numbers

k	The original method			The W Substitution		
	$\|\sigma - \sigma_{h_k}\|_{L^2}$	h_k^n	*iteration*	$\|I_k\sigma - W\, w_{h_k}\|_{L^2}$	h_k^n	*iteration*
1	0.034393554	0.0	11	0.1480089367870	0.0	1
2	0.102470276	0.0	42	0.0136124673160	3.4	1
3	0.043419451	1.2	83	0.0124487509497	0.1	1
4	0.011132593	2.0	128	0.0015444779648	3.0	2
5	0.002731438	2.0	161	0.0001748794751	3.1	3
6	0.000674790	2.0	183	0.0000212825341	3.0	4
7	0.000167626	2.0	198	0.0000026419450	3.0	5
8	0.000041769	2.0	212	0.0000003296587	3.0	8

Outline

- A simple example to illustrate boundary integral equation (BIE) methods
- Regularity theory on the hypersingular boundary integral operators
  - Fast solutions for dynamic poroelasticity
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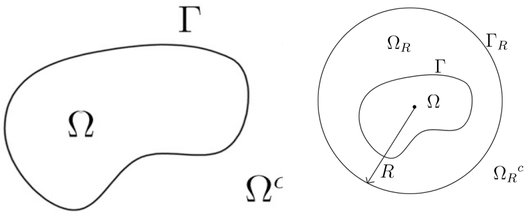
  - Accurate solutions for the fluid-solid transmission problem
  - A new FE-BIE method for the exterior problem

Original Problem

Let us consider the following exterior Dirichlet problem

$$\begin{cases} -\Delta u = f \quad \text{in} \ \Omega^c, \\ \quad u = 0 \ \text{ on } \quad \Gamma, \\ u \quad \text{is bounded} \quad \text{as} \ |\mathbf{x}| \to \infty, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^2$ with smooth boundary Γ, and Ω^C is the complement of $\Omega \cup \Gamma$, f has a compact support in Ω^C and $f \in L^2(\Omega^C)$.



Reduced boundary value problem

$$\begin{cases} -\Delta u = f & \text{in } \Omega_R, \\ u = 0 & \text{on } \Gamma, \\ V\dfrac{\partial u}{\partial n}(x) = (-\dfrac{1}{2}\mathrm{I} + \mathrm{K})u(x) & \text{on } \Gamma_R, \end{cases}$$

where

$$V\frac{\partial u}{\partial n}(x) = \int_{\Gamma_R} E(x,y)\frac{\partial u}{\partial n}(y)\mathrm{d}S_y,$$

$$\mathrm{K}u(x) = \int_{\Gamma_R} \frac{\partial E(x,y)}{\partial n_y} u(y)\mathrm{d}S_y,$$

and

$$E(x,y) = \frac{1}{2\pi}\ln|x-y|.$$

(Johnson-Nédelec,1980)

DtN-FEM

Natural boundary integral methods(Dirichlet-to-Neumann (DtN) methods, artificial boundary integral methods)

- $$\begin{cases} -\Delta u = f \ \text{ in } \ \Omega_R, \\ u = 0 \ \text{ on } \ \Gamma, \\ \dfrac{\partial u}{\partial n} = \dfrac{1}{2\pi R} \displaystyle\int_0^{2\pi} \dfrac{1}{2\sin^2\frac{\theta-\varphi}{2}} u(R,\varphi)\mathrm{d}\varphi \quad \text{on } \Gamma_R. \end{cases}$$

  (Feng, 1980; Feng-Yu, 1982; Han-Ying, 1980)

- $$\begin{cases} -\Delta u = f \ \text{ in } \ \Omega_R, \\ u = 0 \ \text{ on } \ \Gamma, \\ \dfrac{\partial u}{\partial n} = -\displaystyle\sum_{n=1}^{\infty} \dfrac{n}{\pi R} \int_0^{2\pi} u(R,\varphi)\cos(\theta-\varphi)\mathrm{d}\varphi \quad \text{on } \Gamma_R. \end{cases}$$

  (Han-wu, 1985; Yu, 1985; Givoli-Keller, 1989)

Numerical techniques

- Some works on coupling of FEM-BEM: Johnson-Nédelec,1980; Costabel, Boundary Element XI 1987; Han J. Comp. Math. 1990; Sayas SINUM 2009; SIAM Review 2013; Gatica-Hsiao-Sayas Numer. Math. 2012

- Some works on DtN-FEM: Feng, Proceeding of ICM 1983; Han-Wu Math. Comp. 1992; Grote-Keller JCP 1995; Nicholls-Nigam JCP 2004, Numer. Math. 2006; Hsiao-Nigam-Pasiak-X. JCAM 2011; Geng-Yin-X. JCAM 2017; X.-Zhang-Hsiao ANM 2019; Yin-X. Numer. Math. 2021; Wu-X.-Yin Preprint

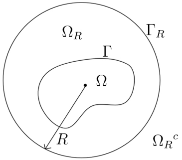
Reduced/nonlocal boundary value problem

Given p^{inc}, find $\mathbf{u} \in \left(C^2(\Omega) \cap C^1(\overline{\Omega})\right)^2$ and $p \in C^2(\Omega_R) \cap C^1(\overline{\Omega_R})$ such that

$$\begin{aligned}
\Delta^*\mathbf{u} + \rho\omega^2\mathbf{u} &= \mathbf{0} \quad \text{in} \quad \Omega, \\
\Delta p + k^2 p &= 0 \quad \text{in} \quad \Omega_R, \\
\omega^2 \rho_f \mathbf{u}\cdot\mathbf{n} &= \frac{\partial}{\partial n}(p + p^{inc}) \quad \text{on} \quad \Gamma, \\
\mathbf{t} &= -\mathbf{n}(p + p^{inc}) \quad \text{on} \quad \Gamma, \\
\frac{\partial p}{\partial n} &= Sp \quad \text{on} \quad \Gamma_R.
\end{aligned}$$

S, $H^s(\Gamma_R) \to H^{s-1}(\Gamma_R)$, is called the DtN mapping (nonlocal) defined as:

$$S\varphi := \sum_{n=0}^{\infty}{}' \frac{kH_n^{(1)'}(kR)}{\pi H_n^{(1)}(kR)} \int_0^{2\pi} \varphi(R,\phi)\cos(n(\theta-\phi))d\phi, \quad \forall \varphi \in H^s(\Gamma_R), 1/2 \le s \in \mathbb{R}.$$



Weak formulation

Given p^{inc}, find $\mathbf{U} = (\mathbf{u}, p) \in \mathcal{H}^1 = \left(H^1(\Omega)\right)^2 \times H^1(\Omega_R)$ such that

$$A(\mathbf{U}, \mathbf{V}) = a_1(\mathbf{u}, \mathbf{v}) + a_2(p, q) + a_3(\mathbf{u}, q) + a_4(p, \mathbf{v}) + b(p, q) = \ell(\mathbf{V}), \quad \forall\, \mathbf{V} = (\mathbf{v}, q) \in \mathcal{H}^1,$$

where

$$\begin{aligned}
a_1(\mathbf{u}, \mathbf{v}) &= \lambda \int_\Omega (\nabla \cdot \mathbf{u})(\nabla \cdot \bar{\mathbf{v}})dx + \frac{\mu}{2} \int_\Omega \left(\nabla \mathbf{u} + (\nabla \mathbf{u})^T\right) : \left(\nabla \bar{\mathbf{v}} + (\nabla \bar{\mathbf{v}})^T\right) dx \\
&\quad -\rho\omega^2 \int_\Omega \mathbf{u} \cdot \bar{\mathbf{v}} dx, \\
a_2(p, q) &= \int_{\Omega_R} \nabla p \cdot \nabla \bar{q} dx - k^2 \int_{\Omega_R} p \bar{q} dx, \\
a_3(\mathbf{u}, q) &= \rho_f \omega^2 \int_\Gamma \mathbf{u} \cdot \mathbf{n} \bar{q} ds, \qquad a_4(p, \mathbf{v}) = \int_\Gamma \mathbf{n} p \cdot \bar{\mathbf{v}} ds, \\
b(p, q) &= -\int_{\Gamma_R} (Sp)\bar{q} ds
\end{aligned}$$

are sesquilinear forms defined on corresponding function spaces, and ℓ, defined by

$$\ell(\mathbf{V}) = \int_\Gamma \frac{\partial p^{inc}}{\partial n} \bar{q} ds - \int_\Gamma \mathbf{n} p^{inc} \cdot \bar{\mathbf{v}} ds \,,$$

is a linear functional on $\mathcal{H}^1$.

Well-posedness

Theorem

The sesquilinear form $A(\mathbf{U},\mathbf{V})$ *satisfies*

$$Re\{A(\mathbf{V},\mathbf{V})\} \geq \alpha\|\mathbf{V}\|^2_{\mathcal{H}^1} - \beta\left(\|\mathbf{v}\|^2_{(H^{1/2+\epsilon}(\Omega))^2} + \|q\|^2_{H^{1/2+\epsilon}(\Omega_R)}\right), \quad \forall\,\mathbf{V} = (\mathbf{v}, q) \in \mathcal{H}^1 ,$$

where $\alpha > 0, \beta \geq 0$ *and* $1/2 > \epsilon > 0$ *are constants independent of* $\mathbf{V}$.

Theorem

Let the interface Γ *and the material parameter* (μ, λ, ρ) *be such that there are no traction free solutions, then the variational equation admits a unique solution* $\mathbf{U} \in \mathcal{H}^1$.

Property of DtN mapping

The truncated DtN mapping is written as, $\forall \varphi \in H^s(\Gamma_R),\ s \geq 1/2,$

$$S^N \varphi := \sum_{n=0}^{N}{}' \frac{k H_n^{(1)\,\prime}(kR)}{\pi H_n^{(1)}(kR)} \int_0^{2\pi} \varphi(R,\phi) \cos(n(\theta-\phi)) d\phi.$$

Theorem (Yin-X. 2021)

Suppose that the DtN mappings S and S^N are defined as above. Let p be a solution of Helmholtz equation outside Ω satisfying either exact DtN or the truncated DtN boundary condition. Then there exists a $N_0 > 0$ such that for all $N > N_0$,

$$\left\| (S - S^N) p \right\|_{H^{s-1}(\Gamma_R)} \leq c q^N \|p\|_{H^{s+t+1/2}(\Omega_R)}, \quad \forall\ t \geq 0,\ s \geq 1/2,$$

where $0 < q < 1$ is a constant independent of N.

Modified weak formulation

We consider the modified variational equation: Find $\mathbf{U}_N = (\mathbf{u}_N, p_N) \in \mathcal{H}^1$,

$$A^N(\mathbf{U}_N, \mathbf{V}) = a_1(\mathbf{u}_N, \mathbf{v}) + a_2(p_N, q) + a_3(\mathbf{u}_N, q) + a_4(p_N, \mathbf{v}) + b^N(p_N, q) = \ell(\mathbf{V})$$

for $\forall\, \mathbf{V} = (\mathbf{v}, q) \in \mathcal{H}^1$, where $b^N(p_N, q) = -\int_{\Gamma_R} (S^N p_N)\bar{q}ds$.

- Boundedness, Gårding's inequality
- Loss of uniqueness? Indeed for modes with $n > N$ the nonlocal boundary condition is just $\partial p/\partial n = 0$ on Γ_R

  No proof for time-harmonic acoustic and elastic wave equations
- Harari and Hughes (1992) suggested choosing $N > kR$
- Grote and Keller (1995) modified the DtN mapping as

$$\frac{\partial u}{\partial n} = (S^N - T^N)u + Tu\ ,$$

  where T is a linear operator satisfying $\mathrm{Im}\{\int_{\Gamma_R} \bar{\mu} T \mu ds\} > 0, \quad \forall \mu \neq 0$

Main results

Theorem (X.-Yin 2021)

Let the surface Γ and the material parameter (μ, λ, ρ) be such that there are no traction free solutions, then there exists a constant $N_0 = N_0 \geq 0$ such that the modified variational equation has at most one solution for $N \geq N_0$.

+ Gårding's inequality

Theorem (X.-Yin 2021)

Let the surface Γ and the material parameter (μ, λ, ρ) be such that there are no traction free solutions, then there exists a constant $N_0 \geq 0$ such that the modified variational equation admits a unique solution $(\mathbf{u}_N, p_N) \in \mathcal{H}^1$ for $N \geq N_0$.

- Not get proved in a discrete function space as done in existing literature

Why uniqueness of modified weak formulation matters?

Why we need the uniqueness for the modified weak formulation under continuous settings?

- Theoretical definition required (good operator)
- Convergence and error analysis for corresponding domain discretization methods solving the reduced boundary value problem
- Applications: eigenvalue/optimal/inverse problems, layered/porous medium, etc.

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Galerkin formulation

Given p^{inc}, find $\mathbf{U}_h = (\mathbf{u}_h, p_h) \in \mathcal{H}_h$, $\mathbf{u}_h = (u_x^h, u_y^h)$ such that

$$A^N(\mathbf{U}_h, \mathbf{V}_h) = B(\mathbf{U}_h, \mathbf{V}_h) + b^N(p_h, q_h) = \ell(\mathbf{V}_h), \quad \forall\, \mathbf{V}_h = (\mathbf{v}_h, q_h) \in \mathcal{H}_h, \mathbf{v}_h = (v_x^h, v_y^h),$$

where

$$B(\mathbf{U}_h, \mathbf{V}_h) = a_1(\mathbf{u}_h, \mathbf{v}_h) + a_2(p_h, q_h) + a_3(\mathbf{u}_h, q_h) + a_4(p_h, \mathbf{v}_h).$$

Remark: It can be shown that the discrete sesquilinear form $A^N(\mathbf{U}_h, \mathbf{V}_h)$ satisfies the BBL-condition as implication of the following:
Gårding's inequality + Uniqueness + Approximation property of $\mathcal{H}_h \Rightarrow$ BBL-condition.

Theorem

Let the surface Γ and the material parameter (μ, λ, ρ) be such that there are no traction free solutions and suppose that the finite element space $\mathcal{H}_h \subset \mathcal{H}^1$ satisfies the standard approximation property, then there exist constants $N_0 \geq 0$ and $h_0 > 0$ such that $A^N(\cdot, \cdot)$ for $0 < h \leq h_0, N \geq N_0$ satisfies the BBL condition in the form

$$\sup_{(\mathbf{0},0) \neq \mathbf{W}_h \in \mathcal{H}_h} \frac{|A^N(\mathbf{V}_h, \mathbf{W}_h)|}{\|\mathbf{W}_h)\|_{\mathcal{H}^1}} \geq \gamma \|\mathbf{V}_h\|_{\mathcal{H}^1}, \quad \forall\, \mathbf{V}_h \in \mathcal{H}_h.$$

Here $\gamma > 0$ is the inf-sup constant independent of h.

Asymptotic error estimates

Theorem

There exist constants $h_0 > 0$ and $N_0 \geq 0$ such that for any $h \in (0, h_0]$ and $N \geq N_0$

$$\begin{aligned} \|\mathbf{U} - \mathbf{U}_h\|_{\mathcal{H}^1} &\leq c \inf_{\mathbf{V}_h \in \mathcal{H}_h} \|\mathbf{U} - \mathbf{V}_h\|_{\mathcal{H}^1} \\ &+ c \sup_{0 \neq w_2 \in S'_h} \frac{|(b(p, w_2) - b^N(p, w_2)|}{\|w_2\|_{H^1(\Omega_R)}} \end{aligned}$$

where $c > 0$ is a constant independent of h and N.

Asymptotic error estimates

Theorem (X.-Yin 2021)

Suppose that $\mathbf{U} \in \mathcal{H}^t$ *for* $2 \le t \in \mathbb{R}$*. Then there exist constants* $h_0 > 0$ *and* $N_0 \ge 0$ *such that for any* $h \in (0, h_0]$ *and* $N \ge N_0$

$$\|\mathbf{U} - \mathbf{U}_h\|_{\mathcal{H}^1} \le c\left\{h^{t-1}\|\mathbf{U}\|_{\mathcal{H}^t} + q^N\|p\|_{H^t(\Omega_R)}\right\},$$

where $c > 0$ *and* $0 < q < 1$ *are constants independent of* h *and* N*.*

Theorem (X.-Yin 2021)

Suppose that $\mathbf{U} \in \mathcal{H}^t$ *for* $2 \le t \in \mathbb{R}$*. Then there exist constants* $h_0 > 0$ *and* $N_0 \ge 0$ *such that for any* $h \in (0, h_0]$ *and* $N \ge N_0$

$$\|\mathbf{U} - \mathbf{U}_h\|_{\mathcal{H}^0} \le c\left\{h^{t}\|\mathbf{U}\|_{\mathcal{H}^t} + q^N\|p\|_{H^t(\Omega_R)}\right\}$$

where $c > 0$ *and* $0 < q < 1$ *are constants independent of* h *and* N*.*

Numerical results

We consider the scattering of a plane incident wave $p^{inc} = e^{ikx\cdot d}$ with direction $d = (1, 0)$ by a disc-shaped elastic body of radius R_0

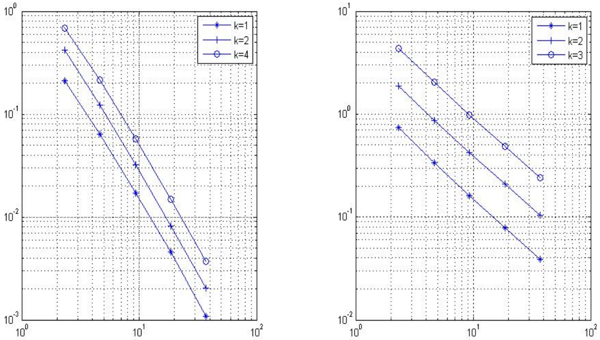

Figure: Nonlocal DtN-FEM: log-log plots for numerical errors (vertical) of **U** vs. $1/h$ (horizontal). Left: $\mathcal{H}^0$-norm; right: $\mathcal{H}^1$-norm.

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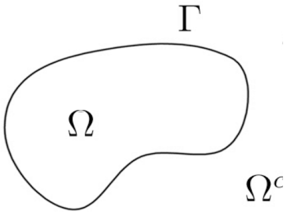
Boundary value problem

- We consider the following exterior Dirichlet problem in 2D ($d = 2$) or 3D ($d = 3$):

$$
\begin{aligned}
-\Delta u &= f && \text{in } \Omega^c := \mathbf{R}^d \setminus (\Omega \cup \Gamma), \\
u &= 0 && \text{on } \Gamma := \partial\Omega, \\
u &= O(1) \quad \text{or} \quad O(|\mathbf{x}|^{-1}) && \text{as } |\mathbf{x}| \to \infty \text{ if } d = 2 \text{ or } 3,
\end{aligned}
$$

where Ω is a bounded domain in $\mathbf{R}^d$, and f is a given function with a compact support near Ω.



Computational domain for the new coupling methods

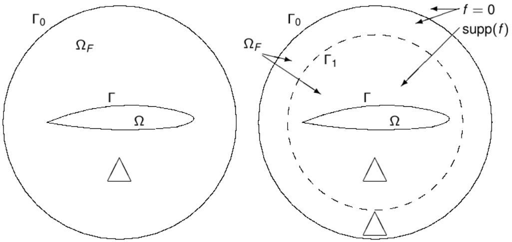


Figure: A conventional boundary integral method (left), and a nonsingular kernel integral method.

New coupling methods (X.-Zhang-Hsiao 2019)

For the original model problem, the newly proposed nonsingular kernel integral method reads: Find $u \in H^1(\Omega_F)$ such that

$$-\Delta u = f \qquad \text{in } \Omega_F := \Omega_0 \setminus (\Omega \cup \Gamma),$$

$$u = 0, \qquad \text{on } \Gamma$$

$$u(\mathbf{x}) = \frac{|\mathbf{x}|^2 - R^2}{2^{d-1}\pi R} \int_{\Gamma_1 = \{|\mathbf{y}|=R\}} \frac{u(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^d} ds_{\mathbf{y}} \quad \mathbf{x} \in \Gamma_0.$$

Here Γ_0 and Γ_1 are two artificial boundaries, circles or spheres, of radius R_0 and R.

$$\begin{cases} -\Delta u = f \quad \text{in } \ \Omega_F, \\ u = 0 \quad \text{on } \ \Gamma, \\ V\dfrac{\partial u}{\partial n}(x) = (-\dfrac{1}{2}\mathrm{I} + \mathrm{K})u(x) \quad \text{on } \Gamma_0, \end{cases}$$

where

$$V\frac{\partial u}{\partial n}(x) = \int_{\Gamma_R} E(x,y)\frac{\partial u}{\partial n}(y)\mathrm{dS}_y, \qquad \mathrm{K}u(x) = \int_{\Gamma_R} \frac{\partial E(x,y)}{\partial n_y} u(y)\mathrm{dS}_y.$$

(Johnson-Nédelec,1980)

Weak formulation

Multiplying a test function and integrating by parts, it follows that

$$a(u,v) = (f,v) \quad \forall v \in H_0^1(\Omega_F),$$
$$u(\mathbf{x}) = Bu(\mathbf{x}) \quad \mathbf{x} \in \Gamma_0,$$

where $H_0^1(\Omega_F) := \{v \in H^1(\Omega_F) \mid v|_{\partial\Omega_F} = 0\}$, and the linear operator $B : H^1(\Omega_F) \to H^{1/2}(\Gamma_0)$ is defined by

$$Bu(\mathbf{x}) := \frac{|\mathbf{x}|^2 - R^2}{2^{d-1}\pi R} \int_{\Gamma_1=\{|\mathbf{y}|=R\}} \frac{u(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^d} ds_{\mathbf{y}}, \quad \forall \mathbf{x} \in \Gamma_0.$$

Here we have standard notations:

$$a(u,v) = \int_{\Omega_F} \nabla u \cdot \nabla v d\mathbf{x},$$
$$(f,v) = \int_{\Omega_F} fv d\mathbf{x}.$$

Galerkin equation

The finite element spaces on $\mathcal{T}_h$ are the P_1 conforming finite element:

$$\begin{aligned} V_{h,0} &= \{v_h \in H_0^1(\Omega_{F,h}) \mid v_h|_K \in P_1 \ \forall K \in \mathcal{T}_h\}, \\ V_h &= \{v_h \in H^1(\Omega_{F,h}) \mid v_h|_\Gamma = 0, \ v_h|_K \in P_1 \ \forall K \mathcal{T}_h\}. \end{aligned}$$

The finite element approximation reads: Find $u_h \in V_h$ such that

$$\begin{aligned} a_h(u_h, v_h) &= (f, v_h)_h \quad \forall v_h \in V_{h,0}, \\ u_h(\mathbf{x}_i) &= B_h u_h(\mathbf{x}_i) \quad \forall \mathbf{x}_i \in \Gamma_{0,h}, \end{aligned}$$

where the bilinear forms are defined by

$$a_h(u_h, v_h) = \int_{\Omega_{F,h}} \nabla u_h \cdot \nabla v_h d\mathbf{x},$$

$$(f, v_h)_h = \int_{\Omega_{F,h}} f v_h d\mathbf{x},$$

and the discrete operator is defined by

$$B_h u(\mathbf{x}_i) = \frac{|\mathbf{x}_i|^2 - R^2}{2^{d-1}\pi R} \int_{\Gamma_{1,h}} \frac{u(\mathbf{y})}{|\mathbf{x}_i - \mathbf{y}|^d} ds_{\mathbf{y}}, \quad \forall \mathbf{x}_i \in \Gamma_{0,h}.$$

Numerical algorithm

In practical computation, we would solve the system of linear equations iteratively while the domain equation can be solved effectively by the multigrid method. The algorithm is defined as follows: Start with an iterate $u_{h,0} = 0$. For $j = 1, 2, \ldots, m$,

1. Let $\hat{u}_{h,j} \in V_h$ be defined by

$$\hat{u}_{h,j}(\mathbf{x}_i) = \begin{cases} (B_h u_{h,j-1})(\mathbf{x}_i) & \text{if } \mathbf{x}_i \in \Gamma_{0,h} \text{ (FE nodes on } \Gamma_0\text{)}, \\ 0 & \text{if } \mathbf{x}_i \in (\Gamma \cup \Omega_F) \text{ (rest FE nodes)}, \end{cases}$$

   where the operator B_h is defined above;
2. Let $u_{h,j} \in V_{h,0}$ solve the equations

$$a_h(u_{h,j}, v_h) = (f, v_h)_h - a_h(\hat{u}_{h,j}, v_h) \quad \forall v_h \in V_{h,0}.$$

3. End the iteration, when $\|u_{h,m} - u_{h,m-1}\|_{H^1} \leq \epsilon$, by defining

$$u_h = u_{h,m} + \hat{u}_{h,m}.$$

Uniqueness and convergence (X.-Zhang-Hsiao 2019)

Theorem

The nonsingular kernel coupling problem has a unique solution which solves the exterior boundary value problem. The system of coupled finite element equations has a unique solution, when the grid size h is sufficiently small.

Theorem

The finite element solution u_h of approximates that u of continuous problem at the optimal order:

$$|u - u_h|_{H^1(\Omega_{F,h})} \leq Ch(\|u\|_{H^2(\Omega_{F,h})} + \|u\|_{W^2_\infty(\Omega_{F,h})}).$$

Theorem

The iterative solutions of converge at the following rate

$$\|u_h - (u_{h,j} + \hat{u}_{h,j})\|_{H^1(\Omega_{F,h})} \leq \delta^j |u_h|_{H^1(\Omega_{F,h})},$$

where $\delta = (1 + Ch^2)R^2/R_0^0 < 1$, R and R_0 are the radii of spheres Γ_1 and Γ_0, respectively.

Numerical tests

We solve a 2D exterior Laplace equation where the exact solution is

$$u(x, y) = -\frac{x}{x^2 + y^2},$$

and the domain is $\Omega^c = \{|\mathbf{x}| > \sqrt{2}\}$.

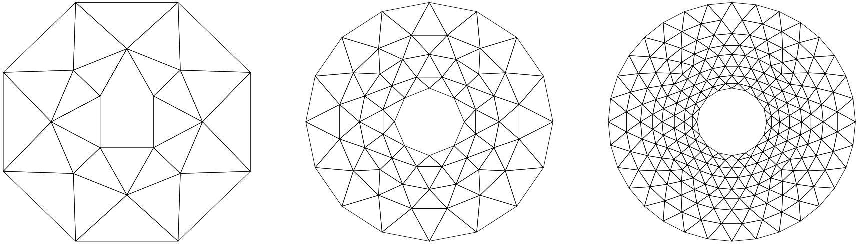


Figure: Level 1, 2 and 3 grids for 2d problem.

Numerical results

The artificial outer boundary Γ_0, and the internal representation boundary Γ_1 are chosen as,

$$\begin{aligned} \Gamma_0 &= \{|\mathbf{x}| = 5\}, \\ \Gamma_1 &= \{|\mathbf{x}| = 2\}. \end{aligned}$$

Table: The error and the order of convergence for 2D problem.

level	$\|u - u_h\|_{L^2}$	h^n	$|u - u_h|_{H^1}$	h^n	# it
1	0.425266	0.00	1.127571	0.00	12
2	0.101300	2.07	0.523291	1.11	15
3	0.025666	1.98	0.257112	1.03	15
4	0.006483	1.99	0.128261	1.00	15
5	0.001631	1.99	0.064117	1.00	15
6	0.000409	2.00	0.032058	1.00	15
7	0.000102	2.00	0.016029	1.00	15
8	0.000026	2.00	0.008014	1.00	15
9	0.000006	2.00	0.004007	1.00	15

Conclusions

- Make a complement to the regularization theory of hypersingular boundary integral operators and apply it to numerics
- Give a proof on the uniqueness on a reduced boundary value problem with the Fourier series DtN integral operator and apply it to numerical error analysis
- Present a new variable substitution technique and a new coupling technique with potentials to apply preconditioning associated to domain discretization methods (multigrid methods)
- Future attention: Investigate accurate/fast schemes for the boundary integral equation methods and their coupling with DG/FE methods for time dependent or harmonic scattering wave problems

Thanks for your attention