

Grammatical Calculus and q -Derivative Grammar

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- 在计算机科学中, 上下文无关文法 (Context-free grammar) 是刻画一类形式语言的文法体系, 对应 N. Chomsky 层级中的 2 型文法;
- 1960 年, N. Chomsky 与 M.P. Shützenberger 合作, 将上下文无关语言 (CFL) 代数化, 建立上下文无关语言的代数表示与枚举定理, 奠定代数语言学基石;
- 1993 年, 陈永川 (W.Y.C. Chen) 首次基于上下文无关文法引入形式微分算子, 建立了文法演算, 用于研究组合对象的计数问题



W.Y.C. Chen, Context-free grammars, differential operators and formal power series, Theoret. Comput. Sci. 117 (1993) 113–129.

- In 1996, Dominique Dumont coined the term “**Grammaires de William Chen**” (William Chen Grammars) to refer to these combinatorial tools. He further developed such grammars for describing several classes of classical combinatorial objects.



D. Dumont, **Grammaires de William Chen** et dérivations dans les arbres et arborescences, Sém. Lothar. Combin. 37 (1996) B37a, 21 pp.



D. Dumont and A. Ramamonjisoa, Grammaire de Ramanujan et Arbres de Cayley, Electron. J. Combin. 3 (1996) R17.

- In 2017, W.Y.C. Chen and Amy M. Fu introduced [the grammatical labeling and grammatical calculus](#), which enabled the derivation of diverse generating functions and combinatorial identities—without relying on complex recurrence relations or differential equations.



W.Y.C. Chen and A.M. Fu, Context-free grammars for permutations and increasing trees, *Adv. in Appl. Math.*, 82 (2017) 58–82.

- In 2019, Shi-Mei Ma, Jun Ma and Yeong-Nan Yeh proposed [the grammar transform](#), which was proven to be useful in the study of γ -positivity of combinatorial polynomials.



Shi-Mei Ma, Jun Ma and Yeong-Nan Yeh, γ -positivity and partial γ -positivity of descent-type polynomials, *J. Combin. Theory Ser. A* 167 (2019) 257–293.

What's the context-free grammar?

- A context-free grammar G over a set $V = \{x, y, z, \dots\}$ of variables is a set of substitution rules replacing a variable in V by a Laurent polynomial of variables in V .

$$G_1 = \{f \rightarrow f\}, G_2 = \{x \rightarrow xy, y \rightarrow xy\}.$$

- For a context-free grammar G over V , the formal derivative D with respect to G is defined as a linear operator acting on Laurent polynomials with variables in V **such that each substitution rule is treated as the common differential rule.**

$$D_{G_1}(f) = f, \quad D_{G_2}(x) = xy = D_{G_2}(y)$$

- The formal derivative D satisfies the following relations:

$$D(u + v) = D(u) + D(v),$$

$$D(uv) = D(u)v + uD(v).$$

What's the context-free grammar?

Hence, it obeys the Leibniz's rule

$$D^n(uv) = \sum_{k=0}^n \binom{n}{k} D^k(u) D^{n-k}(v).$$

For a constant c , we have $D(c) = 0$.

A toy example

- Consider the grammar $G = \{f \rightarrow f\}$. Let D be the formal derivative with respect to G . We see that $D(f) = f$.
- What is $D(f^{-1}) = ?$
- Since $ff^{-1} = 1$, the derivative property demands that

$$D(f)f^{-1} + fD(f^{-1}) = 0,$$

which yields that $D(f^{-1}) = -f^{-1}$.

- In view of the chain rule, we get

$$D^n(f^{-1}) = (-1)^n f^{-1}$$

A toy example

A benchmarking example is the following inverse pair:

$$a_n = \sum_{k=0}^n \binom{n}{k} b_k \quad \Longleftrightarrow \quad b_n = \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} a_k$$

- Define the grammar as

$$G = \{f \rightarrow f, b_i \rightarrow b_{i+1}, \quad i = 0, 1, 2, \dots\}.$$

- Let D denote the formal derivative with respect to G . We see that

$$D(f) = f, \quad D(b_0) = b_1, \quad D^k(b_0) = b_k.$$

- By the Leibniz formula, we get

$$D^n(fb_0) = \sum_{k=0}^n \binom{n}{k} D^{n-k}(f) D^k(b_0) = \sum_{k=0}^n \binom{n}{k} fb_k.$$

A toy example

A benchmarking example is the following inverse pair:

$$a_n = \sum_{k=0}^n \binom{n}{k} b_k \quad \Longleftrightarrow \quad b_n = \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} a_k$$

- Suppose that the relation on the left holds, that is,

$$a_n = f^{-1} D^n (f b_0)$$

Recall that

$$D^n(f^{-1}) = (-1)^n f^{-1}$$

- By definition, we see that

$$\begin{aligned} b_n = D^n(b_0) &= D^n(f^{-1} f b_0) = \sum_{k=0}^n \binom{n}{k} D^{n-k}(f^{-1}) D^k(f b_0) \\ &= \sum_{k=0}^n \binom{n}{k} (-1)^{n-k} a_k. \end{aligned}$$

Descents and ascents

- Let $\mathfrak{S}_n$ be the set of all permutations of $[n] = \{1, 2, \dots, n\}$.
- Let $\sigma = \sigma_1 \sigma_2 \cdots \sigma_n \in \mathfrak{S}_n$, we say that i is a **descent** of σ if $1 \leq i < n$ and $\sigma_i > \sigma_{i+1}$ and i is an **ascent** of σ if $1 \leq i < n$ and $\sigma_i < \sigma_{i+1}$.

$$\sigma = 2 \uparrow 7 \downarrow 1 \uparrow 8 \downarrow 3 \uparrow 6 \downarrow 5 \downarrow 4 \in \mathfrak{S}_8.$$

- Let $\text{des}(\sigma)$ denote the number of descents of σ and $\text{asc}(\sigma)$ denote the number of ascents of σ . We see that $\text{des}(\sigma) = 4$ and $\text{asc}(\sigma) = 3$.
- Let $\sigma \in \mathfrak{S}_n$, it is easy to see that

$$\text{des}(\sigma) + \text{asc}(\sigma) = n - 1.$$

Classical Eulerian polynomials

The Eulerian polynomials $A_n(y)$ are defined by $A_0(y) = 1$ and for $n \geq 1$,

$$A_n(y) = \sum_{\sigma \in \mathfrak{S}_n} y^{\text{des}(\sigma)+1}.$$

The first few values of $A_n(y)$ are given below.

$$A_1(y) = y$$

$$A_2(y) = y + y^2$$

$$A_3(y) = y + 4y^2 + y^3$$

$$A_4(y) = y + 11y^2 + 11y^3 + y^4$$

The generating function of $A_n(y)$

$$\sum_{n=0}^{\infty} A_n(y) \frac{t^n}{n!} = \frac{y-1}{1-y^{-1}e^{(y-1)t}}.$$

Grammatical proof

Let's consider the grammar

$$G = \{x \rightarrow xy, y \rightarrow xy\}.$$

let D_G be the formal derivative with respect to the above grammar G , that is,

$$D(x) = xy, \quad D(y) = xy.$$

Thus, we have

$$\begin{aligned} D^2(x) &= D(xy) = D(x)y + xD(y) = xy^2 + x^2y \\ D(xy^2) &= D(x)y^2 + 2xyD(y) = xy^3 + 2x^2y^2 \\ D(x^2y) &= 2xyD(x) + x^2D(y) = 2x^2y^2 + x^3y \end{aligned}$$

Hence we get

$$D^3(x) = xy^3 + 4x^2y^2 + x^3y.$$

In general, Dumont showed that under the grammar

$$G = \{x \rightarrow xy, y \rightarrow xy\},$$

we have

$$D^n(x) = A_n(x, y) = \sum_{\sigma \in \mathfrak{S}_n} y^{\text{des}(\sigma)+1} x^{\text{asc}(\sigma)+1}.$$

$$n = 1, \quad 0 \text{ } x \text{ } 1 \text{ } y \text{ } 0$$

$$n = 2, \quad 0 \text{ } x \text{ } 1 \text{ } x \text{ } \boxed{2} \text{ } y \text{ } 0, \quad 0 \text{ } x \text{ } \boxed{2} \text{ } y \text{ } 1 \text{ } y \text{ } 0$$

$$n = 3, \quad 0 \text{ } x \text{ } \boxed{3} \text{ } y \text{ } 1 \text{ } x \text{ } 2 \text{ } y \text{ } 0, \quad 0 \text{ } x \text{ } 1 \text{ } x \text{ } \boxed{3} \text{ } y \text{ } 2 \text{ } y \text{ } 0, \quad 0 \text{ } x \text{ } 1 \text{ } x \text{ } 2 \text{ } x \text{ } \boxed{3} \text{ } y \text{ } 0, \\ 0 \text{ } x \text{ } \boxed{3} \text{ } y \text{ } 2 \text{ } y \text{ } 1 \text{ } y \text{ } 0, \quad 0 \text{ } x \text{ } 2 \text{ } x \text{ } \boxed{3} \text{ } y \text{ } 1 \text{ } y \text{ } 0, \quad 0 \text{ } x \text{ } 2 \text{ } y \text{ } 1 \text{ } x \text{ } \boxed{3} \text{ } y \text{ } 0$$

$$A_1(x, y) = D(x) = xy$$

$$A_2(x, y) = D^2(x) = xy^2 + x^2y$$

$$A_3(x, y) = D^3(x) = xy^3 + 4x^2y^2 + x^3y$$

Observe that

$$D(u + v) = D(u) + D(v) \quad D(uv) = D(u)v + uD(v).$$

Hence, it obeys the Leibniz's rule

$$D^n(uv) = \sum_{k=0}^n \binom{n}{k} D^k(u) D^{n-k}(v).$$

For a constant c , we have $D(c) = 0$.

Let

$$\text{Gen}(w, t) = \sum_{n \geq 0} D^n(w) \frac{t^n}{n!}.$$

Then, by the above two relations, we derive that

$$\text{Gen}(u + v, t) = \text{Gen}(u, t) + \text{Gen}(v, t).$$

$$\text{Gen}(uv, t) = \text{Gen}(u, t) \text{Gen}(v, t).$$

Theorem (Chen-Fu)

Let D_G be the formal derivative with respect to the grammar G :

$$G = \{x \rightarrow xy, y \rightarrow xy\}.$$

Then

$$\text{Gen}^{(G)}(x, t) := \sum_{n \geq 0} D_G^n(x) \frac{t^n}{n!} = \frac{x - y}{1 - x^{-1}ye^{(x-y)t}}.$$

Proof. Since $D(x) = xy$ and $D(y) = xy$, we see that $D(x - y) = 0$

$$D(x^{-1}) = -x^{-2}D(x) = -x^{-1}y,$$

$$D(\underline{x^{-1}y}) = -x^{-1}y^2 + y = (x - y)\underline{x^{-1}y}.$$

Hence,

$$D^n(\underline{x^{-1}y}) = (x - y)^n \underline{x^{-1}y}.$$

$$D^n(\underline{x^{-1}y}) = (x-y)^n x^{-1}y$$

$$\begin{aligned} \text{Gen}(x^{-1}, t) &= \sum_{n \geq 0} D^n(x^{-1}) \frac{t^n}{n!} \\ &= x^{-1} - \sum_{n \geq 1} D^{n-1}(x^{-1}y) \frac{t^n}{n!} \\ &= \frac{1 - x^{-1}ye^{(x-y)t}}{x-y} \end{aligned}$$

By $\text{Gen}(x, t)\text{Gen}(x^{-1}, t) = 1$, we get

$$\text{Gen}(x, t) = \frac{x-y}{1 - x^{-1}ye^{(x-y)t}}.$$



W.Y.C. Chen and A.M. Fu, A Context-free grammar for the e-positivity of the trivariate second-order Eulerian polynomials, Discrete Math. 345(1) (2022) 112661.

q -Binomial inverse formula:

$$(i) \ a_n = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q b_k \iff (ii) \ b_n = \sum_{k=0}^n (-1)^{n-k} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\binom{n-k}{2}} a_k.$$

The q -binomial coefficients (Gaussian coefficients) is defined by:

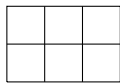
$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \prod_{j=1}^k \frac{1 - q^{n-k+j}}{1 - q^j}.$$

Note that

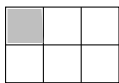
$$\lim_{q \rightarrow 1} \begin{bmatrix} n \\ k \end{bmatrix}_q = \binom{n}{k}.$$

Partition-based combinatorial interpretation

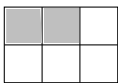
$$\begin{bmatrix} 5 \\ 2 \end{bmatrix} = 1 + q + 2q^2 + 2q^3 + 2q^4 + q^5 + q^6.$$



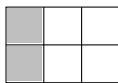
0



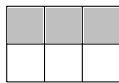
1



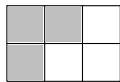
2



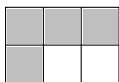
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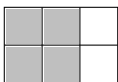
3



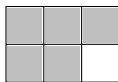
3



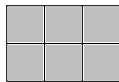
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4



5

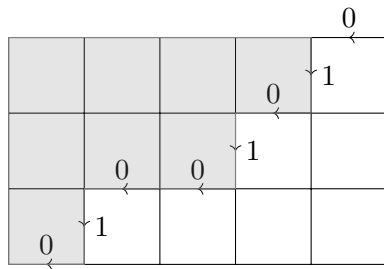


6

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \sum_{m=0}^{k(n-k)} p_m(n-k, k) q^m,$$

where $p_m(n, k)$ counts the number of partitions of m into at most k parts, with largest part $\leq n - k$.

(0, 1)-Sequences-based combinatorial interpretation



$$w = 01010010 \quad (\text{inv}(w) = 8)$$

图 0.1: The lattice path associated with the partition 431

$$\left[\begin{matrix} n \\ k \end{matrix} \right]_q = \sum_{w \in \mathcal{M}(k, n-k)} q^{\text{inv}(w)} = \sum_{w \in \mathcal{M}(k, n-k)} q^{\text{maj}(w)}.$$

Free group and group algebra

- Let S be a (finite or infinite) set of symbols, called master variables.
- To each master variable $s \in S$, we associate an infinite family of indexed non-commutative variables $s_0, s_1, s_2, \dots$
- The total set of variables is denoted by

$$\mathbb{S} := S^{\mathbb{N}} := \{s_i : s \in S, i \in \mathbb{N}\}.$$

- The free group on $\mathbb{S}$, denoted $F(\mathbb{S})$, consists of all reduced words from $\mathbb{S} \cup \mathbb{S}^{-1}$.
- Let q be an indeterminate and let $\mathbb{K}$ be a commutative ring with unity and characteristic zero. The group algebra $\mathbb{K}[q][F(\mathbb{S})]$ on $F(\mathbb{S})$ is denoted by $\mathbb{E}$. Namely, for $E \in \mathbb{E}$,

$$E = \sum_{w \in F(\mathbb{S})} a_w w,$$

where each coefficient a_w lies in $\mathbb{K}[q]$.

Definition. A formal q -grammar G over a commutative ring $\mathbb{K}$ with unity and characteristic zero is a triple

$$G = (S, R, \rho),$$

- S denotes a finite or infinite set of *master variables*;
- A *rule* $R: \mathbb{S} \cup \mathbb{S}^{-1} \rightarrow \mathbb{E}$ is a map that assigns to each variable in $\mathbb{S} \cup \mathbb{S}^{-1}$ an expression in $\mathbb{E}$;
- ρ is an order.

$$R := \{s_0 \mapsto R(s_0), s_1 \mapsto R(s_1), \dots\}.$$

The definition is then extended to the set of formal inverses s_i^{-1} by setting:

$$R(s_i^{-1}) := -s_i^{-1} R(s_i) s_{i+1}^{-1}. \quad (\clubsuit)$$

Definition. Let $G = (S, R, \rho)$ be a q -grammar. The q -derivative operator $D: \mathbb{E} \rightarrow \mathbb{E}$ associated with G is the linear operator defined by:

$$D(w_1 w_2 \cdots w_n) := \sum_{j=1}^n \rho \left(w_1 w_2 \cdots w_{j-1} R(w_j) \uparrow (w_{j+1} \cdots w_n) \right),$$

for each word $w_1 w_2 \cdots w_n$, where each $w_j \in \mathbb{S} \cup \mathbb{S}^{-1}$.



Guo-Niu Han, Kathy Q. Ji, Huan Xiong, q -Derivative grammar, submitted. arXiv:2604.23959. (Section 2: q -derivative)

Example.

$$G_{\text{tan}} = (S = \{x\}, R = \{x_j \rightarrow q^j(1 + x_j x_{j+1})\}, \rho = \text{DIO}).$$

$$D(x_0) = \text{DIO}(1 + x_0 x_1) = 1 + x_1 x_0,$$

$$\begin{aligned} D^2(x_0) &= \text{DIO}(R(x_1) \underline{x_1} + x_1 R(x_0)) \\ &= \text{DIO}(q(1 + x_1 x_2)x_1 + x_1(1 + x_0 x_1)) \\ &= (1 + q)x_1 + x_1 x_1 x_0 + qx_2 x_1 x_1. \end{aligned}$$

$$D(x_0) = \text{LPO}(1 + x_0 x_1) = 1 + x_0 x_1,$$

$$\begin{aligned} D^2(x_0) &= \text{LPO}(R(x_0) \underline{x_2} + x_0 R(x_1)) \\ &= \text{LPO}((1 + x_0 x_1)x_2 + x_0 q(1 + x_1 x_2)) \\ &= qx_0 + x_2 + (1 + q)x_0 x_1 x_2. \end{aligned}$$

Proposition. Let $c \in \mathbb{K}[q]$ and $f \in \mathbb{E}$,

$$D(c) = 0,$$

$$D(cf) = cD(f),$$

$$D(f+g) = D(f) + D(g).$$

Proposition. Let $f, g \in \mathbb{E}$,

$$D(f^{-1}) = -f^{-1}D(f) \uparrow (f^{-1}),$$

$$D(fg) = D(f) \uparrow g + fD(g).$$

$$D(f_1 f_2 \cdots f_n) = \sum_{j=1}^n f_1 f_2 \cdots f_{j-1} D(f_j) \uparrow (f_{j+1} \cdots f_n).$$

Definition (q -Linear grammar). A q -grammar $G = (S, R, \rho)$ is called q -linear if $\rho = \mathbf{KSO}$, and for each master variable $s \in \mathbb{S}$ and any $i \geq 0$, we have

$$R(s_{i+1}) = R(\uparrow s_i) = q \uparrow R(s_i). \quad (\spadesuit)$$

$$G_{\text{tan}} = (S = \{x\}, R = \{x_j \rightarrow q^j(1 + x_j x_{j+1})\}, \rho = \mathbf{DIO}),$$

Observe that

$$R(x_i) = q^i(1 + x_i x_{i+1})$$

and

$$R(x_{i+1}) = q^{i+1}(1 + x_{i+1} x_{i+2}).$$

It can be checked that

$$R(x_{i+1}) = R(\uparrow x_i) = q \uparrow R(x_i).$$

Proposition (q -Leibniz formula). Let D be q -derivative associated with q -linear grammar. For $n \geq 1$ and $f, g \in \mathbb{E}$, we have

$$D^n(fg) = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q D^k(f) \uparrow^k \left(D^{(n-k)}(g) \right).$$

The proof is based on the following key observation:

$$D^n(\uparrow^m f) = q^{nm} \uparrow^m (D^n(f)).$$

q -Binomial inverse formula:

$$(i) \ a_n = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q b_k \iff (ii) \ b_n = \sum_{k=0}^n (-1)^{n-k} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\binom{n-k}{2}} a_k.$$

- Define the following q -linear grammar:

$$G_{\text{binv}} = (\{x, y\}, \{x_j \mapsto q^j x_{j+1}, y_j \mapsto q^j y_{j+1}\}, \text{KSO}).$$

- Let D be the q -derivative associated with G_{binv} . Then for $n \geq 1$,

$$D^n(x_0) = q^{\binom{n}{2}} x_n, \quad D^n(y_0) = y_0.$$

- Observe that

$$D(y_i^{-1}) = -y_i^{-1} D(y_i) \uparrow (y_i^{-1}) = -q^i y_i^{-1} y_i y_{i+1}^{-1} = -q^i y_{i+1}^{-1}.$$

So, by induction, we obtain that for $n \geq 0$,

$$D^n(y_0^{-1}) = (-1)^n q^{\binom{n}{2}} y_n^{-1}.$$

q -Binomial inverse formula:

$$(i) \ a_n = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q b_k \iff (ii) \ b_n = \sum_{k=0}^n (-1)^{n-k} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\binom{n-k}{2}} a_k.$$

Note that G_{binv} is q -linear grammar, so

$$D^n(x_0 y_0) = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q D^k(x_0) \uparrow^k (D^{n-k}(y_0)) = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\binom{k}{2}} x_k y_k.$$

Denote b_i by $q^{\binom{i}{2}} x_i y_i$. If (i) is valid, then

$$D^n(x_0 y_0) = a_n.$$

On the other hand,

$$\begin{aligned}
 b_n &= q^{\binom{n}{2}} x_n y_n = y_n D^n(x_0) \\
 &= y_n D^n(x_0 y_0 y_0^{-1}) \\
 &= y_n \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q D^k(x_0 y_0) \uparrow^k \left(D^{n-k}(y_0^{-1}) \right) \\
 &= y_n \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q a_k (-1)^{n-k} q^{\binom{n-k}{2}} y_n^{-1} \\
 &= \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix}_q (-1)^{n-k} q^{\binom{n-k}{2}} a_k.
 \end{aligned}$$

The converse can be proved similarly.

Research problems on q -grammars

Let $G = (S, R, \rho)$ be a q -grammar and let f be an expression in $\mathbb{E}$. The high order q -derivative associated with G on f can be written in the following form:

$$D^k(f) = \sum_{w \in F(\mathbb{S})} a_w w. \quad (a)$$

Some problems for investigation:

- **Terms:** Characterize the words $w \in F(\mathbb{S})$ appearing in (a) for which $a_w \neq 0$. These words are called *terms* of the q -derivative.
- **Number of terms:** Determine the cardinality of the set of terms.
- **Coefficients:** Obtain explicit formulas for the coefficients a_w .
- **Combinatorial model:** Provide a combinatorial interpretation for the coefficients a_w .

Definition (Evaluation on q -grammars). Let $G = (S, R, \rho)$ be a q -grammar, and let $\mathbb{V}$ be a commutative ring. Recall that $\mathbb{S}$ is the set of all non-commutative variables and $\mathbb{E}$ is the free $\mathbb{K}[q]$ -algebra generated by $\mathbb{S}$.

- An *evaluation* ϕ on a q -grammar is a map $\phi: \mathbb{S} \rightarrow \mathbb{V}$ that sends each non-commutative variable of $\mathbb{S}$ to a commutative element of $\mathbb{V}$.
- The evaluation map ϕ can be uniquely extended to a morphism $\phi: \mathbb{E} \rightarrow \mathbb{V}$ by preserving linearity, multiplicativity and invertibility (i.e., $\phi(x_0^{-1}) := \phi(x_0)^{-1}$).

Research problems on q -grammars (continued)

Let $G = (S, R, \rho)$ be a q -grammar and let f be an expression in $\mathbb{E}$. The high order q -derivative associated with G on f can be written in the following form:

$$D^k(f) = \sum_{w \in F(\mathbb{S})} a_w w. \quad (\text{a})$$

Some problems for investigation:

- Determine **an appropriate evaluation map ϕ** such that $\phi(D^k(f))$ admits a simple closed form;
- Develop a general method, which we refer to as **q -grammar calculus**, to compute the q -exponential generating functions for $\phi(D^k(f))$.

q -Exponential generating function

Let D be the q -derivative associated with a q -grammar G . For $f \in \mathbb{E}$, the q -exponential generating function of $D^n(f)$ is defined by

$$\text{Gen}_q^{(G)}(f; u) = \sum_{n \geq 0} D^n(f) \frac{u^n}{(q; q)_n}.$$

Proposition. For $f, g \in \mathbb{E}$,

$$\text{Gen}_q^{(G)}(f + g; u) = \text{Gen}_q^{(G)}(f; u) + \text{Gen}_q^{(G)}(g; u),$$

$$D_q \text{Gen}_q^{(G)}(f; u) = \text{Gen}_q^{(G)}(D(f); u).$$

A real q -derivative of a function $f(u)$ is defined by

$$D_q f(u) := \frac{f(u) - f(qu)}{u}.$$

Theorem (Product formula). Let $G = (S, R, \rho)$ be a q -grammar, where for each variable $s_i \in \mathbb{S}$ and any $i \geq 0$, we have

$$R(s_{i+1}) = R(\uparrow s_i) = q \uparrow R(s_i). \quad (\spadesuit)$$

Let ϕ be a master-linear evaluation, that is, for any two variables $s_i, s_j \in \mathbb{S}$,

$$\phi(s_i) = \phi(s_j).$$

We have

$$\phi\left(\text{Gen}_q^{(G)}(fg; u)\right) = \phi\left(\text{Gen}_q^{(G)}(f; u)\right) \cdot \phi\left(\text{Gen}_q^{(G)}(g; u)\right).$$

下面的例子是 q -Eulerian 多项式 $A_n^{\text{inv}}(q; x, y)$ 的 q -文法. 定义

$$A_n^{\text{inv}}(q; x, y) = \sum_{\sigma \in S_n} q^{\text{inv}(\sigma)} x^{\text{asc}(\sigma)} y^{\text{des}(\sigma)},$$

这里 $\text{inv}(\sigma)$ 表示 σ 的逆序数, 即

$$\text{inv}(\sigma) = |\{(i, j) : i < j, \sigma_i > \sigma_j\}|,$$

其余的记号如前. $A_n^{\text{inv}}(q; x, y)$ 的前几个值如下:

$$A_1^{\text{inv}}(q; x, y) = xy,$$

$$A_2^{\text{inv}}(q; x, y) = x^2y + qxy^2,$$

$$A_3^{\text{inv}}(q; x, y) = x^3y + (2q + 2q^2)x^2y^2 + q^3xy^3.$$

Eulerian 多项式的文法涉及两个变量 x 和 y , 对于 q -文法, 我们需要生成两个序列 $(x_0, x_1, x_2, \dots)$ 和 $(y_0, y_1, y_2, \dots)$, 并假设 x_i 和 y_i 有如下顺序

$$x_0 < y_0 < x_1 < y_1 < x_2 < y_2 < \dots. \quad (0.1)$$

定义文法

$$G_{\text{inv}} = \{x_j \rightarrow q^j y_j x_{j+1}, \quad y_j \rightarrow q^j y_j x_{j+1} \mid j = 0, 1, 2, \dots\}. \quad (0.2)$$

其中用 ρ 表示把一个单项式按 (0.1) 的顺序整理成标准形式.
定义赋值映射

$$\phi(x_j) = x, \quad \phi(y_j) = y.$$

并设 D 是关于 q -文法 (0.2) 的 q -微分算子, 则

$$\phi(D^n(x_0)) = A_n^{\text{inv}}(q; x, y).$$

例如,

$$\begin{aligned}D(x_0) &= R(x_0) = y_0 x_1, \\D^2(x_0) &= D(y_0 x_1) = R(y_0) x_2 + y_0 R(x_1) = y_0 x_1 x_2 + q y_0 y_1 x_2, \\D^3(x_0) &= D(y_0 x_1 x_2 + q y_0 y_1 x_2) \\&= R(y_0) x_2 x_3 + y_0 R(x_1) x_3 + y_0 x_1 R(x_2) \\&\quad + q R(y_0) y_2 x_3 + q y_0 R(y_1) x_3 + q y_0 y_1 R(x_2) \\&= y_0 x_1 x_2 x_3 + q y_0 y_1 x_2 x_3 + q^2 y_0 x_1 y_2 x_3 \\&\quad + q y_0 x_1 y_2 x_3 + q^2 y_0 y_1 x_2 x_3 + q^3 y_0 y_1 y_2 x_3.\end{aligned}$$

由此可得,

$$\begin{aligned}\phi(D(x_0)) &= xy, \\\phi(D^2(x_0)) &= x^2 y + qxy^2, \\\phi(D^3(x_0)) &= yx^3 + qy^2x^2 + q^2y^2x^2 + qy^2x^2 + q^2y^2x^2 + q^3y^3x.\end{aligned}$$

$$G_{\text{inv}} = \{x_j \rightarrow q^j y_j x_{j+1}, \quad y_j \rightarrow q^j y_j x_{j+1} \mid j = 0, 1, 2, \dots\}.$$

$$\phi(D^n(x_0)) = A_n^{\text{inv}}(q; x, y).$$

$$n = 1, \quad 0 \text{ } x_1 \text{ } 1 \text{ } y_0 \text{ } 0$$

$$n = 2, \quad 0 \text{ } x_2 \text{ } 1 \text{ } x_1 \text{ } \boxed{2} \text{ } y_0 \text{ } 0, \quad 0 \text{ } x_2 \text{ } \boxed{2} \text{ } y_1 \text{ } 1 \text{ } y_0 \text{ } 0$$

$$n = 3, \quad 0 \text{ } x_3 \text{ } \boxed{3} \text{ } y_2 \text{ } 1 \text{ } x_1 \text{ } 2 \text{ } y_0 \text{ } 0, \quad 0 \text{ } x_3 \text{ } 1 \text{ } x_2 \text{ } \boxed{3} \text{ } y_1 \text{ } 2 \text{ } y_0 \text{ } 0, \quad 0 \text{ } x_3 \text{ } 1 \text{ } x_2 \text{ } 2 \text{ } x_1 \text{ } \boxed{3} \text{ } y_0 \text{ } 0, \\ 0 \text{ } x_3 \text{ } \boxed{3} \text{ } y_2 \text{ } 2 \text{ } y_1 \text{ } 1 \text{ } y_0 \text{ } 0, \quad 0 \text{ } x_3 \text{ } 2 \text{ } x_2 \text{ } \boxed{3} \text{ } y_1 \text{ } 1 \text{ } y_0 \text{ } 0, \quad 0 \text{ } x_3 \text{ } 2 \text{ } y_2 \text{ } 1 \text{ } x_1 \text{ } \boxed{3} \text{ } y_0 \text{ } 0$$

$$D(x_0) = x_0 y_1$$

$$D^2(x_0) = y_0 x_1 x_2 + q y_0 y_1 x_2$$

$$D^3(x_0) = y_0 x_1 x_2 x_3 + q y_0 y_1 x_2 x_3 + q^2 y_0 x_1 y_2 x_3 \\ + q y_0 x_1 y_2 x_3 + q^2 y_0 y_1 x_2 x_3 + q^3 y_0 y_1 y_2 x_3$$

$$D^n(x_0) = \sum_{\sigma \in \mathfrak{S}_n} q^{\text{inv}(\sigma)} \omega(\sigma).$$

Theorem ((des, inv) on permutations). Let G_{inv} be the q -grammar defined by

$$G_{\text{inv}} = (\{x, y\}, \{x_j \rightarrow q^j y_j x_{j+1}, y_j \rightarrow q^j y_j x_{j+1}\}, \text{AIO}).$$

Let D be the q -derivative associated with G_{inv} and define the evaluation map ϕ by $\phi(x_j) = x$ and $\phi(y_j) = y$. Then

$$\phi(D^n(x_0)) = \sum_{\sigma \in \mathfrak{S}_n} q^{\text{inv}(\sigma)} x^{\text{asc}(\sigma)} y^{\text{des}(\sigma)},$$

where $\text{des}(\sigma)$ (resp. $\text{asc}(\sigma)$) counts the number of descents (ascents) of σ , and $\text{inv}(\sigma)$ counts the number of inversions of σ .

Theorem. For $i \geq 0$,

$$\phi(\text{Gen}_q^{(G_{\text{inv}})}(x_i; u)) = \frac{x - y}{1 - x^{-1}ye_q((x - y)uq^i)}.$$

Which can recover a result of Stanley:

$$\sum_{n=0}^{\infty} \left(\sum_{\sigma \in \mathfrak{S}_n} q^{\text{inv}(\sigma)} x^{\text{asc}(\sigma)} y^{\text{des}(\sigma)} \right) \frac{u^n}{(q; q)_n} = \frac{x - y}{1 - x^{-1}ye_q((x - y)u)},$$

where $e_q(u)$ is given by

$$e_q(u) = \sum_{n \geq 0} \frac{u^n}{(q; q)_n} = \frac{1}{(u; q)_{\infty}}.$$

Proposition. For $i \geq 0$,

$$D^n(x_i^{-1}) = -q^{ni}x_i^{-1}y_i(x_{i+1} - y_{i+1})^{n-1}.$$

$$\begin{aligned}\phi(\text{Gen}_q^{(G_{\text{inv}})}(x_i^{-1}; u)) &= \sum_{n=0}^{+\infty} \phi(D^n(x_i^{-1})) \frac{u^n}{(q; q)_n} \\ &= x^{-1} - \sum_{n=1}^{+\infty} x^{-1}y(x-y)^{n-1}q^{ni} \frac{u^n}{(q; q)_n} \\ &= x^{-1} - \frac{x^{-1}y}{x-y} (e_q((x-y)uq^i) - 1) \\ &= \frac{1 - x^{-1}ye_q((x-y)uq^i)}{x-y},\end{aligned}$$

which yields the desired generating function because

$$1 = \phi(\text{Gen}_q^{(G_{\text{inv}})}(x_i x_i^{-1}; u)) = \phi(\text{Gen}_q^{(G_{\text{inv}})}(x_i; u)) \phi(\text{Gen}_q^{(G_{\text{inv}})}(x_i^{-1}; u)).$$

Theorem (cycle q -Roselle polynomials). Let $G_{\text{cyc}} = (\{x, y, z, e\}, R, \mathbf{KSO})$ be the q -grammar, where

$$R = \{x_j \rightarrow q^j y_j x_{j+1}, y_j \rightarrow q^j y_j x_{j+1}, z_j \rightarrow q^j y_j x_{j+1}, e_j \rightarrow \underline{\beta} q^j e_j z_{j+1}\}.$$

Let D be the q -derivative associated with G_{cyc} and define the evaluation map ϕ by $\phi(x_j) = x$, $\phi(y_j) = y$, $\phi(z_j) = z$ and $\phi(e_j) = e$. Then

$$\phi(D^n(e_0)) = e \sum_{\sigma \in \mathfrak{S}_n} q^{\text{inv}(\sigma)} x^{\text{iasc}(\sigma)} z^{\text{isol}(\sigma)} y^{\text{des}(\sigma)-1} \beta^{\text{RLmin}(\sigma)}.$$

Theorem

$$\begin{aligned} & \phi(\text{Gen}_q^{(G_{\text{inv}})})(e_0; u) \\ &= \prod_{k=0}^{+\infty} \frac{1 - x^{-1} y e_q((x - y) u q^{k+1})}{1 - \beta u q^k (z - y) - x^{-1} y (1 + \beta u q^k (x - z)) e_q((x - y) u q^{k+1})}, \end{aligned}$$

where

$$e_q(u) = \sum_{n \geq 0} \frac{u^n}{(q; q)_n} = \frac{1}{(u; q)_\infty}.$$

Theorem ((des, inv) on André I permutations). Let G_{AndI} be the q -grammar defined by

$$G_{\text{AndI}} = (\{x, y\}, \{x_j \rightarrow q^j x_j y_{j+1}, y_j \rightarrow q^j x_j\}, \mathbf{AIO}).$$

Let D be the q -derivative associated with G_{AndI} and define the evaluation map ϕ by $\phi(x_j) = x$ and $\phi(y_j) = y$. Then

$$\phi(D^n(x_0)) = E'_{n+1}(q; x, y) = \sum_{\sigma \in \mathcal{T}_{n+1}^I} x^{l(\sigma)} y^{\mu(\sigma)} q^{\text{inv}(\sigma)},$$

where $\mathcal{T}_n^I$ denotes the set of André I trees on $[n]$.

Theorem

For $n \geq 1$,

$$E_{n+1}^I(q; x, y) = yE_n^I(q; x, y) + \sum_{k=0}^{n-2} q^{n-k-1} \begin{bmatrix} n-1 \\ k \end{bmatrix}_q E_{k+1}^I(q; x, y) E_{n-k-1}^I(q; x, y).$$

Theorem ((des, inv) on André II permutations). Let G_{AndII} be the q -grammar defined by

$$G_{\text{AndII}} = (\{x, y\}, \{x_j \rightarrow q^j x_j y_{j+1}, y_j \rightarrow q^{j+1} x_{j+1}\}, \mathbf{AIO}).$$

Let D be the q -derivative associated with G_{AndII} and define the evaluation map ϕ by $\phi(x_j) = x$ and $\phi(y_j) = y$. Then

$$\phi(D^n(x_0)) = E_{n+1}^{\text{II}}(q; x, y) = \sum_{\sigma \in \mathcal{T}_{n+1}^{\text{II}}} x^{l(\sigma)} y^{u(\sigma)} q^{\text{inv}(\sigma)}.$$

Theorem

For $n \geq 1$,

$$E_{n+1}^I(q; x, y)$$

$$= yE_n^I(q; x, y) + \sum_{k=0}^{n-2} q^{2(n-k-1)} \begin{bmatrix} n-1 \\ k \end{bmatrix}_q E_{k+1}^I(q; x, y) E_{n-k-1}^I(q; x, y).$$

Theorem ((des , maj) on permutations) Let G_{maj} be the q -grammar defined by

$$G_{\text{maj}} = (\{x, y\}, \{x_j \mapsto q^j x_0 y_0, y_j \mapsto q^j x_0 y_0\}, \mathbf{LPO}).$$

Let D be the q -derivative associated with G_{maj} and define the evaluation map ϕ by

$$\underline{\phi(x_j) = xq^j} \quad \text{and} \quad \underline{\phi(y_j) = yq^j}.$$

Then

$$\phi(D^n(x_0)) = \sum_{\sigma \in \mathfrak{S}_n} q^{\text{maj}(\sigma)} x^{\text{asc}(\sigma)} y^{\text{des}(\sigma)},$$

where $\text{des}(\sigma)$ (resp. $\text{asc}(\sigma)$) counts the number of descents (ascents) of σ , and $\text{maj}(\sigma)$ equals major index of σ .

♥ This talk is based on the following work:

 Guo-Niu Han, Kathy Q. Ji, Huan Xiong, q -Derivative grammar, submitted. arXiv:2604.23959.

♥ If you are interested in grammar calculus, I recommend the following reference:

 陈永川, 文法演算, 中国科学: 数学, 四川大学数学学科建设 130 周年专辑, 2026.

Thanks for your attention!

