

Partitions and the odd moments of ranks and cranks

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The moments of ranks and cranks

Definition (Atkin and Garvan, 2003)

The k th moment of the rank $N_k(n)$ is defined by

$$N_k(n) = \sum_{m=-\infty}^{+\infty} m^k N(m, n),$$

where $N(m, n)$ denotes the number of partitions of n with rank m .

The k th moment of the crank $M_k(n)$ is defined by

$$M_k(n) = \sum_{m=-\infty}^{+\infty} m^k M(m, n),$$

where $M(m, n)$ denotes the number of partitions of n with crank m .



A.O.L. Atkin and F. Garvan, Relations between the ranks and cranks of partitions, Ramanujan J. 7 (2003) 343–366.

$$\begin{aligned}
 N_k(n) &= \sum_{m=-\infty}^{+\infty} m^k N(m, n) \\
 &= \sum_{m=-\infty}^{+\infty} (-m)^k N(-m, n)
 \end{aligned}$$

⇓ symmetry

$$\begin{aligned}
 &= \sum_{m=-\infty}^{+\infty} (-1)^k m^k N(m, n) \\
 &= (-1)^k N_k(n).
 \end{aligned}$$

This implies that

$$N_{2k+1}(n) = 0.$$

Using the same argument, we could show that

$$M_{2k+1}(n) = 0.$$

The POSITIVE moments of ranks and cranks

Definition (Andrews, Chan and Kim, 2013)

The k th positive moment of the rank $\overline{N}_k(n)$ is defined by

$$\overline{N}_k(n) = \sum_{m=1}^{+\infty} m^k N(m, n).$$

The k th positive moment of the crank $\overline{M}_k(n)$ is defined by

$$\overline{M}_k(n) = \sum_{m=1}^{+\infty} m^k M(m, n).$$



G.E. Andrews, S.H. Chan, and B. Kim, The odd moments of ranks and cranks, J. Combin Theory, Ser. A 120 (2013) 77–91.

Relations

- The new odd moments of ranks and cranks are now nontrivial.
- For even moments of ranks and cranks, we have

$$N_{2k}(n) = 2\overline{N}_{2k}(n), \quad M_{2k}(n) = 2\overline{M}_{2k}(n).$$

The SYMMETRIZED moments of ranks and cranks

Definition (Andrews, 2007)

The k th symmetrized moment of the rank $\eta_k(n)$ is defined by

$$\eta_k(n) = \sum_{m=-\infty}^{+\infty} \binom{m + \lfloor \frac{k-1}{2} \rfloor}{k} N(m, n).$$

Definition (Garvan, 2011)

The k th symmetrized moment of the crank $\mu_k(n)$ is defined by

$$\mu_k(n) = \sum_{m=-\infty}^{+\infty} \binom{m + \lfloor \frac{k-1}{2} \rfloor}{k} M(m, n).$$

These are linear combinations of $N_k(n)$ and $M_k(n)$.



G.E. Andrews, Partitions, Durfee symbols and the Atkin-Garvan moments of ranks, Invent. Math. 169 (2007) 37–73.



F.G. Garvan, Higher order spt-functions, Adv. Math. 228 (2011) 241–265.

$$\begin{aligned}
 \eta_{2k+1}(n) &= \sum_{m=-\infty}^{+\infty} \binom{m+k}{2k+1} N(m, n) \\
 &= \sum_{m=-\infty}^{+\infty} \binom{-m+k}{2k+1} N(-m, n)
 \end{aligned}$$

$\Downarrow$ symmetry

$$\begin{aligned}
 &= \sum_{m=-\infty}^{+\infty} (-1)^{2k+1} \binom{m+k}{2k+1} N(m, n) \\
 &= -\eta_{2k+1}(n).
 \end{aligned}$$

This implies that

$$\eta_{2k+1}(n) = 0.$$

Using the same argument, we could show that

$$\mu_{2k+1}(n) = 0.$$

The combinatorial interpretation of the even symmetrized moments of ranks given by Andrews

To give a combinatorial interpretation of $\eta_{2k}(n)$, Andrews introduced the following function creatively,

$$\begin{aligned}
 & R_k(x_1, x_2, \dots, x_k; q) \\
 &= \sum_{\substack{m_1 > 0 \\ m_2, m_3, \dots, m_k \geq 0}} q^{(m_1 + m_2 + \dots + m_k)^2} \frac{q^{m_1}}{(x_1 q; q)_{m_1} \left(\frac{q}{x_1}; q \right)_{m_1}} \\
 &\times \frac{q^{m_1 + m_2}}{(x_2 q^{m_1}; q)_{m_2 + 1} \left(\frac{q^{m_1}}{x_2}; q \right)_{m_2 + 1}} \times \dots \\
 &\times \frac{q^{m_1 + m_2 + \dots + m_{k-1}}}{(x_{k-1} q^{m_1 + m_2 + \dots + m_{k-2}}; q)_{m_{k-1} + 1} \left(\frac{q^{m_1 + m_2 + \dots + m_{k-2}}}{x_{k-1}}; q \right)_{m_{k-1} + 1}} \\
 &\times \frac{1}{(x_k q^{m_1 + m_2 + \dots + m_{k-1}}; q)_{m_k + 1} \left(\frac{q^{m_1 + m_2 + \dots + m_{k-1}}}{x_k}; q \right)_{m_k + 1}}
 \end{aligned}$$

Using k -fold generalization of q -Whipple's theorem, Andrews showed

$$R_{k+1}(1, 1, \dots, 1; q) = \sum_{n \geq 0} \eta_{2k}(n) q^n.$$

k -marked Durfee symbols

On the other hand, Andrews found the coefficients of q^n in $R_{k+1}(1, 1, \dots, 1; q)$ can be interpreted in terms of k -marked Durfee symbols.

Definition

A k -marked Durfee symbol of n is composed of k pairs of partitions and a subscript, which is defined as

$$\eta = \left(\begin{array}{cccc} \alpha^k, & \alpha^{k-1}, & \dots, & \alpha^1 \\ \beta^k, & \beta^{k-1}, & \dots, & \beta^1 \end{array} \right)_D,$$

where the partitions α^i and β^i must satisfy the following four conditions:

- (1) For $1 \leq i < k$, α^i must be non-empty partition, while α^k and β^i could be empty;
- (2) $\alpha_1^{i-1} \leq \alpha_s^i$ and $\beta_1^{i-1} \leq \alpha_1^{i-1} \leq \beta_s^i$ for $2 \leq i \leq k$;
- (3) $\beta_1^k, \alpha_1^k \leq D$;
- (4) $\sum_{i=1}^k (|\alpha^i| + |\beta^i|) + D^2 = n$.

Note that α_1^i denotes the largest part of α^i and α_s^i denotes the smallest part of α^i .

Example

$$\eta = \left(\begin{array}{ccc} D \geq \overbrace{5_3, 5_3, 4_3}^{\alpha^3} \geq \overbrace{4_2, 3_2, 3_2, 2_2}^{\alpha^2} \geq \overbrace{2_1}^{\alpha^1} \\ D \geq \underbrace{4_3}_{\beta^3} \geq \underbrace{3_2, 2_2, 2_2}_{\beta^2} \geq \underbrace{2_1}_{\beta^1} \end{array} \right)_5$$

is a 3-marked Durfee symbols.

Combinatorial interpretations for $\eta_{2k}(n)$

Theorem (Andrews, 2007)

For $k \geq 1$, $\eta_{2k}(n)$ is equal to the number of $(k+1)$ -marked Durfee symbols of n .



G. E. Andrews, Partitions, Durfee symbols and the Atkin-Garvan moments of ranks, Invent. Math. 169 (2007) 37–73.

Example

$$\begin{aligned}\eta_2(5) &= \sum_{m=-\infty}^{\infty} \binom{m}{2} N(m, 5) \\ &= \binom{4}{2} + \binom{2}{2} + \binom{1}{2} + \binom{0}{2} + \binom{-1}{2} + \binom{-2}{2} + \binom{-4}{2} \\ &= 21.\end{aligned}$$

λ	(5)	$(4, 1)$	$(3, 2)$	$(3, 1^2)$	$(2^2, 1)$	$(2, 1^3)$	(1^5)
$r(\lambda)$	4	2	1	0	-1	-2	-4

$$\begin{aligned}N(4, 5) &= N(2, 5) = N(1, 5) = N(0, 5) = N(-1, 5) \\ &= N(-2, 5) = N(-4, 5) = 1.\end{aligned}$$

We exhibit the 21 2-marked Durfee symbols of 5.

$$\begin{array}{ccc}
 \left(\begin{array}{cccc} 1_2 & 1_2 & 1_2 & 1_1 \end{array} \right)_1 & \left(\begin{array}{cccc} 1_1 & 1_1 & 1_1 & 1_1 \end{array} \right)_1 & \left(\begin{array}{cccc} 1_1 & & & \end{array} \right)_1 \\
 \left(\begin{array}{cccc} 1_2 & 1_1 & 1_1 & \end{array} \right)_1 & \left(\begin{array}{cccc} 1_2 & 1_1 & 1_1 & 1_1 \end{array} \right)_1 & \left(\begin{array}{cccc} 1_2 & 1_1 & & \end{array} \right)_1 \\
 \left(\begin{array}{cccc} 1_2 & 1_2 & 1_1 & \end{array} \right)_1 & \left(\begin{array}{cccc} 1_2 & 1_2 & 1_1 & 1_1 \end{array} \right)_1 & \left(\begin{array}{cccc} 1_2 & 1_2 & 1_1 & \end{array} \right)_1 \\
 \left(\begin{array}{cccc} 1_2 & 1_2 & 1_2 & 1_1 \end{array} \right)_1 & \left(\begin{array}{cccc} 1_1 & 1_1 & 1_1 & \end{array} \right)_1 & \left(\begin{array}{cccc} 1_1 & 1_1 & & \end{array} \right)_1 \\
 \left(\begin{array}{cccc} 1_2 & 1_1 & 1_1 & \end{array} \right)_1 & \left(\begin{array}{cccc} 1_2 & 1_1 & 1_1 & 1_1 \end{array} \right)_1 & \left(\begin{array}{cccc} 1_2 & 1_1 & & \end{array} \right)_1 \\
 \left(\begin{array}{cccc} 1_2 & 1_2 & 1_1 & \end{array} \right)_1 & \left(\begin{array}{cccc} 1_2 & 1_1 & 1_1 & \end{array} \right)_1 & \left(\begin{array}{cccc} 1_2 & 1_1 & & \end{array} \right)_1 \\
 \left(\begin{array}{cccc} 1_1 & & & \end{array} \right)_2 & \left(\begin{array}{cccc} 1_2 & 1_2 & 1_1 & 1_1 \end{array} \right)_1 & \left(\begin{array}{cccc} 1_2 & 1_2 & & 1_1 \end{array} \right)_1
 \end{array}$$

Congruences for $N_2(n)$ and $N_4(n)$

In addition, Andrews gave combinatorial interpretations of the following congruences in terms of k -marked Durfee symbols.

$$N_2(5n + 1) \equiv \eta_2(5n + 1) \equiv 0 \pmod{5}.$$

$$N_2(5n + 4) \equiv \eta_2(5n + 4) \equiv 0 \pmod{5}.$$

$$N_2(7n + 1) \equiv \eta_2(7n + 1) \equiv 0 \pmod{7}.$$

$$N_2(7n + 5) \equiv \eta_2(7n + 5) \equiv 0 \pmod{7}.$$

$$N_4(7n + 1) \equiv \eta_4(7n + 1) \equiv 0 \pmod{7}.$$

$$N_4(7n + 5) \equiv \eta_4(7n + 5) \equiv 0 \pmod{7}.$$

Questions

1. What kind of combinatorial structure is associated with **the odd moments of ranks**?
2. What kind of combinatorial structure is associated with **the moments of cranks**?

In this talk, we will give the answers to these two questions.

The combinatorial interpretation of the odd moments of ranks

The rank for k -marked Durfee symbols

Definition (Andrews, 2007)

Let $\eta = \begin{pmatrix} \alpha^k, & \alpha^{k-1}, & \dots, & \alpha^1 \\ \beta^k, & \beta^{k-1}, & \dots, & \beta^1 \end{pmatrix}_D$ be a k -marked Durfee symbol. The i th rank of η is define by

$$\rho_i(\eta) = \begin{cases} \ell(\alpha^i) - \ell(\beta^i) - 1, & \text{for } 1 \leq i < k, \\ \ell(\alpha^k) - \ell(\beta^k), & \text{for } i = k, \end{cases}$$

where $\ell(\alpha)$ denotes the number of parts of α .



G. E. Andrews, Partitions, Durfee symbols and the Atkin-Garvan momemts of ranks, Invent. Math. 169 (2007) 37–73.

Example

The 3-marked Durfee symbol

$$\eta = \left(\begin{array}{ccc} \overbrace{5_3, 5_3, 4_3}^{\alpha^3} & \overbrace{4_2, 3_2, 3_2, 2_2}^{\alpha^2} & \overbrace{2_1}^{\alpha^1} \\ \underbrace{4_3}_{\beta^3} & \underbrace{3_2, 2_2, 2_2}_{\beta^2} & \underbrace{2_1}_{\beta^1} \end{array} \right)_5$$

- the first rank is $\ell(\alpha^1) - \ell(\beta^1) - 1 = -1$;
- the second rank is $\ell(\alpha^2) - \ell(\beta^2) - 1 = 0$;
- the third rank is $\ell(\alpha^3) - \ell(\beta^3) = 2$.

Symmetry

Let $\mathcal{D}_k(m_1, m_2, \dots, m_k; n)$ denote the number of k -marked Durfee symbols of n with i th rank equal to m_i , Andrews first showed that

Theorem (Andrews, 2007)

$\mathcal{D}_k(m_1, \dots, m_k; n)$ is symmetric in $m_1, m_2, \dots, m_k$.

There are several combinatorial proofs of the above symmetry, see Boulet and Kurungöz, 2011 and Ji, 2011.



C. Boulet and K. Kurungöz, Symmetry of k -marked Durfee symbols, Int. J. Number Theory 7 (2011) 215–230.



Kathy Q. Ji, The combinatorics of k -marked Durfee symbols, Trans. Amer. Math. Soc. 363 (2011) 987–1005.

The SYMMETRIZED POSITIVE moments of ranks

Definition

The k th symmetrized positive moment of the rank $\bar{\eta}_k(n)$ is defined by

$$\bar{\eta}_k(n) = \sum_{m=1}^{\infty} \binom{m + \lfloor \frac{k-1}{2} \rfloor}{k} N(m, n).$$

- The new odd symmetrized moments of ranks are not trivial;
-

$$\eta_{2k}(n) = 2\bar{\eta}_{2k}(n) + \bar{\eta}_{2k-1}(n).$$

The combinatorial interpretation of $\bar{\eta}_k(n)$

Theorem (Chen, Ji and Shen, 2013)

For $k \geq 1$ and $1 \leq p \leq k + 1$, $\bar{\eta}_{2k-1}(n)$ counts the number of $(k + 1)$ -marked Durfee symbols of n with the p th rank equal to zero.

Theorem (Chen, Ji and Shen, 2013)

For $k \geq 1$ and $1 \leq p \leq k + 1$, $\bar{\eta}_{2k}(n)$ counts the number of $(k + 1)$ -marked Durfee symbols of n with the p th rank larger than zero.



W.Y.C. Chen, K.Q. Ji, and E.Y.Y. Shen, k -Marked Durfee symbols and the odd moments of ranks, preprint.

Using this relation

$$\mathcal{D}_k(m_1, \dots, m_p, \dots, m_k; n) = \mathcal{D}_k(m_1, \dots, -m_p, \dots, m_k; n),$$

we see that

Corollary (Chen, Ji and Shen, 2013)

For $k \geq 1$ and $1 \leq p \leq k + 1$, $\bar{\eta}_{2k}(n)$ also counts the number of $(k + 1)$ -marked Durfee symbols of n with the p th rank less than zero.

Using the above three results and

$$\eta_{2k}(n) = 2\bar{\eta}_{2k}(n) + \bar{\eta}_{2k-1}(n),$$

we recover Andrews' result.

Theorem (Andrews, 2007)

For $k \geq 1$, $\eta_{2k}(n)$ counts the number of $(k + 1)$ -marked Durfee symbols of n .

Example

From the definition of $\bar{\eta}_k(n)$, we can calculate that

$$\begin{aligned}\eta_2(5) &= \sum_{m=-\infty}^{\infty} \binom{m}{2} N(m, 5) \\ &= \binom{-4}{2} + \binom{-2}{2} + \binom{-1}{2} + \binom{0}{2} + \binom{1}{2} + \binom{2}{2} + \binom{4}{2} \\ &= 21.\end{aligned}$$

$$\bar{\eta}_2(5) = \sum_{m=1}^{\infty} \binom{m}{2} N(m, 5) = \binom{1}{2} + \binom{2}{2} + \binom{4}{2} = 7.$$

$$\bar{\eta}_1(5) = \bar{M}_1(5) = \sum_{m=1}^{\infty} m N(m, 5) = 1 + 2 + 4 = 7.$$

Obviously, we have

$$\eta_2(5) = 2\bar{\eta}_2(5) + \bar{\eta}_1(5).$$

λ	(5)	(4, 1)	(3, 2)	(3, 1 ²)	(2 ² , 1)	(2, 1 ³)	(1 ⁵)
$r(\lambda)$	4	2	1	0	-1	-2	-4

We exhibit the 21 2-marked Durfee symbols of 5.

$$\begin{array}{ccc}
 \left(\begin{array}{cccc} 1_2 & 1_2 & 1_2 & 1_1 \end{array} \right)_1 & \left(\begin{array}{cccc} 1_1 & 1_1 & 1_1 & 1_1 \end{array} \right)_1 & \left(\begin{array}{cccc} 1_1 & & & \\ & 1_1 & 1_1 & 1_1 \end{array} \right)_1 \\
 \left(\begin{array}{cccc} 1_2 & 1_1 & 1_1 & \\ & & 1_1 & \end{array} \right)_1 & \left(\begin{array}{cccc} 1_2 & 1_1 & 1_1 & 1_1 \end{array} \right)_1 & \left(\begin{array}{cccc} 1_2 & 1_1 & & \\ & & 1_1 & 1_1 \end{array} \right)_1 \\
 \left(\begin{array}{cccc} 1_2 & 1_2 & 1_1 & \\ 1_2 & & & \end{array} \right)_1 & \left(\begin{array}{cccc} 1_2 & 1_2 & 1_1 & 1_1 \end{array} \right)_1 & \left(\begin{array}{cccc} 1_2 & 1_2 & 1_1 & \\ & & & 1_1 \end{array} \right)_1 \\
 \left(\begin{array}{cccc} & & & 1_1 \\ 1_2 & 1_2 & 1_2 & \end{array} \right)_1 & \left(\begin{array}{cccc} 1_1 & 1_1 & 1_1 & \\ & & 1_1 & \end{array} \right)_1 & \left(\begin{array}{cccc} 1_1 & 1_1 & & \\ & & 1_1 & 1_1 \end{array} \right)_1 \\
 \left(\begin{array}{cccc} & 1_1 & 1_1 & \\ 1_2 & & & 1_1 \end{array} \right)_1 & \left(\begin{array}{cccc} & 1_1 & 1_1 & 1_1 \\ 1_2 & & & \end{array} \right)_1 & \left(\begin{array}{cccc} & 1_1 & & \\ 1_2 & & 1_1 & 1_1 \end{array} \right)_1 \\
 \left(\begin{array}{cccc} 1_2 & & 1_1 & \\ 1_2 & 1_2 & & \end{array} \right)_1 & \left(\begin{array}{cccc} 1_2 & 1_1 & 1_1 & \\ 1_2 & & & \end{array} \right)_1 & \left(\begin{array}{cccc} 1_2 & 1_1 & & \\ 1_2 & & & 1_1 \end{array} \right)_1 \\
 \left(\begin{array}{cccc} 1_1 & & & \\ & & & \end{array} \right)_2 & \left(\begin{array}{cccc} & & 1_1 & 1_1 \\ 1_2 & 1_2 & & \end{array} \right)_1 & \left(\begin{array}{cccc} & & & 1_1 \\ 1_2 & 1_2 & & \\ & & & 1_1 \end{array} \right)_1
 \end{array}$$

$\bar{\eta}_1(5) = 7$	$\bar{\eta}_2(5) = 7$	$\bar{\eta}_2(5) = 7$
$\begin{pmatrix} 1_2 & 1_2 & 1_2 & 1_1 \end{pmatrix}_1$	$\begin{pmatrix} 1_1 & 1_1 & 1_1 & 1_1 \end{pmatrix}_1$	$\begin{pmatrix} 1_1 & & & \\ & 1_1 & 1_1 & 1_1 \end{pmatrix}_1$
$\begin{pmatrix} 1_2 & 1_1 & 1_1 & \\ & & 1_1 & \end{pmatrix}_1$	$\begin{pmatrix} 1_2 & 1_1 & 1_1 & 1_1 \end{pmatrix}_1$	$\begin{pmatrix} 1_2 & 1_1 & & \\ & & 1_1 & 1_1 \end{pmatrix}_1$
$\begin{pmatrix} 1_2 & 1_2 & 1_1 & \\ 1_2 & & & \end{pmatrix}_1$	$\begin{pmatrix} 1_2 & 1_2 & 1_1 & 1_1 \end{pmatrix}_1$	$\begin{pmatrix} 1_2 & 1_2 & 1_1 & \\ & & & 1_1 \end{pmatrix}_1$
$\begin{pmatrix} & & & 1_1 \\ 1_2 & 1_2 & 1_2 & \end{pmatrix}_1$	$\begin{pmatrix} 1_1 & 1_1 & 1_1 & \\ & 1_1 & & \end{pmatrix}_1$	$\begin{pmatrix} 1_1 & 1_1 & & \\ & 1_1 & 1_1 & \end{pmatrix}_1$
$\begin{pmatrix} & 1_1 & 1_1 & \\ 1_2 & & 1_1 & \end{pmatrix}_1$	$\begin{pmatrix} & 1_1 & 1_1 & 1_1 \\ 1_2 & & & \end{pmatrix}_1$	$\begin{pmatrix} & 1_1 & & \\ 1_2 & & 1_1 & 1_1 \end{pmatrix}_1$
$\begin{pmatrix} 1_2 & & 1_1 & \\ 1_2 & 1_2 & & \end{pmatrix}_1$	$\begin{pmatrix} 1_2 & 1_1 & 1_1 & \\ 1_2 & & & \end{pmatrix}_1$	$\begin{pmatrix} 1_2 & 1_1 & & \\ 1_2 & & 1_1 & \end{pmatrix}_1$
$\begin{pmatrix} 1_1 & & & \\ & & & \end{pmatrix}_2$	$\begin{pmatrix} & & 1_1 & 1_1 \\ 1_2 & 1_2 & & \end{pmatrix}_1$	$\begin{pmatrix} & & & 1_1 \\ 1_2 & 1_2 & & 1_1 \end{pmatrix}_1$

The main tool

The proof of these results is based on the following partition identity given by my previous paper.

Theorem (Ji, 2011)

For $k \geq 2$, we have

$$\mathcal{D}_k(m_1, m_2, \dots, m_k; n) = \sum_{t_1, \dots, t_{k-1}=0}^{\infty} N \left(\sum_{i=1}^k |m_i| + 2 \sum_{i=1}^{k-1} t_i + k - 1, n \right).$$



Kathy Q. Ji, The combinatorics of k -marked Durfee symbols, Trans. Amer. Math. Soc. 363 (2011) 987–1005.

The combinatorial interpretation of the moments of cranks

If we mimic the combinatorial interpretation of $\eta_{2k}(n)$ due to Andrews, we may first introduce the following function

$$\begin{aligned}
 & C_k(x_1, x_2, \dots, x_k; q) \\
 = & \sum_{\substack{m_1 > 0 \\ m_2, m_3, \dots, m_{k-1} \geq 0}} \frac{q^{m_1}}{(x_1 q; q)_{m_1} \left(\frac{q}{x_1}; q\right)_{m_1}} \times \frac{q^{m_1+m_2}}{(x_2 q^{m_1}; q)_{m_2+1} \left(\frac{q^{m_1}}{x_2}; q\right)_{m_2+1}} \\
 & \times \dots \times \frac{q^{m_1+m_2+\dots+m_{k-1}}}{(x_{k-1} q^{m_1+m_2+\dots+m_{k-2}}; q)_{m_{k-1}+1} \left(\frac{q^{m_1+m_2+\dots+m_{k-2}}}{x_{k-1}}; q\right)_{m_{k-1}+1}} \\
 & \times \frac{(q; q)_{\infty}}{(x_k q^{m_1+m_2+\dots+m_{k-1}}; q)_{\infty} \left(\frac{q^{m_1+m_2+\dots+m_{k-1}}}{x_k}; q\right)_{\infty}}.
 \end{aligned}$$

Using k -fold generalization of q -Whipple's theorem, we could also show that

$$C_{k+1}(1, 1, \dots, 1; q) = \sum_{n \geq 0} \mu_{2k}(n) q^n.$$

But, we could not give a nice combinatorial interpretation of the coefficients of q^n in $C_{k+1}(1, 1, \dots, 1; q)$ since there exists $(q; q)_{\infty}!!$

Dyson bipartitions

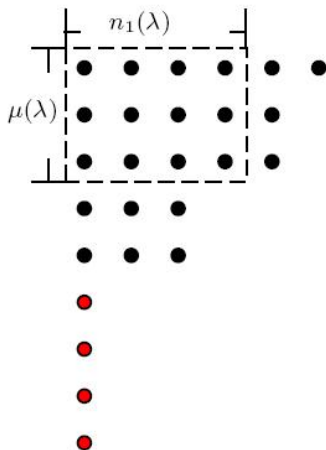
We first introduced a new representation of ordinary partitions, which we call Dyson bipartitions (α, β) since it was inspired by a paper of Dyson in 1989.

If λ is an ordinary partition with **at least one part equal to one**. Let

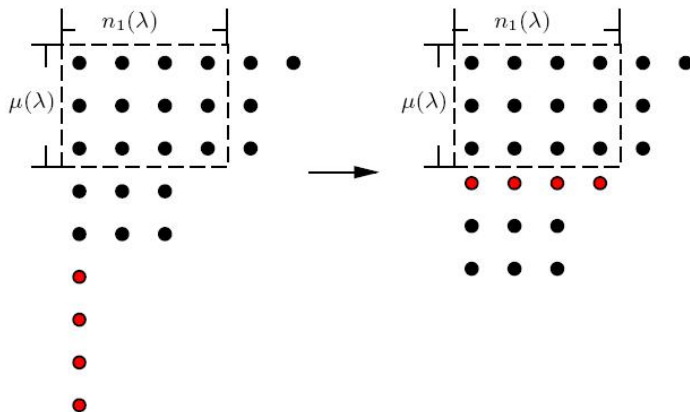
- $n_1(\lambda)$ denote the number of parts equal to one in λ ;
- $\mu(\lambda)$ denote the number of parts in λ larger than $n_1(\lambda)$.

Then the crank of λ is defined to be $\mu(\lambda) - n_1(\lambda)$.

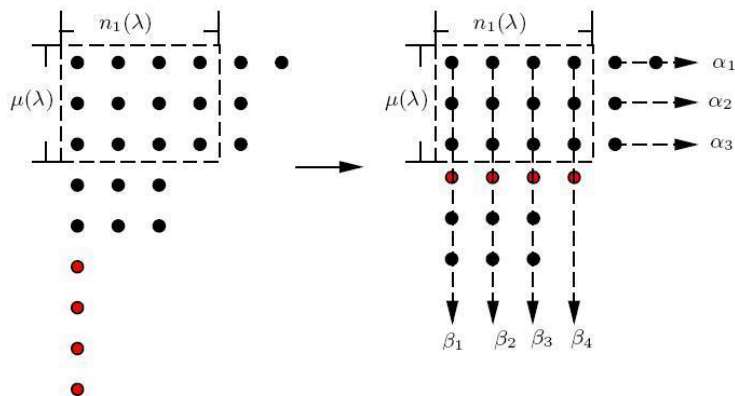
Dyson bipartitions



Dyson bipartitions

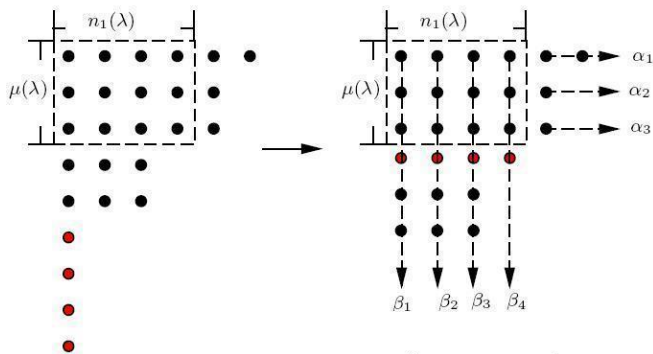


Dyson bipartitions



$$(6, 5, 5, 3, 3, 1, 1, 1, 1) \Rightarrow \begin{pmatrix} 2 & 1 & 1 \\ 6 & 6 & 6 & 4 \end{pmatrix}$$

Dyson bipartitions



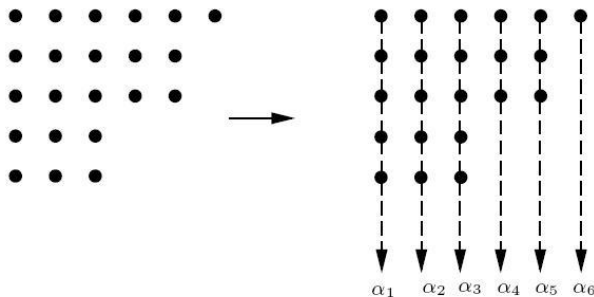
$$(6, 5, 5, 3, 3, 1, 1, 1, 1) \Rightarrow \begin{pmatrix} 2 & 1 & 1 \\ 6 & 6 & 6 & 4 \end{pmatrix}$$

- If $l(\beta) > 1$, then $\beta_1 = \beta_2$, $\beta_s > l(\alpha)$;
- If $l(\beta) = 1$, then $\beta_1 = l(\alpha) + 1$;

The crank of λ is equal to $l(\alpha) - l(\beta)$!!

Dyson bipartitions

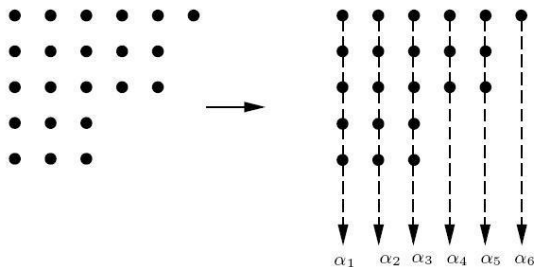
If λ is an ordinary partition with **no part equal to one**, then



$$(6, 5, 5, 3, 3) \Rightarrow \left(5, 5, 5, 3, 3, 1 \right)$$

Dyson bipartitions

If λ is an ordinary partition with **no part equal to one**, then



$$(6, 5, 5, 3, 3) \Rightarrow \left(5, 5, 5, 3, 3, 1 \right)$$

$$l(\beta) = 0 \text{ and } \alpha_1 = \alpha_2.$$

The crank of λ is equal to $\ell(\alpha) - \ell(\beta)!!$

Dyson bipartitions

From the above bijection, we see that

Theorem

The number of Dyson bipartitions (α, β) of n with $\ell(\alpha) - \ell(\beta) = m$ is equal to the number of ordinary partitions of n with crank m , that is $M(m, n)$.

The partition identity

We have shown that $\mu_{2k}(n)$ has the similar partition identity as $\eta_{2k}(n)$.

$$\begin{aligned}\mu_{2k}(n) &= \sum_{m=-\infty}^{+\infty} \binom{m+k-1}{2k} M(m, n) \\ &= \sum_{m_1, \dots, m_{k+1}=-\infty}^{+\infty} \sum_{t_1, \dots, t_k=0}^{+\infty} M\left(\sum_{i=1}^{k+1} |m_i| + 2 \sum_{i=1}^k t_i + k, n\right).\end{aligned}$$

Using the bijections in my previous paper in 2011, we have succeeded to refine Dyson bipartitions to new partitions counted by $\mu_{2k}(n)$ according to the above partition identity.

We call such partitions **k -marked Dyson bipartitions**.

k -marked Dyson bipartitions

Definition

A k -marked Dyson bipartitions of n is composed of k pairs of partitions and a subscript, which is defined as

$$\theta = \left(\begin{array}{cccc} \alpha^k, & \alpha^{k-1}, & \dots, & \alpha^1 \\ \beta^k, & \beta^{k-1}, & \dots, & \beta^1 \end{array} \right)_D,$$

where the partitions α^i and β^i must satisfy the following conditions:

- (1) For $1 \leq i < k$, α^i must be non-empty partition, while α^k and β^i could be empty;
- (2) $\alpha_1^{i-1} \leq \alpha_s^i$ and $\beta_1^{i-1} \leq \alpha_1^{i-1} \leq \beta_s^i$ for $2 \leq i \leq k$;
- (3) $\text{nb}(\eta) = D$;
- (4) If $\alpha^k = \emptyset$, then $\alpha_1^{k-1} = \max\{\alpha_2^{k-1}, \beta_1^{k-1}\}$;
If $\alpha^k \neq \emptyset$ and $l(\alpha^k) = 1$, then $\alpha_1^k = \alpha_1^{k-1}$;
If $\alpha^k \neq \emptyset$ and $l(\alpha^k) > 1$, then $\alpha_1^k = \alpha_2^k$;
- (5) $\sum_{i=1}^k (|\alpha^i| + |\beta^i|) + (l_a(\eta) + D) \times (l_b(\eta) - D) = n$.

The combinatorial interpretation of $\mu_{2k}(n)$

Theorem (Chen, Ji and Shen, 2013)

For $k \geq 1$, $\mu_{2k}(n)$ is equal to the number of $(k+1)$ -marked Dyson bipartitions of n .



W.Y.C. Chen, K.Q. Ji, and E.Y.Y. Shen, Dyson bipartitions and the crank moment function, preprint.

Example

Example

From the definition of $\mu_{2k}(n)$, we have

$$\begin{aligned}\mu_2(5) &= \sum_{m=-\infty}^{+\infty} \binom{m}{2} M(m, 5) \\ &= \binom{5}{2} + \binom{3}{2} + \binom{1}{2} + \binom{0}{2} + \binom{-1}{2} + \binom{-3}{2} + \binom{-5}{2} \\ &= 35.\end{aligned}$$

λ	(5)	$(3, 2)$	$(2^2, 1)$	$(4, 1)$	$(3, 1^2)$	$(2, 1^3)$	(1^5)
crank of λ	5	3	1	0	-1	-3	-5

We exhibit the 35 2-marked Dyson bipartitions of 5.

$$\begin{array}{lll}
 \left(\begin{array}{ccccc} 1_1 & 1_1 & 1_1 & 1_1 & 1_1 \end{array} \right)_0 & \left(\begin{array}{cccc} 1_1 & & & \\ 1_1 & 1_1 & 1_1 & 1_1 \end{array} \right)_0 & \left(\begin{array}{ccccc} 1_2 & 1_1 & 1_1 & 1_1 & 1_1 \end{array} \right)_0 \\
 \left(\begin{array}{ccc} 1_2 & 1_1 & \\ 1_1 & 1_1 & 1_1 \end{array} \right)_0 & \left(\begin{array}{cccc} 1_1 & 1_1 & 1_1 & 1_1 \end{array} \right)_0 & \left(\begin{array}{cccc} 1_1 & & & \\ 1_2 & 1_1 & 1_1 & 1_1 \end{array} \right)_0 \\
 \left(\begin{array}{ccccc} 1_2 & 1_2 & 1_1 & 1_1 & 1_1 \end{array} \right)_0 & \left(\begin{array}{ccc} 1_2 & 1_2 & 1_1 \\ 1_1 & 1_1 & \end{array} \right)_0 & \left(\begin{array}{ccc} 1_1 & 1_1 & 1_1 \\ 1_2 & 1_2 & \end{array} \right)_0 \\
 \left(\begin{array}{cccc} 1_1 & & & \\ 1_2 & 1_2 & 1_1 & 1_1 \end{array} \right)_0 & \left(\begin{array}{ccccc} 1_2 & 1_2 & 1_2 & 1_1 & 1_1 \end{array} \right)_0 & \left(\begin{array}{cccc} 1_2 & 1_2 & 1_2 & 1_1 \\ 1_1 & & & \end{array} \right)_0 \\
 \left(\begin{array}{ccc} 1_1 & 1_1 & \\ 1_2 & 1_2 & 1_2 \end{array} \right)_0 & \left(\begin{array}{cccc} 1_1 & & & \\ 1_2 & 1_2 & 1_2 & 1_1 \end{array} \right)_0 & \left(\begin{array}{ccccc} 1_2 & 1_2 & 1_2 & 1_2 & 1_1 \end{array} \right)_0 \\
 \left(\begin{array}{cccc} 1_1 & & & \\ 1_2 & 1_2 & 1_2 & 1_2 \end{array} \right)_0 & \left(\begin{array}{cc} 1_2 & 1_1 \\ 1_2 & \end{array} \right)_0 & \left(\begin{array}{ccc} 2_1 & 2_1 & 1_1 \end{array} \right)_0 \\
 \left(\begin{array}{cc} 2_1 & \\ 2_1 & 1_1 \end{array} \right)_0 & \left(\begin{array}{ccc} 2_2 & 2_2 & 1_1 \end{array} \right)_0 & \left(\begin{array}{cc} 1_1 & \\ 2_2 & 2_2 \end{array} \right)_0 \\
 \left(\begin{array}{ccc} 2_2 & 2_1 & 1_1 \end{array} \right)_0 & \left(\begin{array}{cc} 2_2 & 2_1 \\ & 1_1 \end{array} \right)_0 & \left(\begin{array}{cc} 2_1 & 1_1 \\ 2_2 & \end{array} \right)_0 \\
 \left(\begin{array}{cc} 2_1 & \\ 2_2 & 1_1 \end{array} \right)_0 & \left(\begin{array}{cc} 2_1 & 1_1 \\ \underline{2_1} & \end{array} \right)_1 & \left(\begin{array}{cccc} 1_1 & 1_1 & 1_1 & 1_1 \\ \underline{1_1} & & & \end{array} \right)_1 \\
 \left(\begin{array}{ccc} 1_1 & 1_1 & \\ 1_1 & 1_1 & 1_1 \end{array} \right)_1 & \left(\begin{array}{cccc} 1_2 & 1_1 & 1_1 & 1_1 \\ \underline{1_1} & & & \end{array} \right)_1 & \left(\begin{array}{ccc} 1_1 & 1_1 & 1_1 \\ 1_2 & 1_1 & \end{array} \right)_1 \\
 \left(\begin{array}{ccc} 1_2 & 1_1 & 1_1 \\ 1_1 & 1_1 & \end{array} \right)_1 & \left(\begin{array}{ccc} 1_1 & 1_1 & \\ 1_2 & 1_1 & 1_1 \end{array} \right)_1 & \left(\begin{array}{cccc} 1_2 & 1_2 & 1_1 & 1_1 \\ \underline{1_1} & & & \end{array} \right)_1 \\
 \left(\begin{array}{ccc} 1_1 & 1_1 & \\ 1_2 & 1_2 & 1_1 \end{array} \right)_1 & \left(\begin{array}{ccc} 1_1 & 1_1 & 1_1 \\ \underline{1_1} & \underline{1_1} & \end{array} \right)_2
 \end{array}$$

Congruences for $M_k(n)$

In addition, we gave combinatorial explanations of the following congruences for $\mu_k(n)$ in terms of k -marked Dyson bipartitions.

$M_2(n) \equiv \mu_2(n) \equiv 0 \pmod{5},$	if $n \equiv 0, 4 \pmod{5}.$
$M_2(n) \equiv \mu_2(n) \equiv 0 \pmod{7},$	if $n \equiv 0, 5 \pmod{7}.$
$M_2(n) \equiv \mu_2(n) \equiv 0 \pmod{11},$	if $n \equiv 0, 6 \pmod{11}.$
$M_4(n) \equiv \mu_4(n) \equiv 0 \pmod{7},$	if $n \equiv 0, 5 \pmod{7}.$
$M_4(n) \equiv \mu_4(n) \equiv 0 \pmod{11},$	if $n \equiv 0, 6 \pmod{11}.$
$M_6(n) \equiv \mu_6(n) \equiv 0 \pmod{11},$	if $n \equiv 0, 6 \pmod{11}.$
$M_8(n) \equiv \mu_8(n) \equiv 0 \pmod{11},$	if $n \equiv 0, 6 \pmod{11}.$

The SYMMETRIZED POSITIVE moments of cranks

Definition

The k th symmetrized positive moment of the crank $\bar{\mu}_k(n)$ is defined by

$$\bar{\mu}_k(n) = \sum_{m=1}^{\infty} \binom{m + \lfloor \frac{k-1}{2} \rfloor}{k} M(m, n).$$

- The new odd symmetrized moments of cranks are not trivial;
-

$$\mu_{2k}(n) = 2\bar{\mu}_{2k}(n) + \bar{\mu}_{2k-1}(n).$$

The combinatorial interpretation of $\bar{\mu}_k(n)$

Theorem (Chen, Ji and Shen, 2013)

For $k \geq 1$ and $1 \leq p \leq k + 1$, $\bar{\mu}_{2k-1}(n)$ is equal to the number of $(k + 1)$ -marked Dyson bipartitions of n with the p th crank equal to zero.

Theorem (Chen, Ji and Shen, 2013)

For $k \geq 1$ and $1 \leq p \leq k + 1$, $\bar{\mu}_{2k}(n)$ is equal to the number of $(k + 1)$ -marked Dyson bipartitions of n with the p th crank greater than zero.

Theorem (Chen, Ji and Shen, 2013)

For $k \geq 1$ and $1 \leq p \leq k + 1$, $\bar{\mu}_{2k}(n)$ is equal to the number of $(k + 1)$ -marked Dyson bipartitions of n with the p th crank less than zero.



W.Y.C. Chen, K.Q. Ji, and E.Y.Y. Shen, Dyson bipartitions and the crank moment function, preprint.



Happy Birthday,
Professor Andrews!