

A Finite Form of Jacobi's Cubic Identity

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Background

Jacobi's cubic identity was stated as:

$$(q; q)_{\infty}^3 = \sum_{k=0}^{\infty} (-1)^k (2k+1) q^{\binom{k+1}{2}}$$

The q -shifted factorial for all integers k is defined by

$$(a; q)_{\infty} := \prod_{k=0}^{\infty} (1 - aq^k) \quad \text{and} \quad (a; q)_k := \frac{(a; q)_{\infty}}{(aq^k; q)_{\infty}}$$

C. Jacobi, *Fundamenta nova theoriae function ellipticarum*, *Mathematische Werke*, 1 (1829) 49–239.

Background

Jacobi's cubic identity has many important applications in combinatorial analysis, number theory and special functions.

- Recently, M.D. Hirschhorn just use this identity to give a simple proof of **two square theorem of Fermat**.

M.D. Hirschhorn, *Jacobi's two-squares theorem and related identities*, J. Ramanujan 3 (1999) 153-158.

Background

- There are also many authors using this identity to prove the **Ramanujan's partition congruences**.

S. Ramanujan, *Congruence properties of partitions*, Math. Zeitschr., 9 (1921) 147–153.

J.A. Ewell, Completion of a Gaussian derivation, Proc. Amer. Math. Soc. 84 (1982), no. 2, 311–314.

G.E. Andrews and R. Roy, *Ramanujan's method in q series congruences*, Electron. J. Combin. 4 (2) 1996.

Background

Jacobi's cubic identity can be deduced from **Jacobi's well-known triple product identity**

$$(z; q)_{\infty} (q/z; q)_{\infty} (q; q)_{\infty} = \sum_{k=-\infty}^{\infty} (-1)^k q^{\binom{k}{2}} z^k.$$

Separate the summation on the right side of the identity and divide both side by $(1 - z)$, we can rewrite it as:

$$(zq; q)_{\infty} (z^{-1}q; q)_{\infty} (q; q)_{\infty} = \sum_{k=0}^{\infty} (-1)^k \frac{(1 - z^{2k+1})}{1 - z} z^{-k} q^{\binom{k+1}{2}},$$

let $z \rightarrow 1$, according to **L'Hopital**, we can get **Jacobi's cubic identity**.

Main Result

We obtain the following **finite form of Jacobi's cubic identity**

$$(q; q)_m (q; q)_n = \sum_{k=-n-1}^m (-1)^k k q^{\binom{k+1}{2}} \begin{bmatrix} m + n + 1 \\ n + k + 1 \end{bmatrix}.$$

Gauss coefficient or the **q -binomial coefficient** is given by

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{(q; q)_n}{(q; q)_k (q; q)_{n-k}}$$

Main Result

Let $n \rightarrow \infty$ and $m \rightarrow \infty$ in above finite form, according to **Tannery's theorem**, we can get:

$$(q; q)_{\infty}^3 = \sum_{k=-\infty}^{\infty} (-1)^k k q^{\binom{k+1}{2}}$$

Separate the right sum of above identity, it can be rewritten as:

$$\begin{aligned} \sum_{k=-\infty}^{\infty} (-1)^k k q^{\binom{k+1}{2}} &= \sum_{k=0}^{\infty} (-1)^k k q^{\binom{k+1}{2}} + \sum_{k=1}^{\infty} (-1)^k (-k) q^{\binom{k}{2}} \\ &= \sum_{k=0}^{\infty} (-1)^k (2k + 1) q^{\binom{k+1}{2}} \end{aligned}$$

Hence, we get the **Jacobi's cubic identity**.

Main Method

Two proofs of the finite form.

One is **analytic method** by using **MacMahon finite form** of Jacobi's triple product identity:

$$(zq; q)_m (z^{-1}; q)_n = \sum_{k=-n}^m (-1)^k q^{\binom{k+1}{2}} z^k \begin{bmatrix} m+n \\ n+k \end{bmatrix}.$$

The other is **combinatorial method** in term of **generalized F -partitions**.

Analytic Method

Replace n by $n + 1$ in **MacMahon's formula**, we get

$$(zq; q)_m (z^{-1}; q)_{n+1} = \sum_{k=-n-1}^m (-1)^k q^{\binom{k+1}{2}} z^k \begin{bmatrix} m + n + 1 \\ n + 1 + k \end{bmatrix}.$$

Let $f(z) = (zq; q)_m (z^{-1}q; q)_n$. Hence, MacMahon's formula can be written as:

$$(1 - z^{-1})f(z) = \sum_{k=-n-1}^m (-1)^k q^{\binom{k+1}{2}} z^k \begin{bmatrix} m + n + 1 \\ n + 1 + k \end{bmatrix}.$$

It should be noted that $f(1) = (q; q)_m (q; q)_n$.

Analytic Method

Differentiating both side of the above identity with respect to z ,

$$\begin{aligned} z^{-2} f(z) + (1 - z^{-1}) f'(z) \\ = \sum_{k=-n-1}^m (-1)^k k q^{\binom{k+1}{2}} z^{k-1} \begin{bmatrix} m + n + 1 \\ n + 1 + k \end{bmatrix}. \end{aligned}$$

Let $z = 1$, the above formula is equal to

$$(q; q)_m (q; q)_n = \sum_{k=-n-1}^m (-1)^k k q^{\binom{k+1}{2}} \begin{bmatrix} m + n + 1 \\ n + k + 1 \end{bmatrix}.$$

Hence we get **finite form of Jacobi's cubic identity**.

Combinatorial Method

We will give a combinatorial interpretation of finite form in term of generalized F -partitions.

$$(q; q)_m (q; q)_n = \sum_{k=-n-1}^m (-1)^k k q^{\binom{k+1}{2}} \begin{bmatrix} m + n + 1 \\ n + k + 1 \end{bmatrix}.$$

More precisely, we will interpret the left side and right side of finite form as the generating function of some restricted generalized F -partitions, respectively. Then the finite form can be transformed to **a key partition identity** which we can prove it by constructing a bijection.

Notation

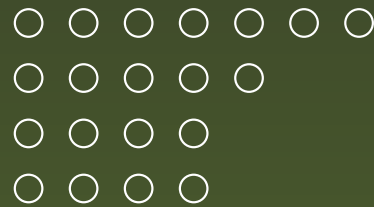
Before giving a combinatorial interpretation of the finite form, we first review some notation.

A partition of a positive integer n is a finite nonincreasing sequence of positive integers $\lambda_1, \lambda_2, \dots, \lambda_r$ such that $\sum_{i=1}^r \lambda_i = n$. The λ_i are called the parts of the partition.

For example, the partitions of 4 are:
(4), (3, 1), (2, 2), (2, 1, 1), (1, 1, 1, 1).

Representations of Partitions

The **graphical representation** (or **Ferrers graph**) of the partition λ of n is a left-justified array of n dots with λ_i dots in the i -th row.



$$\lambda = (7, 5, 4, 4)$$

Generating Function

For a partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_r)$

λ_1 is called **its largest part**.

The sum of parts is called the **weight of the partition λ** denoted by $|\lambda|$.

The number of parts of λ is called the **length of λ** denoted by $l(\lambda)$.

Generalized F -partition

Generalized F -partition is defined as two arrays:

$$\begin{pmatrix} a_1 & a_2 & \cdots & a_r \\ b_1 & b_2 & \cdots & b_s \end{pmatrix}$$

where $(a_1, \cdots, a_r)$ is a partition into distinct positive parts, $(b_1, \cdots, b_s)$ is a partition into distinct parts, **zero is allowed**. We denote such generalized F -partition by λ^f .

The difference of the length of parts in the top and bottom rows is called **size of generalized F -partitions** λ^f denoted by $d(\lambda^f)$, i.e., $d(\lambda^f) = r - s$.

Generating Function of Generalized F -Partition

From the definition of generalized F -partition, it is easy to know that the generating function of generalized F -partition is:

$$\sum_{\lambda^f: \alpha_1 \leq n, \beta_1 \leq m} z^{d(\lambda^f)} q^{|\lambda^f|} = (-zq; q)_n (-z^{-1}; q)_m$$

Consider the **graphical representation** of

$$\lambda^f = \begin{pmatrix} a_1 & a_2 & \cdots & a_r \\ b_1 & b_2 & \cdots & b_s \end{pmatrix}$$

Generalized F -partition

Put r dots on the diagonal, and then put $a_j - 1$ dots in row j to the right of the diagonal and b_j dots in columns $d(\lambda^f) + j$ below the diagonal.

Generalized F -partition

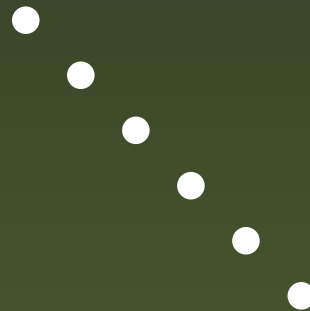
Put r dots on the diagonal, and then put $a_j - 1$ dots in row j to the right of the diagonal and b_j dots in columns $d(\lambda^f) + j$ below the diagonal.

$$\begin{pmatrix} 7 & 5 & 4 & 3 & 2 & 1 \\ * & * & 6 & 5 & 2 & 0 \end{pmatrix}$$

Generalized F -partition

Put r dots on the diagonal, and then put $a_j - 1$ dots in row j to the right of the diagonal and b_j dots in columns $d(\lambda^f) + j$ below the diagonal.

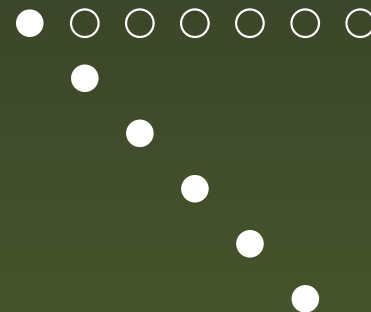
$$\begin{pmatrix} 7 & 5 & 4 & 3 & 2 & 1 \\ * & * & 6 & 5 & 2 & 0 \end{pmatrix}$$



Generalized F -partition

Put r dots on the diagonal, and then put $a_j - 1$ dots in row j to the right of the diagonal and b_j dots in columns $d(\lambda^f) + j$ below the diagonal.

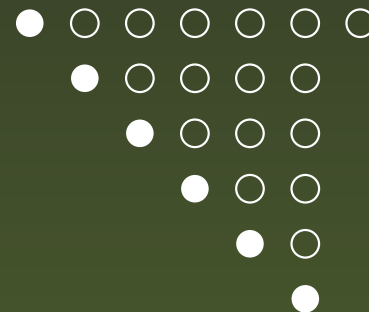
$$\begin{pmatrix} 7 & 5 & 4 & 3 & 2 & 1 \\ * & * & 6 & 5 & 2 & 0 \end{pmatrix}$$



Generalized F -partition

Put r dots on the diagonal, and then put $a_j - 1$ dots in row j to the right of the diagonal and b_j dots in columns $d(\lambda^f) + j$ below the diagonal.

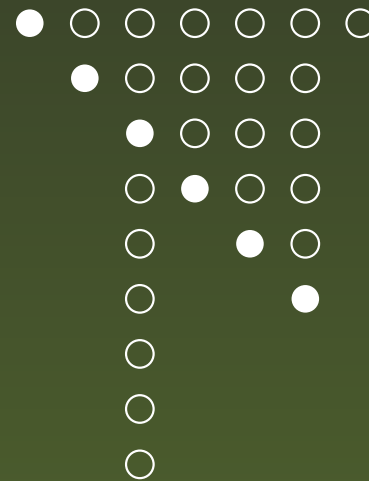
$$\begin{pmatrix} 7 & 5 & 4 & 3 & 2 & 1 \\ * & * & 6 & 5 & 2 & 0 \end{pmatrix}$$



Generalized F -partition

Put r dots on the diagonal, and then put $a_j - 1$ dots in row j to the right of the diagonal and b_j dots in columns $d(\lambda^f) + j$ below the diagonal.

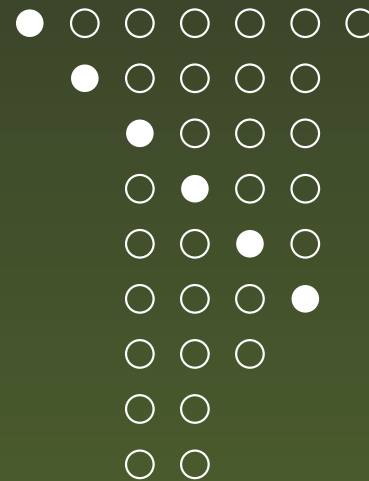
$$\begin{pmatrix} 7 & 5 & 4 & 3 & 2 & 1 \\ * & * & 6 & 5 & 2 & 0 \end{pmatrix}$$



Generalized F -partition

Put r dots on the diagonal, and then put $a_j - 1$ dots in row j to the right of the diagonal and b_j dots in columns $d(\lambda^f) + j$ below the diagonal.

$$\begin{pmatrix} 7 & 5 & 4 & 3 & 2 & 1 \\ * & * & 6 & 5 & 2 & 0 \end{pmatrix}$$



Generating Function of Generalized F -Partition

The generating function of generalized F -partition with size k is :

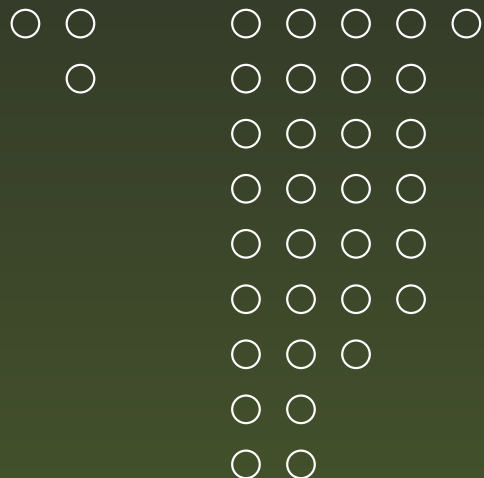
$$\sum_{\substack{\lambda^f : d(\lambda^f)=k \\ \alpha_1 \leq m, \beta_1 \leq n}} q^{|\lambda^f|} = q^{\binom{k+1}{2}} \begin{bmatrix} m+n+1 \\ n+k+1 \end{bmatrix}$$

We will apply **Wright's bijection** to explain this identity. Basic knowledge of partition tell us that $\begin{bmatrix} m+n+1 \\ n+k+1 \end{bmatrix}$ is the generating function of the partitions with at most $n+k+1$ parts, whose maximum part is less than or equal to $m-k$.

Generalized F -Partition

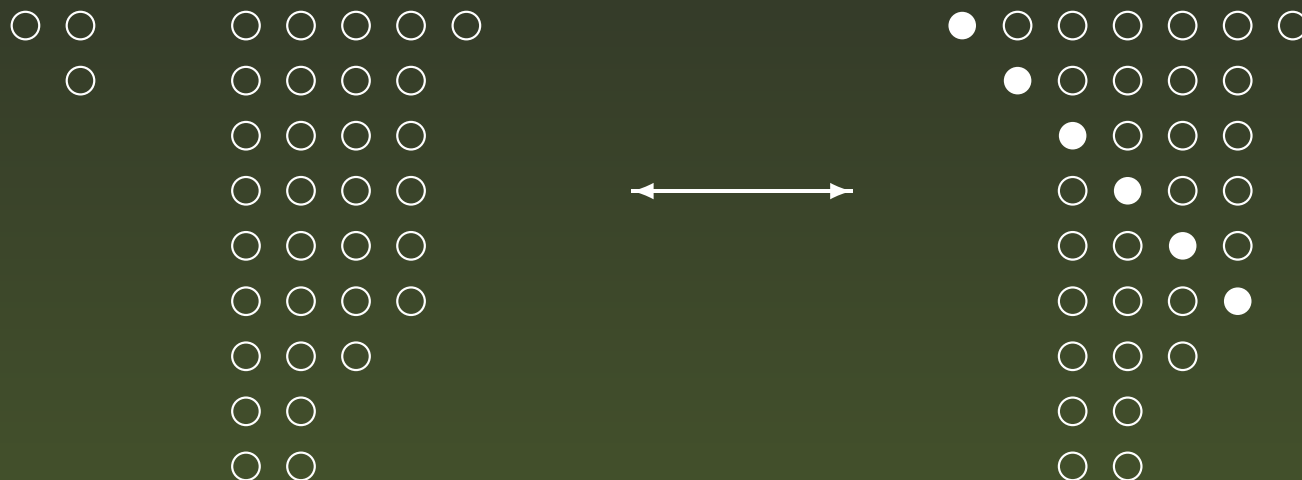
Generalized F -Partition

λ

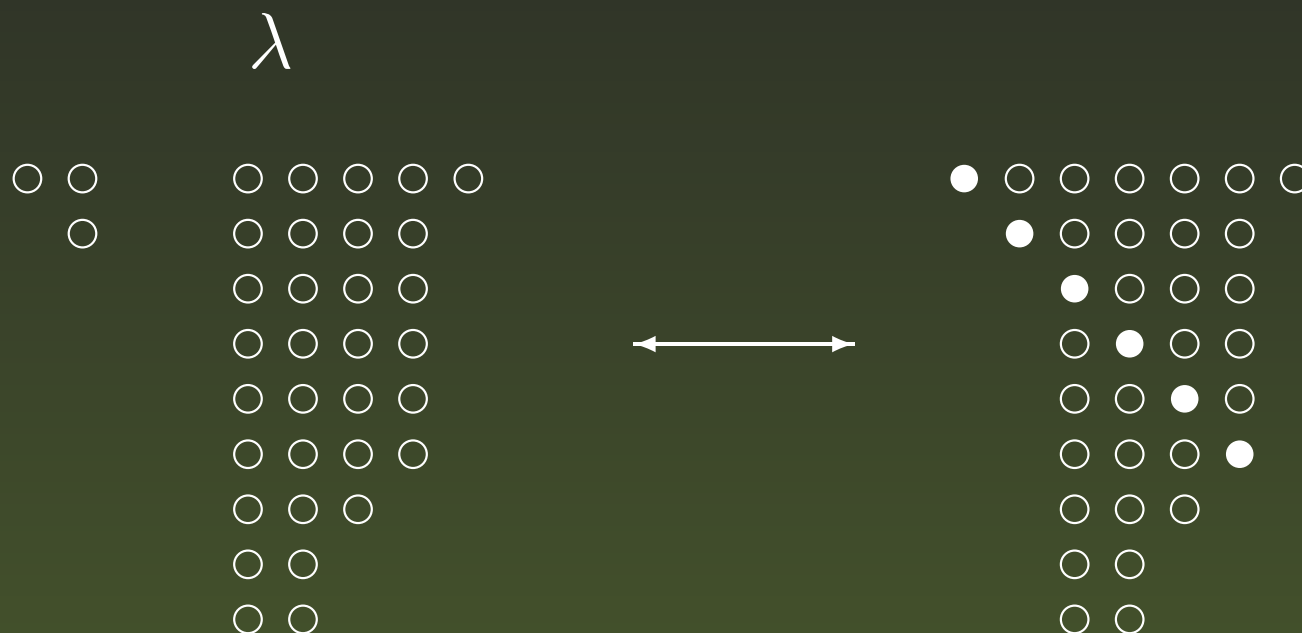


Generalized F -Partition

λ

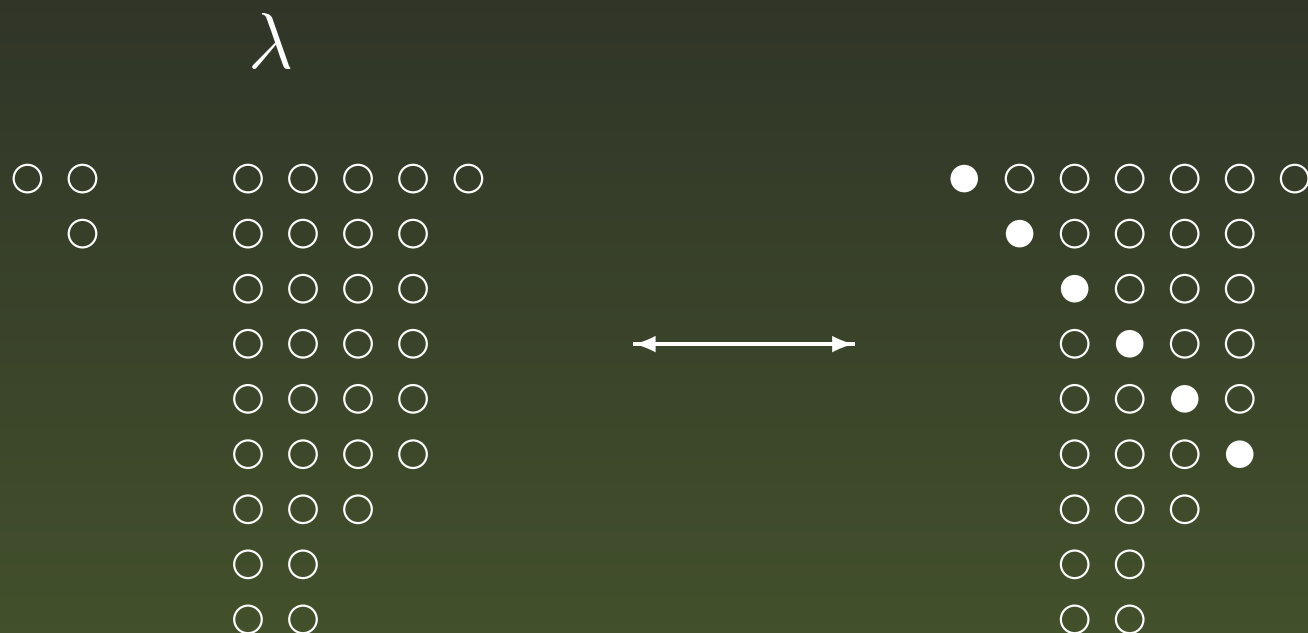


Generalized F -Partition



It is obvious that the right side is diagram of generalized F -partition with size 2.

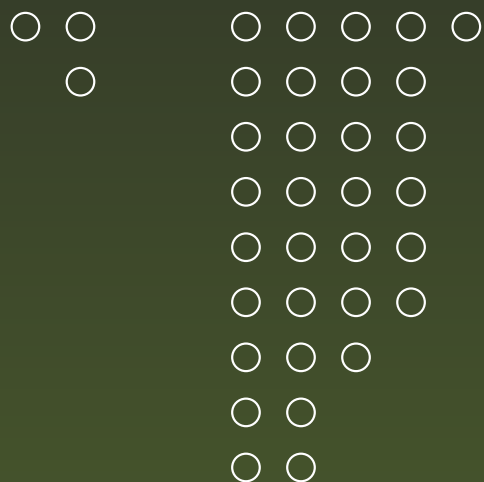
Generalized F -Partition



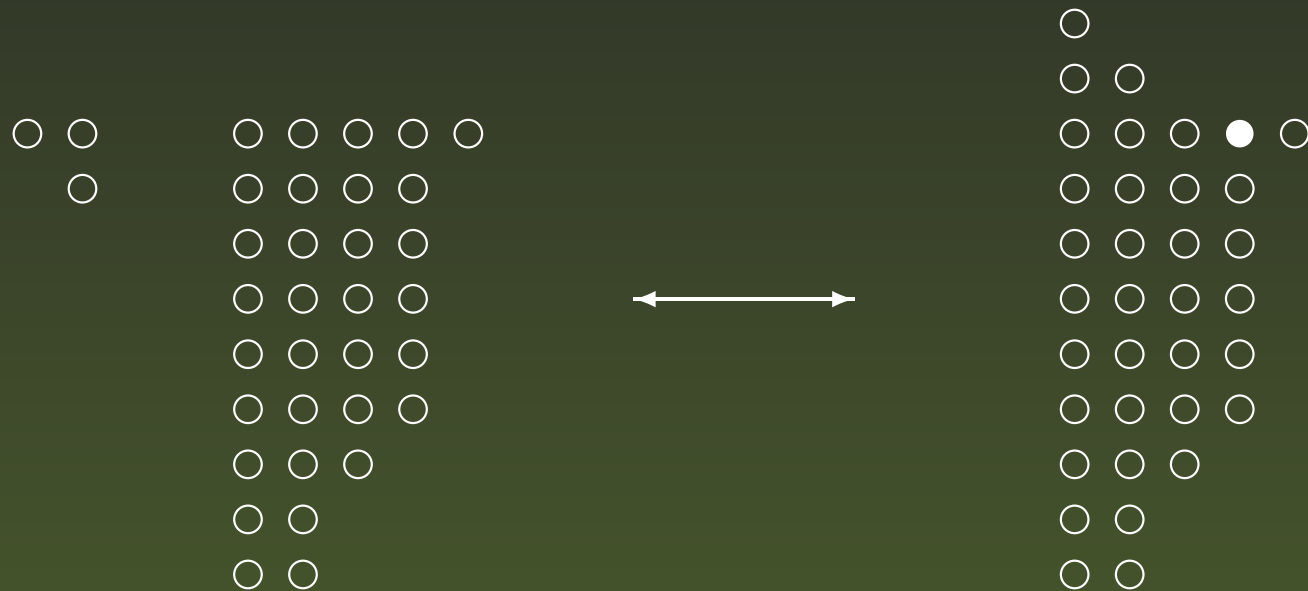
It is obvious that the right side is diagram of generalized F -partition with size 2.

Generalized F -Partition

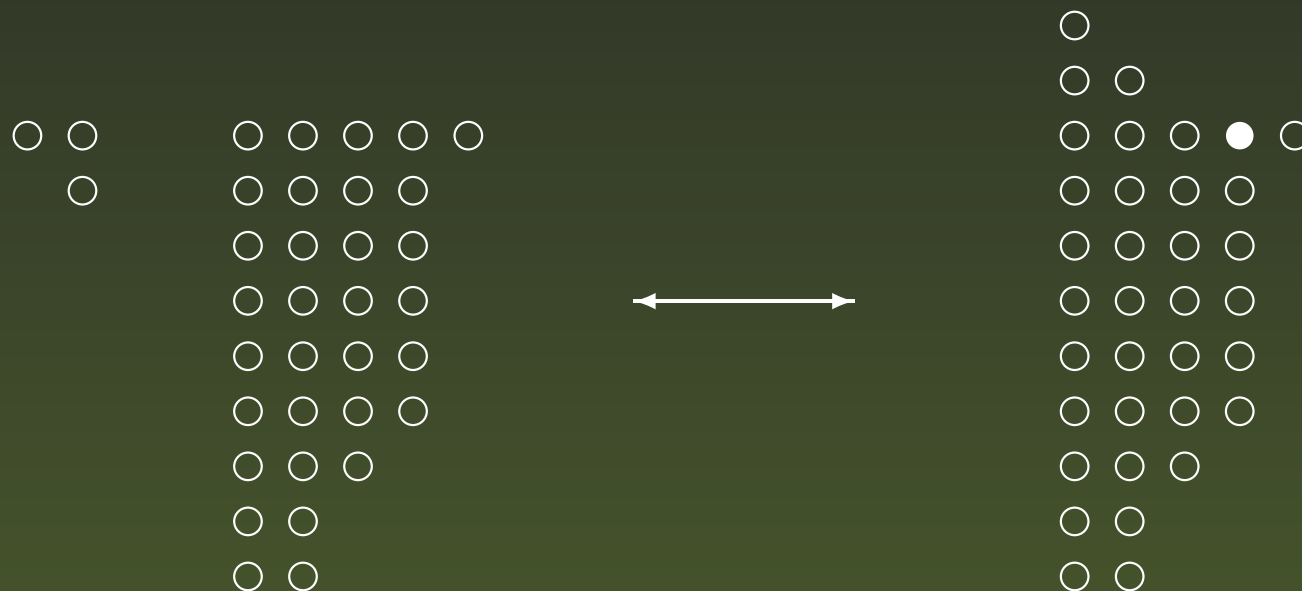
Generalized F -Partition



Generalized F -Partition



Generalized F -Partition



It is obvious that the right side is diagram of generalized F -partition with size -3 .

Combinatorial Interpretation

Let us restate finite form of Jacobi's cubic identity:

$$\sum_{k=-n-1}^m (-1)^k k q^{\binom{k+1}{2}} \begin{bmatrix} m+n+1 \\ n+k+1 \end{bmatrix} = (q; q)_m (q; q)_n.$$

We have know that the left side is the generating function of generalized F -partition with weighted, furthermore **every generalized F -partition repeats its size times**, i.e.,

$$\sum_{k=-n-1}^m (-1)^k k q^{\binom{k+1}{2}} \begin{bmatrix} m+n+1 \\ n+k+1 \end{bmatrix} = \sum_{\lambda^f : \substack{\alpha_1 \leq n \\ \beta_1 \leq m}} (-1)^{d(\lambda^f)} d(\lambda^f) q^{|\lambda^f|}.$$

Combinatorial Interpretation

While the right side is the generating function of generalized F -partition $\lambda^f = (\alpha, \beta)$ with weighted where α and β are all partitions into distinct positive parts, i.e.,

$$(q; q)_m (q; q)_n = \sum_{\lambda^f : \alpha_1 \leq n, \beta_1 \leq m} (-1)^{d(\lambda^f)} q^{|\lambda^f|}$$

Hence the finite form can transform the following partition identity which we call it **key partition identity**:

$$\sum_{\substack{\lambda^f : \alpha_1 \leq n \\ \beta_1 \leq m}} (-1)^{d(\lambda^f)} d(\lambda^f) q^{|\lambda^f|} = \sum_{\substack{\lambda^f : \alpha_1 \leq n \\ \beta_1 \leq m}} (-1)^{d(\lambda^f)} q^{|\lambda^f|}$$

Partition Identity

Now consider two sets A and B .

A is the set of generalized F -partitions $\lambda^f = (\alpha, \beta)$ where β has a part equal to zero. B is the set of generalized F -partitions $\lambda^f = (\alpha, \beta)$ where β has no part equal to zero.

Observe that generalized F -partitions on the left side belong to the set A and B . While generalized F -partitions on the right side just belong to the set B . Hence we can rewrite the key partition identity as follows:

$$\begin{aligned} & \sum_{\lambda^f \in A} (-1)^{d(\lambda^f)} d(\lambda^f) q^{|\lambda^f|} + \sum_{\lambda^f \in B} (-1)^{d(\lambda^f)} d(\lambda^f) q^{|\lambda^f|} \\ &= \sum_{\lambda^f \in B} (-1)^{d(\lambda^f)} d(\lambda^f) q^{|\lambda^f|} \end{aligned}$$

Partition Identity

Supposed that we have known the following **partition identity**, which we will prove later by constructing a bijection:

The number of generalized F -partitions of n belonging to A with size k equals to the number of generalized F -partitions of n belonging to B with size $k + 1$.

In term of generating function, we have

$$\sum_{\lambda^f \in A: d(\lambda^f)=k} q^{|\lambda^f|} = \sum_{\lambda^f \in B: d(\lambda^f)=k+1} q^{|\lambda^f|}$$

Partition Identity

Hence the right side of the second form of **key partition identity**:

$$\begin{aligned}
 & \sum_{\lambda^f \in A: d(\lambda^f)=k} (-1)^k k q^{|\lambda^f|} + \sum_{\lambda^f \in B: d(\lambda^f)=k+1} (-1)^{k+1} (k+1) q^{|\lambda^f|} \\
 &= \sum_{\lambda^f \in B: d(\lambda^f)=k+1} (-1)^k k q^{|\lambda^f|} + \sum_{\lambda^f \in B: d(\lambda^f)=k+1} (-1)^{k+1} (k+1) q^{|\lambda^f|} \\
 &= \sum_{\lambda^f \in B: d(\lambda^f)=k+1} (-1)^{k+1} q^{|\lambda^f|}
 \end{aligned}$$

Thus we prove the **key partition identity**, hence we give a combinatorial proof of the finite form of Jacobi's cubic identity.

Bijection

Finally, we only need to prove **partition identity**, namely,

The number of generalized F -partitions of n belonging to A with size k equals to the number of generalized F -partitions of n belonging to B with size $k + 1$.

Proof. We now construct an involution Φ on the generalized F -partitions $\lambda^f = (\alpha, \beta)$:

- If $\lambda^f \in A$, i.e., β has one part equal to zero, then delete the part equal to zero, shift all other parts of β to left.
- If $\lambda^f \in B$, i.e., β has no part equal to zero, then add a part equal to zero, put it on the leftmost side of the bottom row and move all other parts to right.

Bijection

Bijection

$$\begin{pmatrix} a_1 & \cdots & a_k & a_{k+1} & \cdots & a_{k+s} \\ * & \cdots & * & b_1 & \cdots & 0 \end{pmatrix}$$

Bijection

$$\begin{pmatrix} a_1 & \cdots & a_k & a_{k+1} & \cdots & a_{k+s} \\ * & \cdots & * & b_1 & \cdots & 0 \end{pmatrix}$$

$$\Updownarrow$$

$$\begin{pmatrix} a_1 & a_2 & \cdots & a_{k+1} & a_{k+2} & \cdots & a_{k+s} \\ * & * & \cdots & * & b_1 & \cdots & b_{s-1} \end{pmatrix}$$

Bijection

$$\begin{pmatrix} a_1 & \cdots & a_k & a_{k+1} & \cdots & a_{k+s} \\ * & \cdots & * & b_1 & \cdots & 0 \end{pmatrix}$$

$$\Updownarrow$$

$$\begin{pmatrix} a_1 & a_2 & \cdots & a_{k+1} & a_{k+2} & \cdots & a_{k+s} \\ * & * & \cdots & * & b_1 & \cdots & b_{s-1} \end{pmatrix}$$

Bijection

It is obvious that for the generalized F -partition $\lambda^f \in A$ with size k , its image under this involution is a generalized F -partition belonging to B with size $k + 1$; While for the generalized F -partition $\lambda^f \in B$ with size k , its image under this involution is a generalized F -partition belonging to A with size $k - 1$. Thus we prove this partition identity.



Thanks!