

Modular Nahm sums for symmetrizable matrices with general rank

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The Rogers-Ramanujan identities

Theorem (Rogers-Ramanujan)

For $|q| < 1$,

$$\sum_{n \geq 0} \frac{q^{n^2}}{(1-q)(1-q^2) \cdots (1-q^n)} = \prod_{j=0}^{\infty} \frac{1}{(1-q^{5j+1})(1-q^{5j+4})} \quad (RR1)$$

$$\sum_{n \geq 0} \frac{q^{n^2+n}}{(1-q)(1-q^2) \cdots (1-q^n)} = \prod_{j=0}^{\infty} \frac{1}{(1-q^{5j+2})(1-q^{5j+3})} \quad (RR2).$$

These two identities were first proved by Rogers in 1894 and rediscovered by Ramanujan a few years later.

- **Ramanujan's comment:** It would be difficult to find more beautiful formulas than the Rogers-Ramanujan' identities.
- **Hans Rademacher** clearly was in agreement with Ramanujan's comments.

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- There is a beautiful story about the discovery of the Rogers-Ramanujan identities.
- Rogers' reputation as a mathematician rests almost entirely on the discovery of these two identities.
- how they found these two identities have appeared in Baxter's exact solution to the hard hexagon model in statistical mechanics.

The connection with representation theory of Lie algebras

- The connection between RR identities and the representation theory of Lie algebras was initiated by J. Lepowsky, R.L. Wilson, S. Milne.



J. Lepowsky and S. Milne, Lie algebraic approaches to classical partition identities, *Adv. Math.* 29 (1978), 15–59



J. Lepowsky and R. L. Wilson, A new family of algebras underlying the Rogers-Ramanujan identities, *Proc. Nat. Acad. Sci. USA* 78 (1981) 7254–7258.



J. Lepowsky and R. L. Wilson, The structure of standard modules I: universal algebras and the Rogers–Ramanujan identities, *Invent. Math.* 77 (1984), 199–290.



J. Lepowsky and R. L. Wilson, The structure of standard modules II: the case $A_1^{(1)}$, principal gradation, *Invent. Math.* 79 (1985), 417–442.

Connection with modular forms

- Modular forms: If we look at these two identities from the perspective of modular forms, the product sides of them are modular forms, but it is not clear from the sum sides.

$$\underbrace{\sum_{n=0}^{\infty} \frac{q^{n^2+in}}{(q; q)_n}}_{???} = \frac{1}{\underbrace{(q^i, q^{5-i}; q^5)_{\infty}}_{\text{modular forms}}}, \quad i = 0, 1$$

- Question: What kind of the basic hypergeometric series is a modular form?

Modular functions

Let $\mathbb{H} = \{\tau \in \mathbb{C} : \text{Im}(\tau) > 0\}$. A meromorphic function $f: \mathbb{H} \rightarrow \mathbb{C}$ is called a **modular function** of weight k for $\Gamma' \subset \Gamma$, if it satisfies the following two conditions:

- (1) for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma'$, $f(\gamma\tau) = (c\tau + d)^k f(\tau)$;
- (2) for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, $(c\tau + d)^{-k} f(\gamma\tau)$ has a Fourier expansion of the form

$$(c\tau + d)^{-k} f(\gamma\tau) = \sum_{n=n_\gamma}^{\infty} a(n) q_{w_\gamma}^n,$$

where $a(n_\gamma) \neq 0$, $q_{w_\gamma} = e^{2\pi i \tau / w_\gamma}$, and w_γ is the minimal positive integer h such that

$$\begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix} \in \gamma^{-1} \Gamma_1(N) \gamma.$$

Robins' criterion

For the generalized eta-quotient $f(\tau)$ of the following form:

$$f(\tau) = \prod_{\substack{\delta|N \\ 0 \leq g < \delta}} \eta_{\delta,g}^{r_{\delta,g}}(\tau),$$

where the generalized Dedekind eta-function $\eta_{\delta,g}(\tau)$ is defined by

$$\eta_{\delta,g}(\tau) = q^{\frac{\delta}{2}P_2(\frac{g}{\delta})} \prod_{\substack{n>0 \\ n \equiv g \pmod{\delta}}} (1 - q^n) \prod_{\substack{n>0 \\ n \equiv -g \pmod{\delta}}} (1 - q^n),$$

where $q = e^{2\pi i\tau}$ and

$$P_2(t) = \{t\}^2 - \{t\} + \frac{1}{6}$$

Robins' criterion

Assume that $f(\tau)$ satisfies the following three conditions:

$$(1) w(f) = \sum_{\delta|N} r_{\delta,0} = 0; \quad (2) \text{Ord}_{\infty}(f) = \sum_{\substack{\delta|N \\ 0 \leq g < \delta}} \delta P_2\left(\frac{g}{\delta}\right) r_{\delta,g} = \frac{m_1}{n_1};$$

$$(3) \text{Ord}_0(f) = \sum_{\substack{\delta|N \\ 0 \leq g < \delta}} \frac{N}{\delta} P_2(0) r_{\delta,g} = \frac{m_2}{n_2},$$

where $m_j, n_j \in \mathbb{Z}$, $n_j > 0, j = 1, 2$. Let t and N_0 be the least positive integers such that both tm_1/n_1 and N_0m_2/n_2 are even integers. Then $f(t\tau)$ is a modular function for $\Gamma_1(tN_0N)$.



S. Robins, Generalized Dedekind η -products. In: G.E. Andrews, D.M. Bressoud and L.A. Parson, (eds.) The Rademacher Legacy to Mathematics (University Park, PA, 1992), Contemp. Math., 166, pp. 119–128. Amer. Math. Soc., Providence, RI, 1994.

Nahm's problem

- Suppose $r \geq 1$ is a positive integer, A is a real positive definite symmetric $r \times r$ matrix, B is a vector of length r , and C is a scalar. Define

$$f_{A,B,C}(q) := \sum_{\vec{n}=(n_1,\dots,n_r) \in (\mathbb{Z}_{\geq 0})^r} \frac{q^{\frac{1}{2} \vec{n}^T \cdot A \cdot \vec{n} + \vec{n}^T \cdot B + C}}{(q; q)_{n_1} \cdots (q; q)_{n_r}}$$

- A character in rational conformal field theories (有理共形场论) is often expressed as a Nahm sum and also possesses modularity.
- In general, Nahm sums rarely have modularity.
- **Nahm's Problem 1994, 2007** Describe all such A, B and C with rational entries for which $f_{A,B,C}(q)$ is a modular function.

Rogers-Ramanujan identities and Nahm sum

- Rogers-Ramanujan identities

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q; q)_n} = \frac{1}{(q, q^4; q^5)_{\infty}}, \quad \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q; q)_n} = \frac{1}{(q^2, q^3; q^5)_{\infty}}.$$

- Nahm's sum

$$f_{A,B,C}(q) := \sum_{\vec{n}=(n_1, \dots, n_r) \in (\mathbb{Z}_{\geq 0})^r} \frac{q^{\frac{1}{2} \vec{n}^T \cdot A \cdot \vec{n} + \vec{n}^T \cdot B + C}}{(q; q)_{n_1} \cdots (q; q)_{n_r}}$$

- $(A, \mathbf{b}, c) = (2, 0, -1/60)$ and $(A, \mathbf{b}, c) = (2, 1, 11/60)$ are modular triples.

Vector-valued modular functions

- The function

$$g(\tau) = \left(q^{-\frac{1}{60}} \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q; q)_n}, q^{\frac{11}{60}} \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q; q)_n} \right)^T$$

is a vector-valued modular function which satisfies the following transformation formulas:

$$g(\tau + 1) = \begin{pmatrix} \zeta_{60}^{-1} & 0 \\ 0 & \zeta_{60}^{11} \end{pmatrix} g(\tau), \quad g\left(-\frac{1}{\tau}\right) = \frac{2}{\sqrt{5}} \begin{pmatrix} \sin \frac{2\pi}{5} & \sin \frac{\pi}{5} \\ \sin \frac{\pi}{5} & -\sin \frac{2\pi}{5} \end{pmatrix} g(z),$$

where $\zeta_N = e^{2\pi i/N}$.

Nahm's conjecture

- Nahm made a conjecture which provides a necessary and sufficient condition about the matrix A so that it is the matrix part of some modular triples.
- Zagier made an important progress towards Nahm's problem. He confirmed Nahm's conjecture for $r = 1$ by showing that there are exactly seven modular triples:

$$(1/2, 0, -1/40), (1/2, 1/2, 1/40), (1, 0, -1/48), (1, 1/2, 1/24), \\ (1, -1/2, 1/24), (2, 0, -1/60), (2, 1, 11/60).$$



W. Nahm, Conformal field theory and torsion elements of the Bloch group, in: *Frontiers in Number Theory, Physics and Geometry, II*, Springer-Verlag, 2007, pp. 67–132.



D. Zagier, The dilogarithm function, in: *Frontiers in Number Theory, Physics and Geometry, II*, Springer-Verlag, 2007, pp. 3–65.

Nahm's problem

- For $r = 2$ and 3 , Zagier found many possible modular triples, which were later affirmed by
 - Cao, Rosengren and Wang (2024, JCTA),
 - Cherednik and Feigin (2013, Adv. Math),
 - Vlasenko and Zwegers (2011, CNTP),
 - Wang (2024, Adv. Math., RMS),
 - Zagier (2007).

Nahm sums associated with symmetrizable matrices

- Mizuno (2025) considered generalized Nahm sums associated with symmetrizable matrices, which take the form:

$$\tilde{f}_{A,\mathbf{b},c,\mathbf{d}}(q) := \sum_{\mathbf{n}=(n_1,\dots,n_r)^T \in \mathbb{N}^r} \frac{q^{\frac{1}{2}\mathbf{n}^T A D \mathbf{n} + \mathbf{n}^T \mathbf{b} + c}}{(q^{\mathbf{d}_1}; q^{\mathbf{d}_1})_{n_1} \cdots (q^{\mathbf{d}_r}; q^{\mathbf{d}_r})_{n_r}},$$

where $\mathbf{d} = (d_1, \dots, d_r) \in \mathbb{Z}_{>0}^r$, $\mathbf{b} \in \mathbb{Q}^r$ is a vector and $c \in \mathbb{Q}$ is a scalar.

- Denote by $D := \text{diag}(d_1, \dots, d_r)_{r \times r}$ the $r \times r$ diagonal matrix with diagonal entries $d_1, \dots, d_r$.
- A matrix $A \in \mathbb{Q}^{r \times r}$ is said to be a **symmetrizable matrix** of index $\mathbf{d} = (d_1, \dots, d_r)$ if AD is symmetric positive definite.

Nahm sums associated with symmetrizable matrices

- We call $\tilde{f}_{A,\mathbf{b},\mathbf{c},\mathbf{d}}(q)$ a Nahm sum for the symmetrizable matrix A of index $(d_1, \dots, d_r)$.
- When $\tilde{f}_{A,\mathbf{b},\mathbf{c},\mathbf{d}}(q)$ is a modular function, such a triple $(A, \mathbf{b}, \mathbf{c})$ is called a modular triple of index $(d_1, \dots, d_r)$.
- One of Capparelli's partition identities asserts that

$$\sum_{n_1, n_2 \geq 0} \frac{q^{2n_1^2 + 6n_1n_2 + 6n_2^2}}{(q; q)_{n_1} (q^3; q^3)_{n_2}} = (-q^2, -q^3, -q^4, -q^6; q^6)_{\infty}.$$

- Notice that the above identity reveals that the generalized Nahm sum $\tilde{f}_{A,\mathbf{b},\mathbf{c},\mathbf{d}}(q)$ associated with $A = \begin{pmatrix} 4 & 2 \\ 6 & 4 \end{pmatrix}$, $\mathbf{b} = (0, 0)^T$, $\mathbf{c} = -1/24$ and $\mathbf{d} = (1, 3)$ is modular.

Nahm sums associated with symmetrizable matrices

- Mizuno found that the theory of Nahm sums for symmetric matrices can be analogously applied to Nahm sums for symmetrizable matrices.

A	b	c	d
$\begin{pmatrix} 1 & 1/2 \\ 1 & 1 \end{pmatrix}$	$(0, 0)^T$	$-\frac{3}{56}$	$(1, 2)$
	$(0, 1)^T$	$\frac{1}{56}$	
	$(1, 1)^T$	$\frac{9}{56}$	
$\begin{pmatrix} 3/2 & 1/2 \\ 1 & 1 \end{pmatrix}$	$(-1, 1)^T$	$\frac{9}{40}$	$(1, 2)$
	$(-1/2, 0)^T$	$-\frac{1}{40}$	
	$(0, 1)^T$	$\frac{1}{40}$	

- Warnaar (2003, DM); Li and Wang (2024, RJ), Wang and Wang (2026, TAMS)
- Mizuno (2025, RJ)

Nahm sums associated with symmetrizable matrices

A	b	c	d
$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 2 \\ 1/2 & 1 & 2 \end{pmatrix}$	$(0, 0, 0)^T$	$\frac{7}{88}$	$(2, 2, 1)$
	$(1, 0, 0)^T$	$\frac{1}{88}$	
	$(1, 0, 1)^T$	$\frac{9}{88}$	
	$(1, 2, 2)^T$	$\frac{49}{88}$	
$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 2 \\ 1/2 & 1 & 5/2 \end{pmatrix}$	$(0, 0, -1/2)^T$	$-\frac{5}{72}$	$(2, 2, 1)$
	$(1, 0, -1)^T$	$\frac{1}{8}$	
	$(1, 0, 0)^T$	$\frac{1}{72}$	
	$(1, 2, 1)^T$	$\frac{25}{72}$	

表: $r = 3$

- Wang and Wang (2025, Adv. Math.)



B. Wang and L. Wang, Proofs of Mizuno's conjectures on rank three Nahm sums of index $(1, 2, 2)$, Adv. Math. 477 (2025), Paper No. 110368, 73 pp.

Vector-valued functions associated with the 3×3 matrix A

- For matrix $A = \begin{pmatrix} 1 & 0 & 1/2 \\ 0 & 2 & 1 \\ 1 & 2 & 2 \end{pmatrix}$ of index $\mathbf{d} = (1, 1, 2)$, let

$$\mathbf{b}_0 = (0, 0, 0)^T, \mathbf{b}_1 = (0, 0, 1)^T, \mathbf{b}_2 = (0, 1, 1)^T, \\ \mathbf{b}_3 = (1, 2, 3)^T, \mathbf{b}_4 = (0, 1, 2)^T, \mathbf{b}_5 = (1, 1, 2)^T$$

and

$$c_0 = -\frac{5}{88}, c_1 = -\frac{1}{88}, c_2 = \frac{7}{88}, c_3 = \frac{7}{88}, c_4 = \frac{19}{88}, c_5 = \frac{35}{88}.$$

- Define the vector-valued function

$$\mathbf{g}(\tau) = (\tilde{f}_{A, \mathbf{b}_0, c_0, \mathbf{d}, 0}(q), \tilde{f}_{A, \mathbf{b}_1, c_1, \mathbf{d}, 1}(q), \tilde{f}_{A, \mathbf{b}_2, c_2, \mathbf{d}, 1}(q) + q\tilde{f}_{A, \mathbf{b}_3, c_3, \mathbf{d}, 1}(q), \\ \tilde{f}_{A, \mathbf{b}_4, c_4, \mathbf{d}, 0}(q), \tilde{f}_{A, \mathbf{b}_5, c_5, \mathbf{d}, 0}(q))^T.$$

Vector-valued functions associated with the 3×3 matrix A

- For matrix $A^\vee = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 2 \\ 1/2 & 1 & 2 \end{pmatrix}$ of index $\mathbf{d}^\vee = (2, 2, 1)$. Let

$$\mathbf{b}'_0 = (0, 0, 0)^T, \mathbf{b}'_1 = (0, 0, 1)^T, \mathbf{b}'_2 = (1, 2, 3)^T, \\ \mathbf{b}'_3 = (1, 0, 0)^T, \mathbf{b}'_4 = (1, 0, 1)^T, \mathbf{b}'_5 = (1, 2, 2)^T$$

and

$$c'_0 = -\frac{7}{88}, c'_1 = \frac{25}{88}, c'_2 = \frac{25}{88}, c'_3 = \frac{1}{88}, c'_4 = \frac{9}{88}, c'_5 = \frac{49}{88}.$$

- Define the vector-valued function

$$\mathbf{g}^\vee(\tau) = (\tilde{f}_{A^\vee, \mathbf{b}_0, c_0, \mathbf{d}^\vee}(q), \tilde{f}_{A^\vee, \mathbf{b}_1, c_1, \mathbf{d}^\vee}(q) + \tilde{f}_{A^\vee, \mathbf{b}_2, c_2, \mathbf{d}^\vee}(q), \\ \tilde{f}_{A^\vee, \mathbf{b}_3, c_3, \mathbf{d}^\vee}(q), \tilde{f}_{A^\vee, \mathbf{b}_4, c_4, \mathbf{d}^\vee}(q), \tilde{f}_{A^\vee, \mathbf{b}_5, c_5, \mathbf{d}^\vee}(q))^T.$$

Nahm sums associated with symmetrizable matrices

- Mizuno (2025) conjectured, later confirmed by Wang and Wang (2025, Adv. Math.), that

$$\mathbf{g}\left(-\frac{1}{\tau}\right) = S\mathbf{g}^{\vee}\left(\frac{\tau}{2}\right), \quad \mathbf{g}^{\vee}\left(-\frac{1}{\tau}\right) = 2S\mathbf{g}\left(\frac{\tau}{2}\right),$$

where

$$S = \begin{pmatrix} \alpha_5 & \alpha_4 & \alpha_3 & \alpha_2 & \alpha_1 \\ \alpha_4 & \alpha_1 & -\alpha_2 & -\alpha_5 & -\alpha_3 \\ \alpha_3 & -\alpha_2 & -\alpha_4 & \alpha_1 & \alpha_5 \\ \alpha_2 & -\alpha_5 & \alpha_1 & \alpha_3 & -\alpha_4 \\ \alpha_1 & -\alpha_3 & \alpha_5 & -\alpha_4 & \alpha_2 \end{pmatrix}, \quad \alpha_k = \sqrt{\frac{2}{11}} \sin \frac{k\pi}{11}.$$

- Mizuno called that $\mathbf{g}$ and $\mathbf{g}^{\vee}$ are “Langlands dual” pairs.

Nahm sums for the symmetrizable matrix A of index $\mathbf{d} = (2, \dots, 2, 1)$

Theorem 1 (Du., Ji, Shen and Xu, 2026+)

For $r \geq 2$ and $0 \leq j \leq r$, the Nahm sums $\tilde{f}_{A, \mathbf{b}_j, \mathbf{c}_j, \mathbf{d}}(q^{32r-8})$ for the symmetrizable matrix A of index $\mathbf{d} = (2, \dots, 2, 1)$ are modular functions for $\Gamma_1(128(4r-1)^2)$, where

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 \\ 0 & 2 & 2 & 2 & \cdots & 2 & 2 & 2 \\ 0 & 2 & 4 & 4 & \cdots & 4 & 4 & 4 \\ 0 & 2 & 4 & 6 & \cdots & 6 & 6 & 6 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 2 & 4 & 6 & \cdots & 2(r-3) & 2(r-3) & 2(r-3) \\ 0 & 2 & 4 & 6 & \cdots & 2(r-3) & 2(r-2) & 2(r-2) \\ \frac{1}{2} & 1 & 2 & 3 & \cdots & r-3 & r-2 & r-1 \end{pmatrix}_{r \times r},$$

$$\mathbf{b}_0 = (0, \dots, 0)_{1 \times r}^T, \quad \mathbf{c}_0 = \frac{5-4r}{32r-8},$$

and for $1 \leq j \leq r$,

$$\mathbf{b}_j = (1, 0, \dots, 0, 2, 4, \dots, 2(r-1-j), r-j)_{1 \times r}^T, \quad \mathbf{c}_j = \frac{(4r-4j-1)^2}{32r-8}.$$

Symmetrizable matrix A

$$AD = \begin{pmatrix} 2 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 \\ 0 & 4 & 4 & 4 & \cdots & 4 & 4 & 2 \\ 0 & 4 & 8 & 8 & \cdots & 8 & 8 & 4 \\ 0 & 4 & 8 & 12 & \cdots & 12 & 12 & 6 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \\ 0 & 4 & 8 & 12 & \cdots & 4(r-3) & 4(r-3) & 2(r-3) \\ 0 & 4 & 8 & 12 & \cdots & 4(r-3) & 4(r-2) & 2(r-2) \\ 1 & 2 & 4 & 6 & \cdots & 2(r-3) & 2(r-2) & r-1 \end{pmatrix}_{r \times r}.$$

q -identities corresponding to Nahm sums

Theorem 2 (Du, Ji, Shen and Xu, 2026+)

For $r \geq 2$,

$$(1) \quad \sum_{\mathbf{n}=(n_1, \dots, n_r)^T \in \mathbb{N}^r} \frac{q^{\frac{1}{2}\mathbf{n}^T A D \mathbf{n} + \mathbf{n}^T \mathbf{b}_0}}{(q^2; q^2)_{n_1} (q^2; q^2)_{n_2} \cdots (q^2; q^2)_{n_{r-1}} (q; q)_{n_r}} \\ = \frac{(q^{8r-4}, q^{8r}, q^{16r-4}; q^{16r-4})_{\infty}}{(q, q^3, q^4; q^4)_{\infty}} + q^{\frac{r-1}{2}} \frac{(-q, q^{4r-2}, -q^{4r-1}; -q^{4r-1})_{\infty}}{(q^2, q^2, q^4; q^4)_{\infty}},$$

(2) For $1 \leq j \leq r$,

$$\sum_{\mathbf{n}=(n_1, \dots, n_r)^T \in \mathbb{N}^r} \frac{q^{\frac{1}{2}\mathbf{n}^T A D \mathbf{n} + \mathbf{n}^T \mathbf{b}_j}}{(q^2; q^2)_{n_1} \cdots (q^2; q^2)_{n_{r-1}} (q; q)_{n_r}} \\ = \frac{(q^{2j}, -q^{4r-2j-1}, -q^{4r-1}; -q^{4r-1})_{\infty}}{(q^2, q^2, q^4; q^4)_{\infty}} + q^{\frac{3r-2j-1}{2}} \frac{(q^{4j}, q^{16r-4j-4}, q^{16r-4}; q^{16r-4})_{\infty}}{(q, q^3, q^4; q^4)_{\infty}}.$$

Nahm sums associated with symmetrizable matrix B of index $\mathbf{d} = (2, \dots, 2, 1)$

Theorem 3 (Du, Ji, Shen and Xu, 2026+)

For $r \geq 2$ and $0 \leq j \leq r$, the Nahm sums $\tilde{f}_{B, \mathbf{b}_j, \mathbf{c}_j, \mathbf{d}}(q^{32r-24})$ for the symmetrizable matrix B of index $\mathbf{d} = (2, \dots, 2, 1)$ are modular functions for $\Gamma_1(64(4r-3)^2)$, where

$$B = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 \\ 0 & 2 & 2 & 2 & \cdots & 2 & 2 & 2 \\ 0 & 2 & 4 & 4 & \cdots & 4 & 4 & 4 \\ 0 & 2 & 4 & 6 & \cdots & 6 & 6 & 6 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 2 & 4 & 6 & \cdots & 2(r-3) & 2(r-3) & 2(r-3) \\ 0 & 2 & 4 & 6 & \cdots & 2(r-3) & 2(r-2) & 2(r-2) \\ \frac{1}{2} & 1 & 2 & 3 & \cdots & r-3 & r-2 & r-\frac{1}{2} \end{pmatrix}_{r \times r},$$

$$\mathbf{b}_0 = \left(0, \dots, 0, -\frac{1}{2}\right)_{1 \times r}^T, \quad c_0 = \frac{7-4r}{32r-24},$$

and for $1 \leq j \leq r$,

$$\mathbf{b}_j = (1, 0, \dots, 0, 2, 4, \dots, 2(r-1-j), r-j-1)_{1 \times r}^T, \quad c_j = \frac{(4r-4j-3)^2}{32r-24}.$$

Symmetrizable matrix B

$$BD = \begin{pmatrix} 2 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 \\ 0 & 4 & 4 & 4 & \cdots & 4 & 4 & 2 \\ 0 & 4 & 8 & 8 & \cdots & 8 & 8 & 4 \\ 0 & 4 & 8 & 12 & \cdots & 12 & 12 & 6 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 4 & 8 & 12 & \cdots & 4(r-3) & 4(r-3) & 2(r-3) \\ 0 & 4 & 8 & 12 & \cdots & 4(r-3) & 4(r-2) & 2(r-2) \\ 1 & 2 & 4 & 6 & \cdots & 2(r-3) & 2(r-2) & r - \frac{1}{2} \end{pmatrix}_{r \times r}.$$

q -identities corresponding to Nahm sums

Theorem 4 (Du, Ji, Shen and Xu, 2026+)

For $r \geq 2$,

$$(1) \quad \sum_{\mathbf{n}=(n_1, \dots, n_r)^T \in \mathbb{N}^r} \frac{q^{\frac{1}{2}\mathbf{n}^T B D \mathbf{n} + \mathbf{n}^T \mathbf{b}_0}}{(q^2; q^2)_{n_1} (q^2; q^2)_{n_2} \cdots (q^2; q^2)_{n_{r-1}} (q; q)_{n_r}} \\ = \frac{(q^{8r-8}, q^{8r-4}, q^{16r-12}; q^{16r-12})_{\infty}}{(q, q^3, q^4; q^4)_{\infty}} + q^{\frac{2r-3}{4}} \frac{(-q, q^{4r-4}, -q^{4r-3}; -q^{4r-3})_{\infty}}{(q^2, q^2, q^4; q^4)_{\infty}},$$

(2) For $1 \leq j \leq r$,

$$\sum_{\mathbf{n}=(n_1, \dots, n_r) \in \mathbb{N}^r} \frac{q^{\frac{1}{2}\mathbf{n}^T B D \mathbf{n} + \mathbf{n}^T \mathbf{b}_j}}{(q^2; q^2)_{n_1} (q^2; q^2)_{n_2} \cdots (q^2; q^2)_{n_{r-1}} (q; q)_{n_r}} \\ = \frac{(q^{2j}, -q^{4r-3-2j}, -q^{4r-3}; -q^{4r-3})_{\infty}}{(q^2, q^2, q^4; q^4)_{\infty}} + q^{\frac{6r-4j-5}{4}} \frac{(q^{4j}, q^{16r-4j-12}, q^{16r-12}; q^{16r-12})_{\infty}}{(q, q^3, q^4; q^4)_{\infty}}.$$

Nahm sums associated with symmetrizable matrix $A^\vee$ of index $\mathbf{d}^\vee = (1, \dots, 1, 2)$

Theorem 5 (Du, Ji, Shen and Xu, 2026+)

For $r \geq 2$ and $0 \leq j \leq r$, the Nahm sums $\tilde{f}_{A^\vee, \mathbf{b}_j, \mathbf{c}_j, \mathbf{d}^\vee}(q^{32r-8})$ for the symmetrizable matrix $A^\vee$ of index $\mathbf{d}^\vee = (1, \dots, 1, 2)$ are modular functions for $\Gamma_1(128(4r-1)^2)$, where

$$A^\vee = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 & \frac{1}{2} \\ 0 & 2 & 2 & 2 & \cdots & 2 & 2 & 1 \\ 0 & 2 & 4 & 4 & \cdots & 4 & 4 & 2 \\ 0 & 2 & 4 & 6 & \cdots & 6 & 6 & 3 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 2 & 4 & 6 & \cdots & 2(r-3) & 2(r-3) & r-3 \\ 0 & 2 & 4 & 6 & \cdots & 2(r-3) & 2(r-2) & r-2 \\ 1 & 2 & 4 & 6 & \cdots & 2(r-3) & 2(r-2) & r-1 \end{pmatrix}_{r \times r},$$

$$\mathbf{b}_0 = (1, 1, 2, 3, \dots, r-1)_{1 \times r}^T, \quad \mathbf{c}_0 = \frac{8r^2 - 14r + 5}{32r - 8},$$

and for $1 \leq j \leq r$,

$$\mathbf{b}_j = (0, \dots, 0, 1, 2, \dots, r-j)_{1 \times r}^T, \quad \mathbf{c}_j = \frac{8(r-j)^2 + 4j - 6r + 1}{32r - 8}.$$

q -identities corresponding to Nahm sums

Theorem

For $r \geq 2$,

$$(1) \quad \sum_{\mathbf{n}=(n_1, \dots, n_r)^T \in \mathbb{N}^r} \frac{q^{\frac{1}{2} \mathbf{n}^T A^\vee D^\vee \mathbf{n} + \mathbf{n}^T \mathbf{b}_0}}{(q; q)_{n_1} (q; q)_{n_2} \cdots (q; q)_{n_{r-1}} (q^2; q^2)_{n_r}} = \frac{(q^{\frac{1}{2}}, q^{2r-1}, q^{2r-\frac{1}{2}}; q^{2r-\frac{1}{2}})_\infty}{(q^{\frac{1}{2}}, q^{\frac{3}{2}}, q^2; q^2)_\infty},$$

(2) For $1 \leq j \leq r$,

$$\sum_{\mathbf{n}=(n_1, \dots, n_r)^T \in \mathbb{N}^r} \frac{q^{\frac{1}{2} \mathbf{n}^T A^\vee D^\vee \mathbf{n} + \mathbf{n}^T \mathbf{b}_j}}{(q; q)_{n_1} \cdots (q; q)_{n_{r-1}} (q^2; q^2)_{n_r}} = \frac{(q^j, q^{2r-j-\frac{1}{2}}, q^{2r-\frac{1}{2}}; q^{2r-\frac{1}{2}})_\infty}{(q^{\frac{1}{2}}, q^{\frac{3}{2}}, q^2; q^2)_\infty}.$$



B. Wang and L. Wang, Mizuno's rank three Nahm sums I: identities of index $(1, 1, 2)$, arXiv:2402.06253.

Vector-valued functions corresponding to the matrices A and $A^\vee$

- We introduce the following vector-valued functions $\mathbf{G}_{4r-2}(\tau)$:

$$\mathbf{G}_{4r-2}(\tau) = (g_1(\tau), g_2(\tau), \dots, g_{2r-1}(\tau), g_1^\vee(\tau), g_2^\vee(\tau), \dots, g_{2r-1}^\vee(\tau))^T,$$

where

$$\begin{aligned} g_j(\tau) &= q^{\frac{(4r-4j+1)^2}{32r-8}} \left(q^{\frac{2j-r-1}{2}} \frac{(q^{8r-4j}, q^{8r+4j-4}, q^{16r-4}; q^{16r-4})_\infty}{(q, q^3, q^4; q^4)_\infty} \right. \\ &\quad \left. + \frac{(-q^{2j-1}, q^{4r-2j}, -q^{4r-1}; -q^{4r-1})_\infty}{(q^2, q^2, q^4; q^4)_\infty} \right), \\ g_j^\vee(2\tau) &= \frac{1}{2} q^{\frac{2j^2-2j-2r+1}{16r-4}} \left(\frac{(q^{2r-j}, q^{2r+j-1}, q^{4r-1}; q^{4r-1})_\infty}{(q, q^3, q^4; q^4)_\infty} \right. \\ &\quad \left. + (-1)^{\frac{j(j-1)}{2}} \frac{((-q)^{2r-j}, (-q)^{2r+j-1}, -q^{4r-1}; -q^{4r-1})_\infty}{(-q, -q^3, q^4; q^4)_\infty} \right). \end{aligned}$$

Vector-valued functions corresponding to the matrices A and $A^\vee$

- Let

$$\begin{aligned}\mathbf{g}_{2r-1}(\tau) &= (g_1(\tau), g_2(\tau), \dots, g_{2r-1}(\tau))^T, \\ \mathbf{g}_{2r-1}^\vee(\tau) &= (g_1^\vee(\tau), g_2^\vee(\tau), \dots, g_{2r-1}^\vee(\tau))^T.\end{aligned}$$

Theorem 6 (Du, Ji, Shen and Xu, 2026+)

For $r \geq 2$,

$$\mathbf{g}_{2r-1}\left(-\frac{1}{2\tau}\right) = 2S\mathbf{g}_{2r-1}^\vee(\tau), \quad \mathbf{g}_{2r-1}^\vee\left(-\frac{1}{2\tau}\right) = S\mathbf{g}_{2r-1}(\tau),$$

where $S = (s_{j,k})_{(2r-1) \times (2r-1)}$ and $s_{j,k} = \sqrt{\frac{2}{4r-1}} \cos \frac{(2j-1)(2k-1)\pi}{2(4r-1)}$.

- For $r = 3$, it reduces to “**Langlands dual**” pairs conjectured by Mizuno and proved by Wang and Wang (2025, Adv. Math.).

Vector-valued functions corresponding to the matrices A and $A^\vee$

Theorem 7 (Du, Ji, Shen and Xu, 2026+)

- For $r \geq 2$, $\mathbf{G}_{4r-2}(\tau)$ is a **vector-valued automorphic form** of some multiplier for the group generated by $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix}$.
- In particular, for odd integer $r \geq 3$, $\mathbf{G}_{4r-2}(\tau)$ is a **vector-valued modular function** of some multiplier with respect to the congruence subgroup $\Gamma_0(2)$.

Vector-valued automorphic form

Let Γ' be a discrete subgroup of

$$PSL_2(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{R}, \text{ and } ad - bc > 0 \right\}$$

with $\rho: \Gamma' \rightarrow GL_n(\mathbb{C})$ a representation of Γ' .

- A column vector of meromorphic functions $\mathbf{F}(\tau) = (f_1(\tau), f_2(\tau), \dots, f_n(\tau))^T$ on the upper half-plane is called a **vector-valued automorphic form** of multiplier ρ and integral weight k with respect to Γ' , if it satisfies for $\gamma \in \Gamma'$,

$$\mathbf{F}(\gamma\tau) = \rho(\gamma)(c\tau + d)^k \mathbf{F}(\tau).$$

Vector-valued modular form

- Suppose Γ' is a finite index subgroup of Γ ;
- Let $\mathbf{F}(\tau) = (f_1(\tau), f_2(\tau), \dots, f_n(\tau))^T$ be a vector-valued automorphic form of multiplier ρ and integral weight k for Γ' ;
- If, in addition, for all $\gamma \in \Gamma$ and each component function $f_i(\tau)$, there exists a positive integer $N_{\gamma,i}$ such that

$$f_i(\gamma\tau) = (c\tau + d)^k \sum_n a_n e^{2\pi i n \tau / N_{\gamma,i}}$$

with $a_n = 0$ for $n \ll 0$;

- Then $\mathbf{F}(\tau)$ is called a **vector-valued modular function** of integral weight k associated with ρ for Γ' .

Nahm sums associated with symmetrizable matrix A of index $\mathbf{d} = (2, \dots, 2, 1)$

- We have

$$g_1(\tau) = \tilde{f}_{A, \mathbf{b}_0, \mathbf{c}_0, \mathbf{d}}(\tau),$$

and, for $r \leq j \leq 2r - 1$, we have

$$g_j(\tau) = \tilde{f}_{A, \mathbf{b}_{2r-j}, \mathbf{c}_{2r-j}, \mathbf{d}}(\tau).$$

- We also find

$$g_2(\tau) = \tilde{f}_{A, \mathbf{b}_{r+1}, \mathbf{c}_{r+1}, \mathbf{d}}(\tau) + \tilde{f}_{A, \tilde{\mathbf{b}}_{r+1}, \tilde{\mathbf{c}}_{r+1}, \mathbf{d}}(\tau),$$

Problem

For $r \geq 4$ and $3 \leq j \leq r - 1$, can $g_j(\tau)$ be expressed as a combination of Nahm sums for the symmetrizable matrix A of index $(2, \dots, 2, 1)$?

Nahm sums associated with symmetrizable matrix $A^\vee$ of index $d^\vee = (1, \dots, 1, 2)$

- We have

$$g_{2r-1}^\vee(2\tau) = \frac{1}{2} \left(\tilde{f}_{A^\vee, 2b_0, 2c_0, 2d^\vee}(\tau) + (-1)^{r-1} \tilde{f}_{A^\vee, 2b_0, 2c_0, 2d^\vee} \left(\tau + \frac{1}{2} \right) \right),$$

and, for $2 \leq j \leq 2r-2$ and j is even, we have

$$g_j^\vee(2\tau) = \frac{1}{2} \left(\tilde{f}_{A^\vee, 2b_{r-\frac{j}{2}}, 2c_{r-\frac{j}{2}}, 2d^\vee}(\tau) + (-1)^{\frac{j(j-1)}{2}} \tilde{f}_{A^\vee, 2b_{r-\frac{j}{2}}, 2c_{r-\frac{j}{2}}, 2d^\vee} \left(\tau + \frac{1}{2} \right) \right).$$

Problem

For $r \geq 2$ and $1 \leq j \leq 2r-3$ and j is odd, can $g_j^\vee(2\tau)$ be expressed as a combination of Nahm sums for the symmetrizable matrix $A^\vee$ of index $(1, \dots, 1, 2)$?

Vector-valued functions corresponding to the matrices B and $B^\vee$

• Let

$$\mathbf{H}_{4r-4}(\tau) = (h_1(\tau), h_2(\tau), \dots, h_{2r-2}(\tau), h_1^\vee(\tau), h_2^\vee(\tau), \dots, h_{2r-2}^\vee(\tau))^T,$$

where

$$\begin{aligned} h_j(\tau) &= q^{\frac{(4j-4r+1)^2}{32r-24}} \left(\frac{q^{\frac{4j-2r-1}{4}} (q^{4(2r-1-j)}, q^{8r+4j-8}, q^{16r-12}; q^{16r-12})_\infty}{(q, q^3, q^4; q^4)_\infty} \right. \\ &\quad \left. + \frac{(q^{2(2r-j-1)}, -q^{2j-1}, -q^{4r-3}; -q^{4r-3})_\infty}{(q^2, q^2, q^4; q^4)_\infty} \right), \\ h_j^\vee(2\tau) &= \frac{1}{2} q^{\frac{j^2-j-r+1}{8r-6}} \left(\frac{(q^{2r-j-1}, q^{2r+j-2}, q^{4r-3}; q^{4r-3})_\infty}{(q, q^3, q^4; q^4)_\infty} \right. \\ &\quad \left. + \frac{((-q)^{2r-j-1}, (-q)^{2r+j-2}, -q^{4r-3}; -q^{4r-3})_\infty}{(-q, -q^3, q^4; q^4)_\infty} \right). \end{aligned}$$

Vector-valued functions corresponding to the matrices B and $B^\vee$

- Let

$$\begin{aligned}\mathbf{h}_{2r-2}(\tau) &= (h_1(\tau), h_2(\tau), \dots, h_{2r-2}(\tau))^T, \\ \mathbf{h}_{2r-2}^\vee(\tau) &= (h_1^\vee(\tau), h_2^\vee(\tau), \dots, h_{2r-2}^\vee(\tau))^T.\end{aligned}$$

Theorem 8 (Du, Ji, Shen and Xu, 2026+)

For $r \geq 2$,

$$\mathbf{h}_{2r-2}\left(-\frac{1}{2\tau}\right) = 2\tilde{S} \mathbf{h}_{2r-2}^\vee(\tau), \quad \mathbf{h}_{2r-2}^\vee\left(-\frac{1}{2\tau}\right) = \tilde{S} \mathbf{h}_{2r-2}(\tau),$$

where $\tilde{S} = (\tilde{s}_{j,k})_{(2r-2) \times (2r-2)}$, and $\tilde{s}_{j,k} = \sqrt{\frac{2}{4r-3}} \cos \frac{(2j-1)(2k-1)}{8r-6} \pi$.

Theorem 9 (Du, Ji, Shen and Xu, 2026+)

For $r \geq 2$, $\mathbf{H}_{4r-4}(\tau)$ is a vector-valued automorphic form of some multiplier for the group generated by $\begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 8 & 1 \end{pmatrix}$.

Nahm sums associated with symmetrizable matrix B of index $\mathbf{d} = (2, \dots, 2, 1)$

- We obtain

$$h_1(\tau) = \widetilde{f}_{B, \mathbf{b}_0, c_0, \mathbf{d}}(\tau),$$

and for $r - 1 \leq j \leq 2r - 2$,

$$h_j(\tau) = \widetilde{f}_{B, \mathbf{b}_{2r-j-1}, c_{2r-j-1}, \mathbf{d}}(\tau),$$

where $\mathbf{d} = (2, \dots, 2, 1)$.

Problem

For $r \geq 4$ and $2 \leq j \leq r - 2$, can $h_j(\tau)$ be expressed as a combination of Nahm sums for the symmetrizable matrix B of index $(2, \dots, 2, 1)$?

Nahm sums associated with symmetrizable matrix $B^\vee$ of index $\mathbf{d}^\vee = (1, \dots, 1, 2)$

- We have

$$h_{2r-2}^\vee(2\tau) = \frac{1}{2} \left(\tilde{f}_{B^\vee, 2b_0, 2c_0, 2d^\vee, 0}(\tau) - \tilde{f}_{B^\vee, 2b_0, 2c_0, 2d^\vee, 1}(\tau) \right. \\ \left. + \tilde{f}_{B^\vee, 2b_0, 2c_0, 2d^\vee, 0}\left(\tau + \frac{1}{2}\right) - \tilde{f}_{B^\vee, 2b_0, 2c_0, 2d^\vee, 1}\left(\tau + \frac{1}{2}\right) \right),$$

and for $1 \leq j \leq 2r - 3$ and j is odd, we have

$$h_j^\vee(2\tau) = \frac{1}{2} \left(\tilde{f}_{B^\vee, 2b_{r-\frac{j+1}{2}}, 2c_{r-\frac{j+1}{2}}, 2d^\vee, 0}(\tau) - \tilde{f}_{B^\vee, 2b_{r-\frac{j+1}{2}}, 2c_{r-\frac{j+1}{2}}, 2d^\vee, 1}(\tau) \right. \\ \left. + \tilde{f}_{B^\vee, 2b_{r-\frac{j+1}{2}}, 2c_{r-\frac{j+1}{2}}, 2d^\vee, 0}\left(\tau + \frac{1}{2}\right) - \tilde{f}_{B^\vee, 2b_{r-\frac{j+1}{2}}, 2c_{r-\frac{j+1}{2}}, 2d^\vee, 1}\left(\tau + \frac{1}{2}\right) \right).$$

Problem

For $r \geq 3$ and $2 \leq j \leq 2r - 4$ and j is even, can $h_j^\vee(2\tau)$ be expressed as a combination of Nahm sums for the symmetrizable matrix $B^\vee$ of index $(1, \dots, 1, 2)$?

Thanks for your attention!

