

Bailey Lattices and Analytic Generalizations of Rogers–Ramanujan Identities

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The Rogers-Ramanujan identities

Theorem (Rogers-Ramanujan)

For $|q| < 1$,

$$\sum_{n \geq 0} \frac{q^{n^2}}{(1-q)(1-q^2) \cdots (1-q^n)} = \prod_{j=0}^{\infty} \frac{1}{(1-q^{5j+1})(1-q^{5j+4})} \quad (RR1)$$

$$\sum_{n \geq 0} \frac{q^{n^2+n}}{(1-q)(1-q^2) \cdots (1-q^n)} = \prod_{j=0}^{\infty} \frac{1}{(1-q^{5j+2})(1-q^{5j+3})} \quad (RR2)$$

These two identities were first proved by [Rogers](#) in 1894 and rediscovered by [Ramanujan](#) a few years later.

- [Ramanujan's comment](#): It would be difficult to find more beautiful formulas than the Rogers-Ramanujan' identities.
- [Hans Rademacher](#) clearly was in agreement with Ramanujan's comments.

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- There is a beautiful story about the discovery of the Rogers-Ramanujan identities.
- Rogers' reputation as a mathematician rests almost entirely on the discovery of these two identities.
- how they found these two identities have appeared in Baxter's exact solution to the hard hexagon model in statistical mechanics.

Connections to Other Fields

Rogers–Ramanujan type identities have many important relations with other branches of mathematics or physics:

- **Lie algebra**: Lepowsky, Wilson, Milne (1980s) developed a Lie theoretic approach to establish Rogers–Ramanujan type identities.

Rogers–Ramanujan identities
Andrews–Gordon identity
Bressoud's identity



characters of affine
Kac-Moody Lie algebra

- **Statistical mechanics**: Baxter (1982) used the Rogers–Ramanujan identities to solve the hard-hexagon model, which is a 2-dimensional lattice model of a gas.

Connection with modular forms

- **Modular forms:** If we look at these two identities from the perspective of modular forms, the product sides of them are modular forms, but it is not clear from the sum sides.

$$\underbrace{\sum_{n=0}^{\infty} \frac{q^{n^2+in}}{(q; q)_n}}_{???} = \frac{1}{\underbrace{(q^i, q^{5-i}; q^5)_{\infty}}_{\text{modular forms}}}, \quad i = 0, 1$$

- **Question:** What kind of the basic hypergeometric series is a modular form?

Nahm's problem

- Suppose $r \geq 1$ is a positive integer, A is a real positive definite symmetric $r \times r$ matrix, B is a vector of length r , and C is a scalar. Define

$$f_{A,B,C}(q) := \sum_{\vec{n}=(n_1,\dots,n_r) \in (\mathbb{Z}_{\geq 0})^r} \frac{q^{\frac{1}{2} \vec{n}^T \cdot A \cdot \vec{n} + \vec{n}^T \cdot B + C}}{(q; q)_{n_1} \cdots (q; q)_{n_r}}$$

- **Nahm's Problem 1994, 2007** Describe all such A, B and C with rational entries for which $f_{A,B,C}(q)$ is a modular form.
- The motivation of Nahm's problem comes from physics, and the modular forms $f_{A,B,C}(q)$ are expected to be characters of rational conformal field theories (共形场论).

Definition

A partition of a positive integer n is a finite nonincreasing sequence of positive integers $(\pi_1, \pi_2, \dots, \pi_\ell)$ such that

$$\pi_1 + \pi_2 + \dots + \pi_\ell = n.$$

Example: There are five partitions of 4, which are

$$(4), (3, 1), (2, 2), (2, 1, 1), (1, 1, 1, 1).$$

MacMahon's combinatorial interpretation

Theorem (MacMahon)

- Let $A_1(-; 1, 2, 2; n)$ denote the number of partitions of n into *parts* $\equiv 1, 4 \pmod{5}$ (equivalently, $\not\equiv 0, \pm 2 \pmod{5}$);
- Let $B_1(-; 1, 2, 2; n)$ denote the number of partitions $\pi = (\pi_1, \pi_2, \dots, \pi_\ell)$ of n with $\pi_i - \pi_{i+1} \geq 2$ for $1 \leq i \leq \ell - 1$.

Then for $n \geq 0$,

$$A_1(-; 1, 2, 2; n) = B_1(-; 1, 2, 2; n).$$

$$\begin{aligned} \sum_{n \geq 0} A_1(-; 1, 2, 2; n) q^n &= \frac{1}{(q, q^4; q^5)_\infty} \\ &\stackrel{(RR1)}{=} \sum_{n=0}^{\infty} \frac{q^{n^2} (=1+3+\dots+2n-1)}{(q; q)_n} = \sum_{n \geq 0} B_1(-; 1, 2, 2; n) q^n. \end{aligned}$$

Notation:

$$(a; q)_n = (1-a)(1-aq) \cdots (1-aq^{n-1}) \quad (a; q)_\infty = \lim_{n \rightarrow \infty} (a; q)_n,$$

$$(a_1, a_2, \dots, a_k; q)_n = (a_1; q)_n (a_2; q)_n \cdots (a_k; q)_n.$$

MacMahon's combinatorial interpretation

Theorem (MacMahon)

- Let $A_1(-; 1, 2, 1; n)$ denote the number of partitions of n into *parts* $\equiv 2, 3 \pmod{5}$ (equivalently, $\not\equiv 0, \pm 1 \pmod{5}$);
- Let $B_1(-; 1, 2, 1; n)$ denote the number of partitions $\pi = (\pi_1, \pi_2, \dots, \pi_\ell)$ of n with $\pi_i - \pi_{i+1} \geq 2$ for $1 \leq i \leq \ell - 1$ and $\pi_\ell \geq 2$.

Then for $n \geq 0$,

$$A_1(-; 1, 2, 1; n) = B_1(-; 1, 2, 1; n).$$

$$\sum_{n \geq 0} A_1(-; 1, 2, 1; n) q^n = \frac{1}{(q^2, q^3; q^5)_\infty}$$

$$\stackrel{(RR2)}{=} \sum_{n=0}^{\infty} \frac{q^{n^2+n} \ (=2+4+\dots+2n)}{(q; q)_n} = \sum_{n \geq 0} B_1(-; 1, 2, 1; n) q^n.$$

Gordon's combinatorial generalization

Theorem (Rogers-Ramanujan-Gordon, 1961)

- For $k \geq r \geq 1$, let $A_1(-; 1, k, r; n)$ denote the number of partitions of n into *parts* $\not\equiv 0, \pm r \pmod{2k+1}$.
- For $k \geq r \geq 1$, let $B_1(-; 1, k, r; n)$ denote the number of partitions of n of the form $(\pi_1, \dots, \pi_s)$, where $\pi_j - \pi_{j+k-1} \geq 2$, and at most $r-1$ of the π_j are equal to 1.

Then, for $k \geq r \geq 1$ and $n \geq 0$,

$$A_1(-; 1, k, r; n) = B_1(-; 1, k, r; n).$$

When $k = 2$ and $r = 2, 1$, this identity reduces to (RR1) or (RR2).

Note: Gordon's theorem was independently discovered by G.E. Andrews, see G.E. Andrews, *Some debts I owe*.



B. Gordon, A combinatorial generalization of the Rogers-Ramanujan identities, *Amer. J. Math* 83 (1961) 393–399.

The corresponding q -identity due to Andrews

Andrews found the following q -identity, which can be viewed as a companion to Gordon's partition identity.

Theorem (Andrews, 1974)

For $k \geq r \geq 1$,

$$\sum_{N_1 \geq \dots \geq N_{k-1} \geq 0} \frac{q^{N_1^2 + \dots + N_{k-1}^2 + N_r + \dots + N_{k-1}}}{(q; q)_{N_1 - N_2} \cdots (q; q)_{N_{k-2} - N_{k-1}} (q; q)_{N_{k-1}}}$$
$$= \prod_{\substack{n=1 \\ n \not\equiv 0, \pm r \pmod{2k+1}}}^{\infty} \frac{1}{1 - q^n}.$$

When $k = 2$ and $r = 2, 1$ this identity reduces to (RR1) and (RR1).



G.E. Andrews, An analytic generalization of the Rogers-Ramanujan identities for odd moduli, Proc. Nat. Acad. Sci. USA 71 (1974) 4082–4085.

The sketch of the proof given by Andrews

Andrews first introduced the following function

$$J_{k,r}(0; x; q) = \sum_{n=0}^{\infty} \frac{x^{kn} q^{kn^2 + (k-r+1)n} (-1)^n q^{\binom{n}{2}} (1 - x^r q^{(2n+1)r})}{(q)_n (xq^{n+1})_{\infty}}$$

He then proved that for $1 \leq i \leq k, |q| < 1$,

$$J_{k,r}(0; 1; q) = \sum_{n \geq 0} A_1(-; 1, k, r, n) q^n = \prod_{\substack{n=1 \\ n \neq 0, \pm r \pmod{2k+1}}}^{\infty} \frac{1}{1 - q^n},$$

$$J_{k,r}(0; 1; q) = \sum_{n \geq 0} B_1(-; 1, k, r, n) q^n$$

$$J_{k,r}(0; 1; q) = \sum_{N_1 \geq \dots \geq N_{k-1} \geq 0} \frac{q^{N_1^2 + \dots + N_{k-1}^2 + N_r + \dots + N_{k-1}}}{(q; q)_{N_1 - N_2} \cdots (q; q)_{N_{k-2} - N_{k-1}} (q; q)_{N_{k-1}}}$$

Bressoud's combinatorial generalization

Bressoud observed that Gordon's partition identity only focused on the parts modular the odd numbers. He addressed the case involving the even numbers and derived the following partition identity.

Theorem (Bressoud-Rogers-Ramanujan, 1979)

- For $k \geq r \geq 1$, let $A_0(-; 1, k, r; n)$ denote the number of partitions of n into parts $\not\equiv 0, \pm r \pmod{2k}$.
- For $k \geq r \geq 1$, let $B_0(-; 1, k, r; n)$ denote the number of partitions of n of the form $(\pi_1, \dots, \pi_s)$, where $\pi_j - \pi_{j+k-1} \geq 2$, at most $r-1$ of the π_j are equal to 1.

If $\pi_j - \pi_{j+k-2} \leq 1$, then $\pi_j + \dots + \pi_{j+k-2} \equiv r-1 \pmod{2}$.

Then, for $k \geq r \geq 1$ and $n \geq 0$,

$$A_0(-; 1, k, r; n) = B_0(-; 1, k, r; n).$$



D.M. Bressoud, A generalization of the Rogers-Ramanujan identities for all moduli, J. Combin. Theory, Ser. A 27 (1979) 64–68.

The sketch of the proof given by Bressoud

Based on the following function

$$J_{k,r}(0; x; q) = \sum_{n=0}^{\infty} \frac{x^{kn} q^{kn^2 + (k-r+1)n} (-1)^n q^{\binom{n}{2}} (1 - x^r q^{(2n+1)r})}{(q)_n (xq^{n+1})_{\infty}}$$

Bressoud then proved that for $1 \leq i \leq k, |q| < 1$,

$$(-q; q)_{\infty} J_{(k-1)/2, r/2}(0; 1; q^2) = \sum_{n \geq 0} A_0(-; 1, k, r; n) q^n$$

$$= \prod_{\substack{n=1 \\ n \not\equiv 0, \pm r \pmod{2k}}}^{\infty} \frac{1}{1 - q^n},$$

$$(-q; q)_{\infty} J_{(k-1)/2, r/2}(0; 1; q^2) = \sum_{n \geq 0} B_0(-; 1, k, r; n) q^n.$$

The corresponding q -identity

Bressoud obtained the following q -identity, which can be viewed as a companion to his partition identity.

Theorem (Bressoud, 1980)

For $k \geq r \geq 1$,

$$\sum_{N_1 \geq \dots \geq N_{k-1} \geq 0} \frac{q^{N_1^2 + \dots + N_{k-1}^2 + N_r + \dots + N_{k-1}}}{(q; q)_{N_1 - N_2} \cdots (q; q)_{N_{k-2} - N_{k-1}} (q^2; q^2)_{N_{k-1}}}$$
$$= \prod_{\substack{n=1 \\ n \not\equiv 0, \pm r \pmod{2k}}}^{\infty} \frac{1}{1 - q^n}.$$



D.M. Bressoud, Analytic and combinatorial generalizations of Rogers-Ramanujan identities, Mem. Amer. Math. Soc. 24(227) (1980) 54pp.

$A(n; \text{congruence conditions on parts})$



$B(n; \text{gap conditions on parts})$

Some classical partition identities

- **Euler's partition theorem:** The number of partitions of n into parts $\equiv 1 \pmod{2}$ is equal to the number of partitions $\pi = (\pi_1, \pi_2, \dots, \pi_\ell)$ of n with $\pi_i - \pi_{i+1} \geq 1$ for $1 \leq i \leq \ell - 1$.

$$(-q; q)_\infty = \frac{1}{(q; q^2)_\infty}.$$

- **Schur's theorem:** The number of partitions of n into parts $\equiv \pm 1 \pmod{6}$ is equal to the number of partitions $\pi = (\pi_1, \pi_2, \dots, \pi_\ell)$ of n with $\pi_i - \pi_{i+1} \geq 3$ for $1 \leq i \leq \ell - 1$ with strict inequality if $\pi_i \equiv 0 \pmod{3}$.

$$\sum_{n \geq 0} \frac{q^{3n^2}}{(q^3; q^3)_n} (-q^{2-3n}; q^3)_n (-q^{1+3n}; q^3)_\infty = \frac{1}{(q, q^5; q^6)_\infty}.$$

Some classical partition identities

- **The Göllnitz-Gordon theorem I:** The number of partitions of n into parts $\equiv 1, 4, \text{ or } 7 \pmod{8}$ is equal to the number of partitions $\pi = (\pi_1, \pi_2, \dots, \pi_\ell)$ of n with $\pi_i - \pi_{i+1} \geq 2$ for $1 \leq i \leq \ell - 1$ with strict inequality if $\pi_i \equiv 0 \pmod{2}$.

$$\sum_{n \geq 0} \frac{q^{n^2}(-q; q^2)_n}{(q^2; q^2)_n} = \frac{1}{(q, q^4, q^7; q^8)_\infty}.$$

- **The Göllnitz-Gordon theorem II:** The number of partitions of n into parts $\equiv 3, 4, \text{ or } 5 \pmod{8}$ is equal to the number of partitions $\pi = (\pi_1, \pi_2, \dots, \pi_\ell)$ of n with $\pi_i - \pi_{i+1} \geq 2$ for $1 \leq i \leq \ell - 1$ with strict inequality if $\pi_i \equiv 0 \pmod{2}$. Furthermore, $\pi_\ell \geq 3$.

$$\sum_{n \geq 0} \frac{q^{n^2+2n}(-q; q^2)_n}{(q^2; q^2)_n} = \frac{1}{(q^3, q^4, q^5; q^8)_\infty}.$$

Andrews' combinatorial generalization

Theorem (Andrews-Göllnitz-Gordon, 1967)

- For $k \geq r \geq 1$, let $A_1(1; 2, k, r; n)$ denote the number of partitions of n into *parts* $\not\equiv 2 \pmod{4}$ and $\not\equiv 0, \pm(2r-1) \pmod{4k}$.
- For $k \geq r \geq 1$, let $B_1(1; 2, k, r; n)$ denote the number of partitions of n of the form $(\pi_1, \dots, \pi_s)$ such that *no odd part is repeated*, $\pi_j - \pi_{j+k-1} \geq 2$ with strict inequality if π_j is even, and at most $r-1$ of the π_j are less than or equal to 2.

Then, for $k \geq r \geq 1$ and $n \geq 0$,

$$A_1(1; 2, k, r; n) = B_1(1; 2, k, r; n).$$

When $k = 2$ and $r = 1$, this theorem reduces to GGT-II. When $k = 2$ and $r = 2$, this theorem reduces to GGT-I.



G.E. Andrews, A generalization of the Göllnitz-Gordon partition theorems, Proc. Amer. Math. Soc. 18 (1967) 945–952.

The sketch of the proof

Andrews first introduced the following function

$$H_{k,r}(a; x; q) = \sum_{n=0}^{\infty} \frac{x^{kn} q^{kn^2 + (k-r)n} a^n (1 - x^r q^{2nr}) (axq^{n+1})_{\infty} (a^{-1})_n}{(q)_n (xq^n)_{\infty}}$$

and let

$$J_{k,r}(a; x; q) = H_{k,r}(a; xq; q) - axqH_{k,r-1}(a; xq; q).$$

He then proved that for $1 \leq i \leq k$, $|q| < 1$,

$$\begin{aligned} J_{k,r}(-q^{-1}; 1; q^2) &= \sum_{n \geq 0} A_1(1; 2, k, r; n) q^n \\ &= \frac{(q^2; q^4)_{\infty} (q^{2r-1}, q^{4k-2r+1}, q^{4k}; q^{4k})_{\infty}}{(q; q)_{\infty}}, \end{aligned}$$

$$J_{k,r}(-q^{-1}; 1; q^2) = \sum_{n \geq 0} B_1(1; 2, k, r; n) q^n.$$

The corresponding q -identity

Theorem (Bressoud, equation (3.8))

For $k \geq r \geq 1$,

$$\sum_{N_1 \geq \dots \geq N_{k-1} \geq 0} \frac{(-q^{1-2N_1}; q^2)_{N_1} q^{2(N_1^2 + \dots + N_{k-1}^2 + N_r + \dots + N_{k-1})}}{(q^2; q^2)_{N_1 - N_2} \cdots (q^2; q^2)_{N_{k-2} - N_{k-1}} (q^2; q^2)_{N_{k-1}}} \\ = \frac{(q^2; q^4)_\infty (q^{2r-1}, q^{4k-2r+1}, q^{4k}; q^{4k})_\infty}{(q; q)_\infty}.$$



D.M. Bressoud, Analytic and combinatorial generalizations of Rogers-Ramanujan identities, Mem. Amer. Math. Soc. 24(227) (1980) 54pp.

A further generalization

Assume that $\alpha_1, \alpha_2, \dots, \alpha_\lambda$ and η are integers such that for $1 \leq i \leq \lambda$,
 $0 < \alpha_1 < \dots < \alpha_\lambda < \eta$, and $\alpha_i = \eta - \alpha_{\lambda+1-i}$.

Theorem (Bressoud, Mem. Amer. Math. Soc., 1980)

Let λ , k , r and $j = 0$ or 1 be the integers such that $(2k + j)/2 > r \geq \lambda \geq 0$.
 Then

$$\begin{aligned} & \sum_{N_1 \geq \dots \geq N_{k-1} \geq 0} \frac{q^{\eta(N_1^2 + \dots + N_{k-1}^2 + N_r + \dots + N_{k-1})}}{(q^\eta; q^\eta)_{N_1 - N_2} \cdots (q^\eta; q^\eta)_{N_{k-2} - N_{k-1}} (q^{(2-j)\eta}; q^{(2-j)\eta})_{N_{k-1}}} \\ & \quad \times \prod_{s=1}^{\lambda} (-q^{\eta - \alpha_s - \eta N_s}; q^\eta)_{N_s} \prod_{s=2}^{\lambda} (-q^{\eta - \alpha_s + \eta N_{s-1}}; q^\eta)_{\infty} \\ & = \frac{(-q^{\alpha_1}, \dots, -q^{\alpha_\lambda}; q^\eta)_{\infty} (q^{\eta(r - \frac{\lambda}{2})}, q^{\eta(2k - r - \frac{\lambda}{2} + j)}, q^{\eta(2k - \lambda + j)}; q^{\eta(2k - \lambda + j)})_{\infty}}{(q^\eta; q^\eta)_{\infty}}. \end{aligned}$$

The A_j -partition function

Definition

Let λ , k , r and $j = 0$ or 1 be the integers such that $(2k + j)/2 > r \geq \lambda \geq 0$. Define the partition function $A_j(\alpha_1, \dots, \alpha_\lambda; \eta, k, r, n)$ to be the number of partitions of n into parts congruent to $0, \alpha_1, \dots, \alpha_\lambda \pmod{\eta}$ such that

- If λ is even, then only multiples of η may be repeated and no part is congruent to $0, \pm\eta(r - \lambda/2) \pmod{\eta(2k - \lambda + j)}$;
- If λ is odd and $j = 1$, then only multiples of $\eta/2$ may be repeated, no part is congruent to $\eta \pmod{2\eta}$, and no part is congruent to $0, \pm\eta(2r - \lambda)/2 \pmod{\eta(2k - \lambda + 1)}$;
- If λ is odd and $j = 0$, then only multiples of $\eta/2$ which are not congruent to $\eta(2k - \lambda)/2 \pmod{\eta(2k - \lambda)}$ may be repeated, no part is congruent to $\eta \pmod{2\eta}$, no part is congruent to $0 \pmod{2\eta(2k - \lambda)}$, and no part is congruent to $\pm\eta(2r - \lambda)/2 \pmod{\eta(2k - \lambda)}$.

Remark: Recall that $\alpha_1, \alpha_2, \dots, \alpha_\lambda$ and η are integers such that for $1 \leq i \leq \lambda$, $0 < \alpha_1 < \dots < \alpha_\lambda < \eta$, and $\alpha_i = \eta - \alpha_{\lambda+1-i}$. When λ is odd, observing that

$$\eta = \alpha_{(\lambda+1)/2} + \alpha_{\lambda+1-(\lambda+1)/2} = 2\alpha_{(\lambda+1)/2},$$

we see that η must be even in such case.

The generating function of A_j -partition function

By the definition, it's not difficult to see that for $k \geq r \geq \lambda \geq 0$, $k + j - 1 \geq \lambda$ and $j = 0$ or 1 ,

$$\begin{aligned} & \sum_{n \geq 0} A_j(\alpha_1, \dots, \alpha_\lambda; \eta, k, r, n) q^n \\ &= (-q^{\alpha_1}, \dots, -q^{\alpha_\lambda}; q^\eta)_\infty \\ & \quad \times \frac{(q^{\eta(r - \frac{\lambda}{2})}, q^{\eta(2k - r - \frac{\lambda}{2} + j)}, q^{\eta(2k - \lambda + j)}; q^{\eta(2k - \lambda + j)})_\infty}{(q^\eta; q^\eta)_\infty}. \end{aligned}$$

The B_j -partition function

Definition

Let λ , k , r and $j = 0$ or 1 be the integers such that $(2k + j)/2 > r \geq \lambda \geq 0$. Define $B_j(\alpha_1, \dots, \alpha_\lambda; \eta, k, r, n)$ to be the number of partitions of n of the form $(\pi_1, \dots, \pi_s)$ where $\pi_i \geq \pi_{i+1}$ satisfying the following conditions:

- (1) $\pi_i \equiv 0, \alpha_1, \dots, \alpha_\lambda \pmod{\eta}$;
- (2) Only multiples of η may be repeated;
- (3) $\pi_i - \pi_{i+k-1} \geq \eta$ with strict inequality if $\eta \mid \pi_i$;
- (4) At most $r - 1$ of the π_i are less than or equal to η ;
- (5) If $\pi_i \leq \pi_{i+k-2} + \eta$ with strict inequality if $\eta \nmid \pi_i$, then

$$[\pi_i/\eta] + \dots + [\pi_{i+k-2}/\eta] \equiv r - 1 + V_\pi(\pi_i) \pmod{2 - j},$$

where $V_\pi(N)$ (or $V(N)$ for short) denotes the number of parts not exceeding N which are not divided by η in π and $[]$ denotes the greatest integer function.

Bressoud's conjecture

Conjecture (Bressoud)

Let λ , k , r and $j = 0$ or 1 be the integers such that $(2k + j)/2 > r \geq \lambda \geq 0$.
Then

$$\begin{aligned} & \sum_{n \geq 0} B_j(\alpha_1, \dots, \alpha_\lambda; \eta, k, r; n) q^n \\ &= \sum_{N_1 \geq \dots \geq N_{k-1} \geq 0} \frac{q^{\eta(N_1^2 + \dots + N_{k-1}^2 + N_r + \dots + N_{k-1})}}{(q^\eta; q^\eta)_{N_1 - N_2} \cdots (q^\eta; q^\eta)_{N_{k-2} - N_{k-1}} (q^{(2-j)\eta}; q^{(2-j)\eta})_{N_{k-1}}} \\ & \quad \times \prod_{s=1}^{\lambda} (-q^{\eta - \alpha_s - \eta N_s}; q^\eta)_{N_s} \prod_{s=2}^{\lambda} (-q^{\eta - \alpha_s + \eta N_{s-1}}; q^\eta)_{\infty}. \end{aligned}$$



D.M. Bressoud, Analytic and combinatorial generalizations of Rogers-Ramanujan identities, Mem. Amer. Math. Soc. 24(227) (1980) 54pp.

Bressoud's conjecture II

Conjecture (Bressoud)

Let λ , k , r and $j = 0$ or 1 be the integers such that $(2k + j)/2 > r \geq \lambda \geq 0$. Then for $n \geq 0$,

$$A_j(\alpha_1, \dots, \alpha_\lambda; \eta, k, r, n) = B_j(\alpha_1, \dots, \alpha_\lambda; \eta, k, r, n).$$



D.M. Bressoud, Analytic and combinatorial generalizations of Rogers-Ramanujan identities, Mem. Amer. Math. Soc. 24(227) (1980) 54pp.



S. Kim, Bressoud's conjecture, Adv. Math. 325 (2018) 770–813. ($j = 1$)



Thomas Y. He, Kathy Q. Ji and Alice X.H. Zhao, Overpartitions and Bressoud's conjecture, I, Adv. Math. 404 (2022), Paper No. 108449, 81 pp.



Thomas Y. He, Kathy Q. Ji and Alice X.H. Zhao, Overpartitions and Bressoud's conjecture, II, Europ. J. Combin. 118 (2024) 103937. ($j = 0$)

- Euler's partition theorem: $\lambda = 0, \eta = 1, j = 0, k = 3, r = 2$.
- Shur's theorem: $\lambda = k = r = 2, \alpha_1 = 1, \alpha_2 = 2, \eta = 3, j = 1$.
- Rogers-Ramanujan identities: $\lambda = 0, \eta = 1, j = 1, k = 2, r = 1$ or 2 .
- Rogers-Ramanujan-Gordon identity: $\lambda = 0, \eta = 1, j = 1$.
- Rogers-Ramanujan-Bressoud identity: $\lambda = 0, \eta = 1, j = 0$.
- The Göllnitz-Gordon identity I, II:
 $\lambda = 1, \alpha_1 = 1, \eta = 2, j = 1, k = 2, r = 1$ or 2 .
- Andrews' generalization of the Göllnitz-Gordon identity:
 $\lambda = 1, \alpha_1 = 1, \eta = 2, j = 1$.
- Bressoud's generalization of the Göllnitz-Gordon identity:
 $\lambda = 1, \alpha_1 = 1, \eta = 2, j = 0$.

Bressoud-Göllnitz-Gordon theorem

Theorem (Bressoud-Göllnitz-Gordon)

- For $k \geq r \geq 1$, let $A_0(1; 2, k, r; n)$ denote the number of partitions of n into parts $\not\equiv 2 \pmod{4}$, $\not\equiv 2k-1 \pmod{4k-2}$ may be repeated, no part is multiples of $8k-4$, and $\not\equiv \pm(2r-1) \pmod{4k-2}$.
- For $k \geq r \geq 1$, let $B_0(1; 2, k, r; n)$ denote the number of partitions of n of the form $(\pi_1, \dots, \pi_s)$ such that *no odd part is repeated*, $\pi_j - \pi_{j+k-1} \geq 2$ with strict inequality if π_j is even, at most $r-1$ of the π_j are less than or equal to 2, and if $\pi_j - \pi_{j+k-2} \leq 2$ with strict inequality if π_j is odd, then
$$\pi_j + \dots + \pi_{j+k-2} \equiv r-1 + V_\pi(\pi_j) \pmod{2},$$

where $V_\pi(N)$ (or $V(N)$ for short) denotes the number of odd parts not exceeding N in π .

For $k \geq r \geq 1$ and $n \geq 0$,

$$A_0(1; 2, k, r; n) = B_0(1; 2, k, r; n).$$



D.M. Bressoud, Analytic and combinatorial generalizations of Rogers-Ramanujan identities, Mem. Amer. Math. Soc. 24(227) (1980) 54pp.

The corresponding q -identities

Theorem

For $k \geq r \geq 1$,

$$\sum_{N_1 \geq \dots \geq N_{k-1} \geq 0} \frac{(-q^{1-2N_1}; q^2)_{N_1} q^{2(N_1^2 + \dots + N_{k-1}^2 + N_r + \dots + N_{k-1})}}{(q^2; q^2)_{N_1 - N_2} \cdots (q^2; q^2)_{N_{k-2} - N_{k-1}} (q^4; q^4)_{N_{k-1}}} \\ = \frac{(q^2; q^4)_\infty (q^{2r-1}, q^{4k-2r-1}, q^{4k-2}; q^{4k-2})_\infty}{(q; q)_\infty}.$$



D.M. Bressoud, Analytic and combinatorial generalizations of Rogers-Ramanujan identities, Mem. Amer. Math. Soc. 24(227) (1980) 54pp.

Overpartition

Definition (Corteel and Lovejoy, 2004)

An overpartition π of n is a partition of n in which the first occurrence of a number can be overlined.

For example: There are fourteen overpartitions of 4.

(4)	($\overline{4}$)		
(3, 1)	($\overline{3}$, 1)	(3, $\overline{1}$)	($\overline{3}$, $\overline{1}$)
(2, 2)	($\overline{2}$, 2)		
(2, 1, 1)	($\overline{2}$, 1, 1)	(2, $\overline{1}$, 1)	($\overline{2}$, $\overline{1}$, 1)
(1, 1, 1, 1)	($\overline{1}$, 1, 1, 1)		

The overpartition $(\overline{2}, \overline{1}, 1)$ can be viewed as $((\overline{2}, \overline{1}), (1))$

Let $\bar{p}(n)$ denote the number of overpartitions of n . Then

$$\sum_{n \geq 0} \bar{p}(n) q^n = \frac{(-q; q)_{\infty}}{(q; q)_{\infty}}.$$



An overpartition analogue of Rogers-Ramanujan-Gordon

Theorem (Chen-Shi-Sang, 2013)

- For $k > r \geq 1$, let $\overline{A}_1(-; 1, k, r; n)$ denote the number of overpartitions of n such that non-overlined parts $\not\equiv 0, \pm r \pmod{2k}$, and for $k = r$, let $\overline{A}_1(-; 1, k, k; n)$ denote the number of overpartitions of n into parts not divided by k .
- For $k \geq r \geq 1$, let $\overline{B}_1(-; 1, k, r; n)$ denote the number of overpartitions π of n of the form $(\pi_1, \dots, \pi_s)$, where $\pi_i - \pi_{i+k-1} \geq 1$ with strict inequality if π_i is non-overlined, and at most $r-1$ of the π_i are equal to 1.

Then for $k \geq r \geq 1$ and $n \geq 0$,

$$\overline{A}_1(-; 1, k, r; n) = \overline{B}_1(-; 1, k, r; n).$$



W.Y.C. Chen, D.D.M. Sang and D.Y.H. Shi, The Rogers-Ramanujan-Gordon theorem for overpartitions, Proc. London Math. Soc. 106 (3) (2013) 1371–1393.

The corresponding q -identities

Theorem (Chen-Shi-Sang, 2013)

$$\sum_{N_1 \geq \dots \geq N_{k-1}} \frac{(-q^{1-N_1}; q)_{N_1-1} (1 + q^{N_r}) q^{N_1^2 + \dots + N_{k-1}^2 + N_{r+1} + \dots + N_{k-1}}{(q; q)_{N_1-N_2} \cdots (q; q)_{N_{k-2}-N_{k-1}} (q; q)_{N_{k-1}}}$$
$$= \frac{(-q; q)_\infty (q^r, q^{2k-r}, q^{2k}; q^{2k})_\infty}{(q; q)_\infty}.$$



W.Y.C. Chen, D.D.M. Sang and D.Y.H. Shi, The Rogers-Ramanujan-Gordon theorem for overpartitions, Proc. London Math. Soc. 106 (3) (2013) 1371–1393.

The sketch of the proof

Based on the following function

$$H_{k,r}(a; x; q) = \sum_{n=0}^{\infty} \frac{x^{kn} q^{kn^2 + (k-r)n} a^n (1 - x^r q^{2nr}) (axq^{n+1})_{\infty} (a^{-1})_n}{(q)_n (xq^n)_{\infty}}$$

Chen, Sang and Shi then proved that for $1 \leq i \leq k, |q| < 1$,

$$\begin{aligned} H_{k,r}(-1/q; q; q) &= \sum_{n \geq 0} \bar{A}_1(-; 1, k, r, n) q^n \\ &= \frac{(-q; q)_{\infty} (q^r, q^{2k-r}, q^{2k}; q^{2k})_{\infty}}{(q; q)_{\infty}}, \end{aligned}$$

$$\begin{aligned} H_{k,r}(-1/q; q; q) &= \sum_{n \geq 0} \bar{B}_1(-; 1, k, r, n) q^n \\ &= \sum_{N_1 \geq \dots \geq N_{k-1}} \frac{(-q^{1-N_1}; q)_{N_1-1} (1 + q^{N_r}) q^{N_1^2 + \dots + N_{k-1}^2 + N_{r+1} + \dots + N_{k-1}}{(q; q)_{N_1-N_2} \cdots (q; q)_{N_{k-2}-N_{k-1}} (q; q)_{N_{k-1}}} \end{aligned}$$

An overpartition analogue of Bressoud-Gordon-identities

Theorem (Chen-Sang-Shi)

- For $k \geq r \geq 1$, let $\overline{A}_0(-; 1, k, r; n)$ denote the number of overpartitions of n such that non-overlined parts $\not\equiv 0, \pm r \pmod{2k-1}$;
- For $k \geq r \geq 1$, let $\overline{B}_0(-; 1, k, r; n)$ denote the number of overpartitions $\pi = (\pi_1, \pi_2, \dots, \pi_\ell)$ of n , where $\pi_i \geq \pi_{i+k-1} + 1$ with strict inequality if π_i is non-overlined for $1 \leq i \leq \ell - k + 1$, at most $r - 1$ of the π_i are equal to 1, and for $1 \leq i \leq \ell - k + 2$, if $\pi_i \leq \pi_{i+k-2} + 1$ with strict inequality if π_i is overlined, then $\pi_i + \dots + \pi_{i+k-2} \equiv r - 1 + \overline{V}_\pi(\pi_i) \pmod{2}$,

where $\overline{V}_\pi(N)$ (or $\overline{V}_\pi(N)$ for short) denotes the number of overlined parts not exceeding N in π .

Then, for $k \geq r \geq 1$ and $n \geq 0$,

$$\overline{A}_0(-; 1, k, r; n) = \overline{B}_0(-; 1, k, r; n).$$



W.Y.C. Chen, D.D.M. Sang and D.Y.H. Shi, An overpartition analogue of Bressoud's theorem of Rogers-Ramanujan-Gordon type, Ramanujan J. 37 (2015) 653–679.

The corresponding analytic identities

Theorem (Sang-Shi, 2015)

$$\sum_{N_1 \geq \dots \geq N_{k-1} \geq 0} \frac{q^{N_1^2 + \dots + N_{k-1}^2 + N_r + \dots + N_{k-1}} (1 + q^{-N_r}) (-q^{1-N_1}; q)_{N_1-1}}{(q; q)_{N_1-N_2} \cdots (q; q)_{N_{k-2}-N_{k-1}} (q^2; q^2)_{N_{k-1}}}$$
$$= \frac{(-q; q)_\infty (q^r, q^{2k-r-1}, q^{2k-1}; q^{2k-1})_\infty}{(q; q)_\infty}.$$



D.D.M. Sang and D.Y.H. Shi, An Andrews-Gordon type identity for overpartitions, Ramanujan J. 37 (2015) 653–679.

An overpartition analogue of Bressoud's identities

We also obtain the following analytic form **with the aid of Bailey's pair**. For $k \geq r > \lambda \geq 0$,

$$\begin{aligned}
 & \sum_{N_1 \geq \dots \geq N_{k-1} \geq 0} \frac{q^{\eta(N_1^2 + \dots + N_{k-1}^2 + N_r + \dots + N_{k-1})} (1 + q^{-\eta N_r}) (-q^{\eta - \eta N_{\lambda+1}}; q^\eta)_{N_{\lambda+1}}}{(q^\eta; q^\eta)_{N_1 - N_2} \cdots (q^\eta; q^\eta)_{N_{k-2} - N_{k-1}} (q^{(2-j)\eta}; q^{(2-j)\eta})_{N_{k-1}}} \\
 & \quad \times (-q^{\eta + \eta N_\lambda}; q^\eta)_\infty \prod_{s=1}^{\lambda} (-q^{\eta - \alpha_s - \eta N_s}; q^\eta)_{N_s} \prod_{s=2}^{\lambda} (-q^{\eta - \alpha_s + \eta N_{s-1}}; q^\eta)_\infty \\
 & = (-q^{\alpha_1}, \dots, -q^{\alpha_\lambda}, -q^\eta; q^\eta)_\infty \\
 & \quad \times \frac{(q^{(r - \frac{\lambda}{2})\eta}, q^{(2k - r - \frac{\lambda}{2} + j - 1)\eta}, q^{(2k - \lambda + j - 1)\eta}; q^{(2k - \lambda + j - 1)\eta})_\infty}{(q^\eta; q^\eta)_\infty}.
 \end{aligned}$$

For $k \geq r > \lambda/2 \geq 0$, and for $|q| < 1$, let

$$\begin{aligned}
 & F_{\lambda,k,r}(c_1, c_2; b_1, \dots, b_\lambda; a; q) \\
 &= \frac{(a/b_1, \dots, a/b_\lambda; q)_\infty}{(a; q)_\infty} \sum_{n \geq 0} \frac{(-1)^{n(\lambda+1)} a^{kn} q^{kn^2 - (\lambda+1)\binom{n}{2} + n - rn}}{(q; q)_n (c_1 c_2 b_1 \cdots b_\lambda)^n} \\
 &\times \frac{(a, c_1, c_2, b_1, \dots, b_\lambda; q)_n}{(aq/c_1, aq/c_2, a/b_1, \dots, a/b_\lambda; q)_n} \\
 &\times \left\{ 1 + \frac{(-1)^{\lambda+1} a^r q^{(2r-\lambda)n} (1 - b_1 q^n) \cdots (1 - b_\lambda q^n)}{b_1 \cdots b_\lambda (1 - aq^n/b_1) \cdots (1 - aq^n/b_\lambda)} \right\},
 \end{aligned}$$

$$\begin{aligned}
 & G_{\lambda,k,r}(c_1, c_2; b_1, \dots, b_\lambda; a; q) \\
 &= \lim_{b_{\lambda+1}, \dots, b_{2r+1} \rightarrow \infty} \sum_{N_1 \geq \dots \geq N_{k-1} \geq 0} \frac{(aq/c_1 c_2)_{N_{k-1}} a^{N_1 + \dots + N_{k-1}} q^{N_1^2 + \dots + N_{k-1}^2 - (N_1 + \dots + N_{r-1})}}{(q; q)_{N_1 - N_2} \cdots (q; q)_{N_{k-1}} (aq/c_1, aq/c_2; q)_{N_{k-1}}} \\
 &\times (q^{1-N_1}/b_1; q)_{N_1} \prod_{s=2}^r (a/b_s b_{2r+1-s}; q)_{N_{s-1} - N_s} \\
 &\times \prod_{s=2}^r (q^{1-N_s}/b_s, q^{1-N_s}/b_{2r+1-s}; q)_{N_s} (aq^{N_{s-1}}/b_s, aq^{N_{s-1}}/b_{2r+1-s}; q)_\infty.
 \end{aligned} \tag{0.1}$$

Using the q -difference equations, Bressoud obtained

Theorem (Bressoud, 1980)

For $k > r > \lambda/2 \geq 0$,

$$F_{\lambda,k,r}(c_1, c_2; b_1, \dots, b_\lambda; a; q) = G_{\lambda,k,r}(c_1, c_2; b_1, \dots, b_\lambda; a; q).$$

For $k \geq r > \lambda/2 \geq 0$,

$$\lim_{c_1, c_2 \rightarrow \infty} F_{\lambda,k,r}(c_1, c_2; b_1, \dots, b_\lambda; a; q) = \lim_{c_1, c_2 \rightarrow \infty} G_{\lambda,k,r}(c_1, c_2; b_1, \dots, b_\lambda; a; q).$$



D.M. Bressoud, Analytic and combinatorial generalizations of the Rogers-Ramanujan identities, Mem. Amer. Math. Soc., vol. 227, 1980, 54 pp.

Let $\lambda, k, r, \alpha_1, \dots, \alpha_\lambda$ and η be integers which satisfy

$$(2k+1)/2 > r > \lambda/2 \geq 0, \quad 0 < \alpha_1 < \dots < \alpha_\lambda < \eta$$

and for $1 \leq i \leq \lambda$,

$$\alpha_i = \eta - \alpha_{\lambda+1-i}.$$

Bressoud define two partition functions $A(\alpha_1, \dots, \alpha_\lambda; \eta, k, r, 1; n)$ and $C(\alpha_1, \dots, \alpha_\lambda; \eta, k, r, 1; n)$.

$$\begin{aligned}
& \sum_{n \geq 0} A(\alpha_1, \dots, \alpha_\lambda; \eta, k, r, 1; n) q^n \\
&= \lim_{c_1, c_2 \rightarrow \infty} F_{\lambda, k, r}(c_1, c_2; -q^{\alpha_1}, \dots, -q^{\alpha_\lambda}; q^\eta; q^\eta).
\end{aligned}$$

$$\begin{aligned}
& \sum_{n \geq 0} C(\alpha_1, \dots, \alpha_\lambda; \eta, k, r, 1; n) q^n \\
&= \lim_{c_1, c_2 \rightarrow \infty} G_{\lambda, k, r}(c_1, c_2; -q^{\alpha_1}, \dots, -q^{\alpha_\lambda}; q^\eta; q^\eta).
\end{aligned}$$

From the definition of $F_{\lambda,k,r}(c_1, c_2; -q^{\alpha_1}, \dots, -q^{\alpha_\lambda}; q^\eta; q^\eta)$ and by Jacobi's triple product identity, Bressoud derived that

$$\begin{aligned}
 & \sum_{n \geq 0} A(\alpha_1, \dots, \alpha_\lambda; \eta, k, r, 1; n) q^n \\
 &= \frac{(-q^{\alpha_1}, \dots, -q^{\alpha_\lambda}; q^\eta)_\infty}{(q^\eta; q^\eta)_\infty} \\
 & \times \sum_{n \geq 0} q^{\eta \frac{2k-\lambda+1}{2} \binom{n+1}{2} - \eta \frac{2r-\lambda}{2}} (1 - q^{\eta \frac{(2r-\lambda)(2n+1)}{2}}) \\
 &= (-q^{\alpha_1}, \dots, -q^{\alpha_\lambda}; q^\eta)_\infty \\
 & \times \frac{(q^{\eta(r-\frac{\lambda}{2})}, q^{\eta(2k-r-\frac{\lambda}{2}+1)}, q^{\eta(2k-\lambda+1)}; q^{\eta(2k-\lambda+1)})_\infty}{(q^\eta; q^\eta)_\infty}
 \end{aligned}$$

By the definition of $G_{\lambda,k,r}(c_1, c_2; b_1, \dots, b_\lambda; a; q)$, we see that

$$\begin{aligned}
 & \sum_{n \geq 0} C(\alpha_1, \dots, \alpha_\lambda; \eta, k, r, 1; n) q^n \\
 &= \sum_{N_1 \geq \dots \geq N_{k-1} \geq 0} \frac{q^{\eta(N_1^2 + \dots + N_{k-1}^2 + N_r + \dots + N_{k-1})}}{(q^\eta; q^\eta)_{N_1 - N_2} \cdots (q^\eta; q^\eta)_{N_{k-1}}} \\
 & \times \prod_{s=1}^{\lambda} (q^{\eta - \alpha_s - \eta N_s}; q^\eta)_{N_s} \prod_{s=2}^{\lambda} (q^{\eta - \alpha_s + \eta N_{s-1}}; q^\eta)_{\infty}.
 \end{aligned}$$

Definition (Bailey pairs)

A pair of sequences $(\alpha_n(a; q), \beta_n(a; q))$ is called a Bailey pair relative to $a; q$ if they have the following relation for all $n \geq 0$,

$$\beta_n(a; q) = \sum_{r=0}^{\infty} \frac{\alpha_r(a; q)}{(q; q)_{n-r} (aq; q)_{n+r}}.$$

Some known Bailey pairs

Lemma

We have

$$\alpha_0(1, q) = 1, \quad \alpha_n(1, q) = (-1)^n q^{\binom{n}{2}} (1 + q^n), \quad n \geq 1,$$

$$\beta_0(1, q) = 1, \quad \beta_n(1, q) = 0, \quad n \geq 1$$

is a Bailey's pair relative to $(1; q)$.

Proof. By definition, we have

$$\begin{aligned}
 \beta_n &= \sum_{r=0}^n \frac{\alpha_r}{(q; q)_{n-r}(q; q)_{n+r}} \\
 &= \frac{1}{(q; q)_n(q; q)_n} + \sum_{r=1}^n \frac{(-1)^r q^{\binom{r}{2}} (1+q^r)}{(q; q)_{n-r}(q; q)_{n+r}} \\
 &= \frac{1}{(q; q)_{2n}} \left(\begin{bmatrix} 2n \\ n \end{bmatrix} + \sum_{r=1}^n \begin{bmatrix} 2n \\ n-r \end{bmatrix} (-1)^r q^{\binom{r}{2}} (1+q^r) \right) \\
 &= \frac{1}{(q; q)_{2n}} \sum_{r=-n}^n \begin{bmatrix} 2n \\ n-r \end{bmatrix} (-1)^r q^{\binom{r}{2}}
 \end{aligned}$$

It suffices to show that

$$\sum_{r=-n}^n \begin{bmatrix} 2n \\ n-r \end{bmatrix} (-1)^r q^{\binom{r}{2}} = 0 \quad \text{for } n \geq 1,$$

which follows from the following identity:

$$(z; q)_n = \sum_{k=0}^n (-1)^k q^{\frac{k(k-1)}{2}} \begin{bmatrix} n \\ k \end{bmatrix} z^k.$$

Some known Bailey pairs

Lemma

$$\alpha_0(q; q) = 1, \quad \alpha_n(q; q) = (-1)^n q^{\binom{n}{2}} \frac{1 - q^{2n+1}}{1 - q}, \quad n \geq 1,$$

$$\beta_0(q, q) = 1, \quad \beta_n(q, q) = 0, \quad n \geq 1$$

is a Bailey's pair relative to $(q; q)$.

$$\beta_n = \sum_{r=0}^n \frac{\alpha_r}{(q; q)_{n-r} (q^2; q)_{n+r}}$$

Theorem (Bailey's lemma)

If $(\alpha_n(a, q), \beta_n(a, q))$ is a Bailey pair relative to (a, q) , then $(\alpha'_n(a, q), \beta'_n(a, q))$ is also a Bailey pair relative to (a, q) , where

$$\alpha'_n(a, q) = \frac{(\rho_1; q)_n (\rho_2; q)_n}{(aq/\rho_1; q)_n (aq/\rho_2; q)_n} \left(\frac{aq}{\rho_1 \rho_2} \right)^n \alpha_n(a, q),$$

$$\beta'_n(a, q) = \sum_{j=0}^n \frac{(\rho_1; q)_j (\rho_2; q)_j (aq/\rho_1 \rho_2)_{n-j}}{(aq/\rho_1; q)_n (aq/\rho_2; q)_n (q; q)_{n-j}} \left(\frac{aq}{\rho_1 \rho_2} \right)^j \beta_j(a, q).$$

A corollary of Bailey's lemma

Letting $\rho_1, \rho_2 \rightarrow \infty$, we get

Corollary ($\rho_1, \rho_2 \rightarrow \infty$ in Bailey's lemma)

If (α_n, β_n) is a Bailey pair relative to (a, q) , then (α'_n, β'_n) is also a Bailey pair relative to (a, q) , where

$$\alpha'_n(a, q) = a^n q^{n^2} \alpha_n(a, q)$$

$$\beta'_n(a, q) = \sum_{j=0}^n \frac{a^j q^{j^2}}{(q; q)_{n-j}} \beta_j(a, q).$$

$$\alpha_0(1, q) = 1, \quad \alpha_n(1, q) = (-1)^n q^{\binom{n}{2}} (1 + q^n), \quad n \geq 1,$$

$$\beta_0(1, q) = 1, \quad \beta_n(1, q) = 0, \quad n \geq 1$$

is a Bailey's pair relative to $(1; q)$.

A corollary of Bailey's lemma

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If (α_n, β_n) is a Bailey pair relative to (a, q) , then (α'_n, β'_n) is also a Bailey pair relative to (a, q) , where

$$\alpha'_n(a, q) = a^n q^{n^2} \alpha_n(a, q)$$

$$\beta'_n(a, q) = \sum_{j=0}^n \frac{a^j q^{j^2}}{(q; q)_{n-j}} \beta_j(a, q).$$

$$\alpha_0(1, q) = 1, \quad \alpha_n(1, q) = (-1)^n q^{\binom{n}{2}} (1 + q^n), \quad n \geq 1,$$

$$\beta_0(1, q) = 1, \quad \beta_n(1, q) = 0, \quad n \geq 1$$

is a Bailey's pair relative to $(1; q)$.

The proof of the first Rogers-Ramanujan identity

Applying the first Bailey pair into Bailey's Lemma twice, we get

$$\begin{aligned}\alpha_n^{(2)}(1, q) &= \begin{cases} 1, & n = 0, \\ (-1)^n q^{\frac{5n^2}{2} - \frac{n}{2}} (1 + q^n), & n \geq 1, \end{cases} \\ \beta_n^{(2)}(1, q) &= \sum_{r=0}^n \frac{q^{r^2}}{(q; q)_r (q; q)_{n-r}}. \end{aligned} \quad (0.2)$$

By definition of Bailey's pair and letting $n \rightarrow \infty$

$$\begin{aligned}\sum_{r=0}^{\infty} \frac{q^{r^2}}{(q; q)_r} &= \frac{1}{(q; q)_{\infty}} \left(1 + \sum_{r=1}^{\infty} (-1)^r q^{\frac{5r^2}{2} - \frac{r}{2}} (1 + q^r) \right) \\ &= \frac{1}{(q; q)_{\infty}} \sum_{r=-\infty}^{\infty} q^{\frac{5r^2}{2}} (-q^{-\frac{1}{2}})^r \\ &= \frac{1}{(q; q^5)_{\infty} (q^4; q^5)_{\infty}} \quad (\text{Jacobi's triple product identity}).\end{aligned}$$

which is the first Rogers-Ramanujan identity.

A corollary of Bailey's lemma

Corollary ($\rho_1, \rho_2 \rightarrow \infty$ in Bailey's lemma)

If (α_n, β_n) is a Bailey pair relative to (a, q) , then (α'_n, β'_n) is also a Bailey pair relative to (a, q) , where

$$\alpha'_n(a, q) = a^n q^{n^2} \alpha_n(a, q)$$
$$\beta'_n(a, q) = \sum_{j=0}^n \frac{a^j q^{j^2}}{(q; q)_{n-j}} \beta_j(a, q).$$

$$\alpha_0(q; q) = 1, \quad \alpha_n(q; q) = (-1)^n q^{\binom{n}{2}} \frac{1 - q^{2n+1}}{1 - q}, \quad n \geq 1,$$

$$\beta_0(q, q) = 1, \quad \beta_n(q, q) = 0, \quad n \geq 1$$

is a Bailey's pair relative to $(q; q)$.

Applying the second Bailey pair into Bailey's Lemma twice, we get

$$\alpha_n^{(2)}(q, q) = \begin{cases} 1, & n = 0, \\ (-1)^n q^{\binom{n}{2} + 2n^2 + 2n} \frac{1 - q^{2n+1}}{1 - q}, & n \geq 1, \end{cases}$$

$$\beta_n^{(2)}(q, q) = \sum_{r=0}^n \frac{q^{r^2+r}}{(q; q)_r (q; q)_{n-r}}.$$

By definition of Bailey's pair and let $n \rightarrow \infty$

$$\begin{aligned} \sum_{r=0}^{\infty} \frac{q^{r^2+r}}{(q; q)_r} &= \frac{1}{(q^2; q)_{\infty}} \left(1 + \sum_{r=1}^{\infty} (-1)^r q^{\frac{5}{2}r^2 + \frac{3}{2}r} \frac{1 - q^{2r+1}}{1 - q} \right) \\ &= \frac{1}{(q; q)_{\infty}} \sum_{r=-\infty}^{\infty} q^{\frac{5r^2}{2}} (-q^{-\frac{3}{2}})^r \\ &= \frac{1}{(q^2; q^5)_{\infty} (q^3; q^5)_{\infty}} \quad (\text{Jacobi's triple product identity}). \end{aligned}$$

which is the second Rogers-Ramanujan identity.

Andrews-Gordon's identity

For $k \geq r \geq 1$,

$$\sum_{N_1 \geq \dots \geq N_{k-1} \geq 0} \frac{q^{N_1^2 + \dots + N_{k-1}^2 + N_r + \dots + N_{k-1}}}{(q; q)_{N_1 - N_2} \cdots (q; q)_{N_{k-2} - N_{k-1}} (q; q)_{N_{k-1}}}$$
$$= \prod_{\substack{n=1 \\ n \not\equiv 0, \pm r \pmod{2k+1}}}^{\infty} \frac{1}{1 - q^n}.$$

Lemma (Bailey lattices)

If $(\alpha_n(a, q), \beta_n(a, q))$ is a Bailey pair relative to (a, q) , then $(\alpha'_n(a/q, q), \beta'_n(a/q, q))$ is a Bailey pair relative to $(a/q, q)$, where

$$\alpha'_0 = \alpha_0, \quad \alpha'_n = \frac{(\rho, \sigma; q)_n (a/\rho\sigma)^n}{(a/\rho, a/\sigma; q)_n} (1-a) \left(\frac{\alpha_n}{1-aq^{2n}} - \frac{aq^{2n-2}\alpha_{n-1}}{1-aq^{2n-2}} \right),$$

$$\beta'_n = \sum_{r=0}^n \frac{(\rho, \sigma; q)_r (a/\rho\sigma; q)_{n-r} (a/\rho\sigma)^r}{(q)_{n-r} (a/\rho, a/\sigma; q)_n} \beta_r.$$



A. K. Agarwal, G.E. Andrews and D.M. Bressoud, The Bailey lattice , J. Indian Math. Soc. 51 (1987) 57–73.

A corollary of Bailey lattices

Letting $a = q, \rho, \sigma \rightarrow \infty$, we get

Lemma ($a = q, \rho, \sigma \rightarrow \infty$, in Bailey lattices)

If $(\alpha_n(q, q), \beta_n(q, q))$ is a Bailey pair, then $(\alpha'_n(1, q), \beta'_n(1, q))$ is a Bailey pair, where

$$\alpha'_n(1, q) = (1 - q)q^{n^2} \left[\frac{\alpha_n(q, q)}{1 - q^{2n+1}} - \frac{q^{2n-1}\alpha_{n-1}(q, q)}{1 - q^{2n-1}} \right],$$

$$\beta'_n(1, q) = \sum_{r=0}^n \frac{q^{r^2}}{(q; q)_{n-r}} \beta_r(q, q).$$

The proof of Andrews-Gordon's identity

Applying the second Bailey's pair into the first lemma ($a = q$) $k - i + 1$ times, we have

$$\alpha_n^{(k-i+1)}(q, q) = \begin{cases} 1, & n = 0, \\ \frac{(-1)^n q^{\binom{n}{2} + (k-i+1)n + (k-i+1)n^2} (1 - q^{2n+1})}{1 - q}, & n \geq 1, \end{cases}$$

$$\beta_n^{(k-i+1)}(q, q) = \sum_{n \geq N_i \geq \dots \geq N_{k-1} \geq 0} \frac{q^{N_i^2 + \dots + N_{k-1}^2 + N_i + \dots + N_{k-1}}}{(q; q)_{n - N_i} (q; q)_{N_i - N_{i+1}} \cdots (q; q)_{N_{k-2} - N_{k-1}} (q; q)_{N_{k-1}}}.$$

Applying the above Bailey's pair into the second lemma, we obtain

$$\alpha_n^{(k-i+2)}(1, q) = \begin{cases} 1, & n = 0, \\ (-1)^n q^{\binom{n}{2} + (k-i+1)n + (k-i+2)n^2} (1 + q^{(2i-2k-1)n}), & n \geq 1, \end{cases}$$

$$\beta_n^{(k-i+2)}(1, q) = \sum_{n \geq N_{i-1} \geq \dots \geq N_{k-1} \geq 0} \frac{q^{N_{i-1}^2 + \dots + N_{k-1}^2 + N_{i-1} + \dots + N_{k-1}}}{(q; q)_{n - N_{i-1}} (q; q)_{N_{i-1} - N_i} \cdots (q; q)_{N_{k-2} - N_{k-1}} (q; q)_{N_{k-1}}}$$

Applying the above Bailey's pair into the first lemma ($a = 1$) $i - 2$ 次, we derive that

$$\alpha_n^{(k)}(1, q) = \begin{cases} 1, & n = 0, \\ (-1)^n q^{\binom{n}{2} + (k-i+1)n + kn^2} (1 + q^{(2i-2k-1)n}), & n \geq 1, \end{cases}$$

$$\beta_n^{(k)}(1, q) = \sum_{n \geq N_1 \geq \dots \geq N_{k-1} \geq 0} \frac{q^{N_1^2 + \dots + N_{k-1}^2 + N_i + \dots + N_{k-1}}}{(q; q)_{n-N_1} (q; q)_{N_1-N_2} \cdots (q; q)_{N_{k-2}-N_{k-1}} (q; q)_{N_{k-1}}}.$$

By definition of Bailey's pair and let $n \rightarrow \infty$, we have

$$\begin{aligned}
 & \sum_{N_1 \geq \dots \geq N_{k-1} \geq 0} \frac{q^{N_1^2 + \dots + N_{k-1}^2 + N_i + \dots + N_{k-1}}}{(q; q)_{N_1 - N_2} \cdots (q; q)_{N_{k-2} - N_{k-1}} (q; q)_{N_{k-1}}} \\
 &= \frac{1}{(q; q)_{\infty}} \left(1 + \sum_{r=1}^{\infty} (-1)^r q^{\binom{r}{2} + (k-i+1)r + kr^2} (1 + q^{(2i-2k-1)r}) \right) \\
 &= \frac{1}{(q; q)_{\infty}} \sum_{r=-\infty}^{\infty} (-1)^r q^{\binom{r}{2} + (k-i+1)r + kr^2} \\
 &= \frac{(q^i, q^{2k-i+1}, q^{2k+1}; q^{2k+1})_{\infty}}{(q; q)_{\infty}} \quad (\text{Jacobi's triple product identity}).
 \end{aligned}$$

which is the Andrews-Gordon identity.

An overpartition analogue of the Andrews-Gordon identity

Theorem (Chen-Shi-Sang, 2013)

$$\sum_{N_1 \geq \dots \geq N_{k-1}} \frac{(-q^{1-N_1}; q)_{N_1-1} (1 + q^{N_r}) q^{N_1^2 + \dots + N_{k-1}^2 + N_{r+1} + \dots + N_{k-1}}{(q; q)_{N_1-N_2} \cdots (q; q)_{N_{k-2}-N_{k-1}} (q; q)_{N_{k-1}}}$$
$$= \frac{(-q; q)_\infty (q^r, q^{2k-r}, q^{2k}; q^{2k})_\infty}{(q; q)_\infty}.$$

Change of base in Bailey's pair (I)

Theorem

If A is a real number and $(\alpha_n(1, q), \beta_n(1, q))$ is a Bailey's pair satisfying

$$\alpha_n(1, q) = \begin{cases} 1, & n = 0, \\ (-1)^n q^{An^2} (q^{(A-1)n} + q^{-(A-1)n}), & n \geq 1, \end{cases}$$

Then $(\alpha'_n(1, q), \beta'_n(1, q))$ is also a Bailey's pair, where

$$\alpha'_n(1, q) = \begin{cases} 1, & n = 0, \\ (-1)^n q^{An^2} (q^{An} + q^{-An}), & n \geq 1, \end{cases}$$

$$\beta'_n(1, q) = q^n \beta_n(1, q).$$



D.M. Bressoud, M.E.H. Ismail and D. Stanton, Change of base in Bailey pairs, Ramanujan J. 4 (2000) 435–453.

Change of base in Bailey's pair (II)

Theorem

If A is a real number and $(\alpha_n(1, q), \beta_n(1, q))$ is a Bailey's pair satisfying

$$\alpha_n(1, q) = \begin{cases} 1, & n = 0, \\ (-1)^n q^{An^2} (q^{(A-1)n} + q^{-(A-1)n}), & n \geq 1, \end{cases}$$

Then $(\alpha'_n(1, q), \beta'_n(1, q))$ is also a Bailey's pair, where

$$\alpha'_n(1, q) = \begin{cases} 1, & n = 0, \\ (-1)^n q^{An^2} (q^{An} + q^{-An})(1 + q^n)/2, & n \geq 1, \end{cases}$$
$$\beta'_n(1, q) = (1 + q^n)/2\beta_n(1, q).$$



T.Y. He, K.Q. Ji, and A.Y.F Wang, An overpartition analogue of the Andrews-Gollnitz-Gordon theorem, J. Combin. Theory Ser. A, 166 (2019) 254–296.

Applying Bailey's pair into Bailey's lemma and the result on the change of base $k - i - 1$ times, we arrive at

$$\alpha_n^{(2k-2i-2)}(1, q) = \begin{cases} 1, & n = 0, \\ (-1)^n q^{\frac{2k-2i-1}{2}n^2} (q^{-\frac{2k-2i-1}{2}n} + q^{\frac{2k-2i-1}{2}n}), & n \geq 1, \end{cases}$$

$$\beta_n^{(2k-2i-2)}(1, q) = \sum_{n \geq N_{i+1} \geq \dots \geq N_{k-1} \geq 0} \frac{q^{N_{i+1}^2 + N_{i+2}^2 + \dots + N_{k-1}^2 + N_{i+1} + \dots + N_{k-1}}}{(q; q)_{n-N_{i+1}} (q; q)_{N_{i+1}-N_{i+2}} \cdots (q; q)_{N_{k-2}-N_{k-1}} (q; q)_{N_k}}$$

Plug the above Bailey's pair into the corollary to obtain

$$\alpha_n^{(2k-2i+1)}(1, q) = \begin{cases} 1, & n = 0, \\ (-1)^n q^{\frac{2k-2i+1}{2}n^2} (q^{-\frac{2k-2i+1}{2}n} + q^{\frac{2k-2i+1}{2}n}) (1 + q^n)/2, & n \geq 1, \end{cases}$$

$$\beta_n^{(2k-2i+1)}(1, q) = \sum_{n \geq N_{i+1} \geq \dots \geq N_{k-1} \geq 0} \frac{(1 + q^n) q^{N_{i+1}^2 + N_{i+2}^2 + \dots + N_{k-1}^2 + N_{i+1} + \dots + N_{k-1}}}{2(q; q)_{n-N_{i+1}} (q; q)_{N_{i+1}-N_{i+2}} \cdots (q; q)_{N_{k-2}-N_{k-1}} (q; q)_n}$$

Applying the above Bailey's pair into the corollary $i - 1$ times, we deduce that

$$\alpha_n^{(2k-i)}(1, q) = \begin{cases} 1, & n = 0, \\ (-1)^n q^{\frac{2k-1}{2}n^2} (q^{-\frac{2k-2i+1}{2}n} + q^{\frac{2k-2i-1}{2}n}) (1 + q^n)/2, & n \geq 1, \end{cases}$$

$$\beta_n^{(2k-i)}(1, q) = \sum_{n \geq N_2 \geq \dots \geq N_{k-1} \geq 0} \frac{q^{N_2^2 + N_3^2 + \dots + N_{k-1}^2 + N_i + \dots + N_{k-1}} (1 + q^{-N_i})}{2(q; q)_{n-N_2} (q; q)_{N_2-N_3} \dots (q; q)_{N_{k-2}-N_{k-1}} (q; q)_{N_{k-1}}}.$$

$$\begin{aligned} & \sum_{N_1 \geq \dots \geq N_{k-1}} \frac{(-q^{1-N_1}; q)_{N_1-1} (1 + q^{N_i}) q^{N_1^2 + \dots + N_{k-1}^2 + N_{i+1} + \dots + N_{k-1}}{(q; q)_{N_1-N_2} \dots (q; q)_{N_{k-2}-N_{k-1}} (q; q)_{N_{k-1}}} \\ &= \frac{(-q; q)_\infty (q^i, q^{2k-i}, q^{2k}; q^{2k})_\infty}{(q; q)_\infty}. \end{aligned}$$

Applying the above Bailey's pair into the corollary $i - 1$ times, we deduce that

$$\alpha_n^{(2k-i)}(1, q) = \begin{cases} 1, & n = 0, \\ (-1)^n q^{\frac{2k-1}{2}n^2} (q^{-\frac{2k-2i+1}{2}n} + q^{\frac{2k-2i-1}{2}n}) (1 + q^n)/2, & n \geq 1, \end{cases}$$

$$\beta_n^{(2k-i)}(1, q) = \sum_{n \geq N_2 \geq \dots \geq N_{k-1} \geq 0} \frac{q^{N_2^2 + N_3^2 + \dots + N_{k-1}^2 + N_i + \dots + N_{k-1}} (1 + q^{-N_i})}{2(q; q)_{n-N_2} (q; q)_{N_2-N_3} \cdots (q; q)_{N_{k-2}-N_{k-1}} (q; q)_{N_{k-1}}}.$$

$$\begin{aligned} & \sum_{N_1 \geq \dots \geq N_{k-1}} \frac{(-q^{1-N_1}; q)_{N_1-1} (1 + q^{N_i}) q^{N_1^2 + \dots + N_{k-1}^2 + N_{i+1} + \dots + N_{k-1}}}{(q; q)_{N_1-N_2} \cdots (q; q)_{N_{k-2}-N_{k-1}} (q; q)_{N_{k-1}}} \\ &= \frac{(-q; q)_\infty (q^i, q^{2k-i}, q^{2k}; q^{2k})_\infty}{(q; q)_\infty}. \end{aligned}$$

Theorem (Bailey's Lemma)

If $(\alpha_n(a, q), \beta_n(a, q))$ is a Bailey pair relative to (a, q) , then $(\alpha'_n(a, q), \beta'_n(a, q))$ is also a Bailey pair relative to (a, q) , where

$$\alpha'_n(a, q) = \frac{(\rho_1; q)_n (\rho_2; q)_n}{(aq/\rho_1; q)_n (aq/\rho_2; q)_n} \left(\frac{aq}{\rho_1 \rho_2} \right)^n \alpha_n(a, q),$$
$$\beta'_n(a, q) = \sum_{j=0}^n \frac{(\rho_1; q)_j (\rho_2; q)_j (aq/\rho_1 \rho_2)_{n-j}}{(aq/\rho_1; q)_n (aq/\rho_2; q)_n (q; q)_{n-j}} \left(\frac{aq}{\rho_1 \rho_2} \right)^j \beta_j(a, q).$$

Letting $\rho_1 \rightarrow \infty$ and $\rho_2 = -1$, we have

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$$\beta'_n(a, q) = \sum_{j=0}^n \frac{(\rho_1; q)_j (\rho_2; q)_j (aq/\rho_1 \rho_2)_{n-j}}{(aq/\rho_1; q)_n (aq/\rho_2; q)_n (q; q)_{n-j}} \left(\frac{aq}{\rho_1 \rho_2} \right)^j \beta_j(a, q).$$

Letting $\rho_1 \rightarrow \infty$ and $\rho_2 = -1$, we have

Another corollary of Bailey's lemma

Corollary

If (α_n, β_n) is a Bailey pair relative to (a, q) , then (α'_n, β'_n) is also a Bailey pair relative to (a, q) , where

$$\alpha'_n = \frac{(-1; q)_n a^n q^{\binom{n+1}{2}}}{(-aq; q)_n} \alpha_n,$$
$$\beta'_n = \sum_{r=0}^n \frac{(-1; q)_r a^r q^{\binom{r+1}{2}}}{(-aq; q)_n (q; q)_{n-r}} \beta_r.$$

$$\alpha_n^{(2k-i)}(1, q) = \begin{cases} 1, & n = 0, \\ (-1)^n q^{\frac{2k-1}{2}n^2} (q^{-\frac{2k-2i+1}{2}n} + q^{\frac{2k-2i-1}{2}n}) (1 + q^n)/2, & n \geq 1, \end{cases}$$

$$\beta_n^{(2k-i)}(1, q) = \sum_{n \geq N_2 \geq \dots \geq N_{k-1} \geq 0} \frac{q^{N_2^2 + N_3^2 + \dots + N_{k-1}^2 + N_i + \dots + N_{k-1}} (1 + q^{-N_i})}{2(q; q)_{n-N_2} (q; q)_{N_2-N_3} \cdots (q; q)_{N_{k-2}-N_{k-1}} (q; q)_{N_{k-1}}}.$$

Plug the above Bailey's pair to the corollary to get

$$\alpha_n^{(2k-i+1)}(1, q) = \begin{cases} 1, & n = 0, \\ (-1)^n q^{kn^2} (q^{(i-k)n} + q^{(k-i)n}), & n \geq 1, \end{cases}$$

$$\beta_n^{(2k-i+1)}(1, q) = \sum_{n \geq N_1 \geq \dots \geq N_{k-1} \geq 0} \frac{(-1; q)_{N_1} q^{\binom{N_1+1}{2} + N_2^2 + \dots + N_{k-1}^2 + N_i + \dots + N_{k-1}} (1 + q^{-N_i})}{2(-q; q)_n (q; q)_{n-N_1} (q; q)_{N_1-N_2} \cdots (q; q)_{N_{k-2}-N_{k-1}} (q; q)_{N_{k-1}}}.$$

By definition of Bailey's pair and let $n \rightarrow \infty$, we have

$$\begin{aligned}
 & \sum_{N_1 \geq \dots \geq N_{k-1} \geq 0} \frac{(-q^{1-N_1}; q)_{N_1-1} q^{N_1^2 + \dots + N_{k-1}^2 + N_i + \dots + N_{k-1}} (1 + q^{-N_i})}{(q; q)_{N_1-N_2} \cdots (q; q)_{N_{k-2}-N_{k-1}} (q; q)_{N_{k-1}}} \\
 &= \frac{(-q; q)_{\infty}}{(q; q)_{\infty}} \left(1 + \sum_{r=1}^{\infty} (-1)^r q^{kr^2} (q^{(i-k)n} + q^{(k-i)n}) \right) \\
 &= \frac{(-q; q)_{\infty}}{(q; q)_{\infty}} \sum_{r=-\infty}^{\infty} (-1)^r q^{kr^2 + (i-k)n} \\
 &= \frac{(-q; q)_{\infty} (q^i, q^{2k-i}, q^{2k}; q^{2k})_{\infty}}{(q; q)_{\infty}} \quad (\text{Jacobi's triple product identity}).
 \end{aligned}$$

THANK YOU!