

Recent progress on integer partitions

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joint work with

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背景：分拆理论

- 分拆理论是组合数学的重要分支，即古老，又年轻

- 早期：



Leibniz



Euler



Sylvester

- 近现代及当代：



MacMahon



Hardy



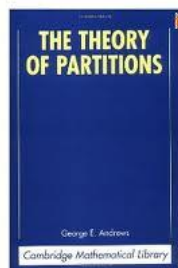
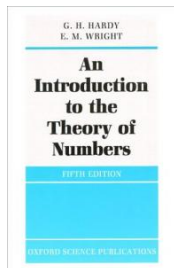
Ramanujan



Andrews

相关专著

- 关于该理论早期最系统地介绍就在由Hardy与Wright编著的《An introduction to the theory of numbers》；
- 美国数学学会前任会长（2009-2010）、美国科学院院士George E. Andrews专门为之编写的专著《The Theory of Partitions》(引用次数高达两千多次)。



Partitions

Definition

A **partition λ of n** is an integer sequence $(\lambda_1, \lambda_2, \dots, \lambda_\ell)$ such that $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_\ell > 0$ and $\sum_{i=1}^{\ell} \lambda_i = n$.

Example: There are **seven** partitions of 5, that is

$(5), (4, 1), (3, 2), (3, 1, 1), (2, 2, 1), (2, 1, 1, 1), (1, 1, 1, 1, 1)$.

Let $p(n)$ denote the number of partitions of n . For the above example, $p(5) = 7$.

Generating function

Theorem (Euler)

$$\sum_{n=0}^{+\infty} p(n)q^n = \prod_{j=1}^{\infty} \frac{1}{1-q^j}.$$

proof: We first consider the following expansion

$$\begin{aligned} \prod_{j=1}^{\infty} \frac{1}{1-q^j} &= (1 + q^1 + q^{1+1} + q^{1+1+1} + \dots) \\ &\quad \times (1 + q^2 + q^{2+2} + q^{2+2+2} + \dots) \\ &\quad \times \dots = \sum_{n=0} (\dots) q^n, \end{aligned}$$

Obviously, the coefficient of q^n in the above expansion is equal to the number of solutions to **Diophantine's equation** $1 \times j_1 + 2 \times j_2 + 3 \times j_3 + \dots = n$. On the other hand, it's easy to see that each solution to such an equation corresponds to a partitions of n .

Euler's partition theorem

Theorem (Euler)

The number of partitions of n with *distinct parts* is equal to the number of partitions of n with *odd parts*.

For example, there are *eleven partitions of 6*:

$(6), (5, 1), (4, 2), (4, 1^2), (3^2), (3, 2, 1), (3, 1^3), (2^3), (2^2, 1^2), (2, 1^4), (1^6)$

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where there are *four partitions with distinct parts*:

$(6), (5, 1), (4, 2), (3, 2, 1).$

Euler's partition theorem

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where there are *four partitions with distinct parts*:

$(6), (5, 1), (4, 2), (3, 2, 1),$

and there are *four partitions with odd parts*:

$(5, 1), (3^2), (3, 1^3), (1^6).$

Example

Let $D(O)$ denote the set of partitions with distinct parts (odd parts). Euler's partition theorem could follow from the following identity:

$$\begin{aligned}\sum_{\lambda \in D} q^{|\lambda|} &= \prod_{j=1}^{\infty} (1 + q^j) \\ &= \prod_{j=1}^{\infty} \frac{(1 + q^j)(1 - q^j)}{(1 - q^j)} \\ &= \prod_{j=1}^{\infty} \frac{(1 - q^{2j})}{(1 - q^j)} \\ &= \prod_{j=1}^{\infty} \frac{1}{(1 - q^{2j-1})} = \sum_{\mu \in O} q^{|\mu|}.\end{aligned}$$

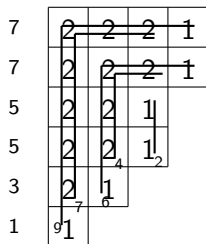
Sylvester's bijection

2-Modular Diagram

7	2	2	2	1
7	2	2	2	1
5	2	2	1	
5	2	2	1	
3	2	1		
1	1			

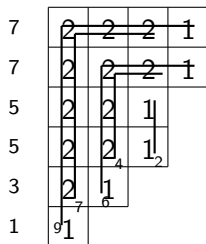
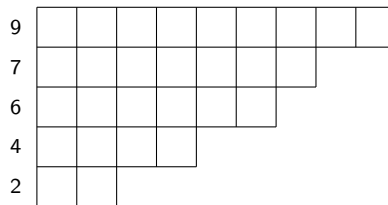
Sylvester's bijection

2-Modular Diagram



Sylvester's bijection

2-Modular Diagram


 λ

 μ

Refinements for Euler's partition theorem

There are many refinements of Euler's partition theorem, such as Sylvester's refinement, Glaisher's refinement, Fine's refinement and Bessenrodt's refinement.



James Sylvester



James Glaisher



Nathan Fine



Christine Bessenrodt

Bessenrodt's refinement

Bessenrodt has shown that Sylvester's bijection implies the following refinement, which is a limiting case of the lecture hall theorem due to Bousquet-Mélou and Erikssonin. Let $\ell_a(\lambda)$ denote the alternating sum of $\lambda = (\lambda_1, \lambda_2, \dots)$, namely,
$$\ell_a(\lambda) = \lambda_1 - \lambda_2 + \lambda_3 - \lambda_4 + \dots$$

Theorem (Bessenrodt, Discrete Math., 1994)

The number of partitions of n into distinct parts with alternating sum ℓ is equal to the number of partitions of n with ℓ odd parts. In terms of generating functions, we have

$$\sum_{\lambda \in D} y^{\ell_a(\lambda)} q^{|\lambda|} = \sum_{\mu \in O} y^{\ell(\mu)} q^{|\mu|}.$$

Example for Bessenrodt's refinement

Euler's partition theorem

- The partitions of 6 with odd parts:

$$(5, 1), (3, 3), (3, 1^3), (1^6).$$

- The partitions of 6 with distinct parts:

$$(6), (5, 1), (4, 2), (3, 2, 1).$$

Example for Bessenrodt's refinement

Euler's partition theorem

- The partitions of 6 with odd parts:

$$(5, 1), (3, 3), (3, 1^3), (1^6).$$

- The partitions of 6 with distinct parts:

$$(6), (5, 1), (4, 2), (3, 2, 1).$$

Bessenrodt's refinement

- The partitions of 6 into two odd parts:

$$(5, 1), (3, 3).$$

- The partitions of 6 into distinct parts with alternating sum equal to two:

$$(4, 2), (3, 2, 1).$$

Glaisher's refinement

From a different angle, Glaisher has also given a refinement. Let $\ell_o(\lambda)$ denote the number of odd parts in λ , and let $n_o(\mu)$ denote the number of different parts in μ with odd multiplicities.

Theorem (Glaisher, Messenger Math., 1883)

The number of partitions of n into distinct parts with k odd parts is equal to the number of partitions of n with odd parts such that there are exactly k different parts repeated odd times. In terms of generating functions, we have

$$\sum_{\lambda \in D} x^{\ell_o(\lambda)} q^{|\lambda|} = \sum_{\mu \in O} x^{n_o(\mu)} q^{|\mu|}.$$

Example for Glaisher's refinement

Euler's partition theorem

- The partitions of 6 with odd parts:

$$(5, 1), (3, 3), (3, 1^3), (1^6).$$

- The partitions of 6 with distinct parts:

$$(6), (5, 1), (4, 2), (3, 2, 1).$$

Glaisher's refinement

- The partitions of 6 into odd parts such that there are exactly two different parts repeated odd times:

$$(5, 1), (3, 1^3).$$

- The partitions of 6 into distinct parts with two odd parts:

$$(5, 1), (3, 2, 1).$$

Remark

Glaisher's refinement could not follow from Sylvester's bijection and it can follow from a new bijection due to Glaisher, which we call [Glaisher's bijection](#).

Two-variable refinement

We gave the following unification of two refinements of Euler's partition theorem respectively due to Bessenrodt and Glaisher.

Theorem (Chen-Gao-Ji-Li, Ramanujan J., to appear)

The number of partitions of n into distinct parts with ℓ odd parts and alternating sum m is equal to the number of partitions of n into exactly m odd parts and ℓ parts repeated odd times. In terms of generating functions, we have

$$\sum_{\lambda \in D} x^{\ell_o(\lambda)} y^{\ell_a(\lambda)} q^{|\lambda|} = \sum_{\mu \in O} x^{n_o(\mu)} y^{\ell(\mu)} q^{|\mu|}.$$

Example for two-variable refinement

Bessenrodt's refinement

- The partitions of 6 into two odd parts : $(5, 1)$, $(3, 3)$.
- The partitions of 6 into distinct parts with alternating sum equal to two: $(4, 2)$, $(3, 2, 1)$.

Glaisher's refinement

- The partitions of 6 into odd parts such that there are exactly **two different parts repeated odd times**: $(5, 1)$, $(3, 1^3)$.
- The partitions of 6 into distinct parts with **two odd parts**: $(5, 1)$, $(3, 2, 1)$.

two-variable refinement

- The partitions of 6 into two odd parts such that there are exactly **two different parts repeated odd times**: $(5, 1)$.
- The partitions of 6 into distinct parts with **two odd parts and alternating sum two**: $(3, 2, 1)$.

Two-variable refinement

Remarks:

- Setting $x = 1$, this refinement reduces to [Bessenrodt's refinement](#) which follows from [Sylvester's bijection](#).
- Setting $y = 1$, this refinement reduces to [Glaisher's refinement](#) which follows from [Glaisher's bijection](#).
- Neither [Sylvester's bijection](#) nor [Glaisher's bijection](#) implies this result.

To prove this two-variable refinement, we construct [another bijection for Euler's partition theorem](#) in which a specialization of Bessenrodt's insertion algorithm (Discrete. Math., 1995) is the main ingredient.

Ramanujan's identities

The following Ramanujan's identities are related to Euler's partition theorem just as anticipated by Andrews.

$$\sum_{n=0}^{\infty} [(-q; q)_{\infty} - (-q; q)_n] = (-q; q)_{\infty} \left[-\frac{1}{2} + \sum_{d=1}^{\infty} \frac{q^d}{1 - q^d} \right] + \frac{1}{2} \sum_{n=0}^{\infty} \frac{q^{\binom{n+1}{2}}}{(-q; q)_n},$$

$$\sum_{n=0}^{\infty} \left[\frac{1}{(q; q^2)_{\infty}} - \frac{1}{(q; q^2)_n} \right] = (-q; q)_{\infty} \left[-\frac{1}{2} + \sum_{d=1}^{\infty} \frac{q^{2d}}{1 - q^{2d}} \right] + \frac{1}{2} \sum_{n=0}^{\infty} \frac{q^{\binom{n+1}{2}}}{(-q; q)_n}.$$

Background

- [Andrews and Berndt \(Notices AMS, 2008\)](#): One of the top ten most fascinating formulas from Ramanujan's Lost Notebook.
- [Andrews \(Adv. Math., 1986\)](#) gave the first algebraic proofs of these two identities and asked for their combinatorial proofs.
- [Chapman \(J. Combin. Theory Ser. A, 2002\)](#) also proposed that it would be interesting to find combinatorial proofs of these two identities.

Rooted partitions

To give combinatorial interpretations of the first sums on the right-hand sides of Ramanujan's identities, we introduce the notion of rooted partitions.

Definition

A **rooted partition** of n can be formally defined as a pair of partitions (λ, μ) , where $|\lambda| + |\mu| = n$ and μ is a nonempty partition with equal parts.

Example

There are **twelve** rooted partitions of 4:

$$\begin{array}{cccc}
 (\emptyset, (4)) & ((1), (3)) & ((3), (1)) & ((2), (2)) \\
 (\emptyset, (2, 2)) & ((1, 1), (2)) & ((2, 1), (1)) & ((2), (1, 1)) \\
 ((1, 1, 1), (1)) & ((1, 1), (1, 1)) & ((1), (1, 1, 1)) & (\emptyset, (1, 1, 1, 1)).
 \end{array}$$

Combinatorial interpretations

We gave combinatorial interpretations of the following two terms with the aid of rooted partitions.

Lemma (Chen-Ji, J. Combin. Theory Ser. A, 2007)

The following relation holds

$$(-q; q)_\infty \sum_{d=1}^{\infty} \frac{q^d}{1 - q^d} = \sum_{\lambda \in O} 2\ell(\lambda)q^{|\lambda|} - \sum_{\mu \in D} \ell(\mu)q^{|\mu|}.$$

Lemma (Chen-Ji, J. Combin. Theory Ser. A, 2007)

The following relation holds

$$(-q; q)_\infty \sum_{d=1}^{\infty} \frac{q^{2d}}{1 - q^{2d}} = \sum_{\lambda \in O} \ell(\lambda)q^{|\lambda|} - \sum_{\mu \in D} \ell(\mu)q^{|\mu|}.$$

Connections to Euler's partition theorem

Based on the above two lemmas, we may reformulate Ramanujan's identities by the following two weighted forms of Euler's partition theorem.

Theorem (Chen-Ji, J. Combin. Theory Ser. A, 2007)

$$\sum_{\mu \in D} \left(\ell(\mu) + \mu_1 + \frac{1 - (-1)^{r(\mu)}}{2} \right) q^{|\mu|} = \sum_{\lambda \in O} 2\ell(\lambda) q^{|\lambda|}.$$

Connections to Euler's partition theorem

Theorem (Chen-Ji, J. Combin. Theory Ser. A, 2007)

$$\sum_{\mu \in D} \left(\ell(\mu) + \frac{1 - (-1)^{r(\mu)}}{2} \right) q^{|\mu|} = \sum_{\lambda \in O} \left(\ell(\lambda) - \frac{\lambda_1 - 1}{2} \right) q^{|\lambda|}.$$

These two weighted forms can be deduced from [weighted forms of Euler's partition theorem](#) coming from Fine's two refinements respectively.

Reviews

- **Math Review:** The paper is clearly written and elegantly demonstrates the interplay between power series and bijective methods.
- **Andrews and Berndt (Notices of AMS, 2008):** “a beautiful paper”, “provided the combinatorial proofs that were sought by ...”
- Recently, **Kathy Ji (Elect. J. Combin., 2008)** give a combinatorial proof of Andrews' identity with the aid of rooted partitions.

印度天才数学家 *Srinivasa Ramanujan* (1887-1920)



Srinivasa Ramanujan (1887-1920)

- Ramanujan 出生于印度一个贫穷的家庭，他没有受过严格的数学训练，却独立发现了将近3900个数学公式。后来被Hardy发现，深受震撼之余，出资赞助他到英国继续他的数学研究。
- Ramanujan 被誉为“印度之子”，与诗人泰戈尔并驾齐驱成为印度最受尊敬和爱戴的人物。
- Hardy亲自设计出一种计算一个人数学天分的评分表，他自己得了25分，Littlewood得了30分，同时代最伟大的德国数学家Hilbert得了80分，而Ramanujan则得了满分100分。

Mock theta function begins with Ramanujan's last letter to Hardy.



Godfrey Hardy (1877-1947)

Early in 1920, three months before his death, Ramanujan wrote his last letter to Hardy. In the course of it he said: “I discovered very interesting functions recently which I call ‘Mock theta-functions’. Unlike the ‘False theta- functions’, they enter into mathematics as beautifully as the ordinary theta-functions. I am sending you with this letter some examples.

Background

The last two pages of Ramanujan's letter consist of lists of definitions of four sets of mock theta-functions with statements of relations connecting members of each of the first three sets. The functions in the various sets are described as being of orders 3, 5 and 7 respectively.

Ramanujan associated an order to his mock theta functions, which was not clearly defined. [Before the work of Zwegers \(2002\)](#), the orders of known mock theta functions included 3, 5, 6, 7, 8, 10.

Ramanujan's notion of order later turned out to correspond to the conductor of the Nebentypus character of the weight $1/2$ harmonic Maass forms which admit Ramanujan's mock theta functions as their holomorphic projections.

On this occasion, I propose to restrict myself to the consideration of [the third order mock theta functions](#).

Background

Ramanujan defined the following three mock theta functions of order 3,

Definition (Ramanujan)

$$f(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q; q)_n^2},$$

$$\phi(q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q^2; q^2)_n},$$

$$\psi(q) = \sum_{n=1}^{\infty} \frac{q^{n^2}}{(q; q^2)_n},$$

where $(a; q^m)_n := \prod_{k=0}^{n-1} (1 - aq^{mk})$ and $(a; q^m)_{\infty} = \lim_{n \rightarrow +\infty} (a; q^m)_n$.

Ramanujan's identities

Ramanujan also found the following two relations.

Theorem (Ramanujan)

$$\begin{aligned}\phi(-q) - 2\psi(-q) &= f(q), \\ \phi(-q) + 2\psi(-q) &= \frac{(q; q)_{\infty}}{(-q; q)_{\infty}^2}.\end{aligned}$$

The first proofs were given by [George Neville Watson](#) ([J. London Math. Soc.](#), 1936). [Nathan Jacob Fine](#) ([Basic Hypergeometric Series and Applications](#), 1988, p. 60) found another proof by using transformation formulas.

Generalizations of Ramanujan's mock theta functions

George Andrews (Quart. J. Math., 1966) defined the following functions as generalizations of Ramanujan's mock theta functions.

Definition (Andrews)

$$f(\alpha; q) = \sum_{n=0}^{\infty} \frac{q^{n^2-n}\alpha^n}{(-q; q)_n(-\alpha; q)_n},$$

$$\phi(\alpha; q) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-\alpha q; q^2)_n},$$

$$\psi(\alpha; q) = \sum_{n=1}^{\infty} \frac{q^{n^2}}{(\alpha; q^2)_n}.$$

When $\alpha = q$, the above functions reduce to Ramanujan's mock theta functions.

Generalizations of Ramanujan's identities

Andrews also extended Ramanujan's identities to the functions $f(\alpha; q)$, $\phi(\alpha; q)$ and $\psi(\alpha; q)$.

Theorem (Andrews, Quart. J. Math., 1966)

$$\begin{aligned}\phi(-\alpha; -q) - (1 + \alpha q^{-1})\psi(-\alpha; -q) &= f(\alpha; q), \\ \phi(-\alpha; -q) + (1 + \alpha q^{-1})\psi(-\alpha; -q) &= \frac{(q; q)_{\infty}}{(-q; q)_{\infty}(-\alpha; q)_{\infty}}.\end{aligned}$$

Clearly, the above identities specialize to Ramanujan's identities by setting $\alpha = q$.

Fine's partition identity for $f(q)$

The connection between Ramanujan's mock theta functions and the theory of partitions was first explored by Fine. Fine derived the following partition identity for $f(q)$ from his transformation formula.

Theorem (Fine, Basic Hypergeometric Series and Applications, 1988, p. 55)

Let $p_{do}(n)$ denote the number of partitions of n into distinct parts with the smallest part being odd. Then

$$(-q; q)_{\infty} f(q) = 1 + 2 \sum_{n \geq 1} p_{do}(n) q^n.$$

Remark: Recently, Chen, Ji and Liu (Quart. J. Math., to appear) gave a simple involution for this identity.

Partition identities for $\phi(-q)$ and $\psi(-q)$

We obtained the following two partition identities for $\phi(-q)$ and $\psi(-q)$ by building two involutions.

Theorem (Chen-Ji-Liu, Quart. J. Math., to appear)

$$(-q; q)\phi(-q) = 1 + \sum_{n=1}^{\infty} p_{do}(n)q^n + \sum_{k=1}^{\infty} (-1)^k q^{k^2},$$

$$2(-q; q)\psi(-q) = - \sum_{n=1}^{\infty} p_{do}(n)q^n + \sum_{k=1}^{\infty} (-1)^k q^{k^2}.$$

Partition identities for $\phi(-q)$ and $\psi(-q)$

The above two partition identities can be restated as the forms analogous to Fine's identity for $f(q)$ by employing Bessenrodt and Pak's partition theorem.

Theorem (Chen-Ji-Liu, Quart. J. Math., to appear)

Let $p_{do}^e(n)$ ($p_{do}^o(n)$) denote the number of partitions of n into even (odd) distinct parts with the smallest part being odd. Then

$$(-q; q)_\infty \phi(-q) = 1 + 2 \sum_{n \geq 1} p_{do}^e(n) q^n,$$

$$(-q; q)_\infty \psi(-q) = - \sum_{n \geq 1} p_{do}^o(n) q^n.$$

Applications

As an application, the above partition identities can lead to Ramanujan's identities. It follows that

$$\begin{aligned}
 & (-q; q)_{\infty} \phi(-q) - 2(-q; q)_{\infty} \psi(-q) \\
 &= 1 + \sum_{n=1}^{\infty} p_{do}(n) q^n + \sum_{k=1}^{\infty} (-1)^k q^{k^2} \\
 &\quad + \sum_{n=1}^{\infty} p_{do}(n) q^n - \sum_{k=1}^{\infty} (-1)^k q^{k^2} \\
 &= 1 + 2 \sum_{n \geq 1} p_{do}(n) q^n \\
 &= (-q; q)_{\infty} f(q),
 \end{aligned}$$

which implies the first Ramanujan's identity by dividing both sides by $(-q; q)_{\infty}$.

Applications

The second Ramanujan's identity could be derived as follows. In terms of two partition identities for $\phi(-q)$ and $\psi(-q)$, we have

$$\begin{aligned}
 & (-q; q)_{\infty} \phi(-q) + 2(-q; q)_{\infty} \psi(-q) \\
 &= 1 + \sum_{n=1}^{\infty} p_{do}(n) q^n + \sum_{k=1}^{\infty} (-1)^k q^{k^2} \\
 &\quad - \sum_{n=1}^{\infty} p_{do}(n) q^n + \sum_{k=1}^{\infty} (-1)^k q^{k^2} \\
 &= 1 + 2 \sum_{k \geq 1} (-1)^k q^{k^2} = \frac{(q; q)_{\infty}}{(-q; q)_{\infty}},
 \end{aligned}$$

where the last equality follows from Gauss' identity:

$$\sum_{k=-\infty}^{\infty} (-1)^k q^{k^2} = \frac{(q; q)_{\infty}}{(-q; q)_{\infty}}.$$

This yields the second Ramanujan's identity after dividing both sides by $(-q; q)_{\infty}$.

Partition identities for $f(\alpha q; q)$, $\phi(-\alpha q; -q)$ and $\psi(-\alpha q; -q)$

Similarly, we derived the following partition identities for $f(\alpha q; q)$, $\phi(-\alpha q; -q)$ and $\psi(-\alpha q; -q)$ based on our involutions which imply Andrews' identities.

Theorem (Chen-Ji-Liu, Quart. J. Math., to appear)

Let P_{do} denote the set of partitions into distinct parts with the smallest part being odd. Then

$$(-\alpha q; q)_{\infty} f(\alpha q; q) = 1 + 2 \sum_{\nu \in P_{do}} \alpha^{l(\nu)} q^{|\nu|},$$

$$(-\alpha q; q)_{\infty} \phi(-\alpha q; -q) = 1 + \sum_{\nu \in P_{do}} \alpha^{l(\nu)} q^{|\nu|} + \sum_{k=1}^{\infty} (-1)^k q^{k^2},$$

$$(-\alpha; q)_{\infty} \psi(-\alpha q; -q) = - \sum_{\nu \in P_{do}} \alpha^{l(\nu)} q^{|\nu|} + \sum_{k=1}^{\infty} (-1)^k q^{k^2}.$$

Partitions: at the interface of q-Series and Modular Forms.

—G.E. Andrews, Ramanujan J. (2003)

The theory of partitions has historically served as a delightful
"testing ground" for some of the deepest developments in the
theory of modular forms

—Ken Ono, at the 08 Harvard-MIT CDM Conference

Thank you !!