

Stanley's Shuffle Theorem: Refinements and Applications

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Permutation statistics

- Let $\mathfrak{S}_n$ be the set of all permutations of $[n] = \{1, 2, \dots, n\}$. Given a permutation $\sigma = \sigma_1 \cdots \sigma_n \in \mathfrak{S}_n$,
- the inversion number $\text{inv}(\sigma)$** of σ counts the number of pairs (i, j) in a permutation such that $i < j$ but $\sigma_i > \sigma_j$.
- A **descent** of σ if $1 \leq i < n$ and $\sigma_i > \sigma_{i+1}$ and i is an **ascent** of σ if $1 \leq i < n$ and $\sigma_i < \sigma_{i+1}$.

$$\sigma = 2 \uparrow 7 \downarrow 1 \uparrow 8 \downarrow 3 \uparrow 6 \downarrow 5 \downarrow 4 \in \mathfrak{S}_8.$$

- The set of descents of σ is called the descent set of σ , denoted $\text{Des}(\sigma)$.
- The **descent number $\text{des}(\sigma)$** is the number of its descents and **the major index $\text{maj}(\sigma)$** is defined to be the sum of its descents. Namely,

$$\text{des}(\sigma) := |\text{Des}(\sigma)|, \quad \text{maj}(\sigma) := \sum_{k \in \text{Des}(\sigma)} k.$$

Permutation statistics

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Example

$$\begin{array}{cccccccc} & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ \sigma & = 2 \uparrow & 7 \downarrow & 1 \uparrow & 8 \downarrow & 3 \uparrow & 6 \downarrow & 5 \downarrow & 4 \\ & 0 & 0 & 2 & 0 & 2 & 2 & 3 & 4 \end{array}$$

so

$$\text{inv}(\sigma) = 13, \quad \text{Des}(\sigma) = \{2, 4, 6, 7\}, \quad \text{des}(\sigma) = 4, \quad \text{maj}(\sigma) = 19.$$

Theorem (MacMahon)

For $n \geq 1$,

$$\begin{aligned}\sum_{\sigma \in \mathfrak{S}_n} q^{\text{inv}(\sigma)} &= \sum_{\sigma \in \mathfrak{S}_n} q^{\text{maj}(\sigma)} = (1+q)(1+q+q^2) \cdots (1+q+\cdots+q^{n-1}) \\ &= [1]_q [2]_q \cdots [n]_q = [n]_q!\end{aligned}\tag{1}$$

where $[n]_q! = [1]_q [2]_q \cdots [n]_q$ with $[k]_q = 1 + q + \cdots + q^{k-1}$.

Remark.

- (a) A permutation statistic whose generating function is given by (1) is called **Mahonian**.
- (b) **Eulerian statistics**, **Stirling statistics**, **descent statistics**, ...

Percy Alexander MacMahon (1854-1929)



- MacMahon was admitted to the Royal Military Academy at the age of 16.
- the President of the London Mathematical Society from 1894 to 1896;
- His two volume “Combinatory analysis”, published in 1915, is the first major book in enumerative combinatorics.
- A classical result of MacMahon states that inversion number and major index have the same distribution over permutations of a given multiset, where major index is named after his rank in the Royal Military Academy.

Proposition

For $\sigma \in \mathfrak{S}_{n-1}$, there is

$$\sum_{\alpha \in \mathfrak{S}_n} q^{\text{inv}(\alpha)} = (1 + q + \cdots + q^{n-1}) q^{\text{inv}(\sigma)}.$$

and

$$\sum_{\alpha \in \mathfrak{S}_n} q^{\text{maj}(\alpha)} = (1 + q + \cdots + q^{n-1}) q^{\text{maj}(\sigma)}.$$

	1	2	3	4	5	6	7	8
σ	$= 2 \uparrow$	$7 \downarrow$	$1 \uparrow$	$8 \downarrow$	$3 \uparrow$	$6 \downarrow$	$5 \downarrow$	4
	0	0	2	0	2	2	3	4

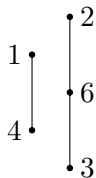


P.A. MacMahon, *Combinatory Analysis*, vol. 1, Cambridge Univ. Press, London, 1915.

Shuffle of permutations

- Let $\mathcal{S}_n$ denote the set of permutations of length n , where a permutation is defined as a sequence of **n distinct integers** (not necessarily restricted to $\{1, 2, \dots, n\}$).
- Given $\pi = 263 \in \mathcal{S}_3$ and $\sigma = 14 \in \mathcal{S}_2$,

$$\pi \sqcup \sigma = \{26314, 26134, 26143, 21463, 21634, 21643, 12463, 14263, 12634, 12643\}.$$



Stanley's shuffle theorem

$$\begin{bmatrix} n+m \\ n \end{bmatrix} = \begin{cases} \frac{(1-q^{n+m})(1-q^{n+m-1}) \cdots (1-q^{m+1})}{(1-q^n)(1-q^{n-1}) \cdots (1-q)}, & n, m \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

Theorem (Stanley's shuffle theorem)

Let $\sigma \in S_m$ and $\pi \in S_n$ be two disjoint permutations with $\text{des}(\sigma) = r$ and $\text{des}(\pi) = s$. Then

$$\sum_{\substack{\alpha \in \sigma \sqcup \pi \\ \text{des}(\alpha) = k}} q^{\text{maj}(\alpha)} = \begin{bmatrix} m - r + s \\ k - r \end{bmatrix} \begin{bmatrix} n - s + r \\ k - s \end{bmatrix} \times q^{\text{maj}(\sigma) + \text{maj}(\pi) + (k-s)(k-r)}.$$



R.P. Stanley, Ordered Structures and Partitions, Mem Amer. Math. Soc., No. 119, American Mathematical Society, 1972, iii+104 pp.

Gessel's shuffle theorem

Theorem (Garsia-Gessel's shuffle theorem)

Let $\sigma \in S_m$ and $\pi \in S_n$ be two disjoint permutations. Then

$$\sum_{\alpha \in \sigma \sqcup \pi} q^{\text{maj}(\alpha)} = \begin{bmatrix} n + m \\ m \end{bmatrix} q^{\text{maj}(\sigma) + \text{maj}(\pi)}.$$

Gessel's shuffle theorem can be derived from Stanley's shuffle theorem by employing q -analogue of the Chu-Vandermonde summation,

$$\sum_{k=0}^h \begin{bmatrix} n \\ k \end{bmatrix} \begin{bmatrix} m \\ h-k \end{bmatrix} q^{(n-k)(h-k)} = \begin{bmatrix} m+n \\ h \end{bmatrix}.$$



A.M. Garsia and I.M. Gessel, Permutation statistics and partitions , Adv. Math. 31 (1979) 288–305.

Shuffle-compatible

Inspired by Stanley's shuffle theorem, Gessel and Zhuang introduced the concept of **shuffle compatible**.

Definition

For any two disjoint permutations π and σ ,
If

$$\{st(\alpha) : \alpha \in \pi \sqcup \sigma\}$$

depends only on $st(\pi)$, $st(\sigma)$, $l(\pi)$, and $l(\sigma)$, then we say that a permutation statistic st is **shuffle-compatible**.

Stanley's shuffle theorem implies that maj and des are shuffle compatible



I.M. Gessel and Y. Zhuang, Shuffle-compatible permutation statistics, *Adv. Math.* 332 (2018) 85–141.

des, maj are shuffle compatible

Stanley's shuffle theorem implies that **maj** and **des** are shuffle compatible.

$$\sum_{\substack{\alpha \in \sigma \sqcup \pi \\ \text{des}(\alpha) = k}} q^{\text{maj}(\alpha)} = \begin{bmatrix} m - r + s \\ k - r \end{bmatrix} \begin{bmatrix} n - s + r \\ k - s \end{bmatrix} \times q^{\text{maj}(\sigma) + \text{maj}(\pi) + (k-s)(k-r)}.$$

Recent developments

- **Gessel and Zhuang** further investigated the shuffle compatibility property of other permutation statistics and posed several conjectures involving shuffle compatibility.

some shuffle-compatible permutation statistics			
Des	des	maj	comaj
Pk	pk	(pk,des)	(des,maj)

- **Adin, Gessel, Reiner and Roichman** introduced cyclic shuffle which yields cyclic quas-symmetric functions.



R.M. Adin, I.M. Gessel, V. Reiner, and Y. Roichman, Cyclic quasi-symmetric functions, Israel J. Math. 243 (2021) 437–500.

- Similarly, based on Adin et al's work, **Domagalski, Liang, Minnich and Sagan** introduced cyclic shuffle compatibility.



R. Domagalski, J. Liang, Q. Minnich and B.E. Sagan, J. Schmidt, A. Sietsema, Cyclic shuffle compatibility, Sémin. Lothar. Combin. 85 ([2020–2021]), Art. B85d, 11 pp.

Recent developments

- Some of Gessel-Zhuang's conjectures were confirmed by Baker-Jarvis and Sagan, Grinberg and Yang and Yan.



D. Baker-Jarvis and B.E. Sagan, Bijective proofs of shuffle compatibility results, Adv. in Appl. Math. 113 (2020) 101973.



D. Grinberg, Shuffle-compatible permutation statistics II: the exterior peak set, Electron. J. Combin. 25 (2018) P4.17.



L. Yang and S.H.F. Yan, On a conjecture concerning shuffle-compatible permutation statistics, Electron. J. Combin. 29 (2022) P3.3.

Known proofs of Stanley's shuffle theorem

- **Stanley** asked for a proof of Stanley's shuffle theorem which avoids the use of the **q -Pfaff-Saalschütz identity**.



R.P. Stanley, Ordered Structures and Partitions, Mem Amer. Math. Soc., No. 119, American Mathematical Society, 1972, iii+104 pp.

- **Goulden**'s proof is obtained by finding bijections for lattice path representations of shuffles which reduce σ and π to canonical permutations, for which the generating function is easily given.



I. P. Goulden, A bijective proof of Stanley's shuffling theorem, Trans. Amer. Math. Soc. 288 (1985) 147–160.

- **Stadler**'s bijection is more elementary, but the inverse of Stadler's map is not very explicit.



J. D. Stadler, Stanley's shuffling theorem revisited, J. Combin. Theory Ser. A 88 (1999) 176–187.

A new proof

- **Ji and Zhang** give an explicit bijective proof of Stanley's shuffle theorem by using the insertion lemma of **Han** and **Haglund, Loehr and Remmel**.



G-N. Han, Calcul denertien,(French)Dissertation, Universit Louis Pasteur, Strasbourg, 1992, Publ. Inst. Rech. Math. Av., 476



J. Haglund, N. Loehr and J.B. Remmel, Statistics on wreath products, perfect matchings, and signed words, European J. Combin. 26 (2005) 835–868.



Kathy Q. Ji and Dax T.X. Zhang, Some refinements of Stanley's shuffle theorem, J. Combin. Theory Ser. A 203 (2024) 105830.

Theorem (MacMahon)

For $n \geq 1$,

$$\sum_{\alpha \in \mathfrak{S}_n} q^{\text{maj}(\alpha)} = (1 + q)(1 + q + q^2) \cdots (1 + q + \cdots + q^{n-1}).$$

For $\sigma \in \mathfrak{S}_{n-1}$,

$$\sum_{\alpha \in \mathfrak{S}_n} q^{\text{maj}(\alpha)} = (1 + q + \cdots + q^{n-1}) q^{\text{maj}(\sigma)}.$$

	1	2	3	4	5	6	7	8
σ	$= 2 \uparrow$	$7 \downarrow$	$1 \uparrow$	$8 \downarrow$	$3 \uparrow$	$6 \downarrow$	$5 \downarrow$	4
	0	0	2	0	2	2	3	4

A special case of Stanley's shuffle theorem

For $\sigma \in \mathcal{S}_n$ with $\text{des}(\sigma) = k$, $r \notin \sigma$,

$$\sum_{\substack{\alpha \in \sigma \wr r \\ \text{des}(\alpha) = k}} q^{\text{maj}(\alpha)} = \begin{bmatrix} k+1 \\ k \end{bmatrix} q^{\text{maj}(\sigma)} = (1 + q + \cdots + q^k) q^{\text{maj}(\sigma)},$$

$$\sum_{\substack{\alpha \in \sigma \wr r \\ \text{des}(\alpha) = k+1}} q^{\text{maj}(\alpha)} = \begin{bmatrix} n-k \\ 1 \end{bmatrix} q^{\text{maj}(\sigma) + k+1} = (q^{k+1} + q^{k+2} + \cdots + q^n) q^{\text{maj}(\sigma)}.$$

Remark. These two relations can be proved by the insertion lemma of Han and Haglund, Loehr and Remmel.

An equivalent form of Stanley's shuffle theorem

Stanley's shuffle theorem

$$\sum_{\substack{\alpha \in \sigma \sqcup \pi \\ \text{des}(\alpha) = k}} q^{\text{maj}(\alpha)} = \begin{bmatrix} m - r + s \\ k - r \end{bmatrix} \begin{bmatrix} n - s + r \\ k - s \end{bmatrix} \times q^{\text{maj}(\sigma) + \text{maj}(\pi) + (k-s)(k-r)}.$$

Equivalent form:

The bijection Φ : $\begin{matrix} \sigma \sqcup \pi \\ \alpha(\text{des}(\alpha) = k) \end{matrix} \Leftrightarrow \begin{matrix} \mathcal{P}_{k-r}(k-s, m) \times \mathcal{P}_{n-k+r}(0, k-s) \\ (\lambda, \mu) \end{matrix}$

s.t.

$$\text{maj}(\alpha) = |\lambda| + |\mu| + \text{maj}(\sigma) + \text{maj}(\pi).$$



Kathy Q. Ji and Dax T.X. Zhang, Some refinements of Stanley's shuffle theorem, J. Combin. Theory Ser. A 203 (2024) 105830.

The first refinement: $(\lambda, \mu) = \Phi(\alpha)$ such that $\mu_{n-k+r} \geq 1$

- Let $\sigma \sqcup_{sb} \pi$ denote the set of shuffles $\alpha = \alpha_1 \cdots \alpha_{n+m}$ of σ and π such that $\alpha_{n+m} = \min\{\pi_n, \sigma_m\}$.
- Given $\pi = 263 \in \mathcal{S}_3$ and $\sigma = 14 \in \mathcal{S}_2$,

$$\pi \sqcup \sigma = \{26314, 26134, 26143, 21463, 21634, 21643, \\ 12463, 14263, 12634, 12643\}.$$

$$\pi \sqcup_{sb} \sigma = \{26143, 21463, 21643, 12463, 14263, 12643\}.$$

Our first refinement of Stanley's shuffle theorem is the following.

The first refinement

Theorem (The first refinement)

Let $\sigma = \sigma_1 \cdots \sigma_m \in \mathcal{S}_m$ and $\pi = \pi_1 \cdots \pi_n \in \mathcal{S}_n$ be two disjoint permutations with $\text{des}(\sigma) = r$ and $\text{des}(\pi) = s$.

- If $\pi_n > \sigma_m$, then

$$\sum_{\substack{\alpha \in \sigma \sqcup sb \pi \\ \text{des}(\alpha) = k}} q^{\text{maj}(\alpha)} = \begin{bmatrix} m - r + s \\ k - r \end{bmatrix} \begin{bmatrix} n - s + r - 1 \\ k - s - 1 \end{bmatrix} \times q^{\text{maj}(\sigma) + \text{maj}(\pi) + n + (k - s - 1)}$$

- If $\pi_n < \sigma_m$, then

$$\sum_{\substack{\alpha \in \sigma \sqcup sb \pi \\ \text{des}(\alpha) = k}} q^{\text{maj}(\alpha)} = \begin{bmatrix} m - r + s - 1 \\ k - r - 1 \end{bmatrix} \begin{bmatrix} n - s + r \\ k - s \end{bmatrix} \times q^{\text{maj}(\sigma) + \text{maj}(\pi) + m + (k - s)(k - r)}$$

A cyclic analogue of Stanley's shuffle theorem

- Adin, Gessel, Reiner and Roichman introduced cyclic shuffle which yields cyclic quas-symmetric functions.
 R.M. Adin, I.M. Gessel, V. Reiner, and Y. Roichman, Cyclic quasi-symmetric functions, Israel J. Math. 243 (2021) 437–500.
- Adin, Gessel, Reiner and Roichman asked a question about looking for a notion of cyclic major index, which provides a bivariate analogue of Theorem 1.2 (Stanley's shuffle theorem).
- This question has been posed by Domagalski, Liang, Minnich, Sagan, Schmidt and Sietsema in [5, Question 4.1] again.
- Ji and Zhang established a cyclic analogue of Stanley's shuffle theorem.
 K.Q. Ji and D.T.X. Zhang, A cyclic analogue of Stanley's shuffling theorem, Electron. J. Combin. 29 (2022) P4.20.

yellow Application 1: q -derangement polynomials on classical Weyl groups

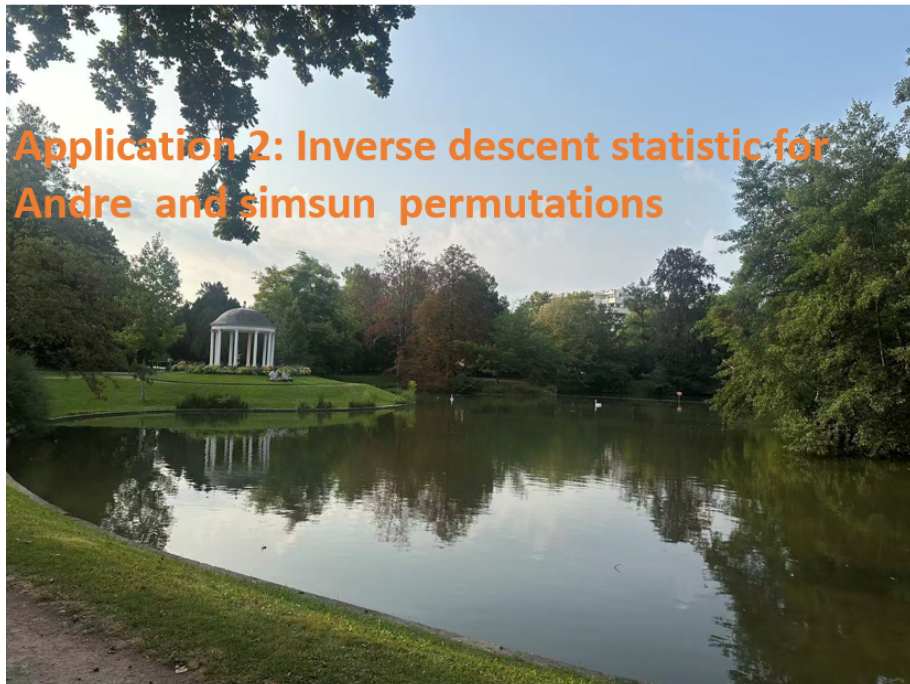


M.L. Wachs, On q -derangement numbers, Proc. Amer. Math. Soc. 106 (1989) 273–278.



Kathy Q. Ji and Dax T.X. Zhang, Signed Mahonian polynomials on derangements in classical Weyl groups, Europ. J. Combin. 124 (2025) 104083.

Application 2: Inverse descent statistic for Andre and simsun permutations



Euler numbers

Euler numbers, denoted by E_n , are a sequence of integers that arise in the Taylor series expansions of $\sec(x) + \tan(x)$, that is,

$$\begin{aligned}\sec(x) + \tan(x) &= \sum_{n \geq 0} E_n \frac{x^n}{n!} \\ &= 1 + 1 \cdot x + 1 \cdot \frac{x^2}{2!} + 2 \cdot \frac{x^3}{3!} + 5 \cdot \frac{x^4}{4!} + 16 \cdot \frac{x^5}{5!} + \cdots\end{aligned}$$

What's combinatorial interpretations of E_n ?

Combinatorial interpretations

The problem of finding **combinatorial interpretations** for special functions is popularized by the **French school**:

M.P. Schützenberger “Every algebraic relation is to be given a **combinatorial counterpart and vice versa**”

D. Foata, X. Viennot, P. Flajolet,
D. Dumont, J. Zeng, G.-N. Han...

& **many other researchers**:

R.P. Stanley, I.M. Gessel, **W.Y.C. Chen**,
Y.-N. Yeh, D. Kim, ...

Combinatorial interpretations of Euler numbers

Their combinatorial significance was cemented by the work of André in the late 19th century.

- **alternating permutations**: $\sigma = \sigma_1 \sigma_2 \dots \sigma_n$, $\sigma_1 > \sigma_2 < \sigma_3 > \dots$.
- **André permutations** (Foata and Schützenberger, 1974)
- **Simsun permutations** (Rodica Simion and Sheila Sundaram, 1996) in a series of studies of homology representations of the symmetric group

André permutations

- The empty word e and any single-letter word are defined as both **André I permutations** and **André II permutations**.
- For a permutation $\sigma = \sigma_1\sigma_2\cdots\sigma_n$ ($n \geq 2$) of length n , decompose it as

$$\sigma = \tau \min(\sigma) \tau'.$$

Then,

- σ is called an **André I permutation** if both τ and τ' are André I permutations and $\max(\tau\tau') \in \tau'$.

$\sigma = 672581439$ is an André I permutation

- σ is called an **André II permutation** if both τ and τ' are André II permutations and $\min(\tau\tau') \in \tau'$.

$\sigma = 693581724$ is an André II permutation

Equidistribution of des, maj and ides

Let AndI_n denote the set of **André I permutations** on $[n]$ and let AndII_n denote the set of **André II permutations** on $[n]$.

Theorem (Han, Ji and Xiong, 2025)

For $n \geq 1$,

$$\sum_{\sigma \in \text{AndI}_n} s^{\text{ides}(\sigma)} t^{\text{des}(\sigma)} q^{\text{maj}(\sigma)} = \sum_{\sigma \in \text{AndII}_n} s^{\text{ides}(\sigma)} t^{\text{des}(\sigma)} q^{\text{maj}(\sigma)}.$$

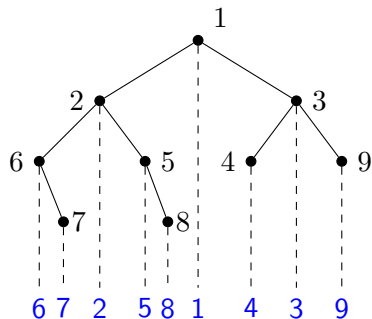
Note that $\text{ides}(\sigma) = \text{des}(\sigma^{-1})$.



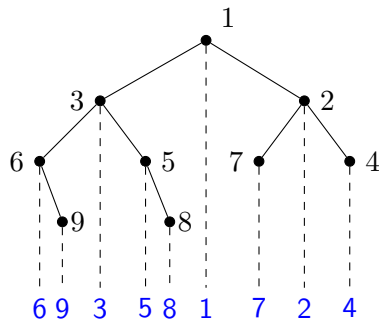
Guoniu Han, Kathy Q. Ji and Huang Xiong, Inverse descent statistic for André and simsun permutations, arXiv:2511.12549.

Permutations and increasing binary trees

André I permutation



André II permutation



A stronger result

Theorem (Han, Ji and Xiong, 2025)

For $n \geq 1$ and any unlabeled binary tree $T \in \text{URL}_n$, we have

$$\sum_{\sigma \in \text{AndI}(T)} s^{\text{idcs}(\sigma)} \stackrel{(a)}{=} \sum_{\sigma \in \text{AndII}(T)} s^{\text{idcs}(\sigma)},$$

where $\text{AndI}(T)$ (resp. $\text{AndII}(T)$) denote the set of n -André I permutations (resp. n -André II permutations) with tree shape T .



Guoniu Han, Kathy Q. Ji and Huang Xiong, Inverse descent statistic for André and simsun permutations, arXiv:2511.12549.

Equivalent transform

- Let $\mathfrak{S}_n$ denote the set of permutations on the set $[n]$.
- Suppose that $\sigma \in \mathfrak{S}_j$ and $\tau \in \mathfrak{S}_k$.

$$\sigma \diamond \tau = \{\mu = \sigma' 1 \tau' \in \mathfrak{S}_{j+k+1} \mid \sigma' \sim \sigma, \tau' \sim \tau\};$$

$$\sigma \Delta \tau = \{\mu = \sigma' 1 \tau' \in \mathfrak{S}_{j+k+1} \mid \sigma' \sim \sigma, \tau' \sim \tau, j+k+1 \in \tau'\};$$

$$\sigma \nabla \tau = \{\mu = \sigma' 1 \tau' \in \mathfrak{S}_{j+k+1} \mid \sigma' \sim \sigma, \tau' \sim \tau, 2 \in \tau'\},$$

where $\sigma' \sim \sigma$ means that reducing the letters of σ' to $\{1, 2, \dots, j\}$ yields σ .

Example

$$\sigma = 35124 \quad \text{and} \quad \tau = 231,$$

$$\mu = 692581473 \in \sigma \diamond \tau.$$

$$\mu = 682571493 \in \sigma \triangle \tau.$$

$$\mu = 93581472 \in \sigma \nabla \tau.$$

$$\text{AndI}(\mathcal{T}) = \bigcup_{\sigma \in \text{AndI}(\mathcal{T}')} \bigcup_{\tau \in \text{AndI}(\mathcal{T}')} \sigma \triangle \tau.$$

$$\text{AndII}(\mathcal{T}) = \bigcup_{\hat{\sigma} \in \text{AndII}(\mathcal{T}')} \bigcup_{\hat{\tau} \in \text{AndII}(\mathcal{T}')} \hat{\sigma} \nabla \hat{\tau}.$$

Example

$$\sigma = 35124 \quad \text{and} \quad \tau = 231,$$

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$$\text{AndI}(\mathcal{T}) = \bigcup_{\sigma \in \text{AndI}(\mathcal{T}')} \bigcup_{\tau \in \text{AndI}(\mathcal{T}'')} \sigma \triangle \tau.$$

$$\text{AndII}(\mathcal{T}) = \bigcup_{\hat{\sigma} \in \text{AndII}(\mathcal{T}')} \bigcup_{\hat{\tau} \in \text{AndII}(\mathcal{T}'')} \hat{\sigma} \nabla \hat{\tau}.$$

Key relation

Let $\sigma, \hat{\sigma} \in \mathcal{S}_j$ ($\text{ides}(\sigma) = \text{ides}(\hat{\sigma})$), and let $\tau, \hat{\tau} \in \mathcal{S}_k$ ($\text{ides}(\tau) = \text{ides}(\hat{\tau})$).

$$\sum_{\mu \in \sigma \Delta \tau} t^{\text{ides}(\mu)} = \sum_{\mu \in \hat{\sigma} \nabla \hat{\tau}} t^{\text{ides}(\mu)},$$

$$\sum_{\mu \in \text{AndI}(T)} t^{\text{ides}(\mu)} = \sum_{\sigma \in \text{AndI}(T^l)} \sum_{\tau \in \text{AndI}(T^r)} \sum_{\mu \in \sigma \Delta \tau} t^{\text{ides}(\mu)}$$

$$\sum_{\mu \in \text{AndII}(T)} t^{\text{ides}(\mu)} = \sum_{\hat{\sigma} \in \text{AndII}(T^l)} \sum_{\hat{\tau} \in \text{AndII}(T^r)} \sum_{\mu \in \hat{\sigma} \nabla \hat{\tau}} t^{\text{ides}(\mu)}.$$

More general result

Theorem (Han, Ji and Xiong, 2025)

Let $\sigma \in \mathfrak{S}_j$ and $\tau \in \mathfrak{S}_k$ be two permutations with $\text{ides}(\sigma) = j'$ and $\text{ides}(\tau) = k'$. Then

$$\sum_{\mu \in \sigma \diamond \tau} t^{\text{ides}(\mu)} = \sum_{i \geq 1} \binom{k - k' + j' + 1}{i + j' - k'} \binom{j - j' + k' - 1}{i - 1} t^{i+j'},$$

$$\sum_{\mu \in \sigma \triangle \tau} t^{\text{ides}(\mu)} = \sum_{i \geq 1} \binom{k - k' + j'}{i + j' - k'} \binom{j - j' + k' - 1}{i - 1} t^{i+j'},$$

$$\sum_{\mu \in \sigma \nabla \tau} t^{\text{ides}(\mu)} = \sum_{i \geq 1} \binom{k - k' + j'}{i + j' - k'} \binom{j - j' + k' - 1}{i - 1} t^{i+j'}.$$

Related to permutation shuffles

$$\sigma = 35124 \quad \text{and} \quad \tau = 231,$$

$$\sigma^{-1} = 34152 \quad \text{and} \quad \tau^{-1} = 312.$$

Set

$$\pi = \sigma^{-1} = 34152 \quad \text{and} \quad \delta = 6978.$$

$$\mu = 692581473 \in \sigma \diamond \tau \Leftrightarrow \mu^{-1} = 639741852 \in \pi \sqcup_I \delta.$$

$$\mu = 682571493 \in \sigma \triangle \tau \Leftrightarrow \mu^{-1} = 639741528 \in \pi \sqcup_{Is} \delta.$$

$$\mu = 93581472 \in \sigma \nabla \tau \Leftrightarrow \mu^{-1} = 693741852 \in \pi \sqcup_{II} \delta.$$

Equivalent form in permutation shuffles

Theorem

Assume that $\pi \in \mathcal{S}_j$ and $\delta \in \mathcal{S}_{k+1}$ are two disjoint permutations, where $\text{des}(\pi) = j'$ and $\text{des}(\delta) = k'$. Moreover, $\delta_1 < \delta_2$ and all of the elements of δ are larger than the elements of π . Then

$$\sum_{\alpha \in \pi \sqcup \delta} t^{\text{des}(\alpha)} = \sum_{i \geq 1} \binom{k+1-k'+j'}{i+j'-k'} \binom{j-j'+k'-1}{i-1} t^{i+j'},$$

$$\sum_{\alpha \in \pi \sqcup \delta} t^{\text{des}(\alpha)} = \sum_{i \geq 1} \binom{k-k'+j'}{i+j'-k'} \binom{j-j'+k'-1}{i-1} t^{i+j'},$$

$$\sum_{\alpha \in \pi \sqcup \delta} t^{\text{des}(\alpha)} = \sum_{i \geq 1} \binom{k-k'+j'}{i+j'-k'} \binom{j-j'+k'-1}{i-1} t^{i+j'}.$$

This can be proved with the aid of the bijection due to Ji and Zhang.

Problem

It would be interesting to give a direct explicit bijection of the following relationship.

$$\sum_{\sigma \in \text{AndI}(T)} s^{\text{idex}(\sigma)} = \sum_{\sigma \in \text{AndII}(T)} s^{\text{idex}(\sigma)},$$

for any unlabeled binary tree T .

References



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The image shows the exterior of a brick building with a white sign that reads "应用数学中心" and "Center for Applied Mathematics". In the foreground, there are green leaves and white cherry blossoms. The text "Thank you!" is overlaid in a stylized blue font with an orange outline.

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Thank you!