

Combinatorial interpretations of congruences for the spt-function

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Integer partitions

Definition

A *partition* of a positive integer n is a finite nonincreasing sequence of positive integers $(\lambda_1, \lambda_2, \dots, \lambda_l)$ such that $\lambda_1 + \lambda_2 + \dots + \lambda_l = n$.

Example: There are five partitions of 4, which are

$$(4), (3, 1), (2, 2), (2, 1, 1), (1, 1, 1, 1).$$

Let $p(n)$ denote the number of partitions of n , we see that $p(4) = 5$. A natural question is

$$p(n) = ??$$

The Hardy-Ramanujan-Rademacher expansion of $p(n)$

- In 1918, Hardy and Ramanujan used the circle method and modular functions to obtain the asymptotic solution:

$$p(n) \sim \frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}}.$$

- In 1937, Rademacher subsequently obtained an exact convergent series solution which yields the Hardy-Ramanujan formula as the first term:

$$p(n) = \frac{1}{\pi\sqrt{2}} \sum_{k=1}^{\infty} A_k(n) \sqrt{k} \frac{d}{dn} \left[\frac{\sinh\left(\frac{\pi}{k} \sqrt{\frac{2}{3}} \left(n - \frac{1}{24}\right)\right)}{\sqrt{n - \frac{1}{24}}} \right],$$

where

$$A_k(n) = \sum_{h=1}^k \delta_{\gcd(h,k),1} \exp \left[\pi i \sum_{j=1}^{k-1} \frac{j}{k} \left(\frac{hj}{k} - \left\lfloor \frac{hj}{k} \right\rfloor - \frac{1}{2} \right) - \frac{2\pi i h n}{k} \right].$$

Ramanujan's congruences for $p(n)$

In 1919, Ramanujan found the the following striking congruences for ordinary partition function $p(n)$.

$$p(5n + 4) \equiv 0 \pmod{5}.$$

$$p(7n + 5) \equiv 0 \pmod{7}.$$

$$p(11n + 6) \equiv 0 \pmod{11}.$$

The prediction of Ramanujan

It appears that there are no equally simple properties for any moduli involving primes other than these three (i.e. $\ell = 5, 7, 11$).



S. Ramanujan, Some properties of $p(n)$, the number of partitions of n , Proc. Cambridge Philos. Soc. 19 (1919) 207–210.



S. Ahlgren and M. Boylan, Arithmetic properties of the partition function, Invent. Math. 153 (2005) 487–502.

Freeman Dyson



- **Dyson's problem:** How to divide the partitions of $5n + 4$ into five classes with the same number of partitions in each class?
- Dyson wanted to find a concrete criterion, so that you could look at any particular partition of $5n + 4$ and use the criterion to tell which of the five equal classes it belonged to.
- Three years later, when he was a sophomore at Cambridge, he found it and called it “rank”.

Dyson's rank

Definition

For a partition λ , its rank $r(\lambda)$ is defined as its largest part minus the number of parts.

For example, $\lambda = (5, 4, 2, 1, 1, 1)$, its rank $r(\lambda) = 5 - 6 = -1$.



F. Dyson, Some guesses in the theory of partitions, Eureka (Cambridge) 8 (1944) 10–15.

Dyson's conjecture

Let $N(i, m, n)$ denote the number of partitions of n with rank $\equiv i \pmod{m}$, Dyson first conjectured and later proved by Atkin and Swinnerton-Dyer.

Theorem (Dyson, 1944, Atkin and Swinnerton-Dyer, 1954)

$$N(k, 5, 5n + 4) = \frac{p(5n + 4)}{5}, \quad 0 \leq k \leq 4,$$

$$N(k, 7, 7n + 5) = \frac{p(7n + 5)}{7}, \quad 0 \leq k \leq 6,$$

which imply $p(5n + 4) \equiv 0 \pmod{5}$ and $p(7n + 5) \equiv 0 \pmod{7}$.



A.O.L. Atkin and P. Swinnerton-Dyer, Some properties of partitions, Proc. London Math. Soc. (3) 4 (1954) 84–106.

Example

	$n = 4$ ($p(4) = 5$)	rank mod 5
$N(1, 5, n)$	(3, 1)	1
$N(2, 5, n)$	(1, 1, 1, 1)	2
$N(3, 5, n)$	(4)	3
$N(4, 5, n)$	(2, 1, 1)	4
$N(5, 5, n)$	(2, 2)	5

	$n = 5$ ($p(5) = 7$)	rank mod 7
$N(1, 7, n)$	(3, 2)	1
$N(2, 7, n)$	(4, 1)	2
$N(3, 7, n)$	(1 ⁵)	3
$N(4, 7, n)$	(5)	4
$N(5, 7, n)$	(2, 1 ³)	5
$N(6, 7, n)$	(2, 2, 1)	6
$N(7, 7, n)$	(3, 1, 1)	7

Dyson's observation

Dyson noted that ranks could not provide an explanation to $p(11n + 6) \equiv 0 \pmod{11}$. He also conjectured that there should be a partition statistic (which he called crank) that would provide the missing combinatorial explanation of the above congruence.

$$p(6) = 11$$

$ \lambda = 6$	$r(\lambda)$	$r(\lambda) \pmod{11}$
(6)	5	5
(5, 1)	4	4
(4, 2)	2	2
(4, 1 ²)	1	1
(3, 3)	1	1
(3, 2, 1)	0	0
(3, 1 ³)	-1	10
(2, 2, 2)	-1	10
(2, 2, 1 ²)	-2	9
(2, 1 ⁴)	-3	8
(1 ⁶)	-5	6

The crank of ordinary partitions

In 1988, Andrews and Garvan finally found the definition of crank in terms of ordinary partitions.

Definition (Andrews and Garvan, 1988)

Let λ be an ordinary partition and let $n_1(\lambda)$ denote the number of ones in λ and $\mu(\lambda)$ denote the number of parts of λ larger than $n_1(\lambda)$. The crank of λ is defined by

$$\text{crank}(\lambda) = \begin{cases} \lambda_1, & \text{if } n_1(\lambda) = 0, \\ \mu(\lambda) - n_1(\lambda), & \text{if } n_1(\lambda) > 0. \end{cases}$$

For example, the crank of $\lambda = (5, 3, 2, 1, 1, 1)$ is

$$\mu(\lambda) - n_1(\lambda) = 1 - 3 = -2.$$

The crank of ordinary partitions

Let $M(i, m, n)$ denote the number of partitions of n with rank $\equiv i \pmod m$. Andrews and Garvan showed that

Theorem (Andrews and Garvan, 1988)

$$M(k, 5, 5n + 4) = \frac{p(5n + 4)}{5}, \quad \text{for } 0 \leq k \leq 4,$$

$$M(k, 7, 7n + 5) = \frac{p(7n + 5)}{7}, \quad \text{for } 0 \leq k \leq 6,$$

$$M(k, 11, 11n + 6) = \frac{p(11n + 6)}{11}, \quad \text{for } 0 \leq k \leq 10,$$

which imply $p(5n + 4) \equiv 0 \pmod 5$, $p(7n + 5) \equiv 0 \pmod 7$ and $p(11n + 6) \equiv 0 \pmod{11}$.



G.E. Andrews and F.G. Garvan, Dyson's crank of a partition, Bull. Amer. Math. Soc. 18 (1988) 167–171.

spt-Function

Definition (Andrews, 2008)

The spt -function $spt(n)$ was defined as the weighted count of partitions of n with respect to the number of occurrences of the minimum part.

By the following table, we may see that $spt(4) = 10$.

						Total
$\lambda \in \mathcal{P}(4)$	(4)	(3, 1)	(2, 2)	(2, 1, 1)	(1, 1, 1, 1)	5
$n_s(\lambda)$	1	1	2	2	4	10

where $n_s(\lambda)$ denotes the number of occurrences of the minimum part in λ . Let $P(n)$ denote the set of ordinary partitions of n , we see that

$$spt(n) = \sum_{\lambda \in P(n)} n_s(\lambda).$$

spt-Congruences

Based on such connection and using the results for $N(m, n)$, Andrews showed that the spt-function satisfies the following remarkable congruences:

Theorem (Andrews, 2008)

$$\text{spt}(5n + 4) \equiv 0 \pmod{5}.$$

$$\text{spt}(7n + 5) \equiv 0 \pmod{7}.$$

$$\text{spt}(13n + 6) \equiv 0 \pmod{13}.$$



G.E. Andrews, The number of smallest parts in the partitions of n , J. Reine Angew. Math. 624 (2008) 133–142.

Problem

Problem

Find a statistic defined on ordinary partitions so that the set of partitions counted by $spt(5n + 4)$ (resp. $spt(7n + 5)$, $spt(13n + 6)$) can be divided into five (resp. seven, thirteen) equinumerous classes.

Some progress

- Andrews, Garvan and Liang (Ramanujan J., 2012) defined the spt-crank of a vector partition which leads to combinatorial interpretations of the congruences of $spt(n) \bmod 5$ and 7 and asked for a definition of the spt-crank for ordinary partitions.
- Chen, Ji and Zang (J. Reine Angew. Math., 2016) provided a solution to the problem of Andrews, Dyson and Rhoades by introducing the spt-crank of a doubly marked partition and constructing a bijection between marked partitions and doubly marked partitions.

Example

S-partition	weight	crank	doubly marked partition	crank
$((1), (1, 1, 1), \emptyset)$	+1	3	$((1, 1, 1, 1), 1, 1)$	3
$((1), (2, 1), \emptyset)$	+1	2	$((2, 1, 1), 1, 1)$	2
$((1), (1, 1), (1))$	+1	1	$((3, 1), 1, 1)$	1
$((1), (3), \emptyset)$	+1	1	$((2, 2), 1, 2)$	1
$((2, 1), (1), \emptyset)$	-1	1		
$((2), (2), \emptyset)$	+1	1		
$((1), (2), (1))$	+1	0	$((2, 2), 1, 1)$	0
$((1), (1), (2))$	+1	0	$((4), 1, 4)$	0
$((3, 1), \emptyset, \emptyset)$	-1	0		
$((4), \emptyset, \emptyset)$	+1	0		

Example

S-partition	weight	crank	doubly marked partition	crank
$((1), (1), (1, 1))$	+1	-1	$((2, 2), 2, 2)$	-1
$((1), \emptyset, (3))$	+1	-1	$((4), 1, 3)$	-1
$((2, 1), \emptyset, (1))$	-1	-1		
$((2), \emptyset, (2))$	+1	-1		
$((1), \emptyset, (2, 1))$	+1	-2	$((4), 1, 2)$	-2
$((1), \emptyset, (1, 1, 1))$	+1	-3	$((4), 1, 1)$	-3

The problem is far from finished!

- It is still interesting to find an spt -crank that can be directly defined on the ordinary partitions.
- There is currently no information on the combinatorial interpretation of the congruences of $spt(n) \bmod 13$.

Thank you!