

Context-Free Grammars and Their Applications

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What's the context-free grammar?

- Context-free grammars, introduced by W.Y.C. Chen in 1993, provide an alternative symbolic mechanism for enumerating combinatorial objects with grammatical labels or patterns.



W.Y.C. Chen, Context-free grammars, differential operators and formal power series, Theoret. Comput. Sci. 117 (1993) 113–129.



D. Dumont, Grammaires de William Chen et derivations dans les arbres et arborescences, Sem. Lothar. Combin., 37 (1996) Art. B37a.

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- Establish the analytic properties (such as, γ -positivity, stable) of combinatorial polynomials
- Construct a bijection.

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What's the context-free grammar?

- A context-free grammar G over a set $V = \{x, y, z, \dots\}$ of variables is a set of substitution rules replacing a variable in V by a Laurent polynomial of variables in V .

$$G_1 = \{f \rightarrow f\}, G_2 = \{x \rightarrow xy, y \rightarrow xy\}.$$

- For a context-free grammar G over V , the formal derivative D with respect to G is defined as a linear operator acting on Laurent polynomials with variables in V such that each substitution rule is treated as the common differential rule.

$$D_{G_1}(f) = f, \quad D_{G_2}(x) = xy = D_{G_2}(y)$$

- The formal derivative D satisfies the following relations:

$$D(u + v) = D(u) + D(v),$$

$$D(uv) = D(u)v + uD(v).$$

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Hence, it obeys the Leibniz's rule

$$D^n(uv) = \sum_{k=0}^n \binom{n}{k} D^k(u) D^{n-k}(v).$$

For a constant c , we have $D(c) = 0$.

A toy example

- Consider the grammar $G = \{f \rightarrow f\}$. Let D be the formal derivative with respect to G . We see that $D(f) = f$.
- What is $D(f^{-1}) = ?$
- Since $ff^{-1} = 1$, the derivative property demands that

$$D(f)f^{-1} + fD(f^{-1}) = 0,$$

which yields that $D(f^{-1}) = -f^{-1}$.

- In view of the chain rule, we get

$$D^n(f^{-1}) = (-1)^n f^{-1}$$

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A benchmarking example is the following inverse pair:

$$a_n = \sum_{k=0}^n \binom{n}{k} b_k \quad \Longleftrightarrow \quad b_n = \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} a_k$$

- Define the grammar as

$$G = \{f \rightarrow f, b_i \rightarrow b_{i+1}, \quad i = 0, 1, 2, \dots\}.$$

- Let D denote the formal derivative with respect to G . We see that

$$D(f) = f, \quad D(b_0) = b_1, \quad D^k(b_0) = b_k.$$

- By the Leibniz formula, we get

$$D^n(fb_0) = \sum_{k=0}^n \binom{n}{k} D^{n-k}(f) D^k(b_0) = \sum_{k=0}^n \binom{n}{k} fb_k.$$

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- Suppose that the relation on the left holds, that is,

$$a_n = f^{-1} D^n (f b_0)$$

- Recall that

$$D^n(f^{-1}) = (-1)^n f^{-1}$$

- By definition, we see that

$$\begin{aligned} b_n = D^n(b_0) &= D^n(f^{-1} f b_0) = \sum_{k=0}^n \binom{n}{k} D^{n-k}(f^{-1}) D^k(f b_0) \\ &= \sum_{k=0}^n \binom{n}{k} (-1)^{n-k} a_k. \end{aligned}$$

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Descents and ascents

- Let $\mathfrak{S}_n$ be the set of all permutations of $[n] = \{1, 2, \dots, n\}$.
- Let $\sigma = \sigma_1 \sigma_2 \cdots \sigma_n \in \mathfrak{S}_n$, we say that i is a **descent** of σ if $1 \leq i < n$ and $\sigma_i > \sigma_{i+1}$ and i is an **ascent** of σ if $1 \leq i < n$ and $\sigma_i < \sigma_{i+1}$.

$$\sigma = 2 \uparrow 7 \downarrow 1 \uparrow 8 \downarrow 3 \uparrow 6 \downarrow 5 \downarrow 4 \in \mathfrak{S}_8.$$

- Let $\text{des}(\sigma)$ denote the number of descents of σ and $\text{asc}(\sigma)$ denote the number of ascents of σ . We see that $\text{des}(\sigma) = 4$ and $\text{asc}(\sigma) = 3$.
- Let $\sigma \in \mathfrak{S}_n$, it is easy to see that

$$\text{des}(\sigma) + \text{asc}(\sigma) = n - 1.$$

Classical Eulerian polynomials

The Eulerian polynomials $A_n(x)$ are defined by $A_0(x) = 1$ and for $n \geq 1$,

$$A_n(x) = \sum_{\sigma \in \mathfrak{S}_n} x^{\text{des}(\sigma)+1}.$$

The first few values of $A_n(x)$ are given below.

$$A_1(x) = x$$

$$A_2(x) = x + x^2$$

$$A_3(x) = x + 4x^2 + x^3$$

$$A_4(x) = x + 11x^2 + 11x^3 + x^4$$

The generating function of $A_n(x)$

$$\sum_{n=0}^{\infty} A_n(x) \frac{t^n}{n!} = \frac{x-1}{1-x^{-1}e^{(x-1)t}}.$$

Grammatical proof

Let's consider the grammar

$$G = \{x \rightarrow xy, y \rightarrow xy\}.$$

let D_G be the formal derivative with respect to the above grammar G , that is,

$$D(x) = xy, \quad D(y) = xy.$$

Thus, we have

$$\begin{aligned} D^2(x) &= D(xy) = D(x)y + xD(y) = xy^2 + x^2y \\ D(xy^2) &= D(x)y^2 + 2xyD(y) = xy^3 + 2x^2y^2 \\ D(x^2y) &= 2xyD(x) + x^2D(y) = 2x^2y^2 + x^3y \end{aligned}$$

Hence we get

$$D^3(x) = xy^3 + 4x^2y^2 + x^3y.$$

In general, Dumont showed that under the grammar

$$G = \{x \rightarrow xy, y \rightarrow xy\},$$

we have

$$D^n(x) = A_n(x, y) = \sum_{\sigma \in \mathfrak{S}_n} x^{\text{des}(\sigma)+1} y^{\text{asc}(\sigma)+1}.$$

$$n = 1, \quad 0 \text{ } y \text{ } 1 \text{ } x \text{ } 0$$

$$n = 2, \quad 0 \text{ } y \text{ } 1 \text{ } y \text{ } \boxed{2} \text{ } x \text{ } 0, \quad 0 \text{ } y \text{ } \boxed{2} \text{ } x \text{ } 1 \text{ } x \text{ } 0$$

$$n = 3, \quad 0 \text{ } y \text{ } \boxed{3} \text{ } x \text{ } 1 \text{ } y \text{ } 2 \text{ } x \text{ } 0, \quad 0 \text{ } y \text{ } 1 \text{ } y \text{ } \boxed{3} \text{ } x \text{ } 2 \text{ } x \text{ } 0, \quad 0 \text{ } y \text{ } 1 \text{ } y \text{ } 2 \text{ } y \text{ } \boxed{3} \text{ } x \text{ } 0, \\ 0 \text{ } y \text{ } \boxed{3} \text{ } x \text{ } 2 \text{ } x \text{ } 1 \text{ } x \text{ } 0, \quad 0 \text{ } y \text{ } 2 \text{ } y \text{ } \boxed{3} \text{ } x \text{ } 1 \text{ } x \text{ } 0, \quad 0 \text{ } y \text{ } 2 \text{ } x \text{ } 1 \text{ } y \text{ } \boxed{3} \text{ } x \text{ } 0$$

$$A_1(x, y) = D(x) = xy$$

$$A_2(x, y) = D^2(x) = xy^2 + x^2y$$

$$A_3(x, y) = D^3(x) = xy^3 + 4x^2y^2 + x^3y$$

Observe that

$$D(u + v) = D(u) + D(v) \quad D(uv) = D(u)v + uD(v).$$

Hence, it obeys the Leibniz's rule

$$D^n(uv) = \sum_{k=0}^n \binom{n}{k} D^k(u) D^{n-k}(v).$$

For a constant c , we have $D(c) = 0$.

Let

$$\text{Gen}(w, t) = \sum_{n \geq 0} D^n(w) \frac{t^n}{n!}.$$

Then, by the above two relations, we derive that

$$\text{Gen}(u + v, t) = \text{Gen}(u, t) + \text{Gen}(v, t).$$

$$\text{Gen}(uv, t) = \text{Gen}(u, t) \text{Gen}(v, t).$$

Theorem (Chen-Fu)

Let D_G be the formal derivative with respect to the grammar G :

$$G = \{x \rightarrow xy, y \rightarrow xy\}.$$

Then

$$\text{Gen}^{(G)}(x, t) := \sum_{n \geq 0} D_G^n(x) \frac{t^n}{n!} = \frac{x - y}{1 - x^{-1}ye^{(x-y)t}}.$$

Proof. Since $D(x) = xy$ and $D(y) = xy$, we see that $D(x - y) = 0$

$$D(x^{-1}) = -x^{-2}D(x) = -x^{-1}y,$$

$$D(\underline{x^{-1}y}) = -x^{-1}y^2 + y = (x - y)\underline{x^{-1}y}.$$

Hence,

$$D^n(\underline{x^{-1}y}) = (x - y)^n x^{-1}y.$$

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$$\begin{aligned} \text{Gen}(x^{-1}, t) &= \sum_{n \geq 0} D^n(x^{-1}) \frac{t^n}{n!} \\ &= x^{-1} - \sum_{n \geq 1} D^{n-1}(x^{-1}y) \frac{t^n}{n!} \\ &= \frac{1 - x^{-1}y e^{(x-y)t}}{x-y} \end{aligned}$$

By $\text{Gen}(x, t) \text{Gen}(x^{-1}, t) = 1$, we get

$$\text{Gen}(x, t) = \frac{x-y}{1 - x^{-1}y e^{(x-y)t}}.$$



W.Y.C. Chen and A.M. Fu, A Context-free grammar for the e-positivity of the trivariate second-order Eulerian polynomials, Discrete Math. 345(1) (2022) 112661.

Symmetry and unimodality

The first few values of

$$A_n(x) = \sum_{\sigma \in \mathfrak{S}_n} x^{\text{des}(\sigma)+1}.$$

are given below.

$$A_3(x) = x + 4x^2 + x^3$$

$$A_4(x) = x + 11x^2 + 11x^3 + x^4$$

$$A_5(x) = x + 26x^2 + 66x^3 + 26x^4 + x^5$$

$$A_6(x) = x + 52x^2 + 302x^3 + 302x^4 + 52x^5 + x^6.$$

- The symmetry can be proved via the following bijection

$$\sigma = \sigma_1 \sigma_2 \cdots \sigma_n \rightarrow \sigma^r = \sigma_n \sigma_{n-1} \cdots \sigma_1$$

- The unimodality can be derived from the conclusion that [the Eulerian polynomials have only real roots.](#)

Theorem (Foata-Schützenberger)

For $n \geq 1$, there exist *nonnegative integers* $\gamma_{n,k}$ such that

$$A_n(x) = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \gamma_{n,k} x^k (1+x)^{n-1-2k}.$$

Remark: (1) This result implies that $A_n(x)$ is symmetric and unidomal.

(2) P. Brändén remarked that if a polynomial $h(x) \in R[x]$ is symmetric and has only real roots, then $h(x)$ is γ -positivity.



D. Foata and M.-P. Schützenberger, *Théorie géométrique des polynômes eulériens*, Lecture Notes in Mathematics, Vol. 138, Springer-Verlag, Berlin, 1970.

Example

The first few values of $A_n(x, y)$ are given below.

$$A_2(x) = 1 + x$$

$$A_3(x) = (1 + x)^2 + 2x$$

$$A_4(x) = (1 + x)^3 + 8x(1 + x)$$

$$A_5(x) = (1 + x)^4 + 22x(1 + x)^2 + 16x^2$$

$$A_6(x) = (1 + x)^5 + 52x(1 + x)^3 + 136x^2(1 + x)$$

Problem: What do the coefficients $\gamma_{n,k}$ count?

The combinatorial interpretations of $\gamma_{n,k}$



D. Foata and V. Strehl, Rearrangements of the symmetric group and enumerative properties of the tangent and secant numbers, Math. Z., **137** (1974), 257–264.



John R. Stembridge, Enriched P -partitions, Trans. Amer. Math. Soc. 349(2) (1997) 763–788.



P. Brändén, Actions on permutations and unimodality of descent polynomials, European J. Combin., **29** (2008), 514–531.



Z. Lin and J. Zeng, The γ -positivity of basic Eulerian polynomials via group actions, J. Combin. Theory, Ser. A, **135** (2015), 112–129.

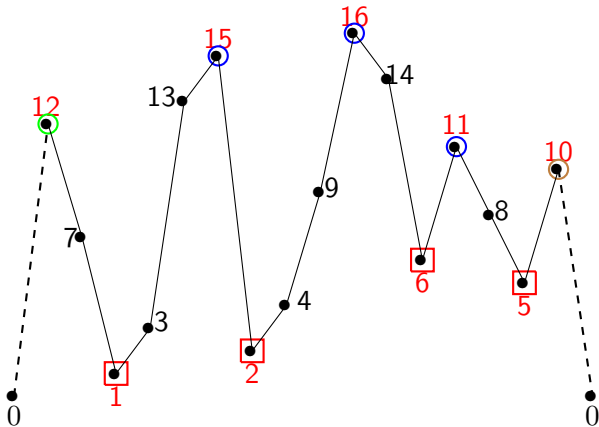


W.Y.C Chen and A.M. Fu, A Context-free grammar for the e -positivity of the trivariate second-order Eulerian polynomials, Discrete Math. 345(1) (2022) 112661.

Descent statistics

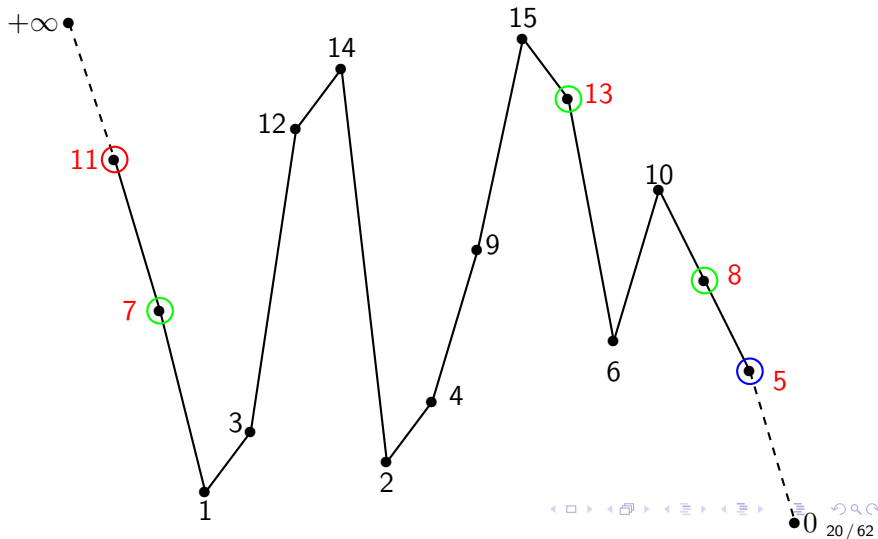
interior peak; left peak; exterior peak; valley

The mountain range view of $\sigma = 12\,7\,1\,3\,13\,15\,2\,4\,9\,16\,14\,6\,11\,8\,5\,10$



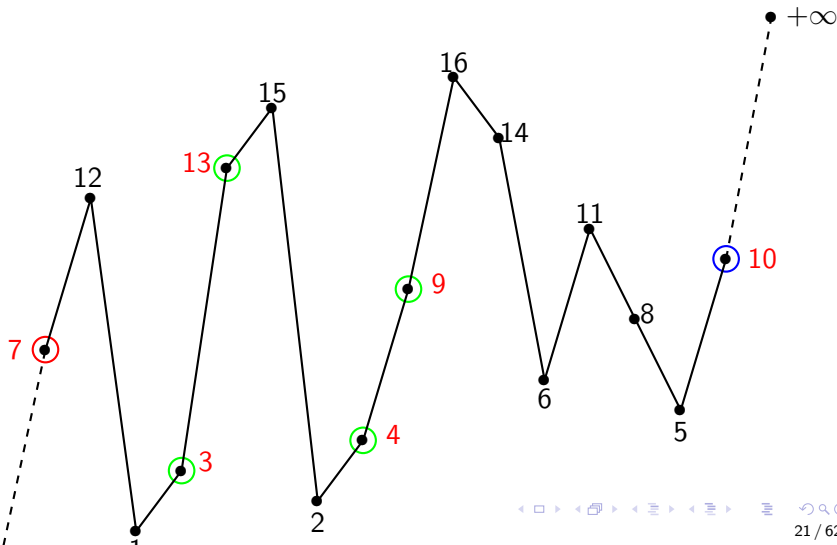
Descent statistics

double descent ($\text{dd}(\sigma)$), left double descent ($\text{ldd}(\sigma)$), left-right double descent ($\text{lrdd}(\sigma)$)



Descent statistics

double ascent ($da(\sigma)$), left double ascent ($lda(\sigma)$), left-right
double ascent ($lrda(\sigma)$)



Two combinatorial interpretations of $\gamma_{n,k}$

For $n \geq 1$, let

$$A_n(x) = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \gamma_{n,k} x^k (1+x)^{n-1-2k}.$$

Then

- (1) $\gamma_{n,k}$ counts the number of permutations in $\mathfrak{S}_n$ with k descents and without double descents.
- (2) $2^{n-1-2k} \gamma_{n,k}$ counts the number of permutations in $\mathfrak{S}_n$ with k peaks.



D. Foata and V. Strehl, Rearrangements of the symmetric group and enumerative properties of the tangent and secant numbers, *Math. Z.*, **137** (1974), 257–264.



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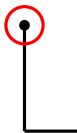
Stembridge's formula

Theorem (Stembridge)

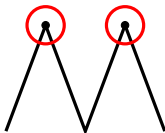
For $n \geq 1$,

$$\sum_{\sigma \in \mathfrak{S}_n} x^{\text{des}(\sigma)} = \left(\frac{1+x}{2} \right)^{n-1} \sum_{\sigma \in \mathfrak{S}_n} \left(\frac{4x}{(1+x)^2} \right)^{M(\sigma)}.$$

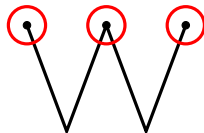
The symbol left peak, peak, exterior peak ($W(\sigma) - 1 = V(\sigma)$)



left peak ($L(\sigma)$)



peak ($M(\sigma)$)



exterior peak ($W(\sigma)$)



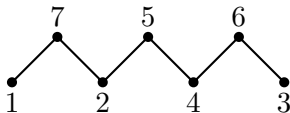
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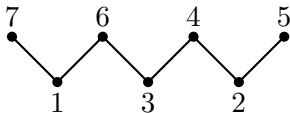
W.Y.C. Chen, The idea of a grammatical calculus, in the International Conference on Enumerative Combinatorics and Applications Virtual ICECA 2023, September 4-6, 2023.

The tangent number and the secant number

- A permutation $\sigma = \sigma_1\sigma_2\cdots\sigma_n \in \mathfrak{S}_n$ is **up-down** if $\sigma_1 < \sigma_2 > \sigma_3 < \sigma_4 > \cdots$.



- A permutation $\sigma = \sigma_1\sigma_2\cdots\sigma_n \in \mathfrak{S}_n$ is **down-up** if $\sigma_1 > \sigma_2 < \sigma_3 > \sigma_4 < \cdots$.



- Let E_n denote the number of up-down permutations of $[n]$. Then

$$\sum_{n \geq 0} E_{2n+1} \frac{t^{2n+1}}{(2n+1)!} = \tan(t) \quad \text{and} \quad \sum_{n \geq 0} E_{2n} \frac{t^{2n}}{(2n)!} = \sec(t).$$

Based on this reason, we usually call E_{2n+1} the **tangent number** and E_{2n} the **secant number**.

Theorem (Euler)

For $n \geq 1$,

$$\sum_{\sigma \in \mathfrak{S}_n} (-1)^{\text{des}(\sigma)} = \begin{cases} (-1)^{\frac{n-1}{2}} E_n, & \text{if } n \text{ is odd,} \\ 0, & \text{if } n \text{ is even.} \end{cases}$$

Theorem (Roselle)

For $n \geq 1$,

$$\sum_{\sigma \in \mathfrak{D}_n} (-1)^{\text{exc}(\sigma)} = \begin{cases} (-1)^{\frac{n}{2}} E_n, & \text{if } n \text{ is even,} \\ 0, & \text{if } n \text{ is odd,} \end{cases}$$

where $\mathfrak{D}_n$ counts the number of permutations in $\mathfrak{S}_n$ without fixed points and $\text{exc}(\sigma)$ counts the number of i for which $\sigma_i > i$.



D.P. Roselle, Permutations by number of rises and successions, Proc. Amer. Math. Soc. 19 (1968) 8–16.

My recent work related to Eulerian polynomials



Kathy Q. Ji, The (α, β) -Eulerian polynomials and descent-Stirling statistics on permutations, Sci China Math., to appear.

This work was inspired by the insightful paper of Carlitz and Scoville.



L. Carlitz and R. Scoville, Generalized Eulerian numbers: combinatorial applications, J. Reine Angew. Math. 265 (1974) 110–137.

I observed that the issue they were unable to fully address can be elegantly addressed through the application of the grammar calculus introduced by Chen.



W.Y.C. Chen, Context-free grammars, differential operators and formal power series, Theoret. Comput. Sci. 117 (1993) 113–129.

Stirling statistics

- The left-to-right maxima ($\text{LRmax}(\sigma) = 3$)

$$\sigma = 27183654$$

- The left-to-right minima ($\text{LRmin}(\sigma) = 2$)

$$\sigma = 27183654$$

- The right-to-left maxima ($\text{RLmax}(\sigma) = 4$)

$$\sigma = 27183654$$

- The right-to-left minima ($\text{RLmin}(\sigma) = 3$)

$$\sigma = 27183654$$

It is well known that

$$\begin{aligned} \sum_{\sigma \in \mathfrak{S}_n} \alpha^{\text{LRmax}(\sigma)} &= \sum_{\sigma \in \mathfrak{S}_n} \alpha^{\text{RLmax}(\sigma)} = \sum_{\sigma \in \mathfrak{S}_n} \alpha^{\text{LRmin}(\sigma)} = \sum_{\sigma \in \mathfrak{S}_n} \alpha^{\text{RLmin}(\sigma)} \\ &= \alpha(\alpha + 1)(\alpha + 2) \cdots (\alpha + n - 1) = n! \binom{\alpha + n - 1}{n}. \end{aligned}$$

The (α, β) -Eulerian polynomials

Carlitz and Scoville considered the following polynomials involving Eulerian-Stirling statistics:

$$A_n(x, y | \alpha, \beta) = \sum_{\sigma \in \mathfrak{S}_{n+1}} x^{\text{asc}(\sigma)} y^{\text{des}(\sigma)} \alpha^{\text{LRmin}(\sigma)-1} \beta^{\text{RLmin}(\sigma)-1}.$$

Theorem (Carlitz and Scoville (Theorem 9))

$$\sum_{n \geq 0} A_n(x, y | \alpha, \beta) \frac{t^n}{n!} = (1 + yF(x, y; t))^\alpha (1 + xF(x, y; t))^\beta,$$

where $F(x, y; t)$ is given by

$$F(x, y; t) = \frac{e^{xt} - e^{yt}}{xe^{yt} - ye^{xt}}.$$



L. Carlitz and R. Scoville, Generalized Eulerian numbers: combinatorial applications, J. Reine Angew. Math. 265 (1974) 110–137.

The polynomial involving descent statistics

Theorem (Carlitz and Scoville (Theorem 2))

$$\sum_{n \geq 0} \left(\sum_{\sigma \in \mathfrak{S}_{n+1}} (u_1 u_2)^{V(\sigma)} u_3^{\text{lra}(\sigma)} u_4^{\text{rdd}(\sigma)} \right) \frac{t^n}{n!} \\ = (1 + yF(x, y; t)) (1 + xF(x, y; t)),$$

where $x + y = u_3 + u_4$ and $xy = u_1 u_2$.

- This result was reformulated by [Goulden and Jackson \(Combinatorial Enumeration, 2004\)](#) and [Stanley \(Enumerative combinatorics, Volume 1\)](#).
- [Fu \(Adv. Appl. Math, 2018\)](#) provided a grammatical proof of this generating function.
- [Pan and Zeng \(Adv. Appl. Math., 2019\)](#) provided *inv* q -analogue of this generating function.

Carlitz-Scoville $\rightarrow$ Stembridge

- Carlitz and Scoville (Theorem 2)

$$\sum_{n \geq 0} \left(\sum_{\sigma \in \mathfrak{S}_{n+1}} (u_1 u_2)^{V(\sigma)} u_3^{\text{lda}(\sigma)} u_4^{\text{rdd}(\sigma)} \right) \frac{t^n}{n!} = (1 + yF(x, y; t)) (1 + xF(x, y; t)),$$

where $x + y = u_3 + u_4$ and $xy = u_1 u_2$.

- Carlitz and Scoville (Theorem 9)

$$\sum_{n \geq 0} A_n(x, y | \alpha, \beta) \frac{t^n}{n!} = (1 + xF(x, y; t))^\alpha (1 + yF(x, y; t))^\beta.$$

Set

$$u_1 = u_2 = \sqrt{xy}, \quad u_3 = u_4 = \frac{x + y}{2}.$$

- J. Stembridge

$$\sum_{\sigma \in \mathfrak{S}_n} x^{\text{des}(\sigma)} = \left(\frac{1+x}{2} \right)^{n-1} \sum_{\sigma \in \mathfrak{S}_n} \left(\frac{4x}{(1+x)^2} \right)^{M(\sigma)}.$$

Main result

Let

$$\begin{aligned} P_n(u_1, u_2, u_3, u_4 | \alpha, \beta) \\ = \sum_{\sigma \in \mathfrak{S}_{n+1}} (u_1 u_2)^{M(\sigma)} u_3^{\text{rda}(\sigma)} u_4^{\text{ldd}(\sigma)} \alpha^{\text{LRmin}(\sigma)-1} \beta^{\text{RLmin}(\sigma)-1}. \end{aligned}$$

Theorem

$$\begin{aligned} \sum_{n \geq 0} P_n(u_1, u_2, u_3, u_4 | \alpha, \beta) \frac{t^n}{n!} \\ = (1 + yF(x, y; t))^{\frac{\alpha+\beta}{2}} (1 + xF(x, y; t))^{\frac{\alpha+\beta}{2}} e^{\frac{1}{2}(\beta-\alpha)(u_3-u_4)t}, \end{aligned}$$

where $x + y = u_3 + u_4$, $xy = u_1 u_2$.



Kathy Q. Ji, The (α, β) -Eulerian polynomials and descent-Stirling statistics on permutations, Sci China Math., to appear.

The (α, β) -Eulerian polynomials

Carlitz and Scoville considered the following polynomials involving Eulerian-Stirling statistics:

$$A_n(x, y | \alpha, \beta) = \sum_{\sigma \in \mathfrak{S}_{n+1}} x^{\text{asc}(\sigma)} y^{\text{des}(\sigma)} \alpha^{\text{LRmin}(\sigma)-1} \beta^{\text{RLmin}(\sigma)-1}.$$

Theorem (Carlitz and Scoville (Theorem 9))

$$\sum_{n \geq 0} A_n(x, y | \alpha, \beta) \frac{t^n}{n!} = (1 + yF(x, y; t))^\alpha (1 + xF(x, y; t))^\beta,$$

where $F(x, y; t)$ is given by

$$F(x, y; t) = \frac{e^{xt} - e^{yt}}{xe^{yt} - ye^{xt}}.$$



L. Carlitz and R. Scoville, Generalized Eulerian numbers: combinatorial applications, J. Reine Angew. Math. 265 (1974) 110–137.

Under the following grammar:

$$G = \{x \rightarrow xy, y \rightarrow xy\}.$$

Dumont showed that

$$D^n(x) = A_n(x, y) = \sum_{\sigma \in \mathfrak{S}_n} x^{\text{des}(\sigma)+1} y^{\text{asc}(\sigma)+1}.$$

Chen and Fu derived that

$$\text{Gen}(x, t) = \frac{x - y}{1 - x^{-1}ye^{(x-y)t}} = x(1 + yF(x, y; t)).$$

Using the similar argument, it can be proved that

$$\text{Gen}(y, t) = y(1 + xF(x, y; t)).$$

It follows that

$$\text{Gen}(xy, t) = xy(1 + xF(x, y; t))(1 + yF(x, y; t))$$

A little trick

Consider the grammar $\tilde{G}$:

$$\tilde{G} = \{a \rightarrow \alpha ay, b \rightarrow \beta bx, x \rightarrow xy, y \rightarrow xy\}.$$

Let $D_{\tilde{G}}$ be the formal derivative with respect to $\tilde{G}$, we have

$$\sum_{n \geq 0} D_{\tilde{G}}^n(ab) \frac{t^n}{n!} = ab(1 + yF(x, y; t))^\alpha (1 + xF(x, y; t))^\beta.$$

Proof. Recall that D_G is the formal derivative with respect to the grammar $G = \{x \rightarrow xy, y \rightarrow xy\}$. Then

$$D_G(x^\alpha) = \alpha x^\alpha y.$$

Set $a = x^\alpha$, we find that

$$D_{\tilde{G}}(a) = D_G(x^\alpha)$$

Observe that $D_{\tilde{G}}(x) = D_G(x)$ and $D_{\tilde{G}}(y) = D_G(y)$, we can use the induction on n to deduce that for $n \geq 0$,

$$D_{\tilde{G}}^n(a) = D_G^n(x^\alpha).$$

It follows that

$$\begin{aligned}\text{Gen}^{\tilde{G}}(a, t) &= \text{Gen}^G(x^\alpha, t) \\ &= x^\alpha(1 + yF(x, y; t))^\alpha \\ &= a(1 + yF(x, y; t))^\alpha.\end{aligned}$$

Using the similar argument, it can be proved that

$$\text{Gen}^{\tilde{G}}(b, t) = b(1 + xF(x, y; t))^\beta.$$

It follows that

$$\text{Gen}^{\tilde{G}}(ab, t) = ab(1 + yF(x, y; t))^\alpha(1 + xF(x, y; t))^\beta.$$

$$\tilde{G} = \{a \rightarrow \alpha ay, b \rightarrow \beta bx, x \rightarrow xy, y \rightarrow xy\}.$$

$$D_{\tilde{G}}^n(ab) = ab \sum_{\sigma \in \mathfrak{S}_{n+1}} y^{\text{des}(\sigma)} x^{\text{asc}(\sigma)} \alpha^{LRmin(\sigma)-1} \beta^{RLmin(\sigma)-1}.$$

$$n = 1,$$

$$0 \mathbf{a} \mathbf{1} \mathbf{b} \mathbf{0}$$

$$n = 2,$$

$$0 \mathbf{a} \mathbf{1} \times \boxed{2}^{\beta} \mathbf{b} \mathbf{0}, \quad 0 \mathbf{a} \boxed{2}^{\alpha} \mathbf{y} \mathbf{1} \mathbf{b} \mathbf{0}$$

$$n = 3,$$

$$0 \mathbf{a} \boxed{3}^{\alpha} \mathbf{y} \mathbf{1} \times 2^{\beta} \mathbf{b} \mathbf{0}, \quad 0 \mathbf{a} \mathbf{1} \times \boxed{3} \mathbf{y} 2^{\beta} \mathbf{b} \mathbf{0}, \quad 0 \mathbf{a} \mathbf{1} \times 2^{\beta} \times \boxed{3}^{\beta} \mathbf{b} \mathbf{0},$$

$$0 \mathbf{a} \boxed{3}^{\alpha} \mathbf{y} 2^{\alpha} \mathbf{y} \mathbf{1} \mathbf{b} \mathbf{0}, \quad 0 \mathbf{a} 2^{\alpha} \times \boxed{3} \mathbf{y} \mathbf{1} \mathbf{b} \mathbf{0}, \quad 0 \mathbf{a} 2^{\alpha} \mathbf{y} \mathbf{1} \times \boxed{3}^{\beta} \mathbf{b} \mathbf{0}$$

$$D_{\tilde{G}}(ab) = ab(\alpha y + \beta x)$$

$$D_{\tilde{G}}^2(ab) = ab(\alpha^2 y^2 + \beta^2 x^2 + 2\alpha\beta xy + \alpha xy + \beta xy).$$

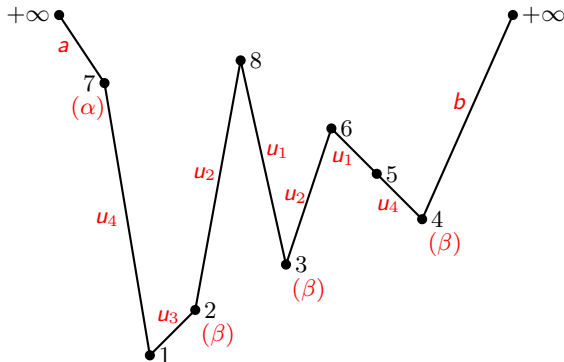
The outline of the proof

Let $D_{\tilde{G}}$ be the formal derivative with respect to the following grammar

$$\tilde{G} = \{a \rightarrow a\alpha u_4, b \rightarrow b\beta u_3, u_4 \rightarrow u_1 u_2, u_3 \rightarrow u_1 u_2, u_1 \rightarrow u_1 u_3, u_2 \rightarrow u_2 u_4\}.$$

Then

$$\begin{aligned} D_{\tilde{G}}^n(ab) &= abP_n(u_1, u_2, u_3, u_4 | \alpha, \beta) \\ &= ab \sum_{\sigma \in \mathfrak{S}_{n+1}} (u_1 u_2)^{M(\sigma)} u_3^{\text{rda}(\sigma)} u_4^{\text{ladd}(\sigma)} \alpha^{\text{LRmin}(\sigma)-1} \beta^{\text{RLmin}(\sigma)-1}. \end{aligned}$$



Let $D_{\tilde{G}_1}$ is the formal derivative with respect to the grammar

$$\tilde{G}_1 = \{u_4 \rightarrow u_1 u_2, u_3 \rightarrow u_1 u_2, u_1 \rightarrow u_1 u_3, u_2 \rightarrow u_2 u_4\}.$$

We first show that

$$\begin{aligned}\text{Gen}^{(\tilde{G}_1)}(u_1 u_2, t) &:= \sum_{n \geq 0} D_{\tilde{G}_1}^n(u_1 u_2) \frac{t^n}{n!} \\ &= xy(1 + yF(x, y; t))(1 + xF(x, y; t))\end{aligned}$$

and

$$\begin{aligned}\text{Gen}^{(\tilde{G}_1)}(u_1^2, t) &:= \text{Gen}^{(\tilde{G}_1)}(u_1 u_2^{-1}, t) \text{Gen}^{(\tilde{G}_1)}(u_1 u_2, t) \\ &= u_1^2(1 + yF(x, y; t))(1 + xF(x, y; t))e^{(u_3 - u_4)t},\end{aligned}$$

where $x + y = u_3 + u_4$ and $xy = u_1 u_2$.

$$\begin{aligned}
& \text{Gen}^{(\tilde{G})}(ab, t) \\
&= \text{Gen}^{(\tilde{G}_1)}(u_2^\alpha u_1^\beta, t) \\
&= (\text{Gen}^{(\tilde{G}_1)}(u_2 u_1, t))^\alpha (\text{Gen}^{(\tilde{G}_1)}(u_1, t))^{\beta-\alpha} \\
&= u_1^\beta u_2^\alpha (1 + yF(x, y; t))^{\frac{\alpha+\beta}{2}} (1 + xF(x, y; t))^{\frac{\alpha+\beta}{2}} e^{\frac{1}{2}(\beta-\alpha)(u_3-u_4)t} \\
&= ab (1 + yF(x, y; t))^{\frac{\alpha+\beta}{2}} (1 + xF(x, y; t))^{\frac{\alpha+\beta}{2}} e^{\frac{1}{2}(\beta-\alpha)(u_3-u_4)t}
\end{aligned}$$

as desired.

Main result

Let

$$\begin{aligned} P_n(u_1, u_2, u_3, u_4 | \alpha, \beta) \\ = \sum_{\sigma \in \mathfrak{S}_{n+1}} (u_1 u_2)^{M(\sigma)} u_3^{\text{rda}(\sigma)} u_4^{\text{ldd}(\sigma)} \alpha^{\text{LRmin}(\sigma)-1} \beta^{\text{RLmin}(\sigma)-1}. \end{aligned}$$

Theorem

$$\begin{aligned} \sum_{n \geq 0} P_n(u_1, u_2, u_3, u_4 | \alpha, \beta) \frac{t^n}{n!} \\ = (1 + yF(x, y; t))^{\frac{\alpha+\beta}{2}} (1 + xF(x, y; t))^{\frac{\alpha+\beta}{2}} e^{\frac{1}{2}(\beta-\alpha)(u_3-u_4)t}, \end{aligned}$$

where $x + y = u_3 + u_4$, $xy = u_1 u_2$.



Kathy Q. Ji, The (α, β) -Eulerian polynomials and descent-Stirling statistics on permutations, submitted. [arXiv:2310.01053](https://arxiv.org/abs/2310.01053)

This result (the problem posed by Carlitz and Scoville) could provide a unify strategy to generalize the results due to the following researchers:

- [P. Brändén](#) (Europ. J. Combin. 2008)
- [F.N. David and D.E. Barton](#) (1962)
- [S. Elizalde and M. Noy](#) (ICM, invited speaker) (Adv. Appl. Math. 2003)
- [R.C. Entringer](#) (Duke Math. J. 1969)
- [D. Foata and V. Strehl](#) (Math. Z., 1974)
- [S. Kitaev](#) (Discrete Appl. Math. 2007)
- [T. K. Petersen](#) (2006)
- [J.R. Stembridge](#) (Trans. Amer. Math. Soc. 1997)
- [Y. Zhuang](#) (J. Combin. Theory Ser. A, 2016; Adv. Appl. Math. 2017)

Theorem (Ji, 2023)

$$\sum_{n \geq 0} P_n(u_1, u_2, u_3, u_4 | \alpha, \beta) \frac{t^n}{n!} \\ = (1 + yF(x, y; t))^{\frac{\alpha+\beta}{2}} (1 + xF(x, y; t))^{\frac{\alpha+\beta}{2}} e^{\frac{1}{2}(\beta-\alpha)(u_3-u_4)t},$$

where $x + y = u_3 + u_4$, $xy = u_1 u_2$.

Theorem (Carlitz and Scoville (Theorem 9))

$$\sum_{n \geq 0} A_n(x, y | \alpha, \beta) \frac{t^n}{n!} = (1 + xF(x, y; t))^\alpha (1 + yF(x, y; t))^\beta.$$

An extension of Stembridge's formula

Theorem

For $n \geq 1$,

$$(1) \sum_{\sigma \in \mathfrak{S}_n} (xy)^{M(\sigma)} \left(\frac{x+y}{2} \right)^{n-2M(\sigma)-1} \alpha^{\text{LRmin}(\sigma)-1} \beta^{\text{RLmin}(\sigma)-1}$$

$$= \sum_{\sigma \in \mathfrak{S}_n} x^{\text{des}(\sigma)} y^{n-\text{des}(\sigma)-1} \left(\frac{\alpha + \beta}{2} \right)^{\text{LRmin}(\sigma) + \text{RLmin}(\sigma) - 2}.$$

$$(2) \sum_{\sigma \in \mathfrak{S}_n} (xy)^{L(\sigma)} \left(\frac{x+y}{2} \right)^{n-2L(\sigma)} \beta^{\text{RLmin}(\sigma)}$$

$$= \sum_{\sigma \in \mathfrak{S}_{n+1}} x^{\text{des}(\sigma)} y^{n-\text{des}(\sigma)} \left(\frac{\beta}{2} \right)^{\text{LRmin}(\sigma) + \text{RLmin}(\sigma) - 2}.$$

γ -positivity of (α, β) -Eulerian polynomial

Recall that

$$A_n \left(x, y \middle| \frac{\alpha + \beta}{2}, \frac{\alpha + \beta}{2} \right) = \sum_{\sigma \in \mathfrak{S}_{n+1}} x^{\text{asc}(\sigma)} y^{\text{des}(\sigma)} \left(\frac{\alpha + \beta}{2} \right)^{\text{LRmax}(\sigma) + \text{RLmax}(\sigma) - 2}.$$

Theorem

Let

$$A_{n-1} \left(x, y \middle| \frac{\alpha + \beta}{2}, \frac{\alpha + \beta}{2} \right) = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} \gamma_{n,k}(\alpha, \beta) (xy)^k (x+y)^{n-2k-1}$$

Then

$$\gamma_{n,k}(\alpha, \beta) = \left(\frac{1}{2} \right)^{n-2k-1} \sum_{\sigma \in \Gamma_{n,k}} \alpha^{\text{LRmin}(\sigma)-1} \beta^{\text{RLmin}(\sigma)-1},$$

where $\Gamma_{n,k}$ denote the set of permutations on $[n]$ with k peaks.

Example

$$A_1 \left(x, y \middle| \frac{\alpha + \beta}{2}, \frac{\alpha + \beta}{2} \right) = \frac{\alpha + \beta}{2} (x + y),$$

$$A_2 \left(x, y \middle| \frac{\alpha + \beta}{2}, \frac{\alpha + \beta}{2} \right) = \left(\frac{\alpha + \beta}{2} \right)^2 (x + y)^2 + (\alpha + \beta)xy,$$

$$A_3 \left(x, y \middle| \frac{\alpha + \beta}{2}, \frac{\alpha + \beta}{2} \right) = \left(\frac{\alpha + \beta}{2} \right)^3 (x + y)^3 \\ + \left(6 \left(\frac{\alpha + \beta}{2} \right)^2 + 2 \left(\frac{\alpha + \beta}{2} \right) \right) xy(x + y),$$

$$A_4 \left(x, y \middle| \frac{\alpha + \beta}{2}, \frac{\alpha + \beta}{2} \right) = \left(\frac{\alpha + \beta}{2} \right)^4 (x + y)^4 \\ + \frac{1}{4} (6(\alpha + \beta)^3 + 8(\alpha + \beta)^2 + 4(\alpha + \beta)) xy(x + y)^2 \\ + (3(\alpha + \beta)^2 + 2(\alpha + \beta)) (xy)^2.$$

Combinatorial proofs of γ -positivity



K.Q. Ji and Z. Lin, The binomial-Stirling-Eulerian polynomials, European J. Combin., 120 (2024) 103962. ($\alpha = \beta$)



William Y.C. Chen and Amy M. Fu, Increasing binary trees and the (α, β) -Eulerian polynomials, arXiv:2404.10331. ($\alpha \neq \beta$)



Yao Dong, Zhicong Lin, Qiongqiong Pan, A q -analog of the Stirling-Eulerian polynomials, arXiv:2404.09777.

An extension of the tangent number and the secant number

Theorem

Let $\mathfrak{S}_n^a$ denote the set of *up-down permutations of $[n]$* . For $n \geq 1$,

$$\begin{aligned} & \sum_{\sigma \in \mathfrak{S}_n} (-1)^{\text{des}(\sigma)} \left(\frac{\alpha + \beta}{2} \right)^{\text{LRmin}(\sigma) + \text{RLmin}(\sigma) - 2} \\ &= \begin{cases} (-1)^{\frac{n-1}{2}} \sum_{\sigma \in \mathfrak{S}_n^a} \alpha^{\text{LRmin}(\sigma)-1} \beta^{\text{RLmin}(\sigma)-1}, & \text{if } n \text{ is odd,} \\ 0, & \text{if } n \text{ is even.} \end{cases} \end{aligned}$$

Let $\mathfrak{S}_n^a$ denote the set of *down-up permutations of $[n]$* . For $n \geq 1$,

$$\begin{aligned} & \sum_{\sigma \in \mathfrak{S}_{n+1}} (-1)^{\text{des}(\sigma)} \left(\frac{\beta}{2} \right)^{\text{LRmin}(\sigma) + \text{RLmin}(\sigma) - 2} \\ &= \begin{cases} (-1)^{\frac{n}{2}} \sum_{\sigma \in \mathfrak{S}_n^a} \beta^{\text{RLmin}(\sigma)}, & \text{if } n \text{ is even,} \\ 0, & \text{if } n \text{ is odd.} \end{cases} \end{aligned}$$

Revisit to the secant number

Theorem

$$\sum_{\sigma \in \mathfrak{S}_{n+1}} (-1)^{\text{des}(\sigma)} \left(\frac{1}{2}\right)^{\text{LRmin}(\sigma) + \text{RLmin}(\sigma) - 2} = \begin{cases} (-1)^{\frac{n}{2}} E_n, & \text{if } n \text{ is even,} \\ 0, & \text{if } n \text{ is odd.} \end{cases}$$

Comparing with Roselle's result, we have

$$\sum_{\sigma \in \mathfrak{S}_{n+1}} (-1)^{\text{des}(\sigma)} \left(\frac{1}{2}\right)^{\text{LRmin}(\sigma) + \text{RLmin}(\sigma) - 2} = \sum_{\sigma \in \mathfrak{D}_n} (-1)^{\text{exc}(\sigma)}.$$

Remark. Chen and Fu provided a beautiful combinatorial proof of the above identity.

Motzkin numbers

The Motzkin numbers, introduced in [1] and denoted M_n , are defined by the following recurrences:

$$M_0 = 1, \quad M_{n+1} = M_n + \sum_{k=0}^{n-1} M_k M_{n-1-k} \quad \text{for } n \geq 0.$$

The following two identities describe the relationships between the Motzkin numbers M_n and the Catalan numbers C_n :

$$(a). \quad M_n = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k} C_k \quad \text{and} \quad (b). \quad C_{n+1} = \sum_{k=0}^n \binom{n}{k} M_k. \quad (0.1)$$



T. Motzkin, Relations between hypersurface cross ratios, and a combinatorial formula for partitions of a polygon, for permanent preponderance, and for non-associative products, Bull. Amer. Math. Soc. 54 (1948) 352–360.



R. Donaghey, Restricted plane tree representations of four Motzkin-Catalan equations, J. Combin. Theory Ser. B 22 (1977) 114–121.

Two-variable Motzkin polynomials

$$M_n(u; v) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k} C_k u^{k+1} v^{n-2k}.$$

Theorem (Sun)

$$\sum_{n \geq 0} M_n(u; v) q^n = \frac{1 - vq - \sqrt{1 - 2vq + (v^2 - 4u)q^2}}{2q^2}.$$



Z. Chen and H. Pan, Identities involving weighted Catalan, Schröder and Motzkin paths, Adv. in Appl. Math. 86 (2017) 81–98.



Z.-W. Sun, Congruences involving generalized central trinomial coefficients, Sci. China Math. 57 (2014) 1375–1400.

Combinatorial interpretation of Motzkin polynomials

- Let $\mathcal{T}_n$ denote the set of plane trees with n edges without young interior vertices. In other words, every interior vertex of a *tip-augmented* plane tree is the parent of an old leaf.
- This kind of plane trees was called by Donaghey as the *tip-augmented* plane trees.

$$M_n(u; v) = \sum_{T \in \mathcal{T}_{n+1}} u^{\text{oleaf}(T)} v^{\text{yleaf}(T)}.$$

For example, the figure below shows four tip-augmented plane trees with four edges, where the old leaves are labeled by u and the young leaves are labeled by v . Therefore, we have

$$M_3(u; v) = uv^3 + 3u^2v.$$

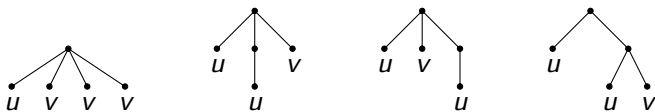


图 0.1: 4-edge tip-augmented plane trees.

Proof of the generating function of $M_n(u; v)$

Theorem

Let D be the formal derivative with respect to the grammar

$$M = \{t \rightarrow t^2v, u \rightarrow 2tuv, v \rightarrow 4tu\}.$$

For $n \geq 1$,

$$D^n\left(\frac{v}{2}\right) = (n+1)!t^nM_{n-1}(u; v).$$



Janet J.W. Dong, Lora R. Du, Kathy Q. Ji and Dax T.X. Zhang, New refinements of Narayana polynomials and Motzkin polynomials, Adv. Appl. Math., to appear.

Proof of the generating function of $M_n(u; v)$

Let $\mathcal{LT}_n$ denote the set of all labeled tip-augmented plane trees with n edges.

$$(n+2)!t^{n+1}M_n(u; v) = \sum_{T \in \mathcal{LT}_{n+1}} u^{\text{oleaf}(T)} v^{\text{yleaf}(T)} t^{\text{edge}(T)}.$$

For $n \geq 1$,

$$D^n\left(\frac{v}{2}\right) = \sum_{T \in \mathcal{LT}_n} u^{\text{oleaf}(T)} v^{\text{yleaf}(T)} t^{\text{edge}(T)},$$

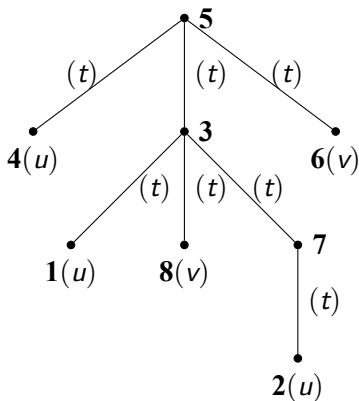


Janet J.W. Dong, Lora R. Du, Kathy Q. Ji and Dax T.X. Zhang, New refinements of Narayana polynomials and Motzkin polynomials, Adv. Appl. Math., to appear.

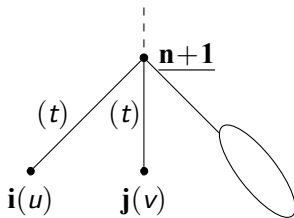
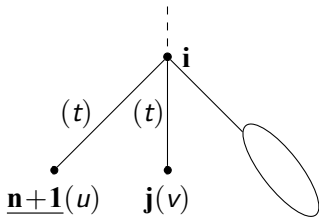
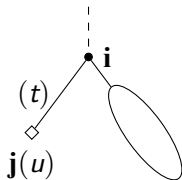
$$M = \{t \rightarrow t^2v, u \rightarrow 2tuv, v \rightarrow 4tu\}$$

For $T \in \mathcal{LT}_n$, we define the grammatical labeling of T as follows:

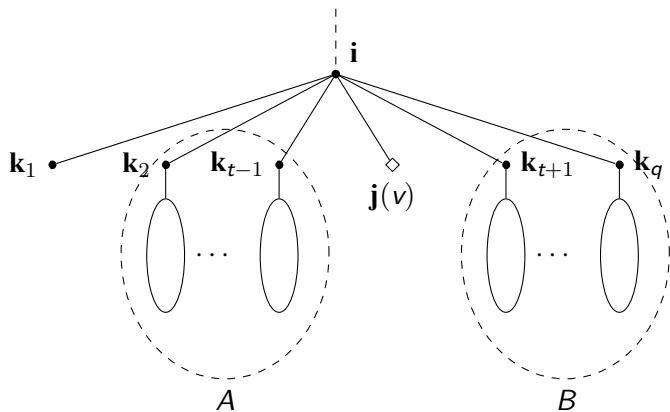
- If the vertex $\mathbf{j}$ of T is an old leaf, then label $\mathbf{j}$ by u ;
- If the vertex $\mathbf{j}$ of T is a young leaf, then label $\mathbf{j}$ by v ;
- All of edges of T are labeled by t .



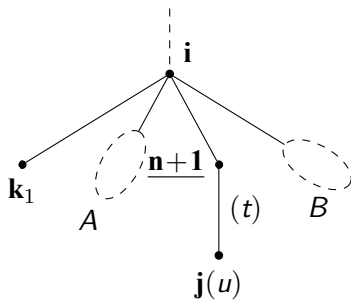
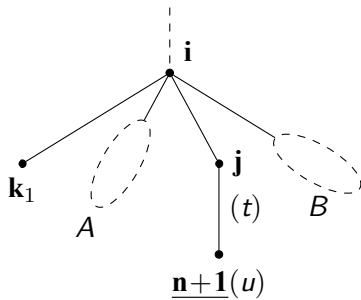
$$u \rightarrow 2tuv$$



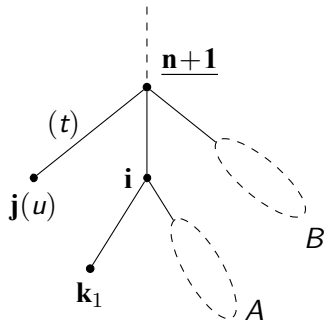
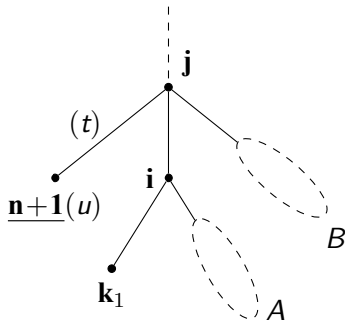
$$v \rightarrow 4tu$$



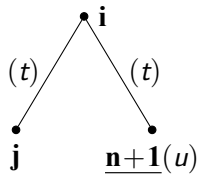
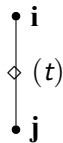
$$v \rightarrow 4tu$$



$$v \rightarrow 4tu$$



$$t \rightarrow t^2 v$$



$$M = \{t \rightarrow t^2v, u \rightarrow 2tuv, v \rightarrow 4tu\}$$

Observe that

$$D(t^{-1}) = -v \quad \text{and} \quad D^2(t^{-1}) = -D(v).$$

$$D(t^{-2}) = -2t^{-1}v, \quad D^2(t^{-2}) = 2v^2 - 8u, \quad \text{and} \quad D^3(t^{-2}) = 0.$$

For $n \geq 2$,

$$D^n(t^{-1}) = -2n!t^{n-1}M_{n-2}(u; v).$$

$$\text{Gen}(t^{-1}; q) = \sum_{n \geq 0} D^n(t^{-1}) \frac{q^n}{n!} = t^{-1} - vq - 2tq^2 \sum_{n \geq 0} M_n(u; v) t^n q^n.$$

$$\text{Gen}(t^{-2}; q) = \sum_{n \geq 0} D^n(t^{-2}) \frac{q^n}{n!} = t^{-2} - 2t^{-1}vq + (v^2 - 4u)q^2.$$

Since

$$\text{Gen}(t^{-2}; q) = \text{Gen}^2(t^{-1}; q),$$

It follows that

$$\text{Gen}(t^{-1}; q) = \sqrt{t^{-2} - 2t^{-1}vq + (v^2 - 4u)q^2}.$$

- Establish the stability of multivariable combinatorial polynomials.



W.Y.C. Chen, R.X.J. Hao and H.R.L. Yang, Context-free grammars and stable multivariate polynomials over Stirling permutations, In: V. Pillwein and C. Schneider (eds.), Algorithmic Combinatorics: Enumerative Combinatorics, Special Functions and Computer Algebra, pp. 109–135, Springer, 2021.



H.R.L. Yang and P.B. Zhang, Stable multivariate the Narayana polynomials and labeled plane trees, arXiv: 2403. 15058v2.

- Construct the bijection between two different kinds of combinatorial objects.



W.Y.C. Chen and A.M. Fu, A grammatical calculus for peaks and runs of permutations, J. Algebraic Combin. 57 (2023) 1139–1162.



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Thank you!