

The (α, β) -Eulerian Polynomials and Descent-Stirling Statistics on Permutations

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The outline of the talk

This talk is based on my recent work posted on the arXiv:



Kathy Q. Ji, The (α, β) -Eulerian polynomials and descent-Stirling statistics on permutations, submitted. [arXiv:2310.01053](#).

This work was inspired by the insightful paper of Carlitz and Scoville.



L. Carlitz and R. Scoville, Generalized Eulerian numbers: combinatorial applications, *J. Reine Angew. Math.* 265 (1974) 110–137.

I observed that the issue they were unable to fully address can be elegantly addressed through the application of the grammar calculus introduced by Chen.



W.Y.C. Chen, Context-free grammars, differential operators and formal power series, *Theoret. Comput. Sci.* 117 (1993) 113–129.

The outline of the talk

This result (the problem posed by Carlitz and Scoville) could provide a unify strategy to generalize the results due to the following researchers:

- [P. Brändén](#) (Europ. J. Combin. 2008)
- [F.N. David and D.E. Barton](#) (1962)
- [S. Elizalde and M. Noy](#) (ICM, invited speaker) (Adv. Appl. Math. 2003)
- [R.C. Entringer](#) (Duke Math. J. 1969)
- [D. Foata and V. Strehl](#) (Math. Z., 1974)
- [S. Kitaev](#) (Discrete Appl. Math. 2007)
- [T. K. Petersen](#) (2006)
- [J.R. Stembridge](#) (Trans. Amer. Math. Soc. 1997)
- [Y. Zhuang](#) (J. Combin. Theory Ser. A, 2016; Adv. Appl. Math. 2017)

Descents and ascents

- Let $\mathfrak{S}_n$ be the set of all permutations of $[n] = \{1, 2, \dots, n\}$.
- Let $\sigma = \sigma_1\sigma_2\cdots\sigma_n \in \mathfrak{S}_n$, we say that i is a descent of σ if $1 \leq i < n$ and $\sigma_i > \sigma_{i+1}$ and i is an ascent of σ if $1 \leq i < n$ and $\sigma_i < \sigma_{i+1}$.

$$\sigma = 2 \uparrow 7 \downarrow 1 \uparrow 8 \downarrow 3 \uparrow 6 \downarrow 5 \downarrow 4 \in \mathfrak{S}_8.$$

- Let $\text{des}(\sigma)$ denote the number of descents of σ and $\text{asc}(\sigma)$ denote the number of ascents of σ . We see that $\text{des}(\sigma) = 4$ and $\text{asc}(\sigma) = 3$.
- Let $\sigma \in \mathfrak{S}_n$, it is easy to see that

$$\text{des}(\sigma) + \text{asc}(\sigma) = n - 1.$$

Classical Eulerian polynomials

The Eulerian polynomials $A_n(x)$ are defined by $A_0(x) = 1$ and for $n \geq 1$,

$$A_n(x) = \sum_{\sigma \in \mathfrak{S}_n} x^{\text{des}(\sigma)+1}.$$

The first few values of $A_n(x)$ are given below.

$$A_1(x) = x$$

$$A_2(x) = x + x^2$$

$$A_3(x) = x + 4x^2 + x^3$$

$$A_4(x) = x + 11x^2 + 11x^3 + x^4$$

$$A_5(x) = x + 26x^2 + 66x^3 + 26x^4 + x^5$$

$$A_6(x) = x + 52x^2 + 302x^3 + 302x^4 + 52x^5 + x^6.$$

Symmetry and unimodality

- The symmetry can be proved via the following bijection

$$\sigma = \sigma_1 \sigma_2 \cdots \sigma_n \rightarrow \sigma^r = \sigma_n \sigma_{n-1} \cdots \sigma_1$$

Example:

$$\sigma = 2 \uparrow 7 \downarrow 1 \uparrow 8 \downarrow 3 \uparrow 6 \downarrow 5 \downarrow 4 \rightarrow \sigma^r = 4 \uparrow 5 \uparrow 6 \downarrow 3 \uparrow 8 \downarrow 1 \uparrow 7 \downarrow 2.$$

- The unimodality can be derived from the conclusion that [the Eulerian polynomials have one real roots](#).

Theorem (Foata-Schützenberger)

For $n \geq 1$, there exist nonnegative integers $\gamma_{n,k}$ such that

$$A_n(x) = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \gamma_{n,k} x^k (1+x)^{n-1-2k}.$$

Remark: (1) This result implies that $A_n(x)$ is symmetric and unidomal.

(2) P. Brändén remarked that if a polynomial $h(x) \in R[x]$ is symmetric and has only real roots, then $h(x)$ is γ -positivity.



D. Foata and M.-P. Schützenberger, *Théorie géométrique des polynômes eulériens*, Lecture Notes in Mathematics, Vol. 138, Springer-Verlag, Berlin, 1970.

Example

The first few values of $A_n(x, y)$ are given below.

$$A_2(x) = 1 + x$$

$$A_3(x) = (1 + x)^2 + 2x$$

$$A_4(x) = (1 + x)^3 + 8x(1 + x)$$

$$A_5(x) = (1 + x)^4 + 22x(1 + x)^2 + 16x^2$$

$$A_6(x) = (1 + x)^5 + 52x(1 + x)^3 + 136x^2(1 + x)$$

Problem: What do the coefficients $\gamma_{n,k}$ count?

The combinatorial interpretations of $\gamma_{n,k}$



D. Foata and V. Strehl, Rearrangements of the symmetric group and enumerative properties of the tangent and secant numbers, Math. Z., **137** (1974), 257–264.



John R. Stembridge, Enriched P -partitions, Trans. Amer. Math. Soc. 349(2) (1997) 763–788.



P. Brändén, Actions on permutations and unimodality of descent polynomials, European J. Combin., **29** (2008), 514–531.



Z. Lin and J. Zeng, The γ -positivity of basic Eulerian polynomials via group actions, J. Combin. Theory, Ser. A, **135** (2015), 112–129.

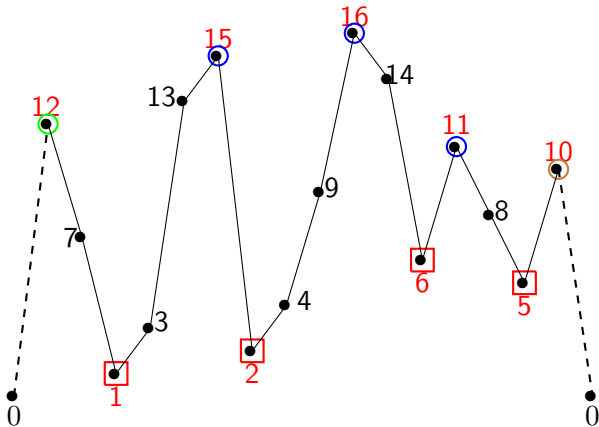


W.Y.C Chen and A.M. Fu, A Context-free grammar for the e -positivity of the trivariate second-order Eulerian polynomials, Discrete Math. 345(1) (2022) 112661.

Descent statistics

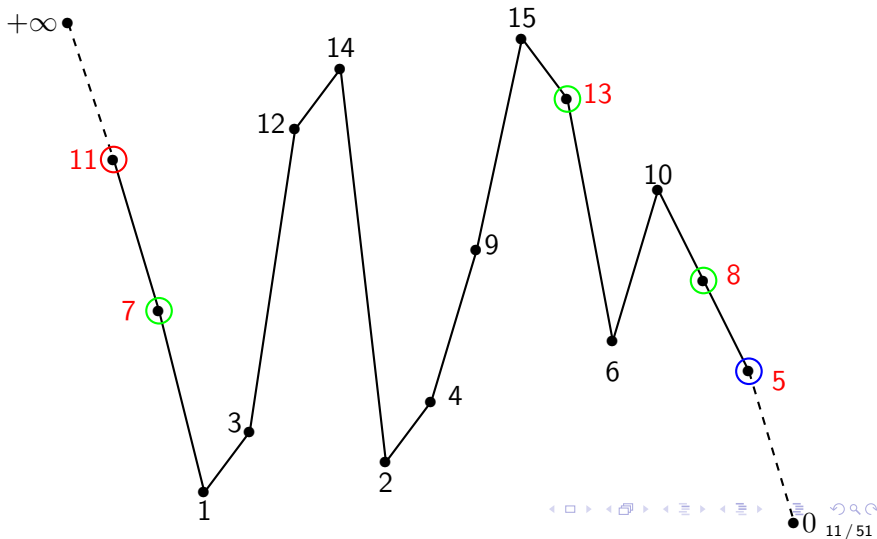
interior peak; left peak; exterior peak; valley

The mountain range view of $\sigma = 12\,7\,1\,3\,13\,15\,2\,4\,9\,16\,14\,6\,11\,8\,5\,10$



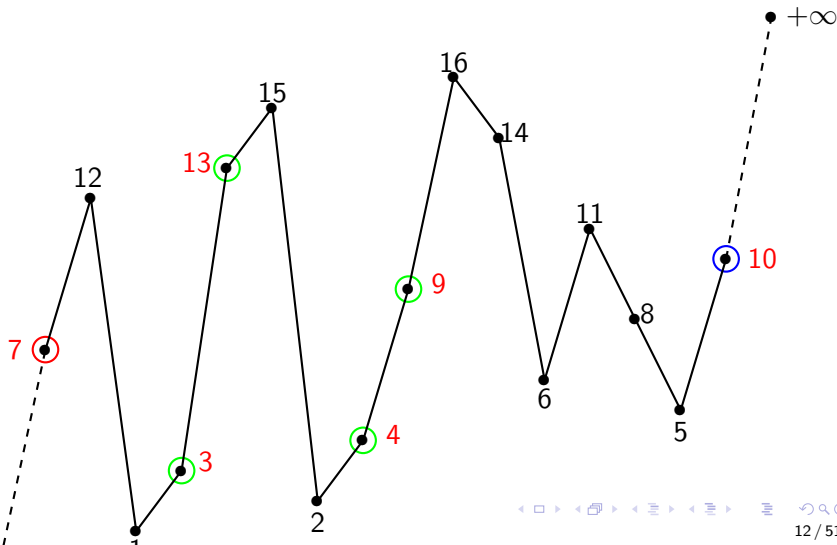
Desecent statistics

double descent ($\text{dd}(\sigma)$), left double descent ($\text{ldd}(\sigma)$), left-right
double descent ($\text{lrdd}(\sigma)$)



Descent statistics

double ascent ($\text{da}(\sigma)$), left double ascent ($\text{lda}(\sigma)$), left-right
double ascent ($\text{lrda}(\sigma)$)



Two combinatorial interpretations of $\gamma_{n,k}$

For $n \geq 1$, let

$$A_n(x) = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \gamma_{n,k} x^k (1+x)^{n-1-2k}.$$

Then

- (1) $\gamma_{n,k}$ counts the number of permutations in $\mathfrak{S}_n$ with k descents and without double descents.
- (2) $2^{n-1-2k} \gamma_{n,k}$ counts the number of permutations in $\mathfrak{S}_n$ with k peaks.



D. Foata and V. Strehl, Rearrangements of the symmetric group and enumerative properties of the tangent and secant numbers, *Math. Z.*, **137** (1974), 257–264.



P. Brändén, Actions on permutations and unimodality of descent polynomials, *European J. Combin.*, **29** (2008), 514–531.

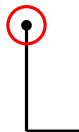
Stembridge's formula

Theorem (Stembridge)

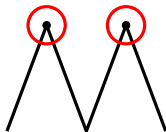
For $n \geq 1$,

$$\sum_{\sigma \in \mathfrak{S}_n} x^{\text{des}(\sigma)} = \left(\frac{1+x}{2} \right)^{n-1} \sum_{\sigma \in \mathfrak{S}_n} \left(\frac{4x}{(1+x)^2} \right)^{M(\sigma)}.$$

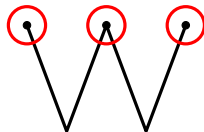
The symbol left peak, peak, exterior peak ($W(\sigma) - 1 = V(\sigma)$)



left peak ($L(\sigma)$)



peak ($M(\sigma)$)



exterior peak ($W(\sigma)$)



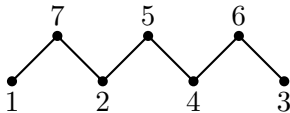
John R. Stembridge, Enriched P -partitions, Trans. Amer. Math. Soc. 349(2) (1997) 763–788.



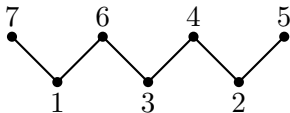
W.Y.C. Chen, The idea of a grammatical calculus, in the International Conference on Enumerative Combinatorics and Applications Virtual ICECA 2023, September 4-6, 2023.

The tangent number and the secant number

- A permutation $\sigma = \sigma_1\sigma_2\cdots\sigma_n \in \mathfrak{S}_n$ is **up-down** if $\sigma_1 < \sigma_2 > \sigma_3 < \sigma_4 > \cdots$.



- A permutation $\sigma = \sigma_1\sigma_2\cdots\sigma_n \in \mathfrak{S}_n$ is **down-up** if $\sigma_1 > \sigma_2 < \sigma_3 > \sigma_4 < \cdots$.



- Let E_n denote the number of up-down permutations of $[n]$. Then

$$\sum_{n \geq 0} E_{2n+1} \frac{t^{2n+1}}{(2n+1)!} = \tan(t) \quad \text{and} \quad \sum_{n \geq 0} E_{2n} \frac{t^{2n}}{(2n)!} = \sec(t).$$

Based on this reason, we usually call E_{2n+1} the tangent number and E_{2n} the secant number.

Theorem (Euler)

For $n \geq 1$,

$$\sum_{\sigma \in \mathfrak{S}_n} (-1)^{\text{des}(\sigma)} = \begin{cases} (-1)^{\frac{n-1}{2}} E_n, & \text{if } n \text{ is odd,} \\ 0, & \text{if } n \text{ is even.} \end{cases}$$

Theorem (Roselle)

For $n \geq 1$,

$$\sum_{\sigma \in \mathfrak{D}_n} (-1)^{\text{exc}(\sigma)} = \begin{cases} (-1)^{\frac{n}{2}} E_n, & \text{if } n \text{ is even,} \\ 0, & \text{if } n \text{ is odd,} \end{cases}$$

where $\mathfrak{D}_n$ counts the number of permutations in $\mathfrak{S}_n$ without fixed points and $\text{exc}(\sigma)$ counts the number of i for which $\sigma_i > i$.



D.P. Roselle, Permutations by number of rises and successions, Proc. Amer. Math. Soc. 19 (1968) 8–16.

Left peak polynomials and the joint polynomials

- Petersen [1, Observation 3.1.2]

$$(1+x)^n \sum_{\sigma \in \mathfrak{S}_n} \left(\frac{4x}{(1+x)^2} \right)^{L(\sigma)} = \sum_{k=1}^n \binom{n}{k} 2^k (1-x)^{n-k} \sum_{\sigma \in \mathfrak{S}_k} x^{\text{des}(\sigma)+1} + (1-x)^n.$$

- Zhuang [2, Theorem 4.2]

$$\sum_{\sigma \in \mathfrak{S}_n} u^{M(\sigma)+1} v^{\text{des}(\sigma)+1} = \left(\frac{1+b}{1+ab} \right)^{n+1} \sum_{\sigma \in \mathfrak{S}_n} a^{\text{des}(\sigma)+1},$$

where

$$a = \frac{(1+v)^2 - 2uv - (1+v)\sqrt{(1+v)^2 - 4uv}}{2uv}$$

and

$$b = \frac{1+v^2 - 2uv - (1-v)\sqrt{(1+v)^2 - 4uv}}{2(1-u)v}.$$



T.K. Petersen, Descents, Peaks, and P -partitions, PhD thesis, Brandeis University, 2006.



Y. Zhuang, Eulerian polynomials and descent statistics, Adv. in Appl. Math. 90 (2017) 86–144.

Stirling statistics

- The left-to-right maxima ($\text{LRmax}(\sigma) = 3$)

$$\sigma = 27183654$$

- The left-to-right minima ($\text{LRmin}(\sigma) = 2$)

$$\sigma = 27183654$$

- The right-to-left maxima ($\text{RLmax}(\sigma) = 4$)

$$\sigma = 27183654$$

- The right-to-left minima ($\text{RLmin}(\sigma) = 3$)

$$\sigma = 27183654$$

It is well known that

$$\begin{aligned} \sum_{\sigma \in \mathfrak{S}_n} \alpha^{\text{LRmax}(\sigma)} &= \sum_{\sigma \in \mathfrak{S}_n} \alpha^{\text{RLmax}(\sigma)} = \sum_{\sigma \in \mathfrak{S}_n} \alpha^{\text{LRmin}(\sigma)} = \sum_{\sigma \in \mathfrak{S}_n} \alpha^{\text{RLmin}(\sigma)} \\ &= \alpha(\alpha+1)(\alpha+2) \cdots (\alpha+n-1) = n! \binom{\alpha+n-1}{n}. \end{aligned}$$

The (α, β) -Eulerian polynomials

Carlitz and Scoville considered the following polynomials involving Eulerian-Stirling statistics:

$$A_n(x, y | \alpha, \beta) = \sum_{\sigma \in \mathfrak{S}_{n+1}} x^{\text{asc}(\sigma)} y^{\text{des}(\sigma)} \alpha^{\text{LRmax}(\sigma)-1} \beta^{\text{RLmax}(\sigma)-1}.$$

Theorem (Carlitz and Scoville (Theorem 9))

$$\sum_{n \geq 0} A_n(x, y | \alpha, \beta) \frac{t^n}{n!} = (1 + xF(x, y; t))^\alpha (1 + yF(x, y; t))^\beta,$$

where $F(x, y; t)$ is given by

$$F(x, y; t) = \frac{e^{xt} - e^{yt}}{xe^{yt} - ye^{xt}}.$$



L. Carlitz and R. Scoville, Generalized Eulerian numbers: combinatorial applications, J. Reine Angew. Math. 265 (1974) 110–137.

The polynomial involving descent statistics

Theorem (Carlitz and Scoville (Theorem 2))

$$\sum_{n \geq 1} \left(\sum_{\sigma \in \mathfrak{S}_n} (u_1 u_2)^{V(\sigma)} u_3^{\text{lra}(\sigma)} u_4^{\text{rdd}(\sigma)} \right) \frac{t^n}{n!} = \frac{e^{xt} - e^{yt}}{xe^{yt} - ye^{xt}} = F(x, y; t),$$

where $x + y = u_3 + u_4$ and $xy = u_1 u_2$.

- This result was reformulated by [Goulden and Jackson \(Combinatorial Enumeration, 2004\)](#) and [Stanley \(Enumerative combinatorics, Volume 1\)](#).
- [Fu \(Adv. Appl. Math, 2018\)](#) provided a grammatical proof of this generating function.
- [Pan and Zeng \(Adv. Appl. Math., 2019\)](#) provided *inv* q -analogue of this generating function.

Reformulation

- Differentiating both sides of the above gf with respect to t ,

$$\sum_{n \geq 0} \left(\sum_{\sigma \in \mathfrak{S}_{n+1}} (u_1 u_2)^{V(\sigma)} u_3^{\text{lda}(\sigma)} u_4^{\text{rdd}(\sigma)} \right) \frac{t^n}{n!} = (1 + yF(x, y; t)) (1 + xF(x, y; t)),$$

where $x + y = u_3 + u_4$ and $xy = u_1 u_2$.

- Comparing with the following gf

$$\begin{aligned} \sum_{n \geq 0} \left(\sum_{\sigma \in \mathfrak{S}_{n+1}} x^{\text{asc}(\sigma)} y^{\text{des}(\sigma)} \alpha^{\text{LRmax}(\sigma)-1} \beta^{\text{RLmax}(\sigma)-1} \right) \frac{t^n}{n!} \\ = (1 + xF(x, y; t))^\alpha (1 + yF(x, y; t))^\beta, \end{aligned}$$

- We obtain

$$\sum_{\sigma \in \mathfrak{S}_n} (u_1 u_2)^{V(\sigma)} u_3^{\text{lda}(\sigma)} u_4^{\text{rdd}(\sigma)} = \sum_{\sigma \in \mathfrak{S}_n} x^{\text{asc}(\sigma)} y^{\text{des}(\sigma)},$$

where $x + y = u_3 + u_4$ and $xy = u_1 u_2$. (This relation can be used to derive the γ -positivity of Eulerian polynomials)

Main result

Let

$$\begin{aligned} P_n(u_1, u_2, u_3, u_4 | \alpha, \beta) \\ = \sum_{\sigma \in \mathfrak{S}_{n+1}} (u_1 u_2)^{M(\sigma)} u_3^{\text{rda}(\sigma)} u_4^{\text{ldd}(\sigma)} \alpha^{\text{LRmin}(\sigma)-1} \beta^{\text{RLmin}(\sigma)-1}. \end{aligned}$$

Theorem

$$\begin{aligned} \sum_{n \geq 0} P_n(u_1, u_2, u_3, u_4 | \alpha, \beta) \frac{t^n}{n!} \\ = (1 + yF(x, y; t))^{\frac{\alpha+\beta}{2}} (1 + xF(x, y; t))^{\frac{\alpha+\beta}{2}} e^{\frac{1}{2}(\beta-\alpha)(u_3-u_4)t}, \end{aligned}$$

where $x + y = u_3 + u_4$, $xy = u_1 u_2$.



Kathy Q. Ji, The (α, β) -Eulerian polynomials and descent-Stirling statistics on permutations, submitted. [arXiv:2310.01053](https://arxiv.org/abs/2310.01053).

The generating function

Theorem

We have

$$\begin{aligned} & \sum_{n \geq 0} P_n(u_1, u_2, u_3, u_4 | \alpha, \beta) \frac{t^n}{n!} \\ &= e^{\frac{1}{2}(\beta - \alpha)(u_3 - u_4)t} \times \left(\cosh \left(\frac{t}{2} \sqrt{(u_3 + u_4)^2 - 4u_1 u_2} \right) \right. \\ & \quad \left. - \frac{u_3 + u_4}{\sqrt{(u_3 + u_4)^2 - 4u_1 u_2}} \sinh \left(\frac{t}{2} \sqrt{(u_3 + u_4)^2 - 4u_1 u_2} \right) \right)^{-(\alpha + \beta)}. \end{aligned}$$

When $\alpha = \beta = -1/2$

Theorem

For $n \geq 1$,

$$2^n \sum_{\sigma \in \mathfrak{S}_{n+1}} (u_1 u_2)^{M(\sigma)} u_3^{\text{rda}(\sigma)} u_4^{\text{ldd}(\sigma)} \left(-\frac{1}{2}\right)^{\text{LRmin}(\sigma) + \text{RLmin}(\sigma) - 2}$$
$$= \begin{cases} ((u_3 + u_4)^2 - 4u_1 u_2)^{\lfloor \frac{n}{2} \rfloor}, & \text{if } n \text{ is even,} \\ -(u_3 + u_4)((u_3 + u_4)^2 - 4u_1 u_2)^{\lfloor \frac{n}{2} \rfloor}, & \text{if } n \text{ is odd.} \end{cases}$$

When $\alpha + \beta = -1$ and $u_3 = u_4$

Theorem

When $\alpha + \beta = -1$ and for $n \geq 1$,

$$\sum_{\sigma \in \mathfrak{S}_{n+1}} u^{M(\sigma)} \alpha^{\text{LRmin}(\sigma)-1} \beta^{\text{RLmin}(\sigma)-1} = \begin{cases} (1-u)^{\lfloor \frac{n}{2} \rfloor}, & \text{if } n \text{ is even,} \\ -(1-u)^{\lfloor \frac{n}{2} \rfloor}, & \text{if } n \text{ is odd.} \end{cases}$$

Theorem

For $n \geq 1$,

$$\sum_{\sigma \in \mathfrak{S}_n} u^{L(\sigma)} (-1)^{\text{RLmin}(\sigma)} = \begin{cases} (1-u)^{\lfloor \frac{n}{2} \rfloor}, & \text{if } n \text{ is even,} \\ -(1-u)^{\lfloor \frac{n}{2} \rfloor}, & \text{if } n \text{ is odd.} \end{cases}$$

An extension of gf for the peak polynomials

Theorem

We have

$$\sum_{n \geq 0} \left(\sum_{\sigma \in \mathfrak{S}_{n+1}} u^{2M(\sigma)} v^{n-2M(\sigma)} \alpha^{\text{LRmin}(\sigma)-1} \beta^{\text{RLmin}(\sigma)-1} \right) \frac{t^n}{n!} \\ = \left(\frac{\sqrt{v^2 - u^2}}{\sqrt{v^2 - u^2} \cosh(t\sqrt{v^2 - u^2}) - v \sinh(t\sqrt{v^2 - u^2})} \right)^{\alpha+\beta}.$$

When $\alpha = \beta = 1$, we recover the gf for the peak polynomials derived by [F.N. David and D.E. Barton \(1962\)](#), [R.C. Entringer \(Duke Math. J. 1969\)](#), [S. Kitaev \(Discrete Appl. Math. 2007\)](#), [Y. Zhuang \(Adv. Appl. Math. 2017\)](#) respectively.

Theorem

We have

$$\sum_{n \geq 0} \left(\sum_{\sigma \in \mathfrak{S}_n} u^{2L(\sigma)} v^{n-2L(\sigma)} \beta^{\text{RLmin}(\sigma)} \right) \frac{t^n}{n!} \\ = \left(\frac{\sqrt{v^2 - u^2}}{\sqrt{v^2 - u^2} \cosh \left(t\sqrt{v^2 - u^2} \right) - v \sinh \left(t\sqrt{v^2 - u^2} \right)} \right)^\beta.$$

Setting $\beta = 1$, we recover the generating function for the left peak polynomials established by Gessel .

An extension of Elizalde-Noy's result (2003)

Theorem

We have

$$\sum_{n \geq 0} \left(\sum_{\sigma \in \mathfrak{S}_n} u^{\text{da}(\sigma)} \alpha^{\text{LRmin}(\sigma)} \right) \frac{t^n}{n!} = e^{\frac{\alpha(1-u)}{2} t} \times \left(\frac{\sqrt{(u+1)^2 - 4}}{\sqrt{(u+1)^2 - 4} \cosh\left(\frac{t}{2} \sqrt{(u+1)^2 - 4}\right) - (1+u) \sinh\left(\frac{t}{2} \sqrt{(u+1)^2 - 4}\right)} \right)^{\alpha}.$$



S. Elizalde and M. Noy, Consecutive patterns in permutations, Adv. in Appl. Math. 30 (2003) 110–125.

An extension of Zhuang's result (2017)

Theorem

We have

$$\sum_{n \geq 0} \left(\sum_{\sigma \in \mathfrak{S}_n} u^{\text{rda}(\sigma)} \beta^{\text{RLmin}(\sigma)} \right) \frac{t^n}{n!} = e^{\frac{\beta(u-1)}{2}t} \times$$
$$\left(\frac{\sqrt{(u+1)^2 - 4}}{\sqrt{(u+1)^2 - 4} \cosh\left(\frac{t}{2}\sqrt{(u+1)^2 - 4}\right) - (1+u) \sinh\left(\frac{t}{2}\sqrt{(u+1)^2 - 4}\right)} \right)^{\beta}.$$
$$\sum_{n \geq 0} \left(\sum_{\sigma \in \mathfrak{S}_{n+1}} u^{\text{rda}(\sigma)} \alpha^{\text{LRmin}(\sigma)-1} \beta^{\text{RLmin}(\sigma)-1} \right) \frac{t^n}{n!} = e^{\frac{(\beta-\alpha)(u-1)}{2}t} \times$$
$$\left(\frac{\sqrt{(u+1)^2 - 4}}{\sqrt{(u+1)^2 - 4} \cosh\left(\frac{t}{2}\sqrt{(u+1)^2 - 4}\right) - (1+u) \sinh\left(\frac{t}{2}\sqrt{(u+1)^2 - 4}\right)} \right)^{\alpha+\beta}.$$

Theorem (Ji, 2023)

$$\sum_{n \geq 0} P_n(u_1, u_2, u_3, u_4 | \alpha, \beta) \frac{t^n}{n!} \\ = (1 + yF(x, y; t))^{\frac{\alpha+\beta}{2}} (1 + xF(x, y; t))^{\frac{\alpha+\beta}{2}} e^{\frac{1}{2}(\beta-\alpha)(u_3-u_4)t},$$

where $x + y = u_3 + u_4$, $xy = u_1 u_2$.

Theorem (Carlitz and Scoville (Theorem 9))

$$\sum_{n \geq 0} A_n(x, y | \alpha, \beta) \frac{t^n}{n!} = (1 + xF(x, y; t))^\alpha (1 + yF(x, y; t))^\beta.$$

An extension of Stembridge's formula

Theorem

For $n \geq 1$,

$$(1) \sum_{\sigma \in \mathfrak{S}_n} (xy)^{M(\sigma)} \left(\frac{x+y}{2} \right)^{n-2M(\sigma)-1} \alpha^{\text{LRmin}(\sigma)-1} \beta^{\text{RLmin}(\sigma)-1}$$

$$= \sum_{\sigma \in \mathfrak{S}_n} x^{\text{des}(\sigma)} y^{n-\text{des}(\sigma)-1} \left(\frac{\alpha + \beta}{2} \right)^{\text{LRmin}(\sigma) + \text{RLmin}(\sigma) - 2}.$$

$$(2) \sum_{\sigma \in \mathfrak{S}_n} (xy)^{L(\sigma)} \left(\frac{x+y}{2} \right)^{n-2L(\sigma)} \beta^{\text{RLmin}(\sigma)}$$

$$= \sum_{\sigma \in \mathfrak{S}_{n+1}} x^{\text{des}(\sigma)} y^{n-\text{des}(\sigma)} \left(\frac{\beta}{2} \right)^{\text{LRmin}(\sigma) + \text{RLmin}(\sigma) - 2}.$$

γ -positivity of (α, β) -Eulerian polynomial

Recall that

$$A_n \left(x, y \middle| \frac{\alpha + \beta}{2}, \frac{\alpha + \beta}{2} \right) = \sum_{\sigma \in \mathfrak{S}_{n+1}} x^{\text{asc}(\sigma)} y^{\text{des}(\sigma)} \left(\frac{\alpha + \beta}{2} \right)^{\text{LRmax}(\sigma) + \text{RLmax}(\sigma) - 2}.$$

Theorem

Let

$$A_{n-1} \left(x, y \middle| \frac{\alpha + \beta}{2}, \frac{\alpha + \beta}{2} \right) = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} \gamma_{n,k}(\alpha, \beta) (xy)^k (x+y)^{n-2k-1}$$

Then

$$\gamma_{n,k}(\alpha, \beta) = \left(\frac{1}{2} \right)^{n-2k-1} \sum_{\sigma \in \Gamma_{n,k}} \alpha^{\text{LRmin}(\sigma)-1} \beta^{\text{RLmin}(\sigma)-1},$$

where $\Gamma_{n,k}$ denote the set of permutations on $[n]$ with k peaks.

When $\alpha = \beta$, the above theorem has been proved via constructing a new group action on permutations in the recent paper joint with Zhicong Lin.

Example

$$A_1 \left(x, y \middle| \frac{\alpha + \beta}{2}, \frac{\alpha + \beta}{2} \right) = \frac{\alpha + \beta}{2} (x + y),$$

$$A_2 \left(x, y \middle| \frac{\alpha + \beta}{2}, \frac{\alpha + \beta}{2} \right) = \left(\frac{\alpha + \beta}{2} \right)^2 (x + y)^2 + (\alpha + \beta)xy,$$

$$A_3 \left(x, y \middle| \frac{\alpha + \beta}{2}, \frac{\alpha + \beta}{2} \right) = \left(\frac{\alpha + \beta}{2} \right)^3 (x + y)^3 \\ + \left(6 \left(\frac{\alpha + \beta}{2} \right)^2 + 2 \left(\frac{\alpha + \beta}{2} \right) \right) xy(x + y),$$

$$A_4 \left(x, y \middle| \frac{\alpha + \beta}{2}, \frac{\alpha + \beta}{2} \right) = \left(\frac{\alpha + \beta}{2} \right)^4 (x + y)^4 \\ + \frac{1}{4} (6(\alpha + \beta)^3 + 8(\alpha + \beta)^2 + 4(\alpha + \beta)) xy(x + y)^2 \\ + (3(\alpha + \beta)^2 + 2(\alpha + \beta)) (xy)^2.$$

An extension of the tangent number and the secant number

Theorem

Let $\mathfrak{S}_n^a$ denote the set of *up-down permutations* of $[n]$. For $n \geq 1$,

$$\begin{aligned} & \sum_{\sigma \in \mathfrak{S}_n} (-1)^{\text{des}(\sigma)} \left(\frac{\alpha + \beta}{2} \right)^{\text{LRmin}(\sigma) + \text{RLmin}(\sigma) - 2} \\ &= \begin{cases} (-1)^{\frac{n-1}{2}} \sum_{\sigma \in \mathfrak{S}_n^a} \alpha^{\text{LRmin}(\sigma)-1} \beta^{\text{RLmin}(\sigma)-1}, & \text{if } n \text{ is odd,} \\ 0, & \text{if } n \text{ is even.} \end{cases} \end{aligned}$$

Let $\mathfrak{S}_n^a$ denote the set of *down-up permutations* of $[n]$. For $n \geq 1$,

$$\begin{aligned} & \sum_{\sigma \in \mathfrak{S}_{n+1}} (-1)^{\text{des}(\sigma)} \left(\frac{\beta}{2} \right)^{\text{LRmin}(\sigma) + \text{RLmin}(\sigma) - 2} \\ &= \begin{cases} (-1)^{\frac{n}{2}} \sum_{\sigma \in \mathfrak{S}_n^a} \beta^{\text{RLmin}(\sigma)}, & \text{if } n \text{ is even,} \\ 0, & \text{if } n \text{ is odd.} \end{cases} \end{aligned}$$

Revisit to the secant number

Theorem

$$\sum_{\sigma \in \mathfrak{S}_{n+1}} (-1)^{\text{des}(\sigma)} \left(\frac{1}{2}\right)^{\text{LRmin}(\sigma) + \text{RLmin}(\sigma) - 2} = \begin{cases} (-1)^{\frac{n}{2}} E_n, & \text{if } n \text{ is even,} \\ 0, & \text{if } n \text{ is odd.} \end{cases}$$

Comparing with Roselle's result, we have

$$\sum_{\sigma \in \mathfrak{S}_{n+1}} (-1)^{\text{des}(\sigma)} \left(\frac{1}{2}\right)^{\text{LRmin}(\sigma) + \text{RLmin}(\sigma) - 2} = \sum_{\sigma \in \mathfrak{D}_n} (-1)^{\text{exc}(\sigma)}.$$

Remark. It would be interesting to give a combinatorial proof of the above identity.

An extension of Zhuang's relation (2017)

Theorem

For $n \geq 1$,

$$\begin{aligned} & \sum_{\sigma \in \mathfrak{S}_n} u^{M(\sigma)} v^{\text{des}(\sigma)} w^{\text{asc}(\sigma)} \alpha^{\text{LRmin}(\sigma) + \text{RLmin}(\sigma) - 2} \\ &= \sum_{\sigma \in \mathfrak{S}_n} x^{\text{des}(\sigma)} y^{\text{asc}(\sigma)} \alpha^{\text{LRmin}(\sigma) + \text{RLmin}(\sigma) - 2}, \end{aligned}$$

where

$$x = \frac{(w + v) - \sqrt{(w + v)^2 - 4uvw}}{2},$$

and

$$y = \frac{(w + v) + \sqrt{(w + v)^2 - 4uvw}}{2}.$$

This relation has been proved via constructing a new group action on permutations in the recent paper joint with Zhicong Lin.



Kathy Q. Ji and Zhicong Lin, The binomial-Stirling-Eulerian polynomials, submitted. arXiv: 2310.04969.

An extension of Peterson's formula

Theorem

$$\begin{aligned}(1) \quad & \sum_{\sigma \in \mathfrak{S}_n} (xy)^{L(\sigma)} \left(\frac{x+y}{2} \right)^{n-2L(\sigma)} \beta^{\text{RLmin}(\sigma)} \\ &= \sum_{k=1}^n \binom{n}{k} \frac{(\beta(y-x))^{n-k}}{2^{n-k}} \left(\sum_{\sigma \in \mathfrak{S}_k} x^{\text{des}(\sigma)+1} y^{\text{asc}(\sigma)} \beta^{\text{LRmin}(\sigma)} \right) + \left(\frac{(\beta(y-x))}{2} \right)^n \\ (2) \quad & \sum_{\sigma \in \mathfrak{S}_{n+1}} (xy)^{M(\sigma)} \left(\frac{x+y}{2} \right)^{n-2M(\sigma)} \alpha^{\text{LRmin}(\sigma)-1} \beta^{\text{RLmin}(\sigma)-1} \\ &= \sum_{k=1}^n \binom{n}{k} \frac{((\alpha+\beta)(y-x))^{n-k}}{2^{n-k}} \left(\sum_{\sigma \in \mathfrak{S}_k} x^{\text{des}(\sigma)+1} y^{\text{asc}(\sigma)} (\alpha+\beta)^{\text{LRmin}(\sigma)} \right) \\ & \quad + \left(\frac{((\alpha+\beta)(y-x))}{2} \right)^n.\end{aligned}$$

Context-free grammar

As an example, let's consider the grammar

$$G = \{x \rightarrow xy, y \rightarrow xy\}.$$

let D_G be the formal derivative with respect to the above grammar G , that is,

$$D(x) = xy, \quad D(y) = xy.$$

Thus, we have

$$\begin{aligned} D^2(x) &= D(xy) = D(x)y + xD(y) = xy^2 + x^2y \\ D(xy^2) &= D(x)y^2 + 2xyD(y) = xy^3 + 2x^2y^2 \\ D(x^2y) &= 2xyD(x) + x^2D(y) = 2x^2y^2 + x^3y \end{aligned}$$

Hence we get

$$D^3(x) = xy^3 + 4x^2y^2 + x^3y.$$



W.Y.C. Chen, Context-free grammars, differential operators and formal power series, Theoret. Comput. Sci. 117 (1993) 113–129.

In general, Dumont showed that

$$D^n(x) = A_n(x, y) = \sum_{\sigma \in \mathfrak{S}_n} x^{\text{des}(\sigma)+1} y^{\text{asc}(\sigma)+1}.$$

$$n = 1, \quad 0 \text{ } y \text{ } 1 \text{ } x \text{ } 0$$

$$n = 2, \quad 0 \text{ } y \text{ } 1 \text{ } y \text{ } \boxed{2} \text{ } x \text{ } 0, \quad 0 \text{ } y \text{ } \boxed{2} \text{ } x \text{ } 1 \text{ } x \text{ } 0$$

$$n = 3, \quad 0 \text{ } y \text{ } \boxed{3} \text{ } x \text{ } 1 \text{ } y \text{ } 2 \text{ } x \text{ } 0, \quad 0 \text{ } y \text{ } 1 \text{ } y \text{ } \boxed{3} \text{ } x \text{ } 2 \text{ } x \text{ } 0, \quad 0 \text{ } y \text{ } 1 \text{ } y \text{ } 2 \text{ } y \text{ } \boxed{3} \text{ } x \text{ } 0, \\ 0 \text{ } y \text{ } \boxed{3} \text{ } x \text{ } 2 \text{ } x \text{ } 1 \text{ } x \text{ } 0, \quad 0 \text{ } y \text{ } 2 \text{ } y \text{ } \boxed{3} \text{ } x \text{ } 1 \text{ } x \text{ } 0, \quad 0 \text{ } y \text{ } 2 \text{ } x \text{ } 1 \text{ } y \text{ } \boxed{3} \text{ } x \text{ } 0$$

$$D(x) = xy$$

$$D^2(x) = xy^2 + x^2y$$

$$D(xy^2) = xy^3 + 2x^2y^2$$

$$D(x^2y) = 2x^2y^2 + x^3y$$

$$D^3(x) = xy^3 + 4x^2y^2 + x^3y$$

Observe that

$$D(u + v) = D(u) + D(v) \quad D(uv) = D(u)v + uD(v).$$

Hence, it obeys the Leibniz's rule

$$D^n(uv) = \sum_{k=0}^n \binom{n}{k} D^k(u) D^{n-k}(v).$$

For a constant c , we have $D(c) = 0$.

Let

$$\text{Gen}(w, t) = \sum_{n \geq 0} D^n(w) \frac{t^n}{n!}.$$

Then, by the above two relations, we derive that

$$\text{Gen}(u + v, t) = \text{Gen}(u, t) + \text{Gen}(v, t).$$

$$\text{Gen}(uv, t) = \text{Gen}(u, t) \text{Gen}(v, t).$$

Theorem (Chen-Fu)

Let D_G be the formal derivative with respect to the grammar G :

$$G = \{x \rightarrow xy, y \rightarrow xy\}.$$

Then

$$\text{Gen}^{(G)}(x, t) := \sum_{n \geq 0} D_G^n(x) \frac{t^n}{n!} = x(1 + yF(x, y; t)).$$

Since $D(x) = xy$ and $D(y) = xy$, we see that $D(x - y) = 0$

$$D(x^{-1}) = -x^{-2}D(x) = -x^{-1}y,$$

$$D(\underline{x^{-1}y}) = -x^{-1}y^2 + y = (x - y)\underline{x^{-1}y}.$$

Hence,

$$D^n(\underline{x^{-1}y}) = (x - y)^n x^{-1}y.$$

$$D^n(\underline{x^{-1}y}) = (x - y)^n x^{-1}y$$

$$\begin{aligned} \text{Gen}(x^{-1}, t) &= \sum_{n \geq 0} D^n(x^{-1}) \frac{t^n}{n!} \\ &= x^{-1} - \sum_{n \geq 1} D^{n-1}(x^{-1}y) \frac{t^n}{n!} \\ &= \frac{1 - x^{-1}ye^{(x-y)t}}{x - y} \end{aligned}$$

By $\text{Gen}(x, t)\text{Gen}(x^{-1}, t) = 1$, we get

$$\begin{aligned} \text{Gen}(x, t) &= \frac{x - y}{1 - x^{-1}ye^{(x-y)t}} \\ &= x(1 + yF(x, y; t)). \end{aligned}$$



W.Y.C. Chen and A.M. Fu, A Context-free grammar for the e-positivity of the trivariate second-order Eulerian polynomials, Discrete Math. 345(1) (2022) 112661.

Using the similar argument, it can be proved that

$$\text{Gen}(y, t) = y(1 + xF(x, y; t)).$$

It follows that

$$\text{Gen}(xy, t) = xy(1 + xF(x, y; t))(1 + yF(x, y; t))$$

A little trick

Theorem

Let $D_{\tilde{G}}$ be the formal derivative with respect to the grammar $\tilde{G}$:

$$\tilde{G} = \{a \rightarrow \alpha ay, x \rightarrow xy, y \rightarrow xy\}.$$

Then

$$\text{Gen}^{\tilde{G}}(a, t) := \sum_{n \geq 0} D_{\tilde{G}}^n(a) \frac{t^n}{n!} = a(1 + yF(x, y; t))^\alpha.$$

Recall that D_G is the formal derivative with respect to the grammar G :

$$G = \{x \rightarrow xy, y \rightarrow xy\}.$$

Then

$$D_G(x^\alpha) = \alpha x^{\alpha-1} y.$$

Set $a = x^\alpha$, we find that

$$D_{\tilde{G}}(a) = D_G(x^\alpha)$$

Observe that $D_{\tilde{G}}(x) = D_{\tilde{G}}(x)$ and $D_{\tilde{G}}(y) = D_{\tilde{G}}(y)$, we can use the induction on n to deduce that for $n \geq 0$,

$$D_{\tilde{G}}^n(a) = D_G^n(x^\alpha).$$

It follows that

$$\begin{aligned}\text{Gen}^{\tilde{G}}(a, t) &= \text{Gen}^G(x^\alpha, t) \\ &= x^\alpha (1 + yF(x, y; t))^\alpha \\ &= a(1 + yF(x, y; t))^\alpha.\end{aligned}$$

$$\tilde{G} = \{a \rightarrow \alpha ay, x \rightarrow xy, y \rightarrow xy\}.$$

$$D_G^n(a) = a \sum_{\sigma \in \mathfrak{S}_n} y^{\text{des}(\sigma)+1} x^{\text{asc}(\sigma)} \alpha^{LRmin(\sigma)}.$$

$$n = 1,$$

$$0 \mathbf{a} 1^\alpha y 0$$

$$n = 2,$$

$$0 \mathbf{a} 1^\alpha x \boxed{2} y 0, \quad 0 \mathbf{a} \boxed{2}^\alpha y 1^\alpha y 0$$

$$n = 3,$$

$$0 \mathbf{a} \boxed{3}^\alpha y 1^\alpha x 2 y 0, \quad 0 \mathbf{a} 1^\alpha x \boxed{3} y 2 y 0, \quad 0 \mathbf{a} 1^\alpha x 2 x \boxed{3} y 0, \\ 0 \mathbf{a} \boxed{3}^\alpha y 2^\alpha y 1^\alpha y 0, \quad 0 \mathbf{a} 2^\alpha x \boxed{3} y 1^\alpha y 0, \quad 0 \mathbf{a} 2^\alpha y 1^\alpha x \boxed{3} y 0$$

$$D_{\tilde{G}}(a) = a\alpha y$$

$$D_{\tilde{G}}^2(a) = a(\alpha^2 y^2 + \alpha xy)$$

$$D_{\tilde{G}}^3(a)(x) = a(\alpha^3 y^3 + 3\alpha^2 xy^2 + \alpha xy^2 + \alpha x^2 y).$$

The outline of the proof

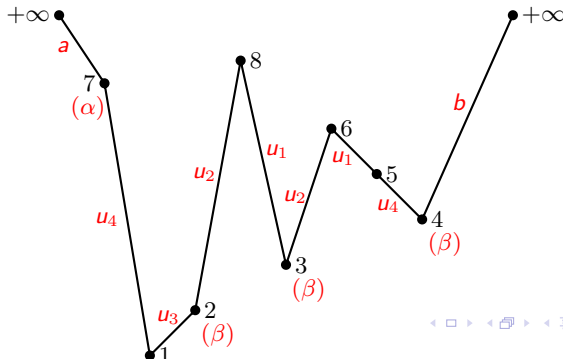
Theorem

Let $D_{\tilde{G}}$ be the formal derivative with respect to the following grammar

$$\tilde{G} = \{a \rightarrow a\alpha u_4, b \rightarrow b\beta u_3, u_4 \rightarrow u_1 u_2, u_3 \rightarrow u_1 u_2, u_1 \rightarrow u_1 u_3, u_2 \rightarrow u_2 u_4\}.$$

Then

$$D_{\tilde{G}}^n(ab) = abP_n(u_1, u_2, u_3, u_4 | \alpha, \beta) = \sum_{\sigma \in \mathfrak{S}_{n+1}} (u_1 u_2)^{M(\sigma)} u_3^{\text{rda}(\sigma)} u_4^{\text{ladd}(\sigma)} \alpha^{\text{LRmin}(\sigma)-1} \beta^{\text{RLmin}(\sigma)}$$



Let $D_{\tilde{G}_1}$ is the formal derivative with respect to the grammar

$$\tilde{G}_1 = \{u_4 \rightarrow u_1 u_2, u_3 \rightarrow u_1 u_2, u_1 \rightarrow u_1 u_3, u_2 \rightarrow u_2 u_4\}.$$

We first show that

$$\begin{aligned}\text{Gen}^{(\tilde{G}_1)}(u_1 u_2, t) &:= \sum_{n \geq 0} D_{\tilde{G}_1}^n(u_1 u_2) \frac{t^n}{n!} \\ &= xy(1 + yF(x, y; t))(1 + xF(x, y; t))\end{aligned}$$

and

$$\begin{aligned}\text{Gen}^{(\tilde{G}_1)}(u_1^2, t) &:= \text{Gen}^{(\tilde{G}_1)}(u_1 u_2^{-1}, t) \text{Gen}^{(\tilde{G}_1)}(u_1 u_2, t) \\ &= u_1^2(1 + yF(x, y; t))(1 + xF(x, y; t))e^{(u_3 - u_4)t},\end{aligned}$$

where $x + y = u_3 + u_4$ and $xy = u_1 u_2$.

$$\begin{aligned}
& \text{Gen}^{(\tilde{G})}(ab, t) \\
&= \text{Gen}^{(\tilde{G}_1)}(u_2^\alpha u_1^\beta, t) \\
&= (\text{Gen}^{(\tilde{G}_1)}(u_2 u_1, t))^\alpha (\text{Gen}^{(\tilde{G}_1)}(u_1, t))^{\beta-\alpha} \\
&= u_1^\beta u_2^\alpha (1 + yF(x, y; t))^{\frac{\alpha+\beta}{2}} (1 + xF(x, y; t))^{\frac{\alpha+\beta}{2}} e^{\frac{1}{2}(\beta-\alpha)(u_3-u_4)t} \\
&= ab (1 + yF(x, y; t))^{\frac{\alpha+\beta}{2}} (1 + xF(x, y; t))^{\frac{\alpha+\beta}{2}} e^{\frac{1}{2}(\beta-\alpha)(u_3-u_4)t}
\end{aligned}$$

as desired.

Main result

Let

$$\begin{aligned} P_n(u_1, u_2, u_3, u_4 | \alpha, \beta) \\ = \sum_{\sigma \in \mathfrak{S}_{n+1}} (u_1 u_2)^{M(\sigma)} u_3^{\text{rda}(\sigma)} u_4^{\text{ldd}(\sigma)} \alpha^{\text{LRmin}(\sigma)-1} \beta^{\text{RLmin}(\sigma)-1}. \end{aligned}$$

Theorem

$$\begin{aligned} \sum_{n \geq 0} P_n(u_1, u_2, u_3, u_4 | \alpha, \beta) \frac{t^n}{n!} \\ = (1 + yF(x, y; t))^{\frac{\alpha+\beta}{2}} (1 + xF(x, y; t))^{\frac{\alpha+\beta}{2}} e^{\frac{1}{2}(\beta-\alpha)(u_3-u_4)t}, \end{aligned}$$

where $x + y = u_3 + u_4$, $xy = u_1 u_2$.



Kathy Q. Ji, The (α, β) -Eulerian polynomials and descent-Stirling statistics on permutations, submitted. [arXiv:2310.01053](https://arxiv.org/abs/2310.01053)

Thank you!

人生也有涯, 而知也无涯. 心有所信, 方能行远.