The (α, β)-Eulerian Polynomials and Descent-Stirling Statistics on Permutations

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Abstract. Carlitz and Scoville introduced the polynomials $A_n(x, y \mid \alpha, \beta)$, which we refer to as the (α, β)-Eulerian polynomials. These polynomials count permutations based on Eulerian-Stirling statistics, including descents, ascents, left-to-right maxima, and right-to-left maxima. Carlitz and Scoville obtained the generating function for $A_n(x, y \mid \alpha, \beta)$. In this paper, we introduce a new family of polynomials, $P_n(u, v, w, z \mid \alpha, \beta)$, defined on descent-Stirling statistics of permutations including valleys, exterior peaks, right double descents, left double ascents, left-to-right maxima, and right-to-left maxima. By employing the grammatical calculus introduced by Chen, we establish a connection between the generating function for $P_n(u, v, w, z \mid \alpha, \beta)$ and the generating function for $A_n(x, y \mid \alpha, \beta)$. Using this connection, we derive the generating function for $P_n(u, v, w, z \mid \alpha, \beta)$, which can be specialized to obtain (α, β)-extensions of generating functions for peaks, left peaks, double ascents, right double ascents and left-right double ascents given by David and Barton, Elizalde and Noy, Entringer, Gessel, Kitaev, and Zhuang. Moreover, we establish two relations between $P_n(u, v, w, z \mid \alpha, \beta)$ and $A_n(x, y \mid \alpha, \beta)$, which enable us to derive (α, β)-extensions of results obtained by Stembridge, Petersen, Brändén, and Zhuang, respectively. We also establish the left peak version of Stembridge's formula and the peak version of Petersen's formula, along with their respective (α, β)-extensions, by utilizing these two relations. Specializing (α, β)-extensions of Stembridge's formula and the left peak version of Stembridge's formula allows us to derive (α, β)-extensions of the tangent and secant numbers.

Keywords: permutations, descents, ascents, peaks, left-to-right maxima, generating functions, the tangent and the secant numbers, context-free grammars

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1 Introduction

The objective of this paper is to investigate the polynomials involving descent-Stirling statistics of permutations. Let $\mathfrak{S}_n$ denote the set of permutations on $[n] := \{1, 2, \ldots, n\}$. We say that i is a descent of $\sigma = \sigma_1 \cdots \sigma_n \in \mathfrak{S}_n$ if $1 \leq i < n$ and $\sigma_i > \sigma_{i+1}$. Let $\mathrm{des}(\sigma)$

count the number of descents of σ. The Eulerian polynomials are defined as: For $n \geq 1$,

$$A_n(x) := \sum_{\sigma \in \mathfrak{S}_n} x^{\mathrm{des}(\sigma)+1}. \tag{1.1}$$

By convention, we set $A_0(x) = 1$. The generating function for $A_n(x)$ is well known:

$$\sum_{n \geq 0} A_n(x) \frac{t^n}{n!} = \frac{1-x}{1-xe^{(1-x)t}}. \tag{1.2}$$

The Eulerian polynomials carry a profound historical legacy and play a pivotal role across diverse combinatorial landscapes. For an extensive analysis, we refer to Petersen [33].

Let us recall Eulerian, Stirling and descent statistics on permutations. A permutation statistic whose generating function is given by (1.1) is called Eulerian. Let $\sigma = \sigma_1 \cdots \sigma_n \in \mathfrak{S}_n$. An index $1 \leq i < n$ for which $\sigma_i < \sigma_{i+1}$ is called an ascent of σ and an index $1 \leq i < n$ for which $\sigma_i > i$ is called an excedance of σ. Let $\mathrm{asc}(\sigma)$ denote the number of ascents of σ and let $\mathrm{exc}(\sigma)$ denote the number of excedances of σ. It is known that $\mathrm{asc}(\sigma)$ and $\mathrm{exc}(\sigma)$ are Eulerian statistics, see MacMahon [28, p.186] and Stanley [39]. Namely,

$$A_n(x) := \sum_{\sigma \in \mathfrak{S}_n} x^{\mathrm{des}(\sigma)+1} = \sum_{\sigma \in \mathfrak{S}_n} x^{\mathrm{asc}(\sigma)+1} = \sum_{\sigma \in \mathfrak{S}_n} x^{\mathrm{exc}(\sigma)+1}. \tag{1.3}$$

A permutation statistic is called a Stirling statistic if

$$\sum_{\sigma \in \mathfrak{S}_n} \alpha^{\mathrm{sst}(\sigma)} = \alpha(\alpha+1)(\alpha+2)\cdots(\alpha+n-1). \tag{1.4}$$

This is because this statistic shares the same generating function as the unsigned Stirling numbers of the first kind, see [39, Proposition 1.3.7]. Here we describe five Stirling statistics. The first one is the number of cycles in a decomposition of σ into disjoint cycles, including those of length 1, which is denoted $\mathrm{cyc}(\sigma)$. For the permutation $\sigma = 2\,7\,1\,8\,3\,6\,5\,4$, its cycle decomposition is $(1\,2\,7\,5\,3)\,(4\,8)\,(6)$, and so $\mathrm{cyc}(\sigma) = 3$. Let $\sigma = \sigma_1 \cdots \sigma_n \in \mathfrak{S}_n$. A left-to-right maximum (resp. a left-to-right minimum) of σ is an element σ_i such that $\sigma_j < \sigma_i$ (resp. $\sigma_j > \sigma_i$) for every $j < i$ and a right-to-left maximum (resp. a right-to-left minimum) of σ is an element σ_i such that $\sigma_j < \sigma_i$ (reps. $\sigma_j > \sigma_i$) for every $j > i$. Let $\mathrm{lrmax}(\sigma)$, $\mathrm{lrmin}(\sigma)$, $\mathrm{rlmax}(\sigma)$ and $\mathrm{rlmin}(\sigma)$ denote the number of left-to-right maxima, left-to-right minima, right-to-left maxima and right-to-left minima of σ, respectively. For $\sigma = 2\,7\,1\,8\,3\,6\,5\,4$, we see that

$$\mathrm{lrmax}(\sigma) = 3,\ \mathrm{lrmin}(\sigma) = 2,\ \mathrm{rlmax}(\sigma) = 4,\ \mathrm{rlmin}(\sigma) = 3.$$

It is well known that

$$\sum_{\sigma \in \mathfrak{S}_n} \alpha^{\mathrm{cyc}(\sigma)} = \sum_{\sigma \in \mathfrak{S}_n} \alpha^{\mathrm{lrmax}(\sigma)} = \sum_{\sigma \in \mathfrak{S}_n} \alpha^{\mathrm{rlmax}(\sigma)} = \sum_{\sigma \in \mathfrak{S}_n} \alpha^{\mathrm{lrmin}(\sigma)} = \sum_{\sigma \in \mathfrak{S}_n} \alpha^{\mathrm{rlmin}(\sigma)}$$
$$= \alpha(\alpha+1)(\alpha+2)\cdots(\alpha+n-1). \tag{1.5}$$

Carlitz and Scoville [6] considered the following polynomials involving Eulerian-Stirling statistics, which we refer to as the (α, β)-Eulerian polynomials: For $n \geq 1$,

$$A_n(x, y \mid \alpha, \beta) = \sum_{\sigma \in \mathfrak{S}_{n+1}} x^{\mathrm{asc}(\sigma)} y^{\mathrm{des}(\sigma)} \alpha^{\mathrm{lrmax}(\sigma)-1} \beta^{\mathrm{rlmax}(\sigma)-1}. \tag{1.6}$$

with the convention that $A_0(x, y \mid \alpha, \beta) = 1$.

The first few values of $A_n(x, y \mid \alpha, \beta)$ are given as follows:

$$\begin{aligned}
A_1(x, y \mid \alpha, \beta) &= \alpha x + \beta y; \\
A_2(x, y \mid \alpha, \beta) &= \alpha^2 x^2 + 2\,\alpha\beta y x + y\alpha x + y\beta x + \beta^2 y^2; \\
A_3(x, y \mid \alpha, \beta) &= \alpha^3 x^3 + 3\,\alpha^2 \beta y x^2 + 3\,\alpha^2 y x^2 + 3\,\alpha\beta y x^2 + \alpha y x^2 + \beta y x^2 \\
&\quad + 3\,\alpha\beta^2 y^2 x + 3\,\alpha\beta y^2 x + 3\,\beta^2 y^2 x + \alpha y^2 x + \beta y^2 x + \beta^3 y^3.
\end{aligned}$$

Carlitz and Scoville [6] obtained the following generating function for $A_n(x, y \mid \alpha, \beta)$:

Theorem 1.1. (Carlitz and Scoville [6, Theorem 9])

$$\sum_{n \geq 0} A_n(x, y \mid \alpha, \beta) \frac{t^n}{n!} = (1 + xF(x, y; t))^{\alpha} (1 + yF(x, y; t))^{\beta}, \tag{1.7}$$

where $F(x, y; t)$ is given by

$$F(x, y; t) = \frac{e^{xt} - e^{yt}}{xe^{yt} - ye^{xt}}. \tag{1.8}$$

Note that Carlitz and Scoville [6] used falls and rises for descents and ascents, respectively. They referred to left-to-right maxima and right-to-left maxima as left upper records and right upper records, respectively.

When $\alpha = 0$, $\beta = 1$ and $x = 1$, $A_n(x, y \mid \alpha, \beta)$ reduces to the classical Eulerian polynomials $A_n(y)$ given by (1.1). Accordingly, we recover the generating function (1.2) for $A_n(y)$ by setting $\alpha = 0$, $\beta = 1$ and $x = 1$ in (1.7).

When $x = y = 1$ and $\beta = 0$, it is not difficult to see that

$$A_n(1, 1 \mid \alpha, 0) = \sum_{\sigma \in \mathfrak{S}_n} \alpha^{\mathrm{lrmax}(\sigma)}, \tag{1.9}$$

and by (1.7), we see that

$$\sum_{n \geq 0} A_n(1, 1 \mid \alpha, 0) \frac{t^n}{n!} = \left(\frac{1}{1-t} \right)^{\alpha}. \tag{1.10}$$

Comparing the coefficients of $t^n/n!$ yields the generating function (1.5).

The (α,β)-Eulerian polynomials $A_n(x,y \mid \alpha,\beta)$ are also connected to a q-analogue of the Eulerian polynomials, introduced by Foata and Schützenberger [20] and later named after Brenti [5]:

$$A_n(x,q) = \sum_{\sigma\in\mathfrak{S}_n} x^{\mathrm{exc}(\sigma)} q^{\mathrm{cyc}(\sigma)}. \tag{1.11}$$

These q-Eulerian polynomials also incorporate Eulerian-Stirling statistics of permutations. They are also linked to the $1/k$-Eulerian polynomials, see Savage and Viswanathan [35] and Ma and Mansour [27] for example. Brenti [5] obtained the following generating function for $A_n(x,q)$ and showed that $A_n(x,q)$ is log-concave and unimodal.

$$1 + \sum_{n\geq 1} A_n(x,q)\frac{t^n}{n!} = \left(\frac{e^{t(x-1)} - x}{1-x}\right)^{-q}. \tag{1.12}$$

Due to the first fundamental transformation of Foata and Schützenberger [20], the q-Eulerian polynomial (1.11) can be seen as a special case of (α,β)-Eulerian polynomials $A_n(x,y \mid \alpha,\beta)$. More precisely, we have

$$A_n(x,q) = \sum_{\sigma\in\mathfrak{S}_n} x^{\mathrm{exc}(\sigma)} q^{\mathrm{cyc}(\sigma)} = \sum_{\sigma\in\mathfrak{S}_n} x^{\mathrm{des}(\sigma)} q^{\mathrm{lrmax}(\sigma)} = A_n(1,x \mid q,0).$$

Thus, the generating function (1.12) for $A_n(x,q)$ can be derived by setting $x=1$ and $\beta=0$ in (1.7), and replacing y and α with x and q, respectively.

The descent statistics are related to the Eulerian statistics, which are permutation statistics that depend only on the descent and length of a permutation, see Gessel and Zhuang[23] and Zhuang [42, 43]. The classical descent statistics include variations of peaks and valleys, double ascents and double descents. In this paper, we adopt the terminology for these descent statistics provided by Gessel and Zhuang [23] and Zhuang [42, 43]. For the detailed definitions of these descent statistics, see Section 2.

Carlitz and Scoville [6] considered the generating function for the polynomials based on the joint distribution of the number of exterior peaks, the number of descents and the number of ascents. Note that exterior peaks are referred to maxima by Carlitz and Scoville [6].

Theorem 1.2. (Carlitz and Scoville [6, Theorem 2]) *Let $W(\sigma)$ denote the number of exterior peaks of σ (see Definition 2.1). Then*

$$\sum_{n\geq 0}\left(\sum_{\sigma\in\mathfrak{S}_{n+1}} p^{W(\sigma)-1} q^{\mathrm{des}(\sigma)} r^{\mathrm{asc}(\sigma)}\right)\frac{t^n}{n!} = (1+yF(x,y;t))(1+xF(x,y;t)), \tag{1.13}$$

where $F(x,y;t)$ is given by (1.8) and

$$x = \frac{(r+q)+\sqrt{(r+r)^2-4pqr}}{2}, \quad y = \frac{(r+q)-\sqrt{(r+q)^2-4pqr}}{2}. \tag{1.14}$$

Goulden and Jackson [24, Exercise 3.3.46] and Stanley [39, Exercise 1.61] reformulated Carlitz and Scoville's result in terms of left double ascents and right double descents, see Definitions 2.8 and 2.9. Note that Goulden and Jackson [24, Exercise 3.3.46] and Stanley [39, Exercise 1.61] referred to left double ascents as double rises, and right double descents as double falls. Valleys are called modified minima by Goulden and Jackson [24, Exercise 3.3.46].

Let $V(\sigma)$, $\mathrm{lda}(\sigma)$ and $\mathrm{rdd}(\sigma)$ denote the number of valleys, left double ascents, right double descents, respectively (see Section 2). Goulden and Jackson [24, Exercise 3.3.46] and Stanley [39, Exercise 1.61] reformulated Theorem 1.2 as

$$\sum_{n\geq 1}\left(\sum_{\sigma\in\mathfrak{S}_n} u^{\mathrm{V}(\sigma)}v^{W(\sigma)-1}w^{\mathrm{lda}(\sigma)}z^{\mathrm{rdd}(\sigma)}\right)\frac{t^n}{n!}=F(x,y;t), \tag{1.15}$$

where $x+y=w+z$ and $xy=uv$.

Fu [22] provided a grammatical proof of (1.15). Pan and Zeng [31] provided *inv* q-analogue of (1.15). Differentiating both sides of (1.15) with respect to t, we have

Theorem 1.3. (Carlitz and Scoville II [6, Theorem 2]) *For $n\geq 1$, define*

$$P_n(u,v,w,z)=\sum_{\sigma\in\mathfrak{S}_{n+1}} u^{\mathrm{V}(\sigma)}v^{W(\sigma)-1}w^{\mathrm{lda}(\sigma)}z^{\mathrm{rdd}(\sigma)}. \tag{1.16}$$

By convention, we set $P_0(u,v,w,z)=1$. Then

$$\sum_{n\geq 0} P_n(u,v,w,z)\frac{t^n}{n!}=\left(1+yF(x,y;t)\right)\left(1+xF(x,y;t)\right), \tag{1.17}$$

where $x+y=w+z$ and $xy=uv$.

Setting $u=z=r$, $v=pq$ and $w=q$ in Theorem 1.3, and using (2.10) and (2.16), one recovers Theorem 1.2.

The main objective of this paper is to investigate the following polynomial, which involves descent-Stirling statistics. For $n\geq 1$,

$$P_n(u,v,w,z\mid\alpha,\beta)=\sum_{\sigma\in\mathfrak{S}_{n+1}} u^{\mathrm{V}(\sigma)}v^{W(\sigma)-1}w^{\mathrm{rdd}(\sigma)}z^{\mathrm{lda}(\sigma)}\alpha^{\mathrm{lrmax}(\sigma)-1}\beta^{\mathrm{rlmax}(\sigma)-1}. \tag{1.18}$$

As a convention, we set $P_0(u,v,w,z\mid\alpha,\beta)=1$.

The first few values of $P_n(u,v,w,z\mid\alpha,\beta)$ are given as follows:

$$P_1(u,v,w,z\mid\alpha,\beta)=\alpha z+\beta w;$$

$$P_2(u,v,w,z\mid\alpha,\beta)=\alpha^2z^2+2\,\alpha\beta wz+\beta^2w^2+\alpha vu+\beta vu;$$

$$\begin{aligned}P_3(u,v,w,z\mid\alpha,\beta)&=\alpha^3z^3+3\,\alpha^2\beta wz^2+3\,\alpha\beta^2w^2z+3\,\alpha\beta uvz+3\,\alpha^2uvz+\alpha uvz\\&\quad+\beta uvz+\beta^3w^3+3\,\beta^2uvw+\alpha uvw+\beta uvw+3\,\alpha\beta uvw.\end{aligned}$$

Analogous to (α, β)-Eulerian polynomials, we refer to the polynomials $P_n(u, v, w, z \mid \alpha, \beta)$ as (α, β)-extensions of the polynomials $P_n(u, v, w, z)$. We will establish the following connection between the generating function for $P_n(u, v, w, z \mid \alpha, \beta)$ and the generating function (1.6) for $A_n(x, y \mid \alpha, \beta)$.

Theorem 1.4. *We have*

$$\sum_{n\geq 0} P_n(u,v,w,z \mid \alpha,\beta)\frac{t^n}{n!} = (1+yF(x,y;t))^{\frac{\alpha+\beta}{2}}\,(1+xF(x,y;t))^{\frac{\alpha+\beta}{2}}\,e^{\frac{1}{2}(\beta-\alpha)(w-z)t}, \tag{1.19}$$

where $x + y = w + z$, $xy = uv$ and $F(x, y; t)$ is given by (1.8).

When $\alpha = \beta = 1$ in Theorem 1.4, we recover Theorem 1.3.

The main technique utilized in this paper is the grammatical approach introduced by Chen [7]. We establish the grammar for $A_n(x, y \mid \alpha, \beta)$ (see Theorem 3.1) and the grammar for $P_n(u, v, w, z \mid \alpha, \beta)$ (see Theorem 3.2). We then derive Theorems 1.1 and 1.4 based on their grammars. The derivations of the generating functions for $A_n(x, y \mid \alpha, \beta)$ and $P_n(u, v, w, z \mid \alpha, \beta)$ become straightforward once their grammars are established. More recently, Xu and Zeng [41] provided a generalization of Theorem 1.4 by employing Foata's fundamental transformation and a cyclic analogue of valley-hopping.

Many consequences can be derived from Theorem 1.4. We first obtain an explicit form of the generating function for $P_n(u, v, w, z \mid \alpha, \beta)$ (see Theorem 4.1). It turns out that this generating function can be specialized to establish (α, β)-extensions of the generating functions for peaks, left peaks, double ascents, right double ascents and left-right double ascents (see Theorems 4.2, 4.3 and 4.4). For more detailed explanations on the generating functions for these statistics, see Section 2. Moreover, the generating function for $P_n(u, v, w, z \mid \alpha, \beta)$ enables us to derive the following explicit expression for $P_n(u, v, w, z \mid \alpha, \beta)$ when $\alpha + \beta = -1$. In particular, we obtain the following interesting enumerative consequences:

Theorem 1.5. *Let $M(\sigma)$ denote the number of peaks of σ. When $\alpha + \beta = -1$ and for $n \geq 1$,*

$$\sum_{\sigma\in\mathfrak{S}_{n+1}} u^{M(\sigma)}\alpha^{\mathrm{lrmin}(\sigma)-1}\beta^{\mathrm{rlmin}(\sigma)-1} = \begin{cases} (1-u)^{\lfloor \frac{n}{2}\rfloor}, & \text{if } n \text{ is even},\\ -(1-u)^{\lfloor \frac{n}{2}\rfloor}, & \text{if } n \text{ is odd}.\end{cases}$$

Theorem 1.6. *Let $L(\sigma)$ denote the number of left peaks of σ. For $n \geq 1$,*

$$\sum_{\sigma\in\mathfrak{S}_{n}} u^{L(\sigma)}(-1)^{\mathrm{rlmin}(\sigma)} = \begin{cases} (1-u)^{\lfloor \frac{n}{2}\rfloor}, & \text{if } n \text{ is even},\\ -(1-u)^{\lfloor \frac{n}{2}\rfloor}, & \text{if } n \text{ is odd}.\end{cases}$$

Theorem 1.7. *Let* $\mathrm{rda}(\sigma)$ *denote the number of right double ascents of* σ*. For* $n \geq 1$*,*

$$2^n \sum_{\sigma\in\mathfrak{S}_{n+1}} u^{\mathrm{rda}(\sigma)} \left(-\frac{1}{2}\right)^{\mathrm{lrmin}(\sigma)+\mathrm{rlmin}(\sigma)-2}$$
$$= \begin{cases} ((1+u)^2-4)^{\lfloor \frac{n}{2} \rfloor}, & \text{if } n \text{ is even,} \\ -(1+u)((1+u)^2-4)^{\lfloor \frac{n}{2} \rfloor}, & \text{if } n \text{ is odd.} \end{cases}$$

By combining Theorems 1.1 and 1.4, we establish two relations between $P_n(u,v,w,z \mid \alpha,\beta)$ and $A_n(x,y \mid \alpha,\beta)$ (see Theorems 6.1 and 6.2). These two relations allow us to derive (α,β)-extensions of formulas originally due to Stembridge, Petersen, Brändén and Zhuang (see Theorems 1.8, 6.4, 6.6 and 6.9). Additionally, we obtain the left peak version of Stembridge's formula and the peak version of Petersen's formula (see Theorems 6.3 and 6.7), along with their (α,β)-extensions (see Theorems 1.9 and 6.5). Here we would like to single out the following two consequences, which can be viewed as the (α,β)-extensions of Stembridge's formula and its left peak counterpart.

Theorem 1.8. *For* $n \geq 1$*,*

$$\sum_{\sigma\in\mathfrak{S}_n} (xy)^{M(\sigma)} \left(\frac{x+y}{2}\right)^{n-2M(\sigma)-1} \alpha^{\mathrm{lrmin}(\sigma)-1}\beta^{\mathrm{rlmin}(\sigma)-1}$$
$$= \sum_{\sigma\in\mathfrak{S}_n} x^{\mathrm{des}(\sigma)} y^{n-\mathrm{des}(\sigma)-1} \left(\frac{\alpha+\beta}{2}\right)^{\mathrm{lrmin}(\sigma)+\mathrm{rlmin}(\sigma)-2}, \tag{1.20}$$

where $M(\sigma)$ *counts the number of peaks of* σ*.*

Theorem 1.9. *For* $n \geq 1$*,*

$$\sum_{\sigma\in\mathfrak{S}_n} (xy)^{L(\sigma)} \left(\frac{x+y}{2}\right)^{n-2L(\sigma)} \beta^{\mathrm{rlmin}(\sigma)}$$
$$= \sum_{\sigma\in\mathfrak{S}_{n+1}} x^{\mathrm{des}(\sigma)} y^{n-\mathrm{des}(\sigma)} \left(\frac{\beta}{2}\right)^{\mathrm{lrmin}(\sigma)+\mathrm{rlmin}(\sigma)-2}, \tag{1.21}$$

where $L(\sigma)$ *counts the number of left peaks of* σ*.*

By setting $\alpha = \beta = 1$ in Theorem 1.8, we recover a formula of Stembridge [40] (see (2.4)), which is known to lead to the following γ-expansion of the Eulerian polynomials:

Theorem 1.10. *For* $n \geq 1$*,*

$$\sum_{\sigma\in\mathfrak{S}_n} x^{\mathrm{des}(\sigma)} = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} 2^{-n+2k+1}\gamma_{n,k} x^k (1+x)^{n-2k-1}, \tag{1.22}$$

where $\gamma_{n,k}$ *counts the number of permutations in* $\mathfrak{S}_n$ *with* k *peaks.*

Note that this result was discovered by several authors, see, for example, Carlitz and Scoville [6], Foata and Schützenberger [20], Foata and Strehl [21] and Shapiro, Woan and Getu [36]. Concerning more γ-positive polynomials arising in enumerative and geometric combinatorics, we refer to two surveys by Athanasiadis [2] and Brändén [4], respectively, and the book by Petersen [33].

In the same vein, Theorem 1.8 can also be reformulated in the following form, which recovers Theorem 1.10 when $\alpha = \beta = 1$ and $y = 1$.

Corollary 1.11. *Let $\gamma_{n,k}$ denote the set of permutations in $\mathfrak{S}_n$ with k peaks. For $n \geq 1$,*

$$A_{n-1}\left(x, y \mid \frac{\alpha+\beta}{2}, \frac{\alpha+\beta}{2}\right) = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} \gamma_{n,k}(\alpha,\beta)(xy)^k(x+y)^{n-2k-1},$$

where

$$\gamma_{n,k}(\alpha,\beta) = \left(\frac{1}{2}\right)^{n-2k-1} \sum_{\sigma \in \Gamma_{n,k}} \alpha^{\mathrm{lrmin}(\sigma)-1}\beta^{\mathrm{rlmin}(\sigma)-1}.$$

The first few γ-expansions of $A_n\left(x, y \mid \frac{\alpha+\beta}{2}, \frac{\alpha+\beta}{2}\right)$ read as

$$\begin{aligned}
A_1\left(x, y \mid \frac{\alpha+\beta}{2}, \frac{\alpha+\beta}{2}\right) &= \frac{\alpha+\beta}{2}(x+y), \\
A_2\left(x, y \mid \frac{\alpha+\beta}{2}, \frac{\alpha+\beta}{2}\right) &= \left(\frac{\alpha+\beta}{2}\right)^2 (x+y)^2 + (\alpha+\beta)xy, \\
A_3\left(x, y \mid \frac{\alpha+\beta}{2}, \frac{\alpha+\beta}{2}\right) &= \left(\frac{\alpha+\beta}{2}\right)^3 (x+y)^3 \\
&\quad + \left(6\left(\frac{\alpha+\beta}{2}\right)^2 + 2\left(\frac{\alpha+\beta}{2}\right)\right)xy(x+y),
\end{aligned}$$

When $\alpha = \beta$, the above result was established through the construction of a new group action on permutations by the author and Lin [29]. The general case was more recently proven by Chen and Fu [12] and Dong, Lin and Pan [14]. Notably, Chen and Fu [12] offered a novel proof of the γ-positivity for this variant of Eulerian polynomials by decomposing an increasing binary tree into a forest.

Specializing Theorems 1.8 and 1.9 enables us to derive (α, β)-extensions of the tangent and secant numbers. Recall that the tangent number E_{2n+1} and the secant number E_{2n} are defined by

$$\sum_{n \geq 0} E_{2n+1} \frac{t^{2n+1}}{(2n+1)!} = \tan(t) \quad \text{and} \quad \sum_{n \geq 0} E_{2n} \frac{t^{2n}}{(2n)!} = \sec(t).$$

A permutation $\sigma = \sigma_1 \cdots \sigma_n \in \mathfrak{S}_n$ is called down-up (or alternating) if $\sigma_1 > \sigma_2 < \sigma_3 > \sigma_4 < \cdots$ and a permutation $\sigma = \sigma_1 \cdots \sigma_n \in \mathfrak{S}_n$ is called up-down (or reverse alternating) if $\sigma_1 < \sigma_2 > \sigma_3 < \sigma_4 > \cdots$. The down-up permutations in $\mathfrak{S}_4$ are $2\,1\,4\,3$,

$3\,1\,4\,2$, $3\,2\,4\,1$, $4\,1\,3\,2$, $4\,2\,3\,1$ and the up-down permutations in $\mathfrak{S}_4$ are $3\,4\,1\,2$, $2\,4\,1\,3$, $2\,3\,1\,4$, $1\,4\,2\,3$, $1\,3\,2\,4$. It is easy to show that the number of down-up permutations of $[n]$ equals the number of up-down permutations of $[n]$.

Let $\mathcal{UD}_n$ denote the set of up-down permutations of $[n]$ and let $\mathcal{DU}_n$ denote the set of down-up permutations of $[n]$. André [1] showed that

$$E_n = |\mathcal{UD}_n| = |\mathcal{DU}_n|.$$

Euler [18] found the following interesting relation: For $n \geq 1$,

$$\sum_{\sigma \in \mathfrak{S}_n} (-1)^{\mathrm{exc}(\sigma)} = \begin{cases} (-1)^{\frac{n-1}{2}} E_n, & \text{if } n \text{ is odd,} \\ 0, & \text{if } n \text{ is even.} \end{cases} \tag{1.23}$$

Roselle [34] obtained the following parallel result to Euler involving secant numbers: For $n \geq 1$,

$$\sum_{\sigma \in \mathcal{D}_n} (-1)^{\mathrm{exc}(\sigma)} = \begin{cases} (-1)^{\frac{n}{2}} E_n, & \text{if } n \text{ is even,} \\ 0, & \text{if } n \text{ is odd,} \end{cases} \tag{1.24}$$

where $\mathcal{D}_n$ counts the number of permutations in $\mathfrak{S}_n$ without fixed points. There are several different q-analogues of (1.23) and (1.24) have been established by [19, 25, 37, 38].

By setting $x = -1$ and $y = 1$ in Theorem 1.8, we obtain the following theorem.

Theorem 1.12. *For $n \geq 1$,*

$$\sum_{\sigma \in \mathfrak{S}_n} (-1)^{\mathrm{des}(\sigma)} \left(\frac{\alpha+\beta}{2}\right)^{\mathrm{lrmin}(\sigma)+\mathrm{rlmin}(\sigma)-2} = \begin{cases} (-1)^{\frac{n-1}{2}} \sum_{\sigma \in \mathcal{UD}_n} \alpha^{\mathrm{lrmin}(\sigma)-1} \beta^{\mathrm{rlmin}(\sigma)-1}, & \text{if } n \text{ is odd,} \\ 0, & \text{if } n \text{ is even.} \end{cases} \tag{1.25}$$

Setting $x = -1$ and $y = 1$ in Theorem 1.9, we have the following theorem.

Theorem 1.13. *For $n \geq 1$,*

$$\sum_{\sigma \in \mathfrak{S}_{n+1}} (-1)^{\mathrm{des}(\sigma)} \left(\frac{\beta}{2}\right)^{\mathrm{lrmin}(\sigma)+\mathrm{rlmin}(\sigma)-2} = \begin{cases} (-1)^{\frac{n}{2}} \sum_{\sigma \in \mathcal{DU}_n} \beta^{\mathrm{rlmin}(\sigma)}, & \text{if } n \text{ is even,} \\ 0, & \text{if } n \text{ is odd.} \end{cases} \tag{1.26}$$

Setting $\alpha = \beta = 1$ in Theorem 1.12, we recover Euler's relation (1.23) with the aid of the first fundamental transformation of Foata and Schützenberger [20]. Setting $\beta = 1$ in Theorem 1.13, we obtain the following identity, which seems to be new,

$$\sum_{\sigma \in \mathfrak{S}_{n+1}} (-1)^{\mathrm{des}(\sigma)} \left(\frac{1}{2}\right)^{\mathrm{lrmin}(\sigma)+\mathrm{rlmin}(\sigma)-2} = \begin{cases} (-1)^{\frac{n}{2}} E_n, & \text{if } n \text{ is even,} \\ 0, & \text{if } n \text{ is odd.} \end{cases} \tag{1.27}$$

Combining (1.24) and (1.27), we obtain the following identity:

$$\sum_{\sigma \in \mathfrak{S}_{n+1}} (-1)^{\mathrm{des}(\sigma)} \left(\frac{1}{2}\right)^{\mathrm{lrmin}(\sigma)+\mathrm{rlmin}(\sigma)-2} = \sum_{\sigma \in \mathcal{D}_n} (-1)^{\mathrm{exc}(\sigma)}. \tag{1.28}$$

A bijective proof of (1.28) was recently provided by Chen and Fu [12]. It would be interesting to give combinatorial proofs of Theorems 1.5, 1.6, 1.7, 1.9, 1.12 and 1.13.

The rest of this paper is organized as follows. Section 2 provides a review of certain classical descent statistics, including peaks and their variations, double ascents (double descents) and their variations. We then collect the generating functions associated with these statistics and their relations with the Eulerian polynomials. Additionally, we present the combinatorial definitions of $A_n(x, y \mid \alpha, \beta)$ and $P_n(u, v, w, z \mid \alpha, \beta)$, involving the left-to-right minima and the right-to-left minima. Section 3 is dedicated to proving the main result of this paper (Theorem 1.4) using the grammatical calculus. We establish the grammar for $A_n(x, y \mid \alpha, \beta)$ (see Theorem 3.1) and the grammar for $P_n(u, v, w, z \mid \alpha, \beta)$ (see Theorem 3.2). We derive Theorems 1.1 and 1.4 based on their grammars. Sections 4, 5 and 6 explore the applications of Theorem 1.4. In Section 4, we first establish an explicit form of the generating function for $P_n(u, v, w, z \mid \alpha, \beta)$ using Theorem 1.4. We then present (α, β)-extensions of some known generating functions related to descent statistics by specializing the generating function for $P_n(u, v, w, z \mid \alpha, \beta)$. Section 5 establishes an explicit expression for $P_n(u, v, w, z \mid \alpha, \beta)$ when $\alpha + \beta = -1$. This result can be specialized to obtain Theorems 1.5, 1.6, and 1.7. In Section 6, we first establish two relations between $P_n(u, v, w, z \mid \alpha, \beta)$ and $A_n\,(x, y \mid \alpha, \beta)$ using Theorem 1.4. We then derive (α, β)-extensions of certain known relations between descent statistics and the Eulerian polynomials established by Stembridge, Petersen, Brändén and Zhuang by specializing these two relations.

2 Descent statistics

In this section, we begin by revisiting classical descent statistics, which encompass variations related to peaks, valleys, double ascents, and double descents. We then collect the generating functions associated with these statistics and their relationships with the Eulerian polynomials. Here we follow Stanley's terminology for peaks and their variations, as described in [39, Exercise 1.61]. For double ascents, double descents and their variations, we adhere to the definitions provided by Zhuang in [42].

2.1 Variations in peaks and related results

Definition 2.1 (Variations in peaks)**.** *Given a permutation $\sigma = \sigma_1 \cdots \sigma_n \in \mathfrak{S}_n$ under the assumption that $\sigma_0 = \sigma_{n+1} = 0$.*

(1) *An index i is called a* **left peak** *of σ if $1 \le i < n$ and $\sigma_{i-1} < \sigma_i > \sigma_{i+1}$.*

(2) *An index i is called an* **interior peak** *(or a peak for short) of σ if $1 < i < n$ and $\sigma_{i-1} < \sigma_i > \sigma_{i+1}$.*

(3) *An index i is called an* **exterior peak** *of σ if $1 \le i \le n$ and $\sigma_{i-1} < \sigma_i > \sigma_{i+1}$.*

(4) *An index i is called a* **valley** *of σ if $1 \le i \le n$ and $\sigma_{i-1} > \sigma_i < \sigma_{i+1}$.*

Let $L(\sigma)$, $M(\sigma)$, $W(\sigma)$ and $V(\sigma)$ denote the number of left peaks, peaks, exterior peaks and valleys of σ, respectively. Note that the symbols $L(\sigma)$, $M(\sigma)$, and $W(\sigma)$ used in this context were introduced by Chen and Fu [9], which are meaningful and easy to remember. The letter L looks like having a peak on the left. It should also be noted that Carlitz and Scoville [6] referred to an exterior peak as maxima, while Goulden and Jackson [24, Exercise 3.3.46] describe them as modified maxima. Similarly, Goulden and Jackson [24, Exercise 3.3.46] termed valleys as modified minima.

For the permutation $\sigma = 7\,1\,3\,8\,5\,9\,6\,2\,4 \in \mathfrak{S}_9$, we see that

$$L(\sigma) = 3, \quad M(\sigma) = 2, \quad W(\sigma) = 4, \quad \mathrm{V}(\sigma) = 3.$$

It is not difficult to show that for $\sigma \in \mathfrak{S}_n$,

$$\sum_{\sigma \in \mathfrak{S}_n} u^{M(\sigma)} = \sum_{\sigma \in \mathfrak{S}_n} u^{\mathrm{V}(\sigma)} = \sum_{\sigma \in \mathfrak{S}_n} u^{W(\sigma)-1}, \tag{2.1}$$

see (2.16). Hence it suffices to consider left peaks and peaks. The generating function for left peaks is attribute to Gessel [30, Sequence A008971], see Zhuang [42, Theorem 10].

Theorem 2.2 (Gessel)**.** *We have*

$$\sum_{n\ge 0} \left(\sum_{\sigma \in \mathfrak{S}_n} u^{L(\sigma)} \right) \frac{t^n}{n!} = \frac{\sqrt{1-u}}{\sqrt{1-u}\cosh\left(t\sqrt{1-u}\right) - \sinh\left(t\sqrt{1-u}\right)}. \tag{2.2}$$

As brought up by Stanley [39], the generating function for peaks can be deduced from an equation of David and Barton [13], see Chen and Fu [10] for more information. The equivalent formula has been found by Entringer [17], Kitaev [26] and Zhuang [42, Theorem 9].

Theorem 2.3. *We have*

$$\sum_{n\ge 0} \left(\sum_{\sigma \in \mathfrak{S}_n} u^{M(\sigma)} \right) \frac{t^n}{n!} = \frac{\sqrt{1-u}\cosh\left(t\sqrt{1-u}\right)}{\sqrt{1-u}\cosh\left(t\sqrt{1-u}\right) - \sinh\left(t\sqrt{1-u}\right)}. \tag{2.3}$$

Stembridge [40] was the first one to explore the connection between peak polynomials and the Eulerian polynomials. In his investigation of enriched P-partitions, he established a notable relation between peak polynomials and the Eulerian polynomials. This relation was later rediscovered by Brändén [3] using the "modified Foata-Strehl action", a variant of a group action on permutations originally defined by Foata and Strehl [21].

Theorem 2.4. (Stembridge [40]) *For* $n \geq 1$,

$$\sum_{\sigma \in \mathfrak{S}_n} x^{\mathrm{des}(\sigma)} = \left(\frac{1+x}{2}\right)^{n-1} \sum_{\sigma \in \mathfrak{S}_n} \left(\frac{4x}{(1+x)^2}\right)^{M(\sigma)}. \tag{2.4}$$

Petersen [32, Observation 3.1.2] established a relation between left peak polynomials and Eulerian polynomials, stated as follows.

Theorem 2.5. (Petersen [32]) *For* $n \geq 1$,

$$(1+x)^n \sum_{\sigma \in \mathfrak{S}_n} \left(\frac{4x}{(1+x)^2}\right)^{L(\sigma)} = \sum_{k=1}^{n} \binom{n}{k} 2^k (1-x)^{n-k} \left(\sum_{\sigma \in \mathfrak{S}_k} x^{\mathrm{des}(\sigma)+1}\right) + (1-x)^n. \tag{2.5}$$

Recently, Zhuang [43] established two relations between the joint polynomials of peaks (or left peaks) and descents and the Eulerian polynomials.

Theorem 2.6. (Zhuang [43, Theorem 4.2]) *For* $n \geq 1$,

$$\sum_{\sigma \in \mathfrak{S}_n} u^{M(\sigma)+1} v^{\mathrm{des}(\sigma)+1} = \left(\frac{1+b}{1+ab}\right)^{n+1} \sum_{\sigma \in \mathfrak{S}_n} a^{\mathrm{des}(\sigma)+1}, \tag{2.6}$$

where

$$a = \frac{(1+v)^2 - 2uv - (1+v)\sqrt{(1+v)^2 - 4uv}}{2uv} \tag{2.7}$$

and

$$b = \frac{1 + v^2 - 2uv - (1-v)\sqrt{(1+v)^2 - 4uv}}{2(1-u)v}. \tag{2.8}$$

Theorem 2.7. (Zhuang [43, Theorem 4.7]) *For* $n \geq 1$,

$$\sum_{\sigma \in \mathfrak{S}_n} u^{L(\sigma)} v^{\mathrm{des}(\sigma)} = \frac{1}{(1+ab)^n} \left(\sum_{k=1}^{n} \binom{n}{k} (1+b)^k (1-a)^{n-k} \sum_{\sigma \in \mathfrak{S}_k} a^{\mathrm{des}(\sigma)+1} + (1-a)^n\right), \tag{2.9}$$

where a *and* b *are defined by* (2.7) *and* (2.8), *respectively.*

Note that grammatical proofs of (2.2), (2.3), (2.4) and (2.5) were recently provided by Chen and Fu [10, 11].

2.2 Variations in double ascents (descents) and related results

Definition 2.8 (Variations in double ascents)**.** *Given a permutation $\sigma = \sigma_1 \cdots \sigma_n \in \mathfrak{S}_n$ under the assumption that σ_0 and $\sigma_{n+1} = +\infty$.*

(1) *An index i is a* **left double ascent** *of σ if $1 \le i < n$ and $\sigma_{i-1} < \sigma_i < \sigma_{i+1}$.*

(2) *An index i is a* **double ascent** *of σ if $1 < i < n$ and $\sigma_{i-1} < \sigma_i < \sigma_{i+1}$.*

(3) *An index i is a* **right double ascent** *of σ if $1 < i \le n$ and $\sigma_{i-1} < \sigma_i < \sigma_{i+1}$.*

(4) *An index i is a* **left-right double ascent** *of σ if $1 \le i \le n$ and $\sigma_{i-1} < \sigma_i < \sigma_{i+1}$.*

Definition 2.9 (Variations in double descents)**.** *Given a permutation $\sigma = \sigma_1 \cdots \sigma_n \in \mathfrak{S}_n$ under the assumption that $\sigma_0 = +\infty$ and $\sigma_{n+1} = 0$.*

(1) *An index i is a* **left double descent** *of σ if $1 \le i < n$ and $\sigma_{i-1} > \sigma_i > \sigma_{i+1}$.*

(2) *An index i is a* **double descent** *of σ if $1 < i < n$ and $\sigma_{i-1} > \sigma_i > \sigma_{i+1}$.*

(3) *An index i is a* **right double descent** *of σ if $1 < i \le n$ and $\sigma_{i-1} > \sigma_i > \sigma_{i+1}$.*

(4) *An index i is a* **left-right double descent** *of σ if $1 \le i \le n$ and $\sigma_{i-1} > \sigma_i > \sigma_{i+1}$.*

Let $\mathrm{lda}(\sigma)(\mathrm{ldd}(\sigma)), \mathrm{da}(\sigma)(\mathrm{dd}(\sigma)), \mathrm{rda}(\sigma))(\mathrm{rdd}(\sigma))$, and $\mathrm{lrda}(\sigma)(\mathrm{lrdd}(\sigma))$ denote the number of left double ascents (left double descents), double ascents (double descents), right double ascents (right double descents), left-right double ascents (left-right double descents) of σ, respectively.

For the permutation $\sigma = 7\,1\,3\,8\,5\,9\,6\,2\,4 \in \mathfrak{S}_9$, we see that

$$\mathrm{da}(\sigma) = 1, \quad \mathrm{lda}(\sigma) = 1, \quad \mathrm{rda}(\sigma) = 2, \quad \mathrm{lrda}(\sigma) = 2$$

and

$$\mathrm{dd}(\sigma) = 1, \quad \mathrm{ldd}(\sigma) = 2, \quad \mathrm{rdd}(\sigma) = 1, \quad \mathrm{lrdd}(\sigma) = 2.$$

By definition, we see that for $\sigma \in \mathfrak{S}_n$,

$$\mathrm{des}(\sigma) = W(\sigma) + \mathrm{rdd}(\sigma) - 1, \qquad \mathrm{asc}(\sigma) = W(\sigma) + \mathrm{lda}(\sigma) - 1 \tag{2.10}$$

and

$$\mathrm{des}(\sigma) = L(\sigma) + \mathrm{dd}(\sigma), \qquad \mathrm{asc}(\sigma) = L(\sigma) + \mathrm{lrda}(\sigma) - 1. \tag{2.11}$$

Let $\sigma = \sigma_1 \cdots \sigma_n \in \mathfrak{S}_n$. The complement of σ is given by

$$\sigma^c = (n+1-\sigma_1)(n+1-\sigma_2)\cdots(n+1-\sigma_n). \tag{2.12}$$

For example, if $\sigma = 7\,1\,3\,8\,5\,6\,2\,4$, then the complement of σ is given by $\sigma^c = 2\,8\,6\,1\,4\,3\,7\,5$.

Evidently, the complement establishes a bijection on $\mathfrak{S}_n$. Additionally, for $\sigma \in \mathfrak{S}_n$, we observe that

$$\mathrm{asc}(\sigma) = \mathrm{des}(\sigma^c), \quad \mathrm{des}(\sigma) = \mathrm{asc}(\sigma^c), \tag{2.13}$$

$$\mathrm{lda}(\sigma) = \mathrm{ldd}(\sigma^c), \quad \mathrm{rdd}(\sigma) = \mathrm{rda}(\sigma^c), \tag{2.14}$$

$$\mathrm{lrmax}(\sigma) = \mathrm{lrmin}(\sigma^c), \quad \mathrm{rlmax}(\sigma) = \mathrm{rlmin}(\sigma^c) \tag{2.15}$$

and

$$W(\sigma) - 1 = \mathrm{V}(\sigma) = \mathrm{M}(\sigma^c). \tag{2.16}$$

Therefore, we only need to consider double ascents, right double ascents and left-right double ascents. By generalizing Gessel's reciprocity formula for noncommutative symmetric functions, Zhuang [42] gave a systematic method for obtaining the generating functions for double ascents, right double ascents and left-right double ascents. Note that the equivalent form of the generating function for double ascents was established by Elizalde and Noy [16]. Elizalde and Noy [16] referred to double descents as proper double descents.

Theorem 2.10. (Elizalde and Noy [16], Zhuang [42, Theorem 12])

$$\sum_{n\geq 0}\left(\sum_{\sigma\in\mathfrak{S}_n} u^{\mathrm{da}(\sigma)}\right)\frac{t^n}{n!} = \frac{ve^{\frac{(1-u)}{2}t}}{v\cosh\left(\frac{1}{2}vt\right) - (1+u)\sinh\left(\frac{1}{2}vt\right)}, \tag{2.17}$$

where $v = \sqrt{(u+1)^2 - 4}$.

Theorem 2.11. (Zhuang [42, Theorem 13])

$$\sum_{n\geq 0}\left(\sum_{\sigma\in\mathfrak{S}_n} u^{\mathrm{rda}(\sigma)}\right)\frac{t^n}{n!} = \frac{v\cosh\left(\frac{1}{2}vt\right) + (1-u)\sinh\left(\frac{1}{2}vt\right)}{v\cosh\left(\frac{1}{2}vt\right) - (1+u)\sinh\left(\frac{1}{2}vt\right)} \tag{2.18}$$

and

$$\sum_{n\geq 0}\left(\sum_{\sigma\in\mathfrak{S}_n} u^{\mathrm{lrda}(\sigma)}\right)\frac{t^n}{n!} = \frac{ve^{\frac{(u-1)}{2}t}}{v\cosh\left(\frac{1}{2}vt\right) - (1+u)\sinh\left(\frac{1}{2}vt\right)}, \tag{2.19}$$

where $v = \sqrt{(u+1)^2 - 4}$.

2.3 Equivalent definitions of $A_n(x, y \mid \alpha, \beta)$ and $P_n(u, v, w, z \mid \alpha, \beta)$

In this subsection, we present the combinatorial definitions of $A_n(x, y \mid \alpha, \beta)$ and $P_n(u, v, w, z \mid \alpha, \beta)$ which involve the left-to-right minima and the right-to-left minima. These definitions are essential for employing the grammatical approach to study these polynomials.

By taking the complements of permutations, and applying (2.13), (2.14) and (2.16), we derive that $A_n(x, y \mid \alpha, \beta)$ can be alternatively interpreted as follows:

$$A_n(x, y \mid \alpha, \beta) = \sum_{\sigma\in\mathfrak{S}_{n+1}} x^{\mathrm{des}(\sigma)} y^{\mathrm{asc}(\sigma)} \alpha^{\mathrm{lrmin}(\sigma)-1} \beta^{\mathrm{rlmin}(\sigma)-1} \tag{2.20}$$

and $P_n(u,v,w,z \mid \alpha,\beta)$ can be interpreted as follows:

$$P_n(u,v,w,z \mid \alpha,\beta) = \sum_{\sigma\in\mathfrak{S}_{n+1}} (uv)^{M(\sigma)} w^{\mathrm{rda}(\sigma)} z^{\mathrm{ldd}(\sigma)} \alpha^{\mathrm{lrmin}(\sigma)-1} \beta^{\mathrm{rlmin}(\sigma)-1}. \tag{2.21}$$

Note that $P_n(u,v,w,z \mid \alpha,\beta)$ encompasses peaks, right double ascents, and left double descents. However, it is worth mentioning that left peaks, left-right double ascents, and double descents can also be characterized by specializing $P_n(u,v,w,z \mid \alpha,\beta)$. More precisely, by setting $\alpha = 0$ in (2.21), we have

$$P_n(u,v,w,z \mid 0,\beta) = \sum_{\substack{\sigma\in\mathfrak{S}_{n+1}\\ \sigma_1=1}} (uv)^{M(\sigma)} w^{\mathrm{rda}(\sigma)} z^{\mathrm{ldd}(\sigma)} \beta^{\mathrm{rlmin}(\sigma)-1}. \tag{2.22}$$

Next, define the following reduction map:

$$\sigma = 1\,\sigma_2 \cdots \sigma_{n+1} \to \overline{\sigma} = (\sigma_2-1)\,(\sigma_3-1)\cdots(\sigma_{n+1}-1). \tag{2.23}$$

It is straightforward to verify that $\overline{\sigma} \in \mathfrak{S}_n$ and that

$$M(\sigma) = \mathrm{L}(\overline{\sigma}),\ \mathrm{rda}(\sigma) = \mathrm{lrda}(\overline{\sigma}),\ \mathrm{ldd}(\sigma) = \mathrm{dd}(\overline{\sigma}),\ \mathrm{rlmin}(\sigma) - 1 = \mathrm{rlmin}(\overline{\sigma}).$$

Moreover, this process is reversible. Consequently, we derive that

$$P_n(u,v,w,z \mid 0,\beta) = \sum_{\sigma\in\mathfrak{S}_n} (uv)^{\mathrm{L}(\sigma)} w^{\mathrm{lrda}(\sigma)} z^{\mathrm{dd}(\sigma)} \beta^{\mathrm{rlmin}(\sigma)}. \tag{2.24}$$

Using the same argument, we deduce that

$$A_n(x,y \mid 0,\beta) = \sum_{\sigma\in\mathfrak{S}_n} x^{\mathrm{des}(\sigma)} y^{\mathrm{asc}(\sigma)+1} \beta^{\mathrm{rlmin}(\sigma)}, \tag{2.25}$$

$$A_n(x,y \mid \alpha,0) = \sum_{\sigma\in\mathfrak{S}_n} x^{\mathrm{des}(\sigma)+1} y^{\mathrm{asc}(\sigma)} \alpha^{\mathrm{lrmin}(\sigma)}. \tag{2.26}$$

3 A grammartical calculus for $A_n(x,y \mid \alpha,\beta)$ and $P_n(u,v,w,z \mid \alpha,\beta)$

The primary objective of this section is to introduce a grammatical approach to the study of the polynomials $A_n(x,y \mid \alpha,\beta)$ and the polynomial $P_n(u,v,w,z \mid \alpha,\beta)$. This technique, which employs context-free grammars to explore combinatorial polynomials, was pioneered by Chen [7]. Note that the derivations of the generating functions for $A_n(x,y \mid \alpha,\beta)$ and $P_n(u,v,w,z \mid \alpha,\beta)$ become quite simple once their grammars are established.

A context-free grammar G over a set $V = \{x,y,z,\ldots\}$ of variables is a set of substitution rules replacing a variable in V by a Laurent polynomial of variables in V. For a context-free grammar G over V, the formal derivative D with respect to G is defined as a linear operator acting on Laurent polynomials with variables in V such that each substitution rule is treated as the common differential rule that satisfies the following relations:

$$D(u+v) = D(u) + D(v), \tag{3.1}$$

$$D(uv) = D(u)v + uD(v). \tag{3.2}$$

Hence, it obeys the Leibniz's rule

$$D^n(uv) = \sum_{k=0}^{n} \binom{n}{k} D^k(u) D^{n-k}(v).$$

For a constant c, we have $D(c) = 0$.

A formal derivative D with respect to G is also associated with an exponential generating function. For a Laurent polynomial w of variables in V, let

$$\mathrm{Gen}^{(G)}(w;t) = \sum_{n\geq 0} D_G^n(w) \frac{t^n}{n!}. \tag{3.3}$$

Then, by (3.1) and (3.2), we derive that

$$\mathrm{Gen}^{(G)}(u+v;t) = \mathrm{Gen}^{(G)}(u;t) + \mathrm{Gen}^{(G)}(v;t). \tag{3.4}$$

$$\mathrm{Gen}^{(G)}(uv;t) = \mathrm{Gen}^{(G)}(u;t)\mathrm{Gen}^{(G)}(v;t). \tag{3.5}$$

For more information on the grammatical calculus, we refer to Chen [7] and Chen and Fu [8, 10].

Dumont [15] showed the following grammar

$$E = \{x \to xy,\ y \to xy\} \tag{3.6}$$

generates the Eulerian polynomials $A_n(x)$ given by (1.1). More precisely, let D_E be the formal derivative with respect to the grammar E given by (3.6), then for $n \geq 1$,

$$D_E^n(y) = \sum_{\sigma\in\mathfrak{S}_n} x^{\mathrm{asc}(\sigma)+1} y^{\mathrm{des}(\sigma)+1}.$$

Chen and Fu [9] showed that

$$\mathrm{Gen}^{(E)}(y;t) := \sum_{n\geq 0} D_E^n(y) \frac{t^n}{n!} = y(1 + xF(x,y;t)), \tag{3.7}$$

where $F(x,y;t)$ is given in (1.8). Together with Dumont's result, they provided a grammatical proof of the generating function (1.2) of $A_n(x)$.

Similarly, it can be shown that

$$\mathrm{Gen}^{(E)}(x;t) := \sum_{n\geq 0} D_E^n(x) \frac{t^n}{n!} = x(1 + yF(x,y;t)). \tag{3.8}$$

In this section, we first show that the following grammar

$$\widetilde{E} = \{a \to a\alpha x,\ b \to b\beta y,\ x \to xy,\ y \to xy\} \tag{3.9}$$

can be used to generate the polynomials $A_n(x,y \mid \alpha,\beta)$. More precisely,

Theorem 3.1. *Let $D_{\widetilde{E}}$ be the formal derivative with respect to the grammar defined in (3.9). For $n \geq 0$,*

$$D_{\widetilde{E}}^{n}(ab) = abA_n(x, y \mid \alpha, \beta). \tag{3.10}$$

Using Theorem 3.1, we then provide a grammatical derivation of the generating function (1.7) for $A_n(x, y \mid \alpha, \beta)$ established by Carlitz and Scoville.

Next, we introduce the grammar for the polynomial $P_n(u, v, w, z \mid \alpha, \beta)$ stated as follows:

Theorem 3.2. *Let $D_{\widetilde{H}}$ be the formal derivative with respect to the grammar*

$$\widetilde{H} = \{a \to a\alpha z,\, b \to b\beta w,\, z \to uv,\, w \to uv,\, u \to uw, v \to vz\}. \tag{3.11}$$

For $n \geq 0$,

$$D_{\widetilde{H}}^{n}(ab) = abP_n(u, v, w, z \mid \alpha, \beta). \tag{3.12}$$

Building on Theorem 3.2, to prove Theorem 1.4, it suffices to demonstrate the following theorem:

Theorem 3.3. *Let $D_{\widetilde{H}}$ be the formal derivative with respect to the grammar defined in (3.11), we have*

$$\mathrm{Gen}^{(\widetilde{H})}(ab;t) = ab\,(1 + yF(x, y; t))^{\frac{\alpha+\beta}{2}}\,(1 + xF(x, y; t))^{\frac{\alpha+\beta}{2}}\,e^{\frac{1}{2}(\beta-\alpha)(w-z)t}, \tag{3.13}$$

where $x + y = w + z$, $xy = uv$ and $F(x, y; t)$ is given by (1.8).

3.1 A grammatical calculus for $A_n(x, y \mid \alpha, \beta)$

We first show Theorem 3.1 by using the grammatical labeling. The notion of a grammatical labeling was introduced by Chen and Fu [8]. Here we adopt the combinatorial definition (2.20) of $A_n(x, y \mid \alpha, \beta)$.

Let $\sigma = \sigma_1 \cdots \sigma_n \in \mathfrak{S}_n$. For $1 \leq i \leq n + 1$, recall that the position i is said to be the position immediately before σ_i, whereas the position $n + 1$ is meant to be the position after σ_n. We adjoin $+\infty$ to σ at both ends so that there are $n + 1$ positions between two adjacent elements. For $1 \leq i \leq n + 1$, we label the position i of σ as follows:

- If $i = 1$, then label it by a;
- If $i = n + 1$, then label it by b;
- If i is an ascent, then label it by y;
- If i is a descent, then label it by x;
- If σ_j is a left-to-right minimum and $\sigma_j \neq 1$, then label α below σ_j;

- If σ_j is a right-to-left minimum and $\sigma_j \neq 1$, then label β below σ_j.

The weight ω of σ is defined to be the product of all the labels.

Below shows the grammatical labeling of a permutation $\sigma = 7\ 1\ 2\ 8\ 3\ 6\ 5\ 4$ whose weight is $\omega(\sigma) = abx^4y^3\alpha\beta^3$. We refer to this labeling scheme of permutations as the $(x, y; \alpha, \beta)$-labeling.

$$\sigma = 7 \quad 1 \quad 2 \quad 8 \quad 3 \quad 6 \quad 5 \quad 4$$

$$\longrightarrow \begin{array}{ccccccccccccccccccc} +\infty & a & 7 & x & 1 & y & 2 & y & 8 & x & 3 & y & 6 & x & 5 & x & 4 & b & +\infty \\ & & \alpha & & & & \beta & & & & \beta & & & & & & \beta & & \end{array}$$

From the definition of the $(x, y; \alpha, \beta)$-labeling, we see that

$$abA_n(x, y \mid \alpha, \beta) = \sum_{\sigma \in \mathfrak{S}_{n+1}} \omega(\sigma). \tag{3.14}$$

Proof of Theorem 3.1. We proceed by induction on n. For $n = 0$, the statement is evident. Assume that this statement holds for $n - 1$. To show that it is valid for n, we represent a permutation $\sigma = \sigma_1 \cdots \sigma_n$ in $\mathfrak{S}_n$ by adjoining $+\infty$ at both ends, resulting in $n + 1$ positions between adjacent elements. These positions allow us to insert $n + 1$ into σ to generate a permutation in $\mathfrak{S}_{n+1}$. Let π be permutation in $\mathfrak{S}_{n+1}$ obtained by inserting the element $n + 1$ into σ at the position i, where $1 \leq i \leq n + 1$. We consider the following four cases:

Case 1: If i is labeled by a in the labeling of σ, that is, $i = 1$, then the first position and the second position of π are labeled by a and x in the labeling of π, respectively. Moreover, $n + 1$ is the left-to-right minimum of π, so label α below $n + 1$. In this case, we find that the change of weights is consistent with the substitution rule $a \to a\alpha x$.

$$\sigma = \begin{array}{cccc} +\infty & a & \sigma_1 & \cdots \\ & & \alpha & \end{array} \Rightarrow \pi = \begin{array}{cccccc} +\infty & a & n+1 & x & \sigma_1 & \cdots \\ & & \alpha & & \alpha & \cdots \end{array}.$$

Case 2: If i is labeled by b in the labeling of σ, that is, $i = n + 1$, then the change of weights caused by the insertion is coded by the rule $b \to b\beta y$.

$$\sigma = \begin{array}{cccc} \cdots & \sigma_n & b & +\infty \\ \cdots & \beta & & \end{array} \Rightarrow \pi = \begin{array}{cccccc} \cdots & \sigma_n & y & n+1 & b & +\infty \\ \cdots & \beta & & \beta & & \end{array}.$$

Case 3: If i is labeled by x in the labeling of σ, then the position i of π is labeled by y and the position $i + 1$ of π is labeled by x in the labeling of π, so the change of weights is consistent with the substitution rule $x \to xy$.

$$\sigma = \quad \cdots \sigma_{i-1}\, x \quad \sigma_i \cdots \Rightarrow \pi = \quad \cdots \sigma_{i-1}\, y \quad n+1 \quad x \quad \sigma_i \cdots.$$

Case 4: If i is labeled by y in the labeling of σ, then the change of weights is in accordance with the rule $y \to xy$, as depicted in the figure below.

$$\sigma = \quad \cdots \sigma_{i-1} \quad y \quad \sigma_i \cdots \Rightarrow \quad \pi = \quad \cdots \sigma_{i-1} \quad y \quad n+1 \quad x \quad \sigma_i \cdots .$$

Summing up all the cases shows that this assertion is valid for n. This completes the proof. ∎

The grammatical derivation for Theorem 1.1. Let $D_{\widetilde{E}}$ be the formal derivative with respect to the grammar (3.9). From Theorem 3.1, we see that

$$\mathrm{Gen}^{(\widetilde{E})}(ab;t) = ab \sum_{n\geq 0} A_n(x,y \mid \alpha,\beta) \frac{t^n}{n!} \tag{3.15}$$

Let D_E be the formal derivative with respect to the grammar (3.6). Suppose that α and β are two fixed numbers. It is easy to check that

$$D_E(y^\alpha) = \alpha y^{\alpha-1} D_E(y) = \alpha y^\alpha x \quad \text{and} \quad D_E(x^\beta) = \beta x^{\beta-1} D_E(x) = \beta x^\beta y. \tag{3.16}$$

Setting $a = y^\alpha$ and $b = x^\beta$. Then, by (3.16), we find that

$$D_E(y^\alpha) = D_{\widetilde{E}}(a) \quad \text{and} \quad D_E(x^\beta) = D_{\widetilde{E}}(b).$$

Moreover, it is easy to check that

$$D_E(x) = D_{\widetilde{E}}(x) \quad \text{and} \quad D_E(y) = D_{\widetilde{E}}(y).$$

Hence we can use the induction on n to deduce that for $n \geq 0$,

$$D_E^n(y^\alpha) = D_{\widetilde{E}}^n(a) \quad \text{and} \quad D_E^n(x^\beta) = D_{\widetilde{E}}^n(b)$$

given that $a = y^\alpha$ and $b = x^\beta$. Consequently, for $n \geq 0$,

$$D_{\widetilde{E}}^n(ab) = D_E^n(x^\beta y^\alpha). \tag{3.17}$$

Hence it follows from (3.5) that

$$\mathrm{Gen}^{(\widetilde{E})}(ab;t) = \mathrm{Gen}^{(E)}(y^\alpha x^\beta;t) = (\mathrm{Gen}^{(E)}(x;t))^\beta (\mathrm{Gen}^{(E)}(y;t))^\alpha. \tag{3.18}$$

Substituting (3.7) and (3.8) into (3.18) and using (3.15), we obtain (1.7). This completes the proof of Theorem 1.1. ∎

3.2 A grammatical labeling of $P_n(u,v,w,z \mid \alpha,\beta)$

To give a grammatical labeling of $P_n(u,v,w,z \mid \alpha,\beta)$, we need to refine the grammatical labeling of permutations introduced in Subsection 3.1. In a similar vein, we take the combinatorial definition (2.21) of $P_n(u,v,w,z \mid \alpha,\beta)$, which incorporates the left-to-right minima and the right-to-left minima.

Let $\sigma = \sigma_1 \cdots \sigma_n \in \mathfrak{S}_n$. For $1 \leq i \leq n+1$, recall that the position i is said to be the position immediately before σ_i, whereas the position $n+1$ is meant to be the position after σ_n. We adjoin $+\infty$ to σ at both ends so that there are $n+1$ positions between two adjacent elements. For $1 \leq i \leq n+1$, we label the position i of σ as follows:

- If $i = 1$, then label it by a;
- If $i = n + 1$, then label it by b;
- If i is a right double ascent, then label the position i by w;
- If $i - 1$ is a left double descent, then label the position i by z;
- If i is a peak, then label the position i by v and label the position $i + 1$ by u;
- If σ_j is a left-to-right minimum and $\sigma_j \neq 1$, then label α below σ_j;
- If σ_j is a right-to-left minimum and $\sigma_j \neq 1$, then label β below σ_j.

The weight ω of σ is defined to be the product of all the labels. This refined labeling scheme of permutations is called the $(u, v, w, z; \alpha, \beta)$-labeling.

For example, the $(u, v, w, z; \alpha, \beta)$-labeling of the permutation $\sigma = 71283654$ is shown below, whose weight is $\omega(\sigma) = abu^2v^2wz\alpha\beta^3$.

$$\begin{array}{l} 7\;\;1\;\;2\;\;8\;\;3\;\;6\;\;5\;\;4 \\ \longrightarrow \begin{array}{ccccccccccccccccc} +\infty & a & 7 & z & 1 & w & 2 & v & 8 & u & 3 & v & 6 & u & 5 & z & 4 & b & +\infty \\ & & \alpha & & & & \beta & & & & \beta & & & & & & \beta & & \end{array} \end{array}$$

From the definition of the $(u, v, w, z; \alpha, \beta)$-labeling, we see that

$$abP_n(u, v, w, z \mid \alpha, \beta) = \sum_{\sigma \in \mathfrak{S}_{n+1}} \omega(\sigma). \tag{3.19}$$

Proof of Theorem 3.2. We use induction on n. The case of $n = 0$ is straightforward. Assume that the statement holds for $n - 1$. To establish its validity for n, we consider a permutation $\sigma = \sigma_1 \cdots \sigma_n$ in $\mathfrak{S}_n$ by adjoining $+\infty$ at both ends so that there are $n + 1$ positions between two adjacent elements. These $n + 1$ positions are available for inserting the element $n + 1$ into σ to generate a permutation in $\mathfrak{S}_{n+1}$. Let π be permutation in $\mathfrak{S}_{n+1}$ obtained by inserting the element $n + 1$ into σ at the position i, where $1 \leq i \leq n + 1$. We consider the following six cases:

Case 1: If i is labeled by a in the labeling of σ, that is, $i = 1$, then the first position and the second position of π are labeled by a and z in the labeling of π, respectively. Moreover, $n + 1$ is the left-to-right minimum of π, so label α below $n + 1$. In this case, we find that the change of weights of π is consistent with the substitution rule $a \to a\alpha z$.

$$\begin{array}{ccccc} \sigma = & +\infty & a & \sigma_1 & \cdots \\ & & & \alpha & \end{array} \Rightarrow \begin{array}{cccccccc} \pi = & +\infty & a & n+1 & z & \sigma_1 & \cdots \\ & & & \alpha & & \alpha & \cdots \end{array}.$$

Case 2: If i is labeled by b in the labeling of σ, that is, $i = n + 1$, then the change of weights caused by the insertion is coded by the rule $b \to b\beta w$.

$$\begin{array}{cccccc} \sigma = & \cdots & \sigma_n & b & +\infty \\ & \cdots & \beta & & \end{array} \Rightarrow \begin{array}{cccccccc} \pi = & \cdots & \sigma_n & w & n+1 & b & +\infty \\ & \cdots & \beta & & \beta & & \end{array}.$$

Case 3: If i is labeled by w in the labeling of σ, then position i of π is labeled by v and position $i+1$ of π is labeled by u in the labeling of π, so the change of weights is consistent with the substitution rule $w \to uv$.

$$\sigma = \cdots \sigma_{i-1}\, w \quad \sigma_i \cdots \Rightarrow \quad \pi = \quad \cdots \sigma_{i-1}\, v \quad n+1 \quad u \quad \sigma_i \cdots .$$

Case 4: If i is labeled by z in the labeling of σ, then the change of weights follows the rule $z \to uv$, as illustrated below.

$$\sigma = \cdots \sigma_{i-1} \quad z \quad \sigma_i \cdots \Rightarrow \pi = \cdots \sigma_{i-1} \quad v \quad n+1 \quad u \quad \sigma_i \cdots .$$

Case 5: If i is labeled by u in the labeling of σ, then the corresponding change of weights is consistent with the rule $u \to uw$, see the figure below.

$$\sigma = \cdots \sigma_{i-2} \quad v \quad \sigma_{i-1} \quad u \quad \sigma_i \cdots \Rightarrow \pi = \cdots \sigma_{i-2} \quad w \quad \sigma_{i-1} \quad v \quad n{+}1 \quad u \quad \sigma_i \cdots .$$

Case 6: If i is labeled by v in the labeling of σ, then the change in weights aligns with the rule $v \to vz$, as depicted below.

$$\sigma = \cdots \sigma_{i-1} \quad v \quad \sigma_i \quad u \quad \sigma_{i+1} \cdots \Rightarrow \pi = \cdots \sigma_{i-1} \quad v \quad n{+}1 \quad u \quad \sigma_i \quad z \quad \sigma_{i+1} \cdots .$$

Adding up all the cases verifies that this statement is true for any n. This completes the proof. ∎

3.3 Proof of Theorem 3.3

In this subsection, we aim to give a grammatical derivation of Theorem 1.4. Utilizing Theorem 3.2, it suffices to prove Theorem 3.3.

Proof of Theorem 3.3. Let D_H be the formal derivative with respect to the grammar

$$H = \{z \to uv,\ w \to uv,\ u \to uw, v \to vz\}. \tag{3.20}$$

We first show that

$$\begin{aligned} \mathrm{Gen}^{(H)}(uv;t) &:= \sum_{n\geq 0} D_H^n(uv)\frac{t^n}{n!} \\ &= xy(1 + yF(x,y;t))(1 + xF(x,y;t)) \end{aligned} \tag{3.21}$$

and

$$\mathrm{Gen}^{(H)}(u^2;t) := u^2(1+yF(x,y;t))(1+xF(x,y;t))e^{(w-z)t}, \tag{3.22}$$

where $x+y=w+z$ and $xy=uv$ and $F(x,y;t)$ is given by (1.8).

Recall that D_E is the formal derivative with respect to the grammar (3.6) and D_H is the formal derivative with respect to the grammar (3.20). If we set $x+y=w+z$ and $xy=uv$, we find that

$$D_E(xy) = xy(x+y) = uv(w+z) = D_H(uv) \tag{3.23}$$

and

$$D_E(x+y) = 2xy = 2uv = D_H(w+z). \tag{3.24}$$

We claim that for $n \geq 1$,

$$D_E^n(xy) = D_H^n(uv). \tag{3.25}$$

From (3.23), it is clear that (3.25) holds for $n=1$. Assume that (3.25) holds for n. Since $D_E^n(xy)$ is symmetric in x,y, we can express $D_E^n(xy)$ as follows:

$$D_E^n(xy) = \sum_{j=1}^{\lfloor \frac{n+2}{2} \rfloor} a_j (xy)^j (x+y)^{n+2-2j}, \tag{3.26}$$

where a_j are integers. By the induction hypothesis, we have

$$D_H^n(uv) = D_E^n(xy),$$

which implies that

$$D_H^n(uv) = \sum_{j=1}^{\lfloor \frac{n+2}{2} \rfloor} a_j (uv)^j (w+z)^{n+2-2j}. \tag{3.27}$$

Applying D_E to (3.26), we obtain

$$\begin{aligned} D_E^{n+1}(xy) = &\sum_{j=1}^{\lfloor \frac{n+2}{2} \rfloor} a_j j (xy)^{j-1} (x+y)^{n-2j+2} D_E(xy) \\ &+ \sum_{j=1}^{\lfloor \frac{n+2}{2} \rfloor} a_j (n-2j+2)(xy)^j (x+y)^{n-2j+1} D_E(x+y). \end{aligned}$$

Since $x+y=w+z$ and $xy=uv$, by applying (3.23) and (3.24), we deduce that

$$\begin{aligned} D_E^{n+1}(xy) = &\sum_{j=1}^{\lfloor \frac{n+2}{2} \rfloor} a_j j (uv)^{j-1} (w+z)^{n-2j+2} D_H(uv) \\ &+ \sum_{j=1}^{\lfloor \frac{n+2}{2} \rfloor} a_j (n-2j+2)(uv)^j (w+z)^{n-2j+1} D_H(w+z). \end{aligned} \tag{3.28}$$

Applying D_H to (3.27), and comparing it with (3.28), we conclude that

$$D_E^{n+1}(xy) = D_H^{n+1}(uv),$$

thus confirming that (3.25) holds for $n+1$. This completes the proof. Therefore, we have

$$\mathrm{Gen}^{(H)}(uv;t) = \mathrm{Gen}^{(E)}(xy;t). \tag{3.29}$$

By applying the multiplicative property (3.5), and using (3.7) and (3.8), we deduce that

$$\mathrm{Gen}^{(E)}(xy;t) = \sum_{n\geq 0} D_E^n(xy)\frac{t^n}{n!} = xy(1+yF(x,y;t))(1+xF(x,y;t)). \tag{3.30}$$

Substituting (3.30) into (3.29), we obtain (3.21).

To prove (3.22), we first observe that

$$D_H(uv^{-1}) = uv^{-1}(w-z). \tag{3.31}$$

Since

$$D_H(w-z) = 0,$$

it follows that for $n \geq 0$,

$$D_H^n(uv^{-1}) = uv^{-1}(w-z)^n.$$

Hence

$$\mathrm{Gen}^{(H)}(uv^{-1};t) := \sum_{n\geq 0} D_H^n(uv^{-1})\frac{t^n}{n!} = uv^{-1}e^{(w-z)t}. \tag{3.32}$$

By the multiplicative property (3.5), we deduce from (3.21) and (3.32) that

$$\begin{aligned}\mathrm{Gen}^{(H)}(u^2;t) &= \mathrm{Gen}^{(H)}(uv^{-1};t)\mathrm{Gen}^{(H)}(uv;t)\\ &= u^2((1+yF(x,y;t))(1+xF(x,y;t)))e^{(w-z)t},\end{aligned}$$

which is (3.22).

Let α, β be two fixed numbers. We see that

$$D_H(v^\alpha) = \alpha v^{\alpha-1}D_H(v) = \alpha v^\alpha z, \tag{3.33}$$

$$D_H(u^\beta) = \beta u^{\beta-1}D_H(u) = \beta u^\beta w. \tag{3.34}$$

Let $D_{\widetilde{H}}$ be the formal derivative with respect to the grammar (3.11). Setting $a = v^\alpha$ and $b = u^\beta$. Then by (3.33) and (3.34), we find that

$$D_H(v^\alpha) = D_{\widetilde{H}}(a) \quad \text{and} \quad D_H(u^\beta) = D_{\widetilde{H}}(b).$$

Moreover, it is easy to check that

$$D_H(u) = D_{\widetilde{H}}(u), \quad D_H(v) = D_{\widetilde{H}}(v), \quad D_H(w) = D_{\widetilde{H}}(w), \quad \text{and} \quad D_H(z) = D_{\widetilde{H}}(z).$$

Hence we can use the induction on n to deduce that for $n \geq 0$,

$$D_H^n(v^\alpha) = D_{\widetilde{H}}^n(a) \quad \text{and} \quad D_H^n(u^\beta) = D_{\widetilde{H}}^n(b),$$

providing that $a = v^\alpha$ and $b = u^\beta$. Consequently, for $n \geq 0$,

$$D_{\widetilde{H}}^n(ab) = D_H^n(v^\alpha u^\beta). \tag{3.35}$$

It follows that

$$\begin{aligned}
\mathrm{Gen}^{(\widetilde{H})}(ab;t) &= \mathrm{Gen}^{(H)}(v^\alpha u^\beta;t) \\
&= (\mathrm{Gen}^{(H)}(vu;t))^\alpha (\mathrm{Gen}^{(H)}(u^2;t))^{\frac{\beta-\alpha}{2}} \\
&= u^\beta v^\alpha \,(1+yF(x,y;t))^{\frac{\alpha+\beta}{2}}\,(1+xF(x,y;t))^{\frac{\alpha+\beta}{2}}\, e^{\frac{1}{2}(\beta-\alpha)(w-z)t} \\
&= ab\,(1+yF(x,y;t))^{\frac{\alpha+\beta}{2}}\,(1+xF(x,y;t))^{\frac{\alpha+\beta}{2}}\, e^{\frac{1}{2}(\beta-\alpha)(w-z)t},
\end{aligned}$$

as desired. This completes the proof of Theorem 3.3. ∎

Note that the generating functions (3.21) and (3.22) in the proof of Theorem 1.4 can also be derived by using the results of Fu [22].

4 The generating function for $P_n(u,v,w,z \mid \alpha,\beta)$

In this section, we derive an explicit form of the generating function for $P_n(u,v,w,z \mid \alpha,\beta)$ using Theorem 1.4. We then explore several consequences of Theorem 4.1, which provide (α,β)-extensions of the generating functions for peaks, left peaks, double ascents, right double ascents and left-right double ascents.

Theorem 4.1. *We have*

$$\sum_{n\geq 0} P_n(u,v,w,z \mid \alpha,\beta)\frac{t^n}{n!} = e^{\frac{1}{2}(\beta-\alpha)(w-z)t} \times \left(\cosh\left(\frac{t}{2}\sqrt{(w+z)^2-4uv}\right) - \frac{w+z}{\sqrt{(w+z)^2-4uv}}\sinh\left(\frac{t}{2}\sqrt{(w+z)^2-4uv}\right)\right)^{-(\alpha+\beta)}.$$

Proof. From Theorem 1.4, we have

$$\sum_{n\geq 0} P_n(u,v,w,z \mid \alpha,\beta)\frac{t^n}{n!} = ((1+xF(x,y;t))(1+yF(x,y;t)))^{\frac{\alpha+\beta}{2}}\, e^{\frac{1}{2}(\beta-\alpha)(w-z)t}, \tag{4.1}$$

where $x+y = w+z$ and $xy = uv$.

Recall that

$$F(x,y;t) = \frac{e^{xt} - e^{yt}}{xe^{yt} - ye^{xt}},$$

and so

$$1 + xF(x,y;t) = \frac{(x-y)e^{xt}}{xe^{yt} - ye^{xt}} = \left(-\frac{x}{y-x}e^{(y-x)t} + \frac{y}{y-x}\right)^{-1}. \tag{4.2}$$

Similarly,

$$1 + yF(x,y;t) = \left(\frac{y}{y-x}e^{(x-y)t} - \frac{x}{y-x}\right)^{-1}. \tag{4.3}$$

Given that $x + y = w + z$ and $xy = uv$, we have

$$x = \frac{(w+z) - \sqrt{(w+z)^2 - 4uv}}{2}, \tag{4.4}$$

$$y = \frac{(w+z) + \sqrt{(w+z)^2 - 4uv}}{2}. \tag{4.5}$$

Thus,

$$y - x = \sqrt{(w+z)^2 - 4uv}, \tag{4.6}$$

$$\frac{x}{y-x} = -\frac{1}{2} + \frac{1}{2}\frac{(w+z)}{\sqrt{(w+z)^2 - 4uv}}, \tag{4.7}$$

$$\frac{y}{y-x} = \frac{1}{2} + \frac{1}{2}\frac{(w+z)}{\sqrt{(w+z)^2 - 4uv}}. \tag{4.8}$$

Substituting (4.6), (4.7) and (4.8) into (4.2), we obtain

$$\begin{aligned}
&-\frac{x}{y-x}e^{(y-x)t} + \frac{y}{y-x} \\
&= \frac{e^{t\sqrt{(w+z)^2-4uv}} + 1}{2} - \frac{(w+z)}{\sqrt{(w+z)^2 - 4uv}}\frac{e^{t\sqrt{(w+z)^2-4uv}} - 1}{2} \\
&= \frac{1}{e^{-\frac{t}{2}\sqrt{(w+z)^2-4uv}}}\left(\cosh\left(\frac{t}{2}\sqrt{(w+z)^2 - 4uv}\right)\right. \\
&\qquad \left. - \frac{(w+z)}{\sqrt{(w+z)^2 - 4uv}}\sinh\left(\frac{t}{2}\sqrt{(w+z)^2 - 4uv}\right)\right).
\end{aligned}$$

Similarly, plugging (4.6), (4.7) and (4.8) into (4.3), we get

$$\begin{aligned}
&-\frac{x}{y-x} + \frac{y}{y-x}e^{(x-y)t} \\
&= \frac{1}{e^{\frac{t}{2}\sqrt{(w+z)^2-4uv}}}\left(\cosh\left(\frac{t}{2}\sqrt{(w+z)^2 - 4uv}\right)\right. \\
&\qquad \left. - \frac{(w+z)}{\sqrt{(w+z)^2 - 4uv}}\sinh\left(\frac{t}{2}\sqrt{(w+z)^2 - 4uv}\right)\right).
\end{aligned}$$

Therefore,

$$(1+xF(x,y;t))(1+yF(x,y;t)) = \left(\cosh\left(\frac{t}{2}\sqrt{(w+z)^2-4uv}\right)\right.$$
$$\left. - \frac{(w+z)}{\sqrt{(w+z)^2-4uv}}\sinh\left(\frac{t}{2}\sqrt{(w+z)^2-4uv}\right)\right)^{-2}, \tag{4.9}$$

where $x+y=w+z$ and $xy=uv$.

Substituting (4.9) into (4.1) yields the generating function for $P_n(u,v,w,z \mid \alpha,\beta)$ as stated in Theorem 4.1. This completes the proof. ∎

Many consequences can be derived from Theorem 4.1. By setting $\alpha = 0$ in Theorem 4.1, and applying (2.24) , we obtain the following result:

Theorem 4.2. *We have*

$$\sum_{n\geq 0}\left(\sum_{\sigma\in\mathfrak{S}_n}(uv)^{L(\sigma)}w^{\mathrm{lrda}(\sigma)}z^{\mathrm{dd}(\sigma)}\beta^{\mathrm{rlmin}(\sigma)}\right)\frac{t^n}{n!} = e^{\frac{1}{2}\beta(w-z)t}\times\left(\cosh\left(\frac{t}{2}\sqrt{(w+z)^2-4uv}\right)\right.$$
$$\left.-\frac{w+z}{\sqrt{(w+z)^2-4uv}}\sinh\left(\frac{t}{2}\sqrt{(w+z)^2-4uv}\right)\right)^{-\beta}.$$

Theorem 4.2 provides a unified extension of the generating functions for left peaks, left-right double ascents and double ascents. More precisely, we obtain

- The β-extension of the generating function (2.2) for left peaks established by Gessel [30, Sequence A008971] (Replace v with u and then set $w=z=v$ in Theorem 4.2.)

$$\sum_{n\geq 0}\left(\sum_{\sigma\in\mathfrak{S}_n}u^{2L(\sigma)}v^{n-2L(\sigma)}\beta^{\mathrm{rlmin}(\sigma)}\right)\frac{t^n}{n!}$$
$$=\left(\frac{\sqrt{v^2-u^2}}{\sqrt{v^2-u^2}\cosh\left(t\sqrt{v^2-u^2}\right)-v\sinh\left(t\sqrt{v^2-u^2}\right)}\right)^{\beta}. \tag{4.10}$$

- The β-extension of the generating function (2.19) for left-right double ascents (Set $u=v=z=1$ and then replace w with u in Theorem 4.2.)

$$\sum_{n\geq 0}\left(\sum_{\sigma\in\mathfrak{S}_n}u^{\mathrm{lrda}(\sigma)}\beta^{\mathrm{rlmin}(\sigma)}\right)\frac{t^n}{n!} = e^{\frac{1}{2}\beta(u-1)t}$$
$$\times\left(\frac{\sqrt{(u+1)^2-4}}{\sqrt{(u+1)^2-4}\cosh\left(\frac{t}{2}\sqrt{(u+1)^2-4}\right)-(1+u)\sinh\left(\frac{t}{2}\sqrt{(u+1)^2-4}\right)}\right)^{\beta}. \tag{4.11}$$

- The α-extension of the generating function (2.17) for double ascents (Set $u = v = w = 1$, then replace z by u in Theorem 4.2, and finally take reverse of a permutation.)

$$\sum_{n\geq 0}\left(\sum_{\sigma\in\mathfrak{S}_n} u^{\mathrm{da}(\sigma)}\alpha^{\mathrm{lrmin}(\sigma)}\right)\frac{t^n}{n!} = e^{\frac{1}{2}\alpha(1-u)t}$$

$$\times\left(\frac{\sqrt{(u+1)^2-4}}{\sqrt{(u+1)^2-4}\cosh\left(\frac{t}{2}\sqrt{(u+1)^2-4}\right)-(1+u)\sinh\left(\frac{t}{2}\sqrt{(u+1)^2-4}\right)}\right)^{\alpha}. \tag{4.12}$$

Replacing v with u and then setting $w = z = v$ in Theorem 4.1 and applying (2.21), we obtain

Theorem 4.3. *We have*

$$\sum_{n\geq 0}\left(\sum_{\sigma\in\mathfrak{S}_{n+1}} u^{2M(\sigma)}v^{n-2M(\sigma)}\alpha^{\mathrm{lrmin}(\sigma)-1}\beta^{\mathrm{rlmin}(\sigma)-1}\right)\frac{t^n}{n!}$$

$$=\left(\frac{\sqrt{v^2-u^2}}{\sqrt{v^2-u^2}\cosh\left(t\sqrt{v^2-u^2}\right)-v\sinh\left(t\sqrt{v^2-u^2}\right)}\right)^{\alpha+\beta}.$$

By setting $v = \alpha = \beta = 1$ in Theorem 4.3, replacing u with $\sqrt{u}$, and performing integration on both sides with respect to t, we retrieve the generating function (2.3) for peaks.

Setting $u = v = z = 1$ and replacing w with u in Theorem 4.1 and using (2.21) gives the following theorem.

Theorem 4.4. *We have*

$$\sum_{n\geq 0}\left(\sum_{\sigma\in\mathfrak{S}_{n+1}} u^{\mathrm{rda}(\sigma)}\alpha^{\mathrm{lrmin}(\sigma)-1}\beta^{\mathrm{rlmin}(\sigma)-1}\right)\frac{t^n}{n!} = e^{\frac{1}{2}(\beta-\alpha)(u-1)t}$$

$$\times\left(\frac{\sqrt{(u+1)^2-4}}{\sqrt{(u+1)^2-4}\cosh\left(\frac{t}{2}\sqrt{(u+1)^2-4}\right)-(1+u)\sinh\left(\frac{t}{2}\sqrt{(u+1)^2-4}\right)}\right)^{\alpha+\beta}.$$

Setting $\alpha = \beta = 1$ in Theorem 4.4 and performing integration on both sides with respect to t, we recover the generating function (2.18) for right double ascents.

5 A formula for $P_n(u, v, w, z \mid \alpha, \beta)$ with $\alpha + \beta = -1$

This section is dedicated to deriving an explicit enumeration for $P_n(u, v, w, z \mid \alpha, \beta)$ when $\alpha + \beta = -1$ by utilizing Theorem 4.1.

Theorem 5.1. *When $\alpha+\beta=-1$ and for $n\geq 1$,*

$$\begin{aligned}
&2^nP_n(u,v,w,z\mid\alpha,\beta) \\
&=\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\binom{n}{2k}((w+z)^2-4uv)^k(\beta-\alpha)^{n-2k}(w-z)^{n-2k} \\
&\quad-(w+z)\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\binom{n}{2k+1}((w+z)^2-4uv)^k(\beta-\alpha)^{n-2k-1}(w-z)^{n-2k-1}.
\end{aligned}\tag{5.1}$$

Proof. When $\alpha+\beta=-1$, and set $r=\sqrt{(w+z)^2-4uv}$, Theorem 4.1 gives

$$\begin{aligned}
&\sum_{n\geq 0}P_n(u,v,w,z\mid\alpha,\beta)\frac{t^n}{n!} \\
&=e^{\frac{1}{2}(\beta-\alpha)(w-z)t}\times\left(\cosh\left(\frac{1}{2}rt\right)-\frac{w+z}{r}\sinh\left(\frac{1}{2}rt\right)\right) \\
&=e^{\frac{1}{2}(\beta-\alpha)(w-z)t}\times\left(\sum_{n\geq 0}\frac{r^{2n}}{2^{2n}}\frac{t^{2n}}{(2n)!}-(w+z)\left(\sum_{n\geq 0}\frac{r^{2n}}{2^{2n+1}}\frac{t^{2n+1}}{(2n+1)!}\right)\right) \\
&=\left(\sum_{n\geq 0}\frac{(\beta-\alpha)^n(w-z)^n}{2^n}\frac{t^n}{n!}\right)\times\left(\sum_{n\geq 0}\frac{r^{2n}}{2^{2n}}\frac{t^{2n}}{(2n)!}-(w+z)\left(\sum_{n\geq 0}\frac{r^{2n}}{2^{2n+1}}\frac{t^{2n+1}}{(2n+1)!}\right)\right) \\
&=\sum_{n\geq 0}\frac{t^n}{n!}\left(\sum_{k\geq 0}\binom{n}{2k}(\beta-\alpha)^{n-2k}(w-z)^{n-2k}\frac{r^{2k}}{2^n}\right. \\
&\qquad\left.-(w+z)\sum_{k\geq 0}\binom{n}{2k+1}(\beta-\alpha)^{n-2k-1}(w-z)^{n-2k-1}\frac{r^{2k}}{2^n}\right).
\end{aligned}$$

Comparing the coefficients of $t^n/n!$ on both sides yields (5.1). This completes the proof. ∎

Setting $w=z=u=1$, replacing v with u in Theorem 5.1, and applying (2.21) yields Theorem 1.5. Theorem 1.6 follows from Theorem 1.5 by setting $\alpha=0$.

By choosing $\alpha=\beta=-1/2$ in Theorem 5.1, we obtain

Theorem 5.2. *For $n\geq 1$,*

$$\begin{aligned}
&2^nP_n\left(u,v,w,z\mid-\frac{1}{2},-\frac{1}{2}\right) \\
&=\begin{cases}((w+z)^2-4uv)^{\lfloor\frac{n}{2}\rfloor}, & \text{if } n \text{ is even},\\ -(w+z)((w+z)^2-4uv)^{\lfloor\frac{n}{2}\rfloor}, & \text{if } n \text{ is odd}.\end{cases}
\end{aligned}$$

Setting $u=v=z=1$, replacing w with u in Theorem 5.2, and employing (2.21) yields Theorem 1.7.

6 Two relations between $P_n(u,v,w,z \mid \alpha,\beta)$ and $A_n(x,y \mid \alpha,\beta)$

In this section, we begin by presenting two relations between $P_n(u,v,w,z \mid \alpha,\beta)$ and $A_n(x,y \mid \alpha,\beta)$ and giving their proofs. Subsequently, we derive several consequences from these connections. These specific derivations will not only yield (α,β)-extensions of the related relations associated with the Eulerian polynomial due to Stembridge, Petersen, Brändén and Zhuang, but will also provide the left peak version of Stembridge's formula, the peak version of Petersen's formula and their (α,β)-extensions.

Theorem 6.1. *For $n \geq 1$,*

$$P_n(u,v,w,z \mid \alpha,\beta) = \sum_{k=0}^{n} \binom{n}{k} A_k\left(x,y \mid \frac{\alpha+\beta}{2}, \frac{\alpha+\beta}{2}\right) \frac{(\beta-\alpha)^{n-k}(w-z)^{n-k}}{2^{n-k}}, \tag{6.1}$$

where $x+y=w+z$ and $xy=uv$.

Proof. Combining Theorem 1.1 and Theorem 1.4, we derive that

$$\begin{aligned}
&\sum_{n\geq 0} P_n(u,v,w,z \mid \alpha,\beta)\frac{t^n}{n!} \\
&= e^{\frac{1}{2}(\beta-\alpha)(w-z)t} \sum_{k\geq 0} A_k\left(x,y \mid \frac{\alpha+\beta}{2}, \frac{\alpha+\beta}{2}\right)\frac{t^k}{k!} \\
&= \left(\sum_{m\geq 0} \frac{(\beta-\alpha)^m(w-z)^m}{2^m}\frac{t^m}{m!}\right)\left(\sum_{k\geq 0} A_k\left(x,y \mid \frac{\alpha+\beta}{2}, \frac{\alpha+\beta}{2}\right)\frac{t^k}{k!}\right) \\
&= \sum_{n\geq 0} \frac{t^n}{n!}\left(\sum_{k=0}^{n} \binom{n}{k} A_k\left(x,y \mid \frac{\alpha+\beta}{2}, \frac{\alpha+\beta}{2}\right)\frac{(\beta-\alpha)^{n-k}(w-z)^{n-k}}{2^{n-k}}\right).
\end{aligned} \tag{6.2}$$

Equating the coefficients of $t^n/n!$ yields the desired result. This completes the proof. ∎

Theorem 6.2. *For $n \geq 1$,*

$$\begin{aligned}
&P_n(u,v,w,z \mid \alpha,\beta) \\
&= \sum_{k=0}^{n} \binom{n}{k} A_k(x,y \mid 0,\alpha+\beta)(\alpha x-\beta y+(\beta-\alpha)w)^{n-k} && (6.3)\\
&= \sum_{k=0}^{n} \binom{n}{k} A_k(x,y \mid \alpha+\beta,0)(\alpha y-\beta x+(\beta-\alpha)w)^{n-k} && (6.4)
\end{aligned}$$

where $x+y=w+z$ and $xy=uv$.

Proof. By (1.8), we see that

$$1 + xF(x, y; t) = \frac{(x-y)e^{xt}}{xe^{yt} - ye^{xt}} \quad \text{and} \quad 1 + yF(x, y; t) = \frac{(x-y)e^{yt}}{xe^{yt} - ye^{xt}}.$$

Hence

$$\begin{aligned}
&((1 + xF(x, y; t))(1 + yF(x, y; t)))^{\frac{\alpha+\beta}{2}} e^{\frac{1}{2}(\beta-\alpha)(w-z)t} \\
&= \left(\frac{x-y}{xe^{yt} - ye^{xt}}\right)^{\alpha+\beta} e^{\left(\frac{(\beta-\alpha)(w-z)}{2} + \frac{(\alpha+\beta)(x+y)}{2}\right)t} \\
&= (1 + yF(x, y; t))^{\alpha+\beta} e^{\left(\frac{(\beta-\alpha)(w-z)}{2} + \frac{(\alpha+\beta)(x-y)}{2}\right)t} \qquad (6.5) \\
&= (1 + xF(x, y; t))^{\alpha+\beta} e^{\left(\frac{(\beta-\alpha)(w-z)}{2} + \frac{(\alpha+\beta)(y-x)}{2}\right)t} \qquad (6.6)
\end{aligned}$$

Substituting (1.7) into (6.5), and using Theorem 1.4, we deduce that

$$\begin{aligned}
&\sum_{n\geq 0} P_n(u, v, w, z \mid \alpha, \beta) \frac{t^n}{n!} \\
&= (1 + yF(x, y; t))^{\alpha+\beta} e^{\left(\frac{(\beta-\alpha)(w-z)}{2} + \frac{(\alpha+\beta)(x-y)}{2}\right)t} \\
&= \left(\sum_{k\geq 0} A_k(x, y \mid 0, \alpha+\beta) \frac{t^k}{k!}\right) \left(\sum_{m\geq 0} (\alpha x - \beta y + (\beta-\alpha)w)^m \frac{t^m}{m!}\right),
\end{aligned}$$

where the last line follows from $x + y = w + z$. Equating the coefficients of $t^n/n!$ yields relation (6.3).

Plugging (1.7) into (6.6), and applying Theorem 1.4, then we derive that

$$\begin{aligned}
&\sum_{n\geq 0} P_n(u, v, w, z \mid \alpha, \beta) \frac{t^n}{n!} \\
&= (1 + xF(x, y; t))^{\alpha+\beta} e^{\left(\frac{(\beta-\alpha)(w-z)}{2} + \frac{(\alpha+\beta)(y-x)}{2}\right)t} \\
&= \left(\sum_{k\geq 0} A_k(x, y \mid \alpha+\beta, 0) \frac{t^k}{k!}\right) \left(\sum_{m\geq 0} (\alpha y - \beta x + (\beta-\alpha)w)^m \frac{t^m}{m!}\right).
\end{aligned}$$

Equating the coefficients of $t^n/n!$ yields the relation (6.4). This completes the proof of Theorem 6.2. ∎

Several consequences can be drawn from Theorems 6.1 and 6.2.

- Setting $w = z$ and $u = v$ in Theorem 6.1, we find that

$$w = z = \frac{x+y}{2} \quad \text{and} \quad u = v = \sqrt{xy}, \qquad (6.7)$$

  and using the combinatorial definitions (2.20) and (2.21) of $A_n(x, y \mid \alpha, \beta)$ and $P_n(u, v, w, z \mid \alpha, \beta)$, respectively, we obtain Theorem 1.8.

- Setting $\alpha = \beta = 1$ and $y = 1$ in Theorem 1.8, we recover relation (2.4) due to Stembridge.
- Choosing $\alpha = 0$ in Theorem 1.8 yields Theorem 1.9.
- Setting $\beta = 1$ and $y = 1$ in Theorem 1.9, we obtain the following consequence, which can be viewed as the left peak version of Stembridge's formula.

  Theorem 6.3. *For* $n \geq 1$,

$$\sum_{\sigma \in \mathfrak{S}_{n+1}} x^{\mathrm{des}(\sigma)} \left(\frac{1}{2}\right)^{\mathrm{lrmin}(\sigma)+\mathrm{rlmin}(\sigma)-2} = \left(\frac{1+x}{2}\right)^n \sum_{\sigma \in \mathfrak{S}_n} \left(\frac{4x}{(1+x)^2}\right)^{L(\sigma)}. \tag{6.8}$$

- Setting $\alpha = \beta$, replacing u with uz, and replacing w with v in Theorem 6.1, applying (2.10), (2.20), and (2.21), we deduce the following consequence, which can be viewed as the α-extension of Zhuang's relation (2.6).

  Theorem 6.4. *For* $n \geq 1$,

$$\begin{aligned} &\sum_{\sigma \in \mathfrak{S}_n} u^{M(\sigma)} v^{\mathrm{des}(\sigma)} w^{\mathrm{asc}(\sigma)} \alpha^{\mathrm{lrmin}(\sigma)+\mathrm{rlmin}(\sigma)-2} \\ &= \sum_{\sigma \in \mathfrak{S}_n} x^{\mathrm{des}(\sigma)} y^{\mathrm{asc}(\sigma)} \alpha^{\mathrm{lrmin}(\sigma)+\mathrm{rlmin}(\sigma)-2}, \end{aligned} \tag{6.9}$$

  where

$$x = \frac{(w+v) - \sqrt{(w+v)^2 - 4uvw}}{2}, \tag{6.10}$$

  and

$$y = \frac{(w+v) + \sqrt{(w+v)^2 - 4uvw}}{2}. \tag{6.11}$$

  Note that (6.10) and (6.11) follow from (4.4) and (4.5) upon replacing u with uw and z with v.

- Setting $\alpha = 1$, $w = 1$, $a = x/y$, and $b = (y-1)/(1-x)$ in Theorem 6.4, we recover relation (2.6) established by Zhuang [43, Theorem 4.2].
- Setting $w = z$ and $u = v$ in (6.4), and using (2.21) and (2.26), we get

  Theorem 6.5. *For* $n \geq 1$,

$$\begin{aligned} &\sum_{\sigma \in \mathfrak{S}_{n+1}} (xy)^{M(\sigma)} \left(\frac{x+y}{2}\right)^{n-2M(\sigma)} \alpha^{\mathrm{lrmin}(\sigma)-1} \beta^{\mathrm{rlmin}(\sigma)-1} \\ &= \sum_{k=1}^{n} \binom{n}{k} \frac{((\alpha+\beta)(y-x))^{n-k}}{2^{n-k}} \left(\sum_{\sigma \in \mathfrak{S}_k} x^{\mathrm{des}(\sigma)+1} y^{\mathrm{asc}(\sigma)} (\alpha+\beta)^{\mathrm{lrmin}(\sigma)} \right) \\ &\quad + \left(\frac{((\alpha+\beta)(y-x))}{2}\right)^n. \end{aligned} \tag{6.12}$$

- By choosing $\alpha = 0$ in Theorem 6.5, and using (2.24), we derive the following β-extension of Petersen's relation, from which we recover relation (2.5) by setting $y = 1$ and $\beta = 1$.

Theorem 6.6. *For $n \geq 1$,*

$$\sum_{\sigma\in\mathfrak{S}_n} (xy)^{L(\sigma)} \left(\frac{x+y}{2}\right)^{n-2L(\sigma)} \beta^{\mathrm{rlmin}(\sigma)}$$
$$= \sum_{k=1}^{n} \binom{n}{k} \frac{(\beta(y-x))^{n-k}}{2^{n-k}} \left(\sum_{\sigma\in\mathfrak{S}_k} x^{\mathrm{des}(\sigma)+1} y^{\mathrm{asc}(\sigma)} \beta^{\mathrm{lrmin}(\sigma)}\right)$$
$$+ \left(\frac{(\beta(y-x))}{2}\right)^{n}. \tag{6.13}$$

- By setting $y = 1$ and $\alpha = \beta = 1$ in Theorem 6.5, we derive the following relation, which can be viewed as the peak version of Petersen's formula (2.5).

Theorem 6.7. *For $n \geq 1$,*

$$\left(\frac{1+x}{2}\right)^{n} \sum_{\sigma\in\mathfrak{S}_{n+1}} \left(\frac{4x}{(1+x)^2}\right)^{M(\sigma)}$$
$$= \sum_{k=1}^{n} \binom{n}{k} (1-x)^{n-k} \sum_{\sigma\in\mathfrak{S}_k} x^{\mathrm{des}(\sigma)+1} 2^{\mathrm{lrmin}(\sigma)} + (1-x)^n.$$

- Replacing u with uw and z with v in (6.4), and applying (2.21) and (2.26), we obtain the following consequence, which can be viewed as an (α, β)-extension of the peak version of Zhuang's relation (2.9).

Theorem 6.8. *For $n \geq 1$,*

$$\sum_{\sigma\in\mathfrak{S}_{n+1}} u^{M(\sigma)} v^{\mathrm{des}(\sigma)} w^{\mathrm{asc}(\sigma)} \alpha^{\mathrm{lrmin}(\sigma)-1} \beta^{\mathrm{rlmin}(\sigma)-1} \tag{6.14}$$
$$= \sum_{k=1}^{n} \binom{n}{k} (\alpha y - \beta x + (\beta-\alpha)w)^{n-k} \left(\sum_{\sigma\in\mathfrak{S}_k} x^{\mathrm{des}(\sigma)+1} y^{\mathrm{asc}(\sigma)} (\alpha+\beta)^{\mathrm{lrmin}(\sigma)}\right)$$
$$+ (\alpha y - \beta x + (\beta-\alpha)w)^{n}, \tag{6.15}$$

where x and y are given by (6.10) and (6.11), respectively.

- Setting $\alpha = 0$ in Theorem 6.8 yields

Theorem 6.9. *For* $n \geq 1$,

$$\sum_{\sigma \in \mathfrak{S}_n} u^{L(\sigma)} v^{\mathrm{des}(\sigma)} w^{n-\mathrm{des}(\sigma)} \beta^{\mathrm{rlmin}(\sigma)}$$
$$= \sum_{k=1}^{n} \binom{n}{k} (\beta(w-x))^{n-k} \left(\sum_{\sigma \in \mathfrak{S}_k} x^{\mathrm{des}(\sigma)+1} y^{\mathrm{asc}(\sigma)} \beta^{\mathrm{lrmin}(\sigma)} \right) + (\beta(w-x))^n , \tag{6.16}$$

where x *and* y *are given by* (6.10) *and* (6.11)*, respectively.*

- Setting $\beta = 1$, $w = 1$, $a = x/y$ and $b = (y-1)/(1-x)$ in Theorem 6.9, we recover the relation (2.9) due to Zhuang [43, Theorem 4.7].

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