



Acta Mathematica Scientia, 2025, **45B**(1): 72–95
https://cstr.cn/32227.14.20250106
https://doi.org/10.1007/s10473-025-0106-x
©Innovation Academy for Precision Measurement Science
and Technology, Chinese Academy of Sciences, 2025

Acta Mathematica Scientia
数学物理学报
http://actams.apm.ac.cn

ON GENERALIZED KISSING NUMBERS OF CONVEX BODIES

Dedicated to the memory of Professor Delin REN

Yiming LI (李奕明)

Center for Applied Mathematics, Tianjin University, Tianjin 300072, China
E-mail: xiaozhuang@tju.edu.cn

Chuanming ZONG (宗传明)*

Center for Applied Mathematics, Tianjin University, Tianjin 300072, China
E-mail: cmzong@tju.edu.cn

Abstract In 1694, Gregory and Newton proposed the problem to determine the kissing number of a rigid material ball. This problem and its higher dimensional generalization have been studied by many mathematicians, including Minkowski, van der Waerden, Hadwiger, Swinnerton-Dyer, Watson, Levenshtein, Odlyzko, Sloane and Musin. In this paper, we introduce and study a further generalization of the kissing numbers for convex bodies and obtain some exact results, in particular for balls in dimensions three, four and eight.

Keywords convex body; generalized kissing number; E_8 lattice

MSC2020 52C17; 11H31

1 Introduction

In 1694, Gregory and Newton discussed the following problem: Can a rigid material ball be brought into contact with thirteen other such balls of the same size? Gregory believed “yes”, while Newton thought “no”. The solution of this problem has a complicated history! Several authors claimed proofs that the largest number of nonoverlapping unit balls which can be brought into contact with a fixed one is twelve (see Hoppe [7], Günter [4], Schütte and van der Waerden [16] and Leech [8]). However, only Schütte and van der Waerden’s proof is complete!

Let K be an n -dimensional convex body and let C be an n -dimensional centrally symmetric convex body centered at the origin of $\mathbb{E}^n$. Let $\kappa(K)$ and $\kappa^*(K)$ denote the translative kissing number and the lattice kissing number of K , respectively. In other words, $\kappa(K)$ is the maximum number of nonoverlapping translates $K + \mathbf{x}$ that can touch K at its boundary, and $\kappa^*(K)$ is defined similarly, with the restriction that the translates are members of a lattice packing of K . It is easy to see that $\kappa^*(K) \leq \kappa(K)$ holds for every convex body K .

Received August 16, 2024. This work was supported by the National Natural Science Foundation of China (12226006, 11921001) and the Natural Key Research and Development Program of China (2018YFA0704701).

*Corresponding author

In 1904, Minkowski [10] defined

$$D(K) = \{\mathbf{x} - \mathbf{y} : \mathbf{x} \in K, \mathbf{y} \in K\}$$

and proved that

$$\kappa^*(K) = \kappa^*(D(K))$$

and

$$\kappa(K) = \kappa(D(K)).$$

Clearly, $D(K)$ is always centrally symmetric and centered at the origin of $\mathbb{E}^n$. Three years later, Minkowski [11] proved that

$$\kappa^*(K) \leq 3^n - 1$$

holds for every n -dimensional convex body K . In 1957, Hadwiger [5] improved Minkowski's upper bound to

$$\kappa(K) \leq 3^n - 1.$$

Let B^n denote the n -dimensional unit ball centered at the origin of $\mathbb{E}^n$. The kissing numbers $\kappa^*(B^n)$ and $\kappa(B^n)$ have been studied by many authors (see [1, 9, 12–14, 18]). The known exact results are summarized in the following Table 1.

Table 1									
n	2	3	4	5	6	7	8	9	24
$\kappa^*(B^n)$	6	12	24	40	72	126	240	272	196560
$\kappa(B^n)$	6	12	24	??	??	??	240	??	196560

It is well-known that each centrally symmetric convex body C centered at the origin defines a metric $\|\cdot\|_C$ on $\mathbb{R}^n$ by

$$\|\mathbf{x}, \mathbf{y}\|_C = \|\mathbf{x} - \mathbf{y}\|_C = \min\{r : r > 0, \mathbf{x} - \mathbf{y} \in rC\}.$$

Especially, we use $\|\cdot\|$ to denote the metric defined by B^n .

Clearly, the kissing numbers $\kappa(C)$ and $\kappa^*(C)$ only consider the closest neighbours of C in translative packings and lattice packings, respectively. In fact, in many physic situations the neighbours in a larger region also have effect on C . For example, in some potential energy models. Therefore, it is reasonable to make following generalizations: For $\alpha \geq 0$, we define $\kappa_\alpha(C)$ to be the maximum number of translates $C + \mathbf{x}$ which can be packed into the region $(3 + \alpha)C \setminus \text{int}(C)$ and define $\kappa_\alpha^*(C)$ to be the maximum number of translates $C + \mathbf{x}$ which can be packed into the region $(3 + \alpha)C \setminus \text{int}(C)$ where all the translative vectors simultaneously belong to a lattice, where $\text{int}(C)$ denotes the interior of C .

In this paper, among other things, we will prove the following results:

Theorem 1 In $\mathbb{E}^2$, we have

$$\kappa_\alpha^*(B^2) = \begin{cases} 6, & 0 \leq \alpha < 2\sqrt{2} - 2, \\ 8, & 2\sqrt{2} - 2 \leq \alpha < 2\sqrt{3} - 2, \\ 12, & \alpha = 2\sqrt{3} - 2. \end{cases}$$

Theorem 2 In $\mathbb{E}^3$, we have

$$\kappa_{\alpha}^*(B^3) = \begin{cases} 12, & 0 \leq \alpha < \frac{4}{3}\sqrt{3} - 2, \\ 14, & \frac{4}{3}\sqrt{3} - 2 \leq \alpha < 2\sqrt{2} - 2, \\ 20, & \alpha = 2\sqrt{2} - 2. \end{cases}$$

Theorem 3 In $\mathbb{E}^4$, we have

$$\kappa_{\alpha}^*(B^4) = \begin{cases} 30, & \sqrt{6} - 2 \leq \alpha < 2\sqrt{2} - 2, \\ 50, & \alpha = 2\sqrt{2} - 2. \end{cases}$$

Theorem 4 In $\mathbb{E}^8$, when $\alpha = 2\sqrt{2} - 2$ we have $\kappa_{\alpha}^*(B^8) = 2400$.

2 Some Basic Lemmas

In 1907, Minkowski [11] studied the lattice kissing number of an n -dimensional convex body and proved the following result.

Lemma 2.1 If K is an n -dimensional convex body, then $\kappa^*(K) \leq 3^n - 1$, where the equality holds if and only if K is a parallelepiped. If C is an n -dimensional centrally symmetric strictly convex body centered at $\mathbf{o}$, then $\kappa^*(C) \leq 2(2^n - 1)$.

For a non-negative number α and a packing lattice Λ of B^n , we define

$$X(\alpha, \Lambda) = \{\mathbf{v} : 2 \leq \|\mathbf{v}\| \leq 2 + \alpha, \mathbf{v} \in \Lambda\}.$$

Next we introduce two technical results which will be frequently used in this paper.

Lemma 2.2 When $0 \leq \alpha < 2\sqrt{2} - 2$, we have

$$\kappa_{\alpha}^*(B^n) \leq 2(2^n - 1).$$

Proof On the contrary, suppose that there is such a positive α which is less than $2\sqrt{2} - 2$ and a suitable lattice Λ satisfying

$$\text{card}\{X(\alpha, \Lambda)\} \geq 2^{n+1}.$$

For convenience, we assume that $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n$ is a basis of Λ and say two lattice vectors

$$\mathbf{v} = z_1\mathbf{a}_1 + z_2\mathbf{a}_2 + \dots + z_n\mathbf{a}_n$$

and

$$\mathbf{v}' = z'_1\mathbf{a}_1 + z'_2\mathbf{a}_2 + \dots + z'_n\mathbf{a}_n$$

are equivalent if $z_i - z'_i \equiv 0 \pmod{2}$ for all $i = 1, 2, \dots, n$. In other words, $\mathbf{v}$ and $\mathbf{v}'$ are equivalent if and only if $\frac{1}{2}(\mathbf{v} - \mathbf{v}') \in \Lambda$. Clearly, this relation divides the points of Λ into 2^n classes.

Since $X(\alpha, \Lambda)$ is centrally symmetric and $\text{card}\{X(\alpha, \Lambda)\} \geq 2^{n+1}$, it contains 2^n lattice points $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_{2^n}$ satisfying

$$\mathbf{v}_i \neq \pm \mathbf{v}_j, \quad i \neq j.$$

If one of the 2^n points, say $\mathbf{v}_1$, is equivalent to $\mathbf{o}$, then we get $\frac{1}{2}\mathbf{v}_1 \in \Lambda$ and $1 \leq \|\frac{1}{2}\mathbf{v}_1\| < \sqrt{2} < 2$, which contradicts the assumption that $B^n + \Lambda$ is a packing.

Since all $\mathbf{o}, \mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_{2^n}$ belong to at most 2^n classes, two of them must belong to the same class. Without loss of generality, we may assume that $\mathbf{v}_1$ and $\mathbf{v}_2$ are equivalent. Then, we have

$$2 \leq \|\mathbf{v}_i\| < 2\sqrt{2}, \quad i = 1, 2 \quad (2.1)$$

and

$$\|\frac{1}{2}(\mathbf{v}_1 \pm \mathbf{v}_2)\| \geq 2. \quad (2.2)$$

By (2.1), we get

$$\|\mathbf{v}_1\|^2 + \|\mathbf{v}_2\|^2 < 16. \quad (2.3)$$

By (2.2), we get

$$\|\mathbf{v}_1\|^2 + \|\mathbf{v}_2\|^2 - 2\langle \mathbf{v}_1, \mathbf{v}_2 \rangle \geq 16 \quad (2.4)$$

and

$$\|\mathbf{v}_1\|^2 + \|\mathbf{v}_2\|^2 + 2\langle \mathbf{v}_1, \mathbf{v}_2 \rangle \geq 16, \quad (2.5)$$

where $\langle \mathbf{v}_1, \mathbf{v}_2 \rangle$ denotes the inner product of $\mathbf{v}_1$ and $\mathbf{v}_2$. Then, by (2.4) and (2.5) we obtain

$$\|\mathbf{v}_1\|^2 + \|\mathbf{v}_2\|^2 \geq 16,$$

which contradicts (2.3). Therefore, for $0 \leq \alpha < 2\sqrt{2} - 2$, we have

$$\kappa_\alpha^*(B^n) \leq 2(2^n - 1).$$

Lemma 2.2 is proved. $\square$

Remark 2.1 Writing

$$X = \{\mathbf{v} : 2 \leq \|\mathbf{v}\| \leq 2\sqrt{2}, \mathbf{v} \in \Lambda\}$$

and repeating the calculations (2.1)–(2.5) one can deduce that two lattice points $\mathbf{v}_1, \mathbf{v}_2 \in X$ satisfying $\mathbf{v}_1 \neq \pm \mathbf{v}_2$ belong to the same equivalent class if and only if

$$\|\mathbf{v}_1\| = 2\sqrt{2}, \quad \|\mathbf{v}_2\| = 2\sqrt{2} \quad \text{and} \quad \langle \mathbf{v}_1, \mathbf{v}_2 \rangle = 0.$$

Lemma 2.3 When $\alpha < 2\sqrt{3} - 2$, the set $X(\alpha, \Lambda)$ contains no four collinear points.

Proof On the contrary, suppose $X(\alpha, \Lambda)$ has four collinear points $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ and $\mathbf{v}_4$. Without loss of generality, we may assume that $n = 2$ and all $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ and $\mathbf{v}_4$ have the same x -coordinates, namely $\mathbf{v}_1 = (x_0, y_1)$, $\mathbf{v}_2 = (x_0, y_2)$, $\mathbf{v}_3 = (x_0, y_3)$ and $\mathbf{v}_4 = (x_0, y_4)$. Furthermore, we may also assume that $x_0 \geq 0$ and

$$y_1 - y_2 = y_2 - y_3 = y_3 - y_4 \geq 2, \quad (2.6)$$

since $B^2 + \Lambda$ is a packing.

If $x_0 \geq \sqrt{3}$, we get

$$y_1 - y_4 \leq 2\sqrt{(\alpha + 2)^2 - x_0^2} < 2\sqrt{(2\sqrt{3})^2 - 3} = 6 \quad (2.7)$$

and therefore

$$y_1 - y_2 = y_2 - y_3 = y_3 - y_4 < 2,$$

which contradicts (2.6).

If $x_0 < \sqrt{3}$, we get

$$y_2 - y_3 \geq 2\sqrt{4 - x_0^2} \quad (2.8)$$

since both $\mathbf{v}_2$ and $\mathbf{v}_3$ belong to Λ . On the other hand, since both $\mathbf{v}_1$ and $\mathbf{v}_4$ belong to $\text{int}(2\sqrt{3}B^2)$, we get

$$y_1 - y_4 < 2\sqrt{12 - x_0^2}. \quad (2.9)$$

By (2.8) and (2.9) one can easily deduce

$$y_1 - y_4 < 2\sqrt{12 - x_0^2} < 6\sqrt{4 - x_0^2} \leq 3(y_2 - y_3), \quad (2.10)$$

which contradicts (2.6).

As a conclusion of the two cases, Lemma 2.3 is proved. $\square$

Remark 2.2 Writing

$$X = \{\mathbf{v} : 2 \leq \|\mathbf{v}\| \leq 2\sqrt{3}, \mathbf{v} \in \Lambda\}$$

and repeating the calculations (2.6)-(2.10) one can deduce that X has four collinear points $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ and $\mathbf{v}_4$ if and only if, up to some rotation,

$$\mathbf{v}_1 = (\sqrt{3}, 3), \quad \mathbf{v}_2 = (\sqrt{3}, 1), \quad \mathbf{v}_3 = (\sqrt{3}, -1) \quad \text{and} \quad \mathbf{v}_4 = (\sqrt{3}, -3).$$

3 Proof of Theorem 1

Theorem 1 In $\mathbb{E}^2$, we have

$$\kappa_\alpha^*(B^2) = \begin{cases} 6, & 0 \leq \alpha < 2\sqrt{2} - 2, \\ 8, & 2\sqrt{2} - 2 \leq \alpha < 2\sqrt{3} - 2, \\ 12, & \alpha = 2\sqrt{3} - 2. \end{cases}$$

Proof When $0 \leq \alpha < 2\sqrt{2} - 2$, by Lemma 2.2 we have $\kappa_\alpha^*(B^2) \leq 6$. Combining with $\kappa_\alpha^*(B^2) \geq \kappa^*(B^2) = 6$, for $0 \leq \alpha < 2\sqrt{2} - 2$ we get

$$\kappa_\alpha^*(B^2) = 6. \quad (3.1)$$

By Remark 2.1, when $\alpha = 2\sqrt{2} - 2$ one can deduce that

$$\kappa_\alpha^*(B^2) = 8, \quad (3.2)$$

where the equality holds when the corresponding lattice Λ is generated by $\mathbf{a}_1 = (2, 0)$, $\mathbf{a}_2 = (0, 2)$. In fact, the optimal lattice is unique up to some rotation.

When $2\sqrt{2} - 2 \leq \alpha < 2\sqrt{3} - 2$, we assume that $B^2 + \Lambda$ is a lattice packing attaining $\kappa_\alpha^*(B^2)$. Then we have

$$\text{card}\{X(\alpha, \Lambda)\} \geq 8.$$

Without loss of generality, by a routine argument we may assume that $\{\mathbf{a}_1, \mathbf{a}_2\}$ is a basis of Λ , $\|\mathbf{a}_1\| = 2$ and $\{\mathbf{a}_1, \mathbf{a}_2\} \subset X(\alpha, \Lambda)$.

For an arbitrary vector $\mathbf{v} \in X(\alpha, \Lambda)$ which is not $\pm\mathbf{a}_1$, the lattice Λ' generated by $\{\mathbf{a}_1, \mathbf{v}\}$ is a sublattice of Λ . Therefore, let $\det(\Lambda)$ denote the determinant of the lattice Λ , we have

$$\det(\Lambda') = g \det(\Lambda),$$

where g is a positive integer. It is easy to see that

$$\det(\Lambda') \leq \|\mathbf{a}_1\| \cdot \|\mathbf{v}\| < 2 \cdot 2\sqrt{3} = 4\sqrt{3} \quad (3.3)$$

and

$$\det(\Lambda) \geq \frac{\omega(B^2)}{\delta^*(B^2)} = 2\sqrt{3}, \quad (3.4)$$

where $\omega(B^2)$ denotes the area of B^2 and $\delta^*(B^2) = \pi/\sqrt{12}$ is the density of the densest lattice packing of B^2 . By (3.3) and (3.4) one can easily deduce that $g = 1$. Consequently, if

$$\mathbf{v} = z_1\mathbf{a}_1 + z_2\mathbf{a}_2,$$

then we must have

$$z_2 = \pm g = \pm 1. \quad (3.5)$$

Since $\pm\mathbf{a}_2 \in X(\alpha, \Lambda)$, by Lemma 2.3 one can deduce that $|z_1| \leq 2$, which means that

$$\mathbf{v} = \pm\mathbf{a}_2, \pm(\mathbf{a}_1 + \mathbf{a}_2), \pm(\mathbf{a}_1 - \mathbf{a}_2), \pm(2\mathbf{a}_1 + \mathbf{a}_2) \text{ or } \pm(2\mathbf{a}_1 - \mathbf{a}_2).$$

If both $(\mathbf{a}_1 + \mathbf{a}_2)$ and $(\mathbf{a}_1 - \mathbf{a}_2)$ belong to $X(\alpha, \Lambda)$, since $2\mathbf{a}_1 + \mathbf{a}_2, \mathbf{a}_1 + \mathbf{a}_2, \mathbf{a}_2, -\mathbf{a}_1 + \mathbf{a}_2$ are collinear, $(2\mathbf{a}_1 + \mathbf{a}_2)$ can not belong to $X(\alpha, \Lambda)$. Similarly, since $2\mathbf{a}_1 - \mathbf{a}_2, \mathbf{a}_1 - \mathbf{a}_2, -\mathbf{a}_2, -\mathbf{a}_1 - \mathbf{a}_2$ are collinear, $(2\mathbf{a}_1 - \mathbf{a}_2) \notin X(\alpha, \Lambda)$. On the other hand, if both $(2\mathbf{a}_1 + \mathbf{a}_2)$ and $(2\mathbf{a}_1 - \mathbf{a}_2)$ belong to $X(\alpha, \Lambda)$, by convexity one can deduce that $2\mathbf{a}_1 \in X(\alpha, \Lambda)$, which contradicts to $X(\alpha, \Lambda) \subset \text{int}(2\sqrt{3}B^2)$. As a conclusion of these two cases, we get

$$\text{card}\{X(\alpha, \Lambda)\} \leq 8.$$

Combining with (3.2), for $2\sqrt{2} - 2 \leq \alpha < 2\sqrt{3} - 2$ we get

$$\kappa_\alpha^*(B^2) = 8. \quad (3.6)$$

Finally, we deal with the case $\alpha = 2\sqrt{3} - 2$. If $X(\alpha, \Lambda)$ contains no four collinear points and (3.5) holds, by previous arguments we still obtain

$$\text{card}\{X(\alpha, \Lambda)\} \leq 8.$$

Therefore, the necessary condition for $\text{card}\{X(\alpha, \Lambda)\} \geq 10$ is either $X(\alpha, \Lambda)$ contains four collinear points or (3.5) does not hold.

If $X(\alpha, \Lambda)$ has four collinear vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ and $\mathbf{v}_4$, by Remark 2.2 we may assume that

$$\mathbf{v}_1 = (\sqrt{3}, 3), \quad \mathbf{v}_2 = (\sqrt{3}, 1), \quad \mathbf{v}_3 = (\sqrt{3}, -1), \quad \mathbf{v}_4 = (\sqrt{3}, -3).$$

In this case, it is easy to verify that Λ is generated by $\mathbf{a}_1 = (\sqrt{3}, 1), \mathbf{a}_2 = (\sqrt{3}, -1)$ and

$$\text{card}\{X(\alpha, \Lambda)\} = 12.$$

If there is a point $\mathbf{v} = z_1\mathbf{a}_1 + z_2\mathbf{a}_2 \in X(\alpha, \Lambda)$ with $z_2 = \pm 2$, by repeating (3.3) and (3.4) we get

$$\det(\Lambda) = \frac{\omega(B^2)}{\delta^*(B^2)} = 2\sqrt{3}.$$

In this case, Λ is the densest packing lattice of B^2 and therefore

$$\text{card}\{X(\alpha, \Lambda)\} = 12.$$

As a conclusion of the two cases, for $\alpha = 2\sqrt{3} - 2$ we have

$$\kappa_{\alpha}^*(B^2) = 12, \quad (3.7)$$

and the corresponding lattice Λ is generated by $\mathbf{a}_1 = (\sqrt{3}, 1)$ and $\mathbf{a}_2 = (\sqrt{3}, -1)$, up to some rotation.

The theorem follows from (3.1), (3.6) and (3.7). $\square$

In 2003, Zong [23] proved the following result.

Lemma 3.1 For every two-dimensional centrally symmetric convex domain C there is a parallelogram with vertices $\mathbf{o}$, $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_1 + \mathbf{v}_2$ such that

$$\|\mathbf{o}, \mathbf{v}_1\|_C = \|\mathbf{o}, \mathbf{v}_2\|_C = 2$$

and

$$2 \leq \|\mathbf{v}_1, \mathbf{v}_2\|_C = \|\mathbf{o}, \mathbf{v}_1 + \mathbf{v}_2\|_C \leq 2\sqrt{2}.$$

This result has the following corollary.

Corollary 3.1 When $\alpha = 2\sqrt{2} - 2$, $\kappa_{\alpha}^*(C) \geq 8$ holds for every two-dimensional centrally symmetric convex domain C .

4 Proof of Theorem 2

Lemma 4.1 When $\frac{4}{3}\sqrt{3} - 2 \leq \alpha < 2\sqrt{2} - 2$, we have $\kappa_{\alpha}^*(B^3) = 14$.

Proof Let Λ be the lattice generated by $\mathbf{a}_1 = (-\frac{2}{3}\sqrt{6}, \frac{2}{3}\sqrt{3}, 0)$, $\mathbf{a}_2 = (\frac{2}{3}\sqrt{6}, \frac{2}{3}\sqrt{3}, 0)$ and $\mathbf{a}_3 = (0, \frac{2}{3}\sqrt{3}, \frac{2}{3}\sqrt{6})$. When $\alpha = \frac{4}{3}\sqrt{3} - 2$, one can verify that $\text{card}\{X(\alpha, \Lambda)\} = 14$. Combining with Lemma 2.2, for $\frac{4}{3}\sqrt{3} - 2 \leq \alpha < 2\sqrt{2} - 2$ we have $\kappa_{\alpha}^*(B^3) = 14$. Lemma 4.1 is proved. $\square$

Lemma 4.2 Let $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ be three linearly independent vectors of $X(\alpha, \Lambda)$ and let Λ' denote the lattice generated by them. If $0 \leq \alpha < \frac{4}{3}\sqrt{3} - 2$, then $\det(\Lambda')/\det(\Lambda) \leq 2$.

Proof Since $\alpha < \frac{4}{3}\sqrt{3} - 2$, we have

$$\det(\Lambda') \leq \|\mathbf{v}_1\| \cdot \|\mathbf{v}_2\| \cdot \|\mathbf{v}_3\| < \left(\frac{4}{3}\sqrt{3}\right)^3.$$

On the other hand, let $\delta^*(B^3)$ denote the density of the densest three-dimensional lattice sphere packing which is $\pi/\sqrt{18}$, we have

$$\det(\Lambda) \geq \frac{\text{vol}(B^3)}{\delta^*(B^3)} = \frac{\frac{4}{3}\pi}{\frac{\pi}{\sqrt{18}}} = 4\sqrt{2}.$$

Therefore, we get

$$\det(\Lambda')/\det(\Lambda) < \left(\frac{4}{3}\sqrt{3}\right)^3 / 4\sqrt{2} < 3.$$

In other words, $\det(\Lambda')/\det(\Lambda)$ only can take two values, one or two. Lemma 4.2 is proved. $\square$

Lemma 4.3 If $\alpha < \frac{4}{3}\sqrt{3} - 2$ and $\text{card}\{X(\alpha, \Lambda)\} = 14$, then Λ has a basis $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ such that

$$X(\alpha, \Lambda) = \{\pm\mathbf{a}_1, \pm\mathbf{a}_2, \pm\mathbf{a}_3, \pm(\mathbf{a}_1 + \mathbf{a}_2), \pm(\mathbf{a}_1 + \mathbf{a}_3), \pm(\mathbf{a}_2 + \mathbf{a}_3), \pm(\mathbf{a}_1 + \mathbf{a}_2 + \mathbf{a}_3)\}.$$

Proof Suppose

$$X(\alpha, \Lambda) = \{\pm\mathbf{v}_1, \pm\mathbf{v}_2, \pm\mathbf{v}_3, \pm\mathbf{v}_4, \pm\mathbf{v}_5, \pm\mathbf{v}_6, \pm\mathbf{v}_7\}.$$

By Remark 2.1, $\mathbf{v}_i$ cannot be equivalent to $\pm\mathbf{v}_j$ for $i \neq j$, since $\alpha < \frac{4}{3}\sqrt{3} - 2$. Therefore, for a basis $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ of Λ , we may assume $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5, \mathbf{v}_6, \mathbf{v}_7$ are equivalent to $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_1 + \mathbf{a}_2, \mathbf{a}_1 + \mathbf{a}_3, \mathbf{a}_2 + \mathbf{a}_3, \mathbf{a}_1 + \mathbf{a}_2 + \mathbf{a}_3$, respectively.

It is easy to verify that, we can expand to a basis of Λ based on $\{\mathbf{v}_1, \mathbf{v}_2\}$. Without loss of generality, we assume $\mathbf{v}_1 = \pm\mathbf{a}_1, \mathbf{v}_2 = \pm\mathbf{a}_2$. By Lemma 4.2, we have $\mathbf{v}_3 = z_1\mathbf{a}_1 + z_2\mathbf{a}_2 \pm \mathbf{a}_3$, where z_1, z_2 are even. Since $\{\mathbf{a}_1, \mathbf{a}_2, z_1\mathbf{a}_1 + z_2\mathbf{a}_2 \pm \mathbf{a}_3\}$ is also a basis of Λ , we assume $\mathbf{v}_3 = \pm\mathbf{a}_3$ without loss of generality.

Therefore, for $\mathbf{v} = z_1\mathbf{a}_1 + z_2\mathbf{a}_2 + z_3\mathbf{a}_3$ belonging to $X(\alpha, \Lambda)$, we have $\|z_i\| \leq 2$ for $i = 1, 2, 3$, by Lemma 4.2. Thus, we have

$$\begin{aligned} \mathbf{v}_4 &= \pm\mathbf{a}_1 \pm \mathbf{a}_2 + z\mathbf{a}_3, & z &= 0 \text{ or } \pm 2, \\ \mathbf{v}_5 &= \pm\mathbf{a}_1 + z'\mathbf{a}_2 \pm \mathbf{a}_3, & z' &= 0 \text{ or } \pm 2, \\ \mathbf{v}_6 &= z''\mathbf{a}_1 \pm \mathbf{a}_2 \pm \mathbf{a}_3, & z'' &= 0 \text{ or } \pm 2, \end{aligned}$$

and $\mathbf{v}_7 = \pm\mathbf{a}_1 \pm \mathbf{a}_2 \pm \mathbf{a}_3$.

Suppose one of z, z', z'' is ± 2 , without loss of generality, say $z = \pm 2$. Furthermore, we may assume $\mathbf{v}_4 = \pm(\mathbf{a}_1 + \mathbf{a}_2 + 2\mathbf{a}_3)$, since the sign of $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ does not change $\pm\mathbf{v}_1, \pm\mathbf{v}_2, \pm\mathbf{v}_3$.

It is easy to verify that

$$\begin{aligned} \det(\mathbf{a}_1 - \mathbf{a}_3, \mathbf{a}_2, \mathbf{a}_1 + \mathbf{a}_2 + 2\mathbf{a}_3) &= \begin{vmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{vmatrix} \det(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3) = 3 \det(\Lambda), \\ \det(\mathbf{a}_1, \mathbf{a}_1 + 2\mathbf{a}_2 + \mathbf{a}_3, \mathbf{a}_1 + \mathbf{a}_2 + 2\mathbf{a}_3) &= \begin{vmatrix} 1 & 0 & 0 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{vmatrix} \det(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3) = 3 \det(\Lambda), \\ \det(\mathbf{a}_1, \mathbf{a}_1 + 2\mathbf{a}_2 - \mathbf{a}_3, \mathbf{a}_1 + \mathbf{a}_2 + 2\mathbf{a}_3) &= \begin{vmatrix} 1 & 0 & 0 \\ 1 & 2 & -1 \\ 1 & 1 & 2 \end{vmatrix} \det(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3) = 5 \det(\Lambda), \\ \det(\mathbf{a}_1, -\mathbf{a}_1 + 2\mathbf{a}_2 + \mathbf{a}_3, \mathbf{a}_1 + \mathbf{a}_2 + 2\mathbf{a}_3) &= \begin{vmatrix} 1 & 0 & 0 \\ -1 & 2 & 1 \\ 1 & 1 & 2 \end{vmatrix} \det(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3) = 3 \det(\Lambda), \\ \det(\mathbf{a}_1, -\mathbf{a}_1 + 2\mathbf{a}_2 - \mathbf{a}_3, \mathbf{a}_1 + \mathbf{a}_2 + 2\mathbf{a}_3) &= \begin{vmatrix} 1 & 0 & 0 \\ -1 & 2 & -1 \\ 1 & 1 & 2 \end{vmatrix} \det(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3) = 5 \det(\Lambda), \end{aligned}$$

where $\det(\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3)$ denotes the determinant of the lattice which is generated by $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$. By Lemma 4.2, $\mathbf{v}_5$ cannot be $\pm(\mathbf{a}_1 - \mathbf{a}_3), \pm(\mathbf{a}_1 + 2\mathbf{a}_2 + \mathbf{a}_3), \pm(\mathbf{a}_1 + 2\mathbf{a}_2 - \mathbf{a}_3), \pm(-\mathbf{a}_1 + 2\mathbf{a}_2 + \mathbf{a}_3), \pm(-\mathbf{a}_1 + 2\mathbf{a}_2 - \mathbf{a}_3)$. Therefore, $\mathbf{v}_5 = \pm(\mathbf{a}_1 + \mathbf{a}_3)$. By the same deduction, we have $\mathbf{v}_6 = \pm(\mathbf{a}_2 + \mathbf{a}_3)$.

For $\mathbf{v}_7$, since

$$\det(\mathbf{a}_1, \mathbf{a}_1 + \mathbf{a}_2 - \mathbf{a}_3, \mathbf{a}_1 + \mathbf{a}_2 + 2\mathbf{a}_3) = \begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & -1 \\ 1 & 1 & 2 \end{vmatrix} \det(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3) = 3 \det(\Lambda),$$

$$\det(-\mathbf{a}_1, \mathbf{a}_1 - \mathbf{a}_2 + \mathbf{a}_3, \mathbf{a}_1 + \mathbf{a}_2 + 2\mathbf{a}_3) = \begin{vmatrix} -1 & 0 & 0 \\ 1 & -1 & 1 \\ 1 & 1 & 2 \end{vmatrix} \det(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3) = 3 \det(\Lambda),$$

$$\det(-\mathbf{a}_2, \mathbf{a}_1 - \mathbf{a}_2 - \mathbf{a}_3, \mathbf{a}_1 + \mathbf{a}_2 + 2\mathbf{a}_3) = \begin{vmatrix} 0 & -1 & 0 \\ 1 & -1 & -1 \\ 1 & 1 & 2 \end{vmatrix} \det(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3) = 3 \det(\Lambda),$$

$\mathbf{v}_7$ cannot be $\pm(\mathbf{a}_1 + \mathbf{a}_2 - \mathbf{a}_3)$, $\pm(\mathbf{a}_1 - \mathbf{a}_2 + \mathbf{a}_3)$, $\pm(\mathbf{a}_1 - \mathbf{a}_2 - \mathbf{a}_3)$, by Lemma 4.2. Therefore, $\mathbf{v}_7 = \pm(\mathbf{a}_1 + \mathbf{a}_2 + \mathbf{a}_3)$, which means that

$$X(\alpha, \Lambda) = \{\pm\mathbf{a}_1, \pm\mathbf{a}_2, \pm\mathbf{a}_3, \pm(\mathbf{a}_1 + \mathbf{a}_2 + 2\mathbf{a}_3), \pm(\mathbf{a}_1 + \mathbf{a}_3), \pm(\mathbf{a}_2 + \mathbf{a}_3), \pm(\mathbf{a}_1 + \mathbf{a}_2 + \mathbf{a}_3)\}.$$

By choose $\mathbf{a}'_1 = -(\mathbf{a}_1 + \mathbf{a}_3)$, $\mathbf{a}'_2 = \mathbf{a}_3$, $\mathbf{a}'_3 = \mathbf{a}_1 + \mathbf{a}_2 + \mathbf{a}_3$, one can verify that Lemma 4.3 holds in this case.

On the other hand, if $z = z' = z'' = 0$, we have

$$\begin{aligned} \mathbf{v}_4 &= \pm(\mathbf{a}_1 + \mathbf{a}_2) \text{ or } \pm(\mathbf{a}_1 - \mathbf{a}_2), \\ \mathbf{v}_5 &= \pm(\mathbf{a}_1 + \mathbf{a}_3) \text{ or } \pm(\mathbf{a}_1 - \mathbf{a}_3), \\ \mathbf{v}_6 &= \pm(\mathbf{a}_2 + \mathbf{a}_3) \text{ or } \pm(\mathbf{a}_2 - \mathbf{a}_3). \end{aligned}$$

Since the sign of $\mathbf{a}_2$ does not change $\pm\mathbf{v}_1$, $\pm\mathbf{v}_2$, $\pm\mathbf{v}_3$, we assume $\mathbf{v}_4 = \pm(\mathbf{a}_1 + \mathbf{a}_2)$, without loss of generality. Furthermore, since the sign of $\mathbf{a}_3$ does not change $\pm\mathbf{v}_1$, $\pm\mathbf{v}_2$, $\pm\mathbf{v}_3$, $\pm\mathbf{v}_4$, we may assume

$$\mathbf{v}_5 = \pm(\mathbf{a}_1 + \mathbf{a}_3), \mathbf{v}_6 = \pm(\mathbf{a}_2 + \mathbf{a}_3)$$

or

$$\mathbf{v}_5 = \pm(\mathbf{a}_1 + \mathbf{a}_3), \mathbf{v}_6 = \pm(\mathbf{a}_2 - \mathbf{a}_3).$$

Combining with

$$\mathbf{v}_7 = \pm(\mathbf{a}_1 + \mathbf{a}_2 + \mathbf{a}_3), \pm(\mathbf{a}_1 + \mathbf{a}_2 - \mathbf{a}_3), \pm(\mathbf{a}_1 - \mathbf{a}_2 + \mathbf{a}_3) \text{ or } \pm(\mathbf{a}_1 - \mathbf{a}_2 - \mathbf{a}_3),$$

we obtain that in this case, $X(\alpha, \Lambda)$ is one of the following sets:

- (1) $\{\pm\mathbf{a}_1, \pm\mathbf{a}_2, \pm\mathbf{a}_3, \pm(\mathbf{a}_1 + \mathbf{a}_2), \pm(\mathbf{a}_1 + \mathbf{a}_3), \pm(\mathbf{a}_2 + \mathbf{a}_3), \pm(\mathbf{a}_1 + \mathbf{a}_2 + \mathbf{a}_3)\};$
- (2) $\{\pm\mathbf{a}_1, \pm\mathbf{a}_2, \pm\mathbf{a}_3, \pm(\mathbf{a}_1 + \mathbf{a}_2), \pm(\mathbf{a}_1 + \mathbf{a}_3), \pm(\mathbf{a}_2 + \mathbf{a}_3), \pm(\mathbf{a}_1 + \mathbf{a}_2 - \mathbf{a}_3)\};$
- (3) $\{\pm\mathbf{a}_1, \pm\mathbf{a}_2, \pm\mathbf{a}_3, \pm(\mathbf{a}_1 + \mathbf{a}_2), \pm(\mathbf{a}_1 + \mathbf{a}_3), \pm(\mathbf{a}_2 + \mathbf{a}_3), \pm(\mathbf{a}_1 - \mathbf{a}_2 + \mathbf{a}_3)\};$
- (4) $\{\pm\mathbf{a}_1, \pm\mathbf{a}_2, \pm\mathbf{a}_3, \pm(\mathbf{a}_1 + \mathbf{a}_2), \pm(\mathbf{a}_1 + \mathbf{a}_3), \pm(\mathbf{a}_2 + \mathbf{a}_3), \pm(\mathbf{a}_1 - \mathbf{a}_2 - \mathbf{a}_3)\};$
- (5) $\{\pm\mathbf{a}_1, \pm\mathbf{a}_2, \pm\mathbf{a}_3, \pm(\mathbf{a}_1 + \mathbf{a}_2), \pm(\mathbf{a}_1 + \mathbf{a}_3), \pm(\mathbf{a}_2 - \mathbf{a}_3), \pm(\mathbf{a}_1 + \mathbf{a}_2 + \mathbf{a}_3)\};$
- (6) $\{\pm\mathbf{a}_1, \pm\mathbf{a}_2, \pm\mathbf{a}_3, \pm(\mathbf{a}_1 + \mathbf{a}_2), \pm(\mathbf{a}_1 + \mathbf{a}_3), \pm(\mathbf{a}_2 - \mathbf{a}_3), \pm(\mathbf{a}_1 + \mathbf{a}_2 - \mathbf{a}_3)\};$
- (7) $\{\pm\mathbf{a}_1, \pm\mathbf{a}_2, \pm\mathbf{a}_3, \pm(\mathbf{a}_1 + \mathbf{a}_2), \pm(\mathbf{a}_1 + \mathbf{a}_3), \pm(\mathbf{a}_2 - \mathbf{a}_3), \pm(\mathbf{a}_1 - \mathbf{a}_2 + \mathbf{a}_3)\};$
- (8) $\{\pm\mathbf{a}_1, \pm\mathbf{a}_2, \pm\mathbf{a}_3, \pm(\mathbf{a}_1 + \mathbf{a}_2), \pm(\mathbf{a}_1 + \mathbf{a}_3), \pm(\mathbf{a}_2 - \mathbf{a}_3), \pm(\mathbf{a}_1 - \mathbf{a}_2 - \mathbf{a}_3)\}.$

Since

$$\det(\mathbf{a}_1 + \mathbf{a}_3, \mathbf{a}_2 + \mathbf{a}_3, -(\mathbf{a}_1 + \mathbf{a}_2 - \mathbf{a}_3)) = \begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ -1 & -1 & 1 \end{vmatrix} \det(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3) = 3 \det(\Lambda),$$

$$\det(\mathbf{a}_1 + \mathbf{a}_2, \mathbf{a}_2 + \mathbf{a}_3, \mathbf{a}_1 - \mathbf{a}_2 + \mathbf{a}_3) = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & -1 & 1 \end{vmatrix} \det(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3) = 3 \det(\Lambda),$$

$$\det(\mathbf{a}_1 + \mathbf{a}_2, \mathbf{a}_1 + \mathbf{a}_3, \mathbf{a}_1 - \mathbf{a}_2 - \mathbf{a}_3) = \begin{vmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & -1 & -1 \end{vmatrix} \det(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3) = 3 \det(\Lambda),$$

$$\det(\mathbf{a}_2 - \mathbf{a}_3, \mathbf{a}_1 + \mathbf{a}_3, \mathbf{a}_1 - \mathbf{a}_2 - \mathbf{a}_3) = \begin{vmatrix} 0 & 1 & -1 \\ 1 & 0 & 1 \\ 1 & -1 & -1 \end{vmatrix} \det(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3) = 3 \det(\Lambda),$$

by Lemma 4.2, $X(\alpha, \Lambda)$ cannot be the sets (2), (3), (4), (8).

For set (1), Lemma 4.3 already holds; For set (5), by choosing $\mathbf{a}'_1 = \mathbf{a}_1 + \mathbf{a}_2$, $\mathbf{a}'_2 = -\mathbf{a}_2$, $\mathbf{a}'_3 = \mathbf{a}_3$, it can be verified that Lemma 4.3 holds; For set (6), by choosing $\mathbf{a}'_1 = \mathbf{a}_1$, $\mathbf{a}'_2 = \mathbf{a}_2 - \mathbf{a}_3$, $\mathbf{a}'_3 = \mathbf{a}_3$, it can be verified that Lemma 4.3 holds; For set (7), by choosing $\mathbf{a}'_1 = \mathbf{a}_1$, $\mathbf{a}'_2 = \mathbf{a}_2$, $\mathbf{a}'_3 = -(\mathbf{a}_1 + \mathbf{a}_3)$, it can be verified that Lemma 4.3 holds.

As a conclusion of two cases, Lemma 4.3 is proved. $\square$

Lemma 4.4 When $0 \leq \alpha < \frac{4}{3}\sqrt{3} - 2$, we have $\kappa_\alpha^*(B^3) = 12$.

Proof On the contrary, suppose there exists a packing lattice Λ of B^3 and $0 \leq \alpha < \frac{4}{3}\sqrt{3} - 2$ satisfies $\text{card}\{X(\alpha, \Lambda)\} = 14$. By Lemma 4.3, there exist a basis $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ of Λ such that

$$X(\alpha, \Lambda) = \{\pm \mathbf{a}_1, \pm \mathbf{a}_2, \pm \mathbf{a}_3, \pm(\mathbf{a}_1 + \mathbf{a}_2), \pm(\mathbf{a}_1 + \mathbf{a}_3), \pm(\mathbf{a}_2 + \mathbf{a}_3), \pm(\mathbf{a}_1 + \mathbf{a}_2 + \mathbf{a}_3)\}.$$

Without loss of generality, we suppose that $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_1 + \mathbf{a}_2$ lie in the plane

$$\{(v_1, v_2, v_3) : v_3 = 0\}$$

and

$$\mathbf{a}_1 + \mathbf{a}_2 = (0, c, 0), \quad \mathbf{a}_1 = (-a, b, 0), \quad \mathbf{a}_2 = (a, c - b, 0).$$

Then we have:

$$\frac{16}{3} > \|\mathbf{a}_1 + \mathbf{a}_2\|^2 = c^2 \geq 4, \quad (4.1)$$

$$\frac{16}{3} > \|\mathbf{a}_1\|^2 = a^2 + b^2 \geq 4, \quad (4.2)$$

$$\frac{16}{3} > \|\mathbf{a}_2\|^2 = a^2 + b^2 + c^2 - 2bc \geq 4. \quad (4.3)$$

Let $\mathbf{a}_3 = (v_1, v_2, v_3)$ and denote $\|\mathbf{a}_3\|^2 = D_1$, $\|\mathbf{a}_1 + \mathbf{a}_2 + \mathbf{a}_3\|^2 = D_2$, then we have

$$\frac{16}{3} > D_1 = v_1^2 + v_2^2 + v_3^2 \geq 4, \quad (4.4)$$

$$\frac{16}{3} > D_2 = v_1^2 + v_2^2 + v_3^2 + c^2 + 2cv_2 \geq 4. \quad (4.5)$$

By (4.4) and (4.5), we have

$$v_2 = \frac{D_2 - D_1 - c^2}{2c}. \quad (4.6)$$

By (4.4) and (4.6), we have

$$\begin{aligned} \frac{16}{3} > \|\mathbf{a}_1 + \mathbf{a}_3\|^2 &= (v_1 - a)^2 + (v_2 + b)^2 + v_3^2 \\ &= D_1 + a^2 + b^2 - 2av_1 + 2bv_2 \end{aligned}$$

and

$$\begin{aligned} \frac{16}{3} > \|\mathbf{a}_2 + \mathbf{a}_3\|^2 &= (v_1 + a)^2 + (v_2 + c - b)^2 + v_3^2 \\ &= D_1 + a^2 + 2av_1 + 2v_2(c - b) + 2bv_2 + b^2 + c^2 - 2bc \\ &= D_1 + a^2 + b^2 + 2av_1 + 2bv_2 + (2v_2 + c)(c - 2b) \\ &= D_1 + a^2 + b^2 + 2av_1 + 2bv_2 + \frac{D_2 - D_1}{c}(c - 2b). \end{aligned}$$

Using (4.6) again, we obtain

$$\begin{aligned} \frac{16}{3} > \frac{\|\mathbf{a}_1 + \mathbf{a}_3\|^2 + \|\mathbf{a}_2 + \mathbf{a}_3\|^2}{2} \\ &= D_1 + a^2 + b^2 + 2bv_2 + \frac{D_2 - D_1}{2c}(c - 2b) \\ &= D_1 + a^2 + b^2 + \frac{b}{c}(D_2 - D_1) - bc + \frac{D_2 - D_1}{2} - \frac{b}{c}(D_2 - D_1) \\ &= \frac{D_1 + D_2}{2} + a^2 + b^2 - bc. \end{aligned} \quad (4.7)$$

On the other hand, by (4.2) and (4.3), we have $2(a^2 + b^2 - bc) + c^2 \geq 8$. Therefore, by (4.1), we get

$$a^2 + b^2 - bc \geq 4 - \frac{c^2}{2} > \frac{4}{3}, \quad (4.8)$$

together with (4.4) and (4.5), we have

$$\frac{D_1 + D_2}{2} + a^2 + b^2 - bc > 4 + \frac{4}{3} = \frac{16}{3},$$

which contradicts (4.7).

Therefore, when $0 \leq \alpha < \frac{4}{3}\sqrt{3} - 2$, we have $\kappa_\alpha^*(B^3) \leq 12$. Combining with $\kappa_\alpha^*(B^3) \geq \kappa^*(B^3) = 12$, for $0 \leq \alpha < \frac{4}{3}\sqrt{3} - 2$ we have $\kappa_\alpha^*(B^3) = 12$. Lemma 4.4 is proved. $\square$

Remark 4.1 For $\alpha = \frac{4}{3}\sqrt{3} - 2$, by repeating (4.1)–(4.8) one can deduce that

$$\text{card}\{X(\alpha, \Lambda)\} = 14$$

if and only if the lattice Λ is generated by $\mathbf{a}_1 = (-\frac{2}{3}\sqrt{6}, \frac{2}{3}\sqrt{3}, 0)$, $\mathbf{a}_2 = (\frac{2}{3}\sqrt{6}, \frac{2}{3}\sqrt{3}, 0)$ and $\mathbf{a}_3 = (0, \frac{2}{3}\sqrt{3}, \frac{2}{3}\sqrt{6})$, up to some rotation.

Lemma 4.5 When $\alpha = 2\sqrt{2} - 2$, we have $\kappa_\alpha^*(B^3) = 20$.

Proof Let Λ be the lattice generated by $\mathbf{a}_1 = (2, 0, 0)$, $\mathbf{a}_2 = (0, 2, 0)$ and $\mathbf{a}_3 = (1, 0, \sqrt{3})$. When $\alpha = 2\sqrt{2} - 2$, one can verify that

$$\text{card}\{X(\alpha, \Lambda)\} = 20.$$

By Remark 2.1, for a packing lattice Λ of B^3 and $\alpha = 2\sqrt{2} - 2$, to let $\text{card}\{X(\alpha, \Lambda)\} \geq 20$, a necessary condition is there exist $\mathbf{a}_1, \mathbf{a}_2 \in \Lambda$ such that

$$\|\mathbf{a}_1\| = \|\mathbf{a}_2\| = 2, \quad \|\mathbf{a}_1 + \mathbf{a}_2\| = \|\mathbf{a}_1 - \mathbf{a}_2\| = 2\sqrt{2}.$$

Without loss of generality, we suppose $\mathbf{a}_1 = (2, 0, 0)$, $\mathbf{a}_2 = (0, 2, 0)$. It is obvious to see that we can expand a basis of Λ based on $\{\mathbf{a}_1, \mathbf{a}_2\}$.

Suppose $\mathbf{a}_1, \mathbf{a}_2$ and $\mathbf{a}_3 = (v_1, v_2, v_3)$ is a basis of Λ . We assume $v_3 > 0$, without loss of generality. Since

$$\det(\Lambda) \geq \frac{\text{vol}(B^3)}{\delta^*(B^3)} = \frac{\frac{4}{3}\pi}{\frac{\pi}{\sqrt{18}}} = 4\sqrt{2},$$

we have $v_3 \geq \sqrt{2}$ and the equality holds if and only if Λ is the densest packing lattice of B^3 . In this case one can verify that $\text{card}\{X(\alpha, \Lambda)\} = 18 < 20$. Therefore, we have $v_3 > \sqrt{2}$. Which means that, for

$$\mathbf{v} = z_1\mathbf{a}_1 + z_2\mathbf{a}_2 + z_3\mathbf{a}_3 \in X(\alpha, \Lambda),$$

we have $z_3 = 0$ or ± 1 .

Since

$$\text{card}\{\mathbf{v} : \mathbf{v} = z_1\mathbf{a}_1 + z_2\mathbf{a}_2 \in X(\alpha, \Lambda)\} = 8$$

and $X(\alpha, \Lambda)$ is centrally symmetric, to let $\text{card}\{X(\alpha, \Lambda)\} \geq 20$, we have

$$\text{card}\{\mathbf{v} : \mathbf{v} = z_1\mathbf{a}_1 + z_2\mathbf{a}_2 + \mathbf{a}_3 \in X(\alpha, \Lambda)\} \geq 6.$$

Therefore, there exist two of them which are equivalent. Replace $\mathbf{a}_3$ by the mid-point of them, we may further assume $\|\mathbf{a}_3\|^2 = v_1^2 + v_2^2 + v_3^2 = 4$, by Remark 2.1.

Without loss of generality, we suppose $v_1, v_2 \geq 0$. Since $\|\mathbf{a}_3 - \mathbf{a}_1\| \geq 2$, $\|\mathbf{a}_3 - \mathbf{a}_2\| \geq 2$, by routine computation we have $0 \leq v_1 \leq 1$, $0 \leq v_2 \leq 1$.

For a lattice vector

$$\mathbf{v} = z_1\mathbf{a}_1 + z_2\mathbf{a}_2 + \mathbf{a}_3 \in X(\alpha, \Lambda),$$

we have

$$\|\mathbf{v}\|^2 = 4z_1^2 + 4z_1v_1 + 4z_2^2 + 4z_2v_2 + v_1^2 + v_2^2 + v_3^2 \leq (2\sqrt{2})^2,$$

which means

$$z_1^2 + z_1v_1 + z_2^2 + z_2v_2 \leq 1. \quad (4.9)$$

By routine computation, a necessary condition for (4.9) is $|z_1| \leq 1$, $|z_2| \leq 1$. For $(z_1, z_2) = (1, 0)$: $\mathbf{a}_1 + \mathbf{a}_3 \in X(\alpha, \Lambda)$ if and only if

$$v_1 = 0. \quad (4.9.1)$$

For $(z_1, z_2) = (1, -1)$: $\mathbf{a}_1 - \mathbf{a}_2 + \mathbf{a}_3 \in X(\alpha, \Lambda)$ if and only if

$$1 + v_1 - v_2 \leq 0. \quad (4.9.2)$$

For $(z_1, z_2) = (0, 1)$: $\mathbf{a}_2 + \mathbf{a}_3 \in X(\alpha, \Lambda)$ if and only if

$$v_2 = 0. \quad (4.9.3)$$

For $(z_1, z_2) = (-1, 1)$: $-\mathbf{a}_1 + \mathbf{a}_2 + \mathbf{a}_3 \in X(\alpha, \Lambda)$ if and only if

$$1 - v_1 + v_2 \leq 0. \quad (4.9.4)$$

For $(z_1, z_2) = (-1, -1)$: $-\mathbf{a}_1 - \mathbf{a}_2 + \mathbf{a}_3 \in X(\alpha, \Lambda)$ if and only if

$$1 - v_1 - v_2 \leq 0. \quad (4.9.5)$$

For $(z_1, z_2) = (1, 1)$, since $1 + v_1 + 1 + v_2 > 1$, we have

$$\mathbf{a}_1 + \mathbf{a}_2 + \mathbf{a}_3 \notin X(\alpha, \Lambda). \quad (4.9.6)$$

Obviously, (4.9.1) and (4.9.4) cannot hold simultaneously, (4.9.2) and (4.9.3) cannot hold simultaneously. Combining with (4.9.6), we have

$$\text{card}\{\mathbf{v} : \mathbf{v} = z_1\mathbf{a}_1 + z_2\mathbf{a}_2 + \mathbf{a}_3 \in X(\alpha, \Lambda)\} \leq 6.$$

Therefore, for $\alpha = 2\sqrt{2} - 2$ we get $\text{card}\{X(\alpha, \Lambda)\} \leq 20$.

To let $\text{card}\{X(\alpha, \Lambda)\} = 20$, we must have: one of (4.9.1) and (4.9.4) holds, one of (4.9.2) and (4.9.3) holds, and (4.9.5) holds. By routine computation one can deduce that $\mathbf{a}_3 = (0, 1, \sqrt{3})$ or $(1, 0, \sqrt{3})$. Therefore, when $\alpha = 2\sqrt{2} - 2$, we have $\kappa_\alpha^*(B^3) = 20$, and the equality holds if and only if the corresponding lattice Λ is generated by $\mathbf{a}_1 = (2, 0, 0)$, $\mathbf{a}_2 = (0, 2, 0)$ and $\mathbf{a}_3 = (1, 0, \sqrt{3})$, up to some rotation. Lemma 4.5 is proved. $\square$

Lemma 4.1, Lemma 4.4 and Lemma 4.5 together yields the following theorem.

Theorem 2 In $\mathbb{E}^3$, we have

$$\kappa_\alpha^*(B^3) = \begin{cases} 12, & 0 \leq \alpha < \frac{4}{3}\sqrt{3} - 2, \\ 14, & \frac{4}{3}\sqrt{3} - 2 \leq \alpha < 2\sqrt{2} - 2, \\ 20, & \alpha = 2\sqrt{2} - 2. \end{cases}$$

Remark 4.2 In fact, by repeating the calculations (4.9.1)–(4.9.6), one can deduce that for $\alpha = 2\sqrt{2} - 2$, $\text{card}\{X(\alpha, \Lambda)\} = 18$ if and only if Λ is generated by $\mathbf{a}_1 = (2, 0, 0)$, $\mathbf{a}_2 = (0, 2, 0)$ and $\mathbf{a}_3 = (1, 1, \sqrt{2})$, or $\mathbf{a}_1 = (2, 0, 0)$, $\mathbf{a}_2 = (0, 2, 0)$ and $\mathbf{a}_3 = (0, 0, 2)$, up to some rotation.

We end this section by a problem as following.

Problem 4.1 When $\alpha = \frac{4}{3}\sqrt{3} - 2$, is it true that $\kappa_\alpha^*(C) \geq 14$ holds for every three-dimensional centrally symmetric convex body C ?

5 Kissing Numbers of Convex Bodies

Although the concept of $\kappa_\alpha^*(B^3)$ itself is interesting, it can also lead to determine the lattice kissing numbers of convex bodies which were geometrically similar to B^3 . To this end, we present the following theorem:

Theorem 5.1 For a 3-dimensional centrally symmetric convex body C centered at $\mathbf{o}$, if $B^3 \subset C \subset \text{int}(\frac{2}{3}\sqrt{3}B^3)$ holds, then we have $\kappa^*(C) = 12$.

Proof Let $C + \Lambda$ be a lattice packing attaining $\kappa^*(C)$ and

$$X = \{\mathbf{v}_1, \dots, \mathbf{v}_{\kappa^*(C)}\} = \partial(2C) \cap \Lambda,$$

where $\partial(2C)$ denotes the boundary of $2C$. Since $2B^3 \subset 2C \subset \text{int}(\frac{4}{3}\sqrt{3}B^3)$, we have $2 \leq \|\mathbf{v}_i\| < \frac{4}{3}\sqrt{3}$ holds for all $i = 1, 2, \dots, \kappa^*(C)$. Since Λ is also a packing lattice of B^3 , by Lemma 4.3 we get

$$\kappa^*(C) = \text{card}X \leq 12.$$

On the other hand, since $\kappa^*(C) \geq 12$ holds for all 3-dimensional centrally symmetric convex body C (see [17]), we have $\kappa^*(C) = 12$.

Theorem 5.1 is proved. $\square$

We now give several convex bodies as examples which lattice kissing numbers can be determined by Theorem 5.1.

Example 5.1 We take $\tau = \frac{\sqrt{5}+1}{2}$ and define

$$\begin{aligned} P_d &= \{(v_1, v_2, v_3) : |\tau v_1| + |v_2| \leq 1, |\tau v_2| + |v_3| \leq 1, |\tau v_3| + |v_1| \leq 1\}, \\ P_i &= \{(v_1, v_2, v_3) : |v_1| + |v_2| + |v_3| \leq 1, |\tau v_1| + |\frac{1}{\tau} v_3| \leq 1, |\tau v_2| + |\frac{1}{\tau} v_1| \leq 1, \\ &\quad |\tau v_3| + |\frac{1}{\tau} v_2| \leq 1\}. \end{aligned}$$

Usually, P_d and P_i are called a dodecahedron and an icosahedron, respectively. Define

$$P_{tri} = (1 + \tau)P_i \cap (4/3 + \tau)P_d.$$

Usually, P_{tri} is called a truncated icosahedron.

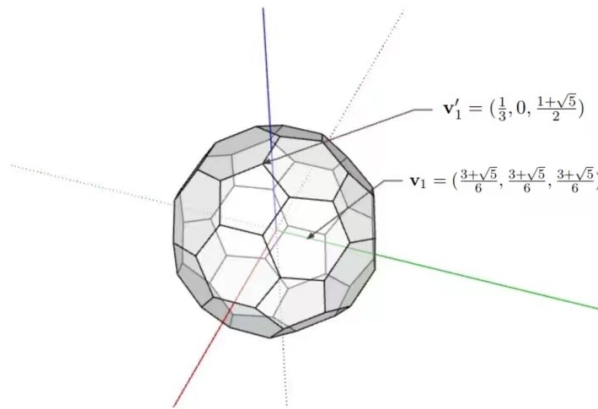


Figure 1 Truncated icosahedron P_{tri}

By routine computation, we have $\|v_1\|B^3 \subset P_{tri} \subset \|v'_1\|B^3$, where $v_1 = (\frac{3+\sqrt{5}}{6}, \frac{3+\sqrt{5}}{6}, \frac{3+\sqrt{5}}{6})$, $v'_1 = (\frac{1}{3}, 0, \frac{1+\sqrt{5}}{2})$, see Figure 1. Since

$$B^3 \subset \frac{1}{\|v_1\|}P_{tri} \subset \frac{\|v'_1\|}{\|v_1\|}B^3 = 1.0929 \dots B^3 \subset \text{int}\left(\frac{4}{3}\sqrt{3}B^3\right),$$

by Theorem 5.1 we have

$$\kappa^*(P_{tri}) = \kappa^*\left(\frac{1}{\|v_1\|}P_{tri}\right) = 12.$$

Example 5.2 We define

$$\begin{aligned} P_{rtc} &= \{(v_1, v_2, v_3) : |\tau v_1| \leq 1, |\tau v_2| \leq 1, |\tau v_3| \leq 1, |\frac{1}{2}v_1| + |\frac{\tau}{2}v_2| + |\frac{\tau+1}{2}v_3| \leq 1, \\ &\quad |\frac{\tau}{2}v_1| + |\frac{\tau+1}{2}v_2| + |\frac{1}{2}v_3| \leq 1, |\frac{\tau+1}{2}v_1| + |\frac{1}{2}v_2| + |\frac{\tau}{2}v_3| \leq 1\}. \end{aligned}$$

Usually, P_{rtc} is called a rhombic triacontahedron. Define

$$P_{rid} = (3\tau + 2)P_{rtc} \cap (4\tau + 1)P_i \cap (3(1 + \tau))P_d,$$

$$P_{trid} = (5\tau + 4)P_{rtc} \cap (6\tau + 3)P_i \cap (5(1 + \tau))P_d.$$

Usually, P_{rid} and P_{trid} are called a rhombic icosidodecahedron and a truncated icosidodecahedron, respectively.

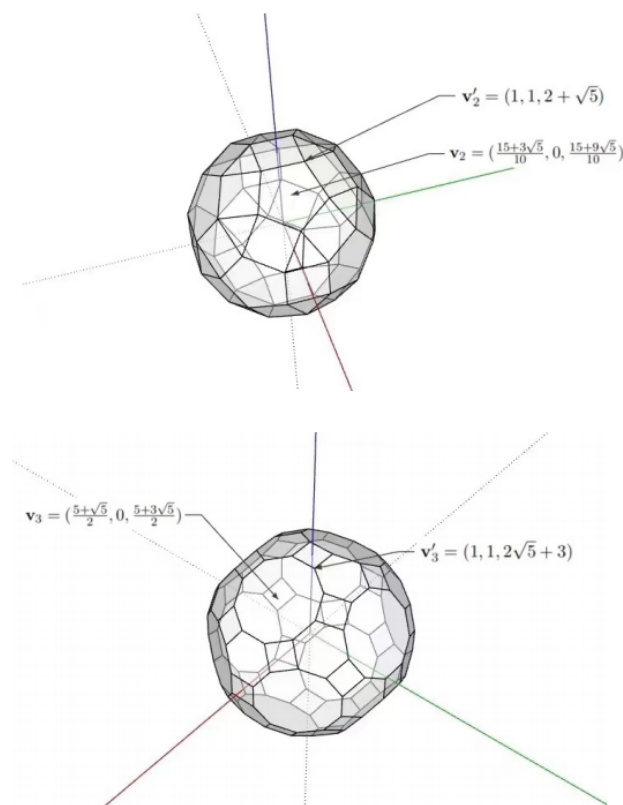


Figure 2 Rhombic icosidodecahedron P_{rid} and truncated icosidodecahedron P_{trid}

By routine computation, we have

$$\|\mathbf{v}_2\|B^3 \subset P_{rid} \subset \|\mathbf{v}'_2\|B^3,$$

where $\mathbf{v}_2 = (\frac{15+3\sqrt{5}}{10}, 0, \frac{15+9\sqrt{5}}{10})$, $\mathbf{v}'_2 = (1, 1, 2 + \sqrt{5})$, see Figure 2. Since

$$B^3 \subset \frac{1}{\|\mathbf{v}_2\|}P_{rid} \subset \frac{\|\mathbf{v}'_2\|}{\|\mathbf{v}_2\|}B^3 = 1.0815 \cdots B^3 \subset \text{int}\left(\frac{4}{3}\sqrt{3}B^3\right),$$

by Theorem 5.1 we have

$$\kappa^*(P_{rid}) = \kappa^*\left(\frac{1}{\|\mathbf{v}_2\|}P_{rid}\right) = 12.$$

By routine computation, we have

$$\|\mathbf{v}_3\|B^3 \subset P_{trid} \subset \|\mathbf{v}'_3\|B^3,$$

where $\mathbf{v}_3 = (\frac{5+\sqrt{5}}{2}, 0, \frac{5+3\sqrt{5}}{2})$, $\mathbf{v}'_3 = (1, 1, 2\sqrt{5} + 3)$, see Figure 2. Since

$$B^3 \subset \frac{1}{\|\mathbf{v}_3\|}P_{trid} \subset \frac{\|\mathbf{v}'_3\|}{\|\mathbf{v}_3\|}B^3 = 1.1050 \cdots B^3 \subset \text{int}\left(\frac{4}{3}\sqrt{3}B^3\right),$$

by Theorem 5.1 we have

$$\kappa^*(P_{trid}) = \kappa^*\left(\frac{1}{\|\mathbf{v}_3\|}P_{trid}\right) = 12.$$

Example 5.3 We use the configuration of snub dodecahedron given by Henk [6], denote it by P_{sd} . By routine computation, we have

$$\|\mathbf{v}_4\|B^3 \subset P_{sd} \subset \|\mathbf{v}'_4\|B^3,$$

where

$$\mathbf{v}_4 = (-0.9661\dots, 0, 1.5632\dots), \quad \mathbf{v}'_4 = (-0.3477\dots, -0.3069\dots, 1.9454\dots),$$

see Figure 3. Since

$$B^3 \subset \frac{1}{\|\mathbf{v}_4\|}P_{sd} \subset \frac{\|\mathbf{v}'_4\|}{\|\mathbf{v}_4\|}B^3 = 1.0883\dots B^3 \subset \text{int}\left(\frac{4}{3}\sqrt{3}B^3\right),$$

by Theorem 5.1 we have

$$\kappa^*(P_{sd}) = \kappa^*\left(\frac{1}{\|\mathbf{v}_4\|}P_{sd}\right) = 12.$$

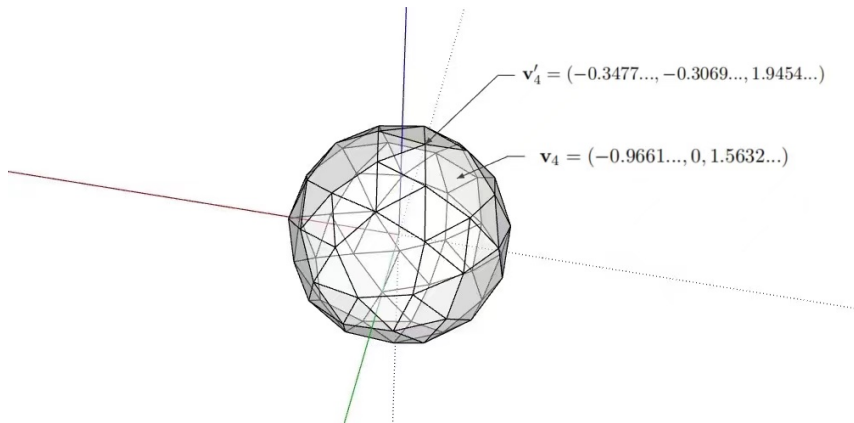


Figure 3 Snub dodecahedron P_{sd}

Example 5.4 Define

$$B_p^3 = \{(v_1, v_2, v_3) : |v_1|^p + |v_2|^p + |v_3|^p \leq 1\}.$$

Usually, B_p^3 is called a L_p unit ball in 3-dimension. When $p_1 \leq p_2$, it is well known that

$$B_{p_1}^3 \subset B_{p_2}^3.$$

For $p \geq 2$ and a point $\mathbf{v} = (v_1, v_2, v_3) \in B_p^3$ where $v_1, v_2, v_3 \geq 0$, we have

$$v_1^p + v_2^p + v_3^p \leq 1.$$

According to Power-Mean Inequality, we have

$$v_1^2 + v_2^2 + v_3^2 \leq \left(\left(\frac{1}{3}\right)^{\frac{1}{p}}\right)^2 \times 3 = \left(\frac{1}{3}\right)^{\frac{2}{p}} \times 3.$$

Therefore we have

$$B^3 \subset B_p^3 \subset \sqrt{3} \times \left(\frac{1}{3}\right)^{\frac{1}{p}} B^3.$$

By routine computation, when $2 \leq p < \frac{\ln 3}{\ln 3 - \ln 2}$ we get $\kappa^*(B_p^3) = 12$ by Theorem 5.1.

For $p < 2$, by the same deduction we obtain

$$\sqrt{3} \times \left(\frac{1}{3}\right)^{\frac{1}{p}} B^3 \subset B_p^3 \subset B^3,$$

which means

$$B^3 \subset \frac{1}{\sqrt{3} \times \left(\frac{1}{3}\right)^{\frac{1}{p}}} B_p^3 \subset \frac{1}{\sqrt{3} \times \left(\frac{1}{3}\right)^{\frac{1}{p}}} B^3.$$

By routine computation, when $2 > p > \frac{\ln 3}{\ln 2}$ we get $\kappa^*(B_p^3) = 12$ by Theorem 5.1.

As a conclusion of the two cases, when $\frac{\ln 3}{\ln 3 - \ln 2} > p > \frac{\ln 3}{\ln 2}$, we have $\kappa^*(B_p^3) = 12$.

Remark 5.1 Let Λ be the lattice generated by $\mathbf{a}_1 = (2, 0, 0)$, $\mathbf{a}_2 = (0, 2, 0)$ and $\mathbf{a}_3 = (1, 1, 1)$, then one can verify that: when $p = \frac{\ln 3}{\ln 2}$, Λ is a packing lattice of B_p^3 and

$$\text{card}\{\partial(2B_p^3) \cap \Lambda\} = 14.$$

On the other hand, since B_p^3 is a strictly convex body when $1 < p < \infty$, combining with Lemma 2.2, when $p = \frac{\ln 3}{\ln 2}$ we have $\kappa^*(B_p^3) = 14$.

6 Proof of Theorem 4

For $\alpha = 2\sqrt{2} - 2$ and a packing lattice Λ of B^n , we have the following lemma.

Lemma 6.1 One equivalent class of Λ can contain at most n pairs of vectors of $X(\alpha, \Lambda)$.

Proof Suppose

$$\pm \mathbf{v}_1, \dots, \pm \mathbf{v}_i \in X(\alpha, \Lambda)$$

belong to the same equivalent class, $i \geq 2$. By Remark 2.1, we have $\|\mathbf{v}_i\| = 2\sqrt{2}$ holds for all i and $\langle \mathbf{v}_i, \mathbf{v}_j \rangle = 0$ holds for all $i \neq j$. Therefore, one equivalent class of Λ can contain at most n pairs of vectors of $X(\alpha, \Lambda)$.

Lemma 6.1 is proved. $\square$

Denote the numbers of equivalent classes of Λ which contain exactly i pairs of vectors of $X(\alpha, \Lambda)$ by m_i . We define a collection of sets

$$C(X(\alpha, \Lambda)) = \left\{ A = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} : \mathbf{v}_2 = \frac{1}{2}(\mathbf{v}_1 + \mathbf{v}_3) \text{ and } \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \in X(\alpha, \Lambda) \right\}.$$

By estimate $\text{card}\{C(X(\alpha, \Lambda))\}$ in two different ways, we prove the following lemma.

Lemma 6.2

$$\sum_{i=2}^n 2i(i-1)m_i \leq \kappa^*(B^{n-1}) \cdot m_1.$$

Proof For a set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} \in C(X(\alpha, \Lambda))$, by Remark 2.1, we have

$$\|\mathbf{v}_2\| = 2, \|\mathbf{v}_1 - \mathbf{v}_2\| = \|\mathbf{v}_3 - \mathbf{v}_2\| = 2, \|\mathbf{v}_1\| = \|\mathbf{v}_3\| = 2\sqrt{2}.$$

We assume $\mathbf{v}_2 = (0, 0, \dots, 0, 2)$, without loss of generality. Then one can easily deduce that $\mathbf{v}_1$ and $\mathbf{v}_3$ must lie in the $(n-1)$ -dimensional hyperplane

$$\pi_0 : \{(v_1, v_2, \dots, v_{n-1}, v_n) : v_n = 2\}.$$

It is obvious that

$$\text{card}\{\mathbf{v} : \|\mathbf{v} - \mathbf{v}_2\| = 2, \mathbf{v} \in X(\alpha, \Lambda) \cap \pi_0\} \leq \kappa^*(B^{n-1}),$$

which means

$$\text{card}\{A \in C(X(\alpha, \Lambda)) : \mathbf{v}_2 \in A\} \leq \kappa^*(B^{n-1})/2.$$

For a vector $\mathbf{v} \in X(\alpha, \Lambda)$ of length 2, by Remark 2.1, $\mathbf{v}$ cannot be equivalent to any vector of $X(\alpha, \Lambda)$, besides $\pm \mathbf{v}$. Therefore, we have

$$\text{card}\{\mathbf{v} : \|\mathbf{v}\| = 2, \mathbf{v} \in X(\alpha, \Lambda)\} \leq 2m_1.$$

Consequently, we get

$$\text{card}\{C(X(\alpha, \Lambda))\} \leq \kappa^*(B^{n-1}) \cdot m_1. \quad (6.1)$$

On the other hand, by the definition of $C(X(\alpha, \Lambda))$, a set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ belongs to it if and only if $\mathbf{v}_1, \mathbf{v}_3 \in X(\alpha, \Lambda)$ are equivalent and $\mathbf{v}_1 \neq \pm \mathbf{v}_3$. For an equivalent class which contains $i \geq 2$ pairs of vectors of $X(\alpha, \Lambda)$, denote it by X_{i1} . By enumeration we have

$$\text{card}\{\{\mathbf{v}_1, \mathbf{v}_3\} : \{\mathbf{v}_1, \mathbf{v}_3\} \subset X_{i1}, \mathbf{v}_1 \neq \pm \mathbf{v}_3\} = 2i(i-1).$$

Therefore, we get

$$\text{card}\{C(X(\alpha, \Lambda))\} = \sum_{i=2}^n 2i(i-1)m_i. \quad (6.2)$$

By (6.1) and (6.2), Lemma 6.2 is proved. $\square$

Theorem 4 In $\mathbb{E}^8$, when $\alpha = 2\sqrt{2} - 2$ we have $\kappa_\alpha^*(B^8) = 2400$.

Proof For $\alpha = 2\sqrt{2} - 2$, it is well known (see [2]) that

$$\text{card}\{X(\alpha, \sqrt{2}E_8)\} = 240 + 2160 = 2400,$$

where

$$E_8 = \left\{ (v_1, v_2, \dots, v_8) : 2v_i \in Z; v_i - v_j \in Z; \sum v_i \in 2Z \right\}.$$

Suppose that there is a suitable lattice Λ satisfying $\text{card}\{X(\alpha, \Lambda)\} \geq 2400$, which means

$$8m_8 + 7m_7 + \dots + 2m_2 + m_1 \geq 1200. \quad (6.3)$$

Since there are at most $2^8 - 1 = 255$ equivalent classes which can contain the vectors of $X(\alpha, \Lambda)$, we have

$$m_8 + m_7 + \dots + m_1 \leq 255. \quad (6.4)$$

For $n = 8$ case, we restate Lemma 6.2 as

$$112m_8 + 84m_7 + 60m_6 + 40m_5 + 24m_4 + 12m_3 + 4m_2 \leq 126m_1 \quad (6.5)$$

by substituting $\kappa^*(B^7) = 126$.

By (6.3) and (6.4), we have $7m_8 + 6m_7 + \dots + m_2 \geq 945$, multiply both sides by 34, we have

$$238m_8 + 204m_7 + 170m_6 + 136m_5 + 102m_4 + 68m_3 + 34m_2 \geq 32130. \quad (6.6)$$

By (6.4) and (6.5), one can deduce that

$$238m_8 + 210m_7 + 186m_6 + 166m_5 + 150m_4 + 138m_3 + 130m_2 \leq 32130. \quad (6.7)$$

By (6.6) and (6.7), we obtain $m_7 = m_6 = \dots = m_2 = 0$. Combining with (6.3), (6.4) and (6.5), we have

$$\begin{cases} 8m_8 + m_1 \geq 1200, \\ m_8 + m_1 \leq 255, \\ 112m_8 \leq 126m_1. \end{cases}$$

By routine computation, one can easily deduce that $m_1 = 120$, $m_8 = 135$. Furthermore, in this case the equality in (6.1) holds, which means that

$$\text{card}\{\mathbf{v} : \|\mathbf{v}\| = 2, \mathbf{v} \in X(\alpha, \Lambda)\} = 2m_1 = 240.$$

Since $\kappa^*(B^8) = 240$ and the corresponding lattice must be $\sqrt{2}E_8$, up to some rotation (see [21]), for $\alpha = 2\sqrt{2} - 2$ we have $\kappa_\alpha^*(B^8) = 2400$, and the equality can be attained if and only if the corresponding lattice $\Lambda = \sqrt{2}E_8$, up to rotation and reflection.

Theorem 4 is proved. $\square$

Based on this proof, we may make the following conjecture.

Conjecture 6.1 In $\mathbb{E}^8$, when $\alpha = 2\sqrt{2} - 2$ we have $\kappa_\alpha(B^8) = 2400$.

Let Λ_{24} denote the Leech lattice (see [2]). When $\alpha = 2\sqrt{2} - 2$, we have

$$\text{card}\{X(\alpha, \Lambda_{24})\} = 196560 + 16773120 + 398034000 = 415003680.$$

This observation supports the following conjecture.

Conjecture 6.2 In $\mathbb{E}^{24}$, when $\alpha = 2\sqrt{2} - 2$ we have $\kappa_\alpha^*(B^{24}) = 415003680$.

7 Proof of Theorem 3

Theorem 3 In $\mathbb{E}^4$, we have

$$\kappa_\alpha^*(B^4) = \begin{cases} 30, & \sqrt{6} - 2 \leq \alpha < 2\sqrt{2} - 2, \\ 50, & \alpha = 2\sqrt{2} - 2. \end{cases}$$

Proof As usual, we write

$$\begin{aligned} A_n &= \left\{ (v_0, v_1, v_2, \dots, v_n) : v_i \in \mathbb{Z}; \sum v_i = 0 \right\}, \\ D_n &= \left\{ (v_1, v_2, \dots, v_n) : v_i \in \mathbb{Z}; \sum v_i \in 2\mathbb{Z} \right\}. \end{aligned}$$

Furthermore, we denote the *dual lattice* of A_n by A_n^* , namely

$$A_n^* = \{\mathbf{v} : \langle \mathbf{v}, \mathbf{u} \rangle \in \mathbb{Z} \text{ for all } \mathbf{u} \in A_n\}.$$

When $\alpha = \sqrt{6} - 2$, one can verify that $\text{card}\{X(\alpha, \sqrt{5}A_4^*)\} = 30$. Combining with Lemma 2.2, for $\sqrt{6} - 2 \leq \alpha < 2\sqrt{2} - 2$ we have $\kappa_\alpha^*(B^4) = 30$.

For $\alpha = 2\sqrt{2} - 2$, let Λ be the lattice generated by $\mathbf{a}_1 = (2, 0, 0, 0)$, $\mathbf{a}_2 = (0, 2, 0, 0)$, $\mathbf{a}_3 = (1, 0, \sqrt{3}, 0)$, $\mathbf{a}_4 = (0, 1, \frac{2}{3}\sqrt{3}, \frac{\sqrt{5}}{\sqrt{3}})$. One can verify that

$$\text{card}\{X(\alpha, \Lambda)\} = 50. \quad (7.1)$$

Suppose that there exists a packing lattice Λ of B^4 satisfying $\text{card}\{X(\alpha, \Lambda)\} \geq 52$. We still denote the numbers of equivalent classes of Λ which contain exactly i pairs of vectors of $X(\alpha, \Lambda)$ by m_i .

If $m_4 \neq 0$, by Remark 2.1, we may assume

$$\mathbf{v}_1 = (2\sqrt{2}, 0, 0, 0), \quad \mathbf{v}_2 = (0, 2\sqrt{2}, 0, 0), \quad \mathbf{v}_3 = (0, 0, 2\sqrt{2}, 0), \quad \mathbf{v}_4 = (0, 0, 0, 2\sqrt{2})$$

belong to $X(\alpha, \Lambda)$ and $\frac{1}{2}(\mathbf{v}_i \pm \mathbf{v}_j)$, $i \neq j$ belong to $X(\alpha, \Lambda)$, without loss of generality. In this case, lattice Λ is generated by

$$\mathbf{a}_1 = (\sqrt{2}, \sqrt{2}, 0, 0), \quad \mathbf{a}_2 = (\sqrt{2}, -\sqrt{2}, 0, 0), \quad \mathbf{a}_3 = (\sqrt{2}, 0, \sqrt{2}, 0), \quad \mathbf{a}_4 = (\sqrt{2}, 0, 0, \sqrt{2}),$$

which means $\Lambda = \sqrt{2}D_4$. One can verify that

$$\text{card}\{X(\alpha, \sqrt{2}D_4)\} = 24 + 24 = 48 < 52,$$

therefore $m_4 = 0$. Since $\sqrt{2}D_4$ is the unique densest packing lattice for B^4 , up to rotation and reflection (see [21]), from now on we suppose

$$\det(\Lambda) > 8. \quad (7.2)$$

If $m_3 = 0$ and for every vector $\mathbf{v} \in X(\alpha, \Lambda)$ which length is 2 we have

$$\text{card}\{A \in C(X(\alpha, \Lambda)) : \mathbf{v} \in A\} < \kappa^*(B^3)/2 = 6,$$

by restate (6.1), (6.2), (6.3) and (6.4) for $n = 4$, we obtain

$$\begin{cases} 4m_2 \leq 10m_1, \\ 2m_2 + m_1 \geq 26, \\ m_2 + m_1 \leq 15, \end{cases}$$

which admits no solution. Therefore, we have $m_3 \neq 0$ or there exist a vector $\mathbf{v} \in X(\alpha, \Lambda)$ which length is 2 satisfy

$$\text{card}\{A \in C(X(\alpha, \Lambda)) : \mathbf{v} \in A\} = \kappa^*(B^3)/2 = 6.$$

If $m_3 \neq 0$, by Remark 2.1, we assume

$$\mathbf{v}_1 = (2, -2, 0, 0), \quad \mathbf{v}_2 = (2, 2, 0, 0), \quad \mathbf{v}_3 = (0, 0, 2\sqrt{2}, 0)$$

belong to $X(\alpha, \Lambda)$ and $\frac{1}{2}(\mathbf{v}_i \pm \mathbf{v}_j)$, $i \neq j$ belong to $X(\alpha, \Lambda)$, without loss of generality. Therefore, the basis of lattice Λ can be expanded by

$$\mathbf{a}_1 = (2, 0, 0, 0), \quad \mathbf{a}_2 = (0, 2, 0, 0), \quad \mathbf{a}_3 = (1, 1, \sqrt{2}, 0).$$

On the other hand, if there exist a vector $\mathbf{v} \in X(\alpha, \Lambda)$ which length is 2 satisfy

$$\text{card}\{A \in C(X(\alpha, \Lambda)) : \mathbf{v} \in A\} = \kappa^*(B^3)/2 = 6,$$

then there exist a three-dimensional subspace H_0 satisfy

$$\text{card}\{\mathbf{v} : \mathbf{v} \in H_0 \cap \Lambda, \|\mathbf{v}\| = 2\} = \kappa^*(B^3) = 12.$$

Therefore, we may suppose $H_0 = \{(v_1, v_2, v_3, v_4) : v_4 = 0\}$ and the three-dimensional lattice $H_0 \cap \Lambda$ is generated by

$$\mathbf{a}_1 = (2, 0, 0, 0), \quad \mathbf{a}_2 = (0, 2, 0, 0), \quad \mathbf{a}_3 = (1, 1, \sqrt{2}, 0),$$

without loss of generality.

As a conclusion of two cases above, we set a basis of lattice Λ by

$$\mathbf{a}_1 = (2, 0, 0, 0), \quad \mathbf{a}_2 = (0, 2, 0, 0), \quad \mathbf{a}_3 = (1, 1, \sqrt{2}, 0), \quad \mathbf{a}_4 = (v_1, v_2, v_3, v_4)$$

and $v_1 \geq 0, v_2 \geq 0, v_3 \geq 0, v_4 \geq 0$ without loss of generality. Furthermore, by (7.2) we have $v_4 > \sqrt{2}$. Therefore, for a vector

$$\mathbf{v} = z_1\mathbf{a}_1 + z_2\mathbf{a}_2 + z_3\mathbf{a}_3 + z_4\mathbf{a}_4 \in X(\alpha, \Lambda),$$

we have $z_4 = 0$ or ± 1 .

By Remark 4.2,

$$\text{card}\{\mathbf{v} : \mathbf{v} = z_1\mathbf{a}_1 + z_2\mathbf{a}_2 + z_3\mathbf{a}_3 \in X(\alpha, \Lambda)\} = 18.$$

Since $X(\alpha, \Lambda)$ is centrally symmetric, we have

$$\text{card}\{\mathbf{v} : \mathbf{v} = z_1\mathbf{a}_1 + z_2\mathbf{a}_2 + z_3\mathbf{a}_3 + \mathbf{a}_4 \in X(\alpha, \Lambda)\} \geq 17,$$

which means that there exist two of them is equivalent. Replace $\mathbf{a}_4$ by the mid-point of them, we may further assume

$$\|\mathbf{a}_4\|^2 = v_1^2 + v_2^2 + v_3^2 + v_4^2 = 4,$$

by Remark 2.1.

By routine computation, besides $z_1\mathbf{a}_1 + z_2\mathbf{a}_2 + z_3\mathbf{a}_3$, vector which belong to $X(\alpha, \Lambda)$ must be one of the following form:

$$\begin{aligned} z_1\mathbf{a}_1 + z_2\mathbf{a}_2 \pm \mathbf{a}_4, & \quad z_1\mathbf{a}_1 + z_2\mathbf{a}_2 \pm (\mathbf{a}_3 + \mathbf{a}_4), \\ z_1\mathbf{a}_1 + z_2\mathbf{a}_2 \pm (\mathbf{a}_3 - \mathbf{a}_4), & \quad z_1\mathbf{a}_1 + z_2\mathbf{a}_2 \pm (2\mathbf{a}_3 - \mathbf{a}_4). \end{aligned}$$

To let $\text{card}\{X(\alpha, \Lambda)\} \geq 52$, there exist one form above have at least ten vectors which belongs to $X(\alpha, \Lambda)$. Without loss of generality, we suppose

$$\text{card}\{\mathbf{v} : \mathbf{v} = z_1\mathbf{a}_1 + z_2\mathbf{a}_2 \pm \mathbf{a}_4 \in X(\alpha, \Lambda)\} \geq 10.$$

Combining with

$$\text{card}\{\mathbf{v} : \mathbf{v} = z_1\mathbf{a}_1 + z_2\mathbf{a}_2 \in X(\alpha, \Lambda)\} = 8,$$

by Lemma 4.4 and Remark 4.2, we may assume

$$v_1 = 0, \quad v_2 = 0, \quad v_3^2 + v_4^2 = 4$$

or

$$v_1 = 0, \quad v_2 = 1, \quad v_3^2 + v_4^2 = 3$$

without loss of generality.

For case $v_1 = 0, v_2 = 0, v_3^2 + v_4^2 = 4$, by routine computation we have:

$$\begin{aligned} \text{card}\{\mathbf{v} : \mathbf{v} = z_1\mathbf{a}_1 + z_2\mathbf{a}_2 \pm \mathbf{a}_4 \in X(\alpha, \Lambda)\} &= 10, \\ \text{card}\{\mathbf{v} : \mathbf{v} = z_1\mathbf{a}_1 + z_2\mathbf{a}_2 \pm (\mathbf{a}_3 + \mathbf{a}_4) \in X(\alpha, \Lambda)\} &= \begin{cases} 8, & v_3 = 0, \\ 0, & v_3 \neq 0, \end{cases} \\ \text{card}\{\mathbf{v} : \mathbf{v} = z_1\mathbf{a}_1 + z_2\mathbf{a}_2 \pm (\mathbf{a}_3 - \mathbf{a}_4) \in X(\alpha, \Lambda)\} &= 8, \\ \text{card}\{\mathbf{v} : \mathbf{v} = z_1\mathbf{a}_1 + z_2\mathbf{a}_2 \pm (2\mathbf{a}_3 - \mathbf{a}_4) \in X(\alpha, \Lambda)\} &= \begin{cases} 2, & v_3 \geq 1/\sqrt{2}, \\ 0, & v_3 = 0. \end{cases} \end{aligned}$$

Therefore, in this case we have $\text{card}\{X(\alpha, \Lambda)\} \leq 44$.

For case $v_1 = 0, v_2 = 1, v_3^2 + v_4^2 = 3$, since $\|\mathbf{a}_3 - \mathbf{a}_4\| \geq 2$, we have $v_3 \leq 1/\sqrt{2}$. By routine computation we have:

$$\begin{aligned} \text{card}\{\mathbf{v} : \mathbf{v} = z_1\mathbf{a}_1 + z_2\mathbf{a}_2 \pm \mathbf{a}_4 \in X(\alpha, \Lambda)\} &= 12, \\ \text{card}\{\mathbf{v} : \mathbf{v} = z_1\mathbf{a}_1 + z_2\mathbf{a}_2 \pm (\mathbf{a}_3 + \mathbf{a}_4) \in X(\alpha, \Lambda)\} &= 4, \end{aligned}$$

$$\begin{aligned} \text{card}\{\mathbf{v} : \mathbf{v} = z_1\mathbf{a}_1 + z_2\mathbf{a}_2 \pm (\mathbf{a}_3 - \mathbf{a}_4) \in X(\alpha, \Lambda)\} &= \begin{cases} 12, & v_3 = 1/\sqrt{2}, \\ 4, & v_3 < 1/\sqrt{2}, \end{cases} \\ \text{card}\{\mathbf{v} : \mathbf{v} = z_1\mathbf{a}_1 + z_2\mathbf{a}_2 \pm (2\mathbf{a}_3 - \mathbf{a}_4) \in X(\alpha, \Lambda)\} &= \begin{cases} 4, & v_3 = 1/\sqrt{2}, \\ 0, & v_3 < 1/\sqrt{2}. \end{cases} \end{aligned}$$

Therefore, in this case we have $\text{card}\{X(\alpha, \Lambda)\} \leq 50$.

As a conclusion of these two cases and (7.1), for $\alpha = 2\sqrt{2} - 2$ we have $\kappa_\alpha^*(B^4) = 50$. Theorem 3 is proved. $\square$

Remark 7.1 It is interesting to see that the $\sqrt{2}D_4$ lattice is not the optimal lattice in this case. Let Λ be the lattice generated by $\mathbf{a}_1 = (2, 0, 0, 0)$, $\mathbf{a}_2 = (0, 2, 0, 0)$, $\mathbf{a}_3 = (1, 0, \sqrt{3}, 0)$ and $\mathbf{a}_4 = (0, 1, 0, \sqrt{3})$. It is easy to show that, when $\alpha = 2\sqrt{2} - 2$,

$$\text{card}\{X(\alpha, \Lambda)\} = \text{card}\{X(\alpha, \sqrt{2}D_4)\} = 48.$$

8 A Link Between $\kappa_\alpha^*(B^n)$ and $\gamma^*(B^n)$

In 1964, Erdős and Rogers [3] studied the star number of the lattice covering for a convex body and proved the following result.

Theorem 8.1 Let C be an $\mathbf{o}$ -symmetric strictly convex body and Λ a covering lattice of C in $\mathbb{E}^n$. Then the star number of the covering $\{C + \mathbf{v} : \mathbf{v} \in \Lambda\}$ is at least $2^{n+1} - 1$, where the star number is the numbers of the translates of C by lattice vectors, including C , which intersect the body C .

Let $\gamma^*(B^n)$ be the lattice packing-covering constant of B^n , namely

$$\gamma^*(B^n) = \min_{\Lambda} \{r : rB^n + \Lambda \text{ is a covering of } \mathbb{E}^n\},$$

where Λ is a lattice such that $B^n + \Lambda$ is a packing in $\mathbb{E}^n$. For more details about $\gamma^*(B^n)$, we refer to [22].

There exist a strong relation between $\kappa_\alpha^*(B^n)$ and $\gamma^*(B^n)$:

Theorem 8.2 For a given dimension n_0 , suppose $\gamma^*(B^{n_0}) = \sqrt{2} - \beta$ for a positive number β . Then for $\alpha \in [2\sqrt{2} - 2\beta - 2, 2\sqrt{2} - 2)$ we have $\kappa_\alpha^*(B^{n_0}) = 2^{n_0+1} - 2$. Which means that, if $\kappa_\alpha^*(B^{n_0}) < 2^{n_0+1} - 2$ holds for $\alpha < 2\sqrt{2} - 2$, then we have $\gamma^*(B^{n_0}) \geq \sqrt{2}$.

Proof We assume that $B^{n_0} + \Lambda$ is a lattice packing attaining $\gamma^*(B^{n_0}) = \sqrt{2} - \beta$ for a positive β . For convenience, let

$$X = \{\mathbf{v} : 2 \leq \|\mathbf{v}\| \leq 2\sqrt{2} - 2\beta, \mathbf{v} \in \Lambda\}.$$

It is easy to see that the star number of the covering configuration $(\sqrt{2} - \beta)B^{n_0} + \Lambda$ is $\text{card}X + 1$. By Theorem 8.1 we have $\text{card}X \geq 2^{n_0+1} - 2$. Combining with Lemma 2.2, we get $\text{card}X = 2^{n_0+1} - 2$.

Therefore, for $\alpha \in [2\sqrt{2} - 2\beta - 2, 2\sqrt{2} - 2)$, we have $\kappa_\alpha^*(B^{n_0}) = 2^{n_0+1} - 2$. Theorem 8.2 is proved. $\square$

Remark 8.1 Notice that $\gamma^*(B^5) > \sqrt{2}$, see [22]. However, when $\alpha = 2\sqrt{9/5} - 2$, one can verify that

$$\text{card}\{X(\alpha, \sqrt{24/5}A_5^*)\} = 62.$$

Combining with Lemma 2.2, when $\alpha \in [2\sqrt{9/5} - 2, 2\sqrt{2} - 2)$ we have $\kappa_\alpha^*(B^5) = 62$.

Corollary 8.1 In [15], Schürmann and Vallentin improved the former best known result (see [22]) $\gamma^*(B^6) \leq \sqrt{2}$ to

$$\gamma^*(B^6) \leq 2\sqrt{2\sqrt{798} - 56} = 1.411081242 \dots$$

Therefore, by Theorem 8.2, for $\alpha \in [0.8222, 2\sqrt{2} - 2)$ we have $\kappa_\alpha^*(B^6) = 126$.

For a packing lattice Λ of B^n and an $\alpha < 2\sqrt{2} - 2$, the sufficient and necessary condition for

$$\text{card}\{X(\alpha, \Lambda)\} = 2^{n+1} - 2$$

is each equivalent class of Λ , except the class which contain $\mathbf{o}$, must contain a pair of vectors of $X(\alpha, \Lambda)$. It is reasonable to imagine that, this condition is hard to satisfy in high dimensions. If so, the following conjecture make sense.

Conjecture 8.1 There are infinity numbers of dimension n such that, when $\alpha < 2\sqrt{2} - 2$ we have $\kappa_\alpha^*(B^n) < 2^{n+1} - 2$. Especially, when $\alpha < 2\sqrt{2} - 2$ we have $\kappa_\alpha^*(B^8) < 510$ and $\kappa_\alpha^*(B^{24}) < 33554430$.

Remark 8.2 If Conjecture 8.1 is true, by Theorem 8.2, we have $\gamma^*(B^8) = \sqrt{2}$ and $\gamma^*(B^{24}) = \sqrt{2}$, which give an affirmative answer for Zong's Conjecture 3.1 in [22].

We write

$$E_7 = \left\{ \mathbf{v} : \mathbf{v} \in E_8; \sum v_i = 0 \right\}$$

and

$$E_6 = \left\{ \mathbf{v} : \mathbf{v} \in E_8; \sum v_i = v_7 + v_8 = 0 \right\}.$$

When $\alpha = 2\sqrt{2} - 2$, we have (see [2]) that

$$\text{card}\{X(\alpha, \sqrt{2}D_5)\} = 130,$$

$$\text{card}\{X(\alpha, \sqrt{2}E_6)\} = 342$$

and

$$\text{card}\{X(\alpha, \sqrt{2}E_7)\} = 882.$$

To end this article, we list some known results of $\kappa_\alpha^*(B^n)$ as the following Table 2.

Table 2		
n	$\kappa_\alpha^*(B^n)$ for $\alpha < 2\sqrt{2} - 2$	$\kappa_\alpha^*(B^n)$ for $\alpha = 2\sqrt{2} - 2$
2	≤ 6 (can be attained)	$= 8$
3	≤ 14 (can be attained)	$= 20$
4	≤ 30 (can be attained)	$= 50$
5	≤ 62 (can be attained)	≥ 130
6	≤ 126 (can be attained)	≥ 342
7	≤ 254 (??)	≥ 882
8	≤ 510 (??)	$= 2400$
24	≤ 33554430 (??)	≥ 415003680

Conflict of Interest The authors declare no conflict of interest.

References

- [1] Bannai E, Sloane N J A. Uniqueness of certain spherical codes. *Canadian J Math*, 1981, **33**: 437–449
- [2] Conway J H, Sloane N J A. *Sphere Packings, Lattices and Groups*. New York: Springer-Verlag, 1988
- [3] Erdős P, Rogers C A. The star number of coverings of space with convex bodies. *Acta Arith*, 1964, **9**: 41–45
- [4] Günter S. Ein stereometrisches problem. *Grunert Arch*, 1875, **57**: 209–215
- [5] Hadwiger H. Über Treffanzahlen bei translationgleichen Eikörpern. *Arch Math*, 1957, **8**: 212–213
- [6] Henk M. Electronic Geometry Model No. 2001.02.060 (2001-04-27).
http://www.eg-models.de/models/Lattices_and_Packings/Densest_Polytopes/2001.02.060/_applet.html
- [7] Hoppe R. Bestimmung der grössten Anzahl gleich grosser Kugeln, welche sich auf eine Kugel von demselben Radius, wie die übrigen, auflegen lassen. *Grunert Arch*, 1874, **56**: 302–313
- [8] Leech J. The problem of the thirteen spheres. *Math Gazette*, 1956, **40**: 22–23
- [9] Levenshtein V I. Boundaries for packings in n -dimensional Euclidean space. *Dokl Akad Nauk SSSR*, 1979, **245**: 1299–1303
- [10] Minkowski H. Dichteste gitterförmige Lagerung kongruenter Körper. *Nachr K Ges Wiss Göttingen*, 1904, 311–355
- [11] Minkowski H. *Diophantische Approximationen*. Leipzig: Teubner, 1907
- [12] Musin O R. The problem of twenty-five spheres. *Russian Math Surveys*, 2003, **58**: 794–795
- [13] Musin O R. The kissing number in four dimensions. *Ann of Math*, 2008, **168**: 1–32
- [14] Odlyzko A M, Sloane N J A. New bounds on the number of unit spheres that can touch a unit sphere in n dimensions. *J Combin Theory Ser A*, 1979, **26**: 210–214
- [15] Schürmann A, Vallentin F. Computational approaches to lattice packing and covering problems. *Discrete Comput Geom*, 2006, **35**: 73–116
- [16] Schütte K, van der Waerden B L. Das Problem der dreizehn Kugel. *Math Ann*, 1953, **125**: 325–334
- [17] Swinnerton-Dyer H P F. Extremal lattices of convex bodies. *Math Proc Cambridge Philos Soc*, 1953, **49**: 161–162
- [18] Watson G L. The number of minimum points of a positive quadratic form. *Dissertationes Math*, 1971, **84**: 1–42
- [19] Zong C. The kissing numbers of tetrahedra. *Discrete Comput Geom*, 1996, **15**: 239–252
- [20] Zong C. *Strange Phenomena in Convex and Discrete Geometry*. New York: Springer-Verlag, 1996
- [21] Zong C. *Sphere Packings*. New York: Springer-Verlag, 1999
- [22] Zong C. From deep holes to free planes. *Bull Amer Math Soc*, 2002, **39**: 533–555
- [23] Zong C. Simultaneous packing and covering in three-dimensional Euclidean space. *J London Math Soc*, 2003, **67**: 29–40