

DETECTING NONTRIVIAL PRODUCTS IN THE STABLE HOMOTOPY RING OF SPHERES VIA THE THIRD MORAVA STABILIZER ALGEBRA

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ABSTRACT. Let $p \geq 7$ be a prime number. Let $S(3)$ denote the third Morava stabilizer algebra. In recent years, Kato-Shimomura and Gu-Wang-Wu found several families of nontrivial products in the stable homotopy ring of spheres $\pi_*(S)$ using $H^{*,*}(S(3))$. In this paper, we determine all nontrivial products in $\pi_*(S)$ of the Greek letter family elements $\alpha_s, \beta_s, \gamma_s$ and Cohen's elements ζ_n which are detectable by $H^{*,*}(S(3))$. In particular, we show $\beta_1 \gamma_s \zeta_n \neq 0 \in \pi_*(S)$, if $n \equiv 2 \pmod 3$, $s \not\equiv 0, \pm 1 \pmod p$.

1. INTRODUCTION

The computation of the ring of stable homotopy groups of spheres, denoted as $\pi_*(S)$, is one of the fundamental problems in algebraic topology. The Adams-Novikov spectral sequence (ANSS) based on the Brown-Peterson spectrum BP is an incredibly powerful tool for computing the p -component of $\pi_*(S)$, where p is a prime number. The E_2 -page of the ANSS is of the form $Ext_{BP_*BP}^{s,t}(BP_*, BP_*)$ and has been extensively studied in low dimensions.

For $s = 1$, $Ext_{BP_*BP}^{1,*}(BP_*, BP_*)$ is generated by $\alpha_{kp^n/n+1}$ for $n \geq 0$, and $p \nmid k \geq 1$ [15].

For $s = 2$, $Ext_{BP_*BP}^{2,*}(BP_*, BP_*)$ is generated by $\beta_{kp^n/j, i+1}$ for suitable (n, k, j, i) [11, 12].

For $s = 3$, only partial results of $Ext_{BP_*BP}^{3,*}(BP_*, BP_*)$ are known (see, for example, [13, 14, 18]). Nonetheless, a construction of a family of linearly independent elements denoted as $\gamma_{s_3/s_2, s_1}$ in $Ext_{BP_*BP}^{3,*}(BP_*, BP_*)$ has been achieved [11].

Through the computations of $Ext_{BP_*BP}^{s,t}(BP_*, BP_*)$ in low dimensions, numerous nontrivial elements in $\pi_*(S)$ can be obtained. In particular, for $p \geq 7$, there are the Greek letter family elements, denoted as α_s, β_s , and γ_s with $s \geq 1$ [11, 15, 19, 20]. These families are represented by elements of the same name in $Ext_{BP_*BP}^{1,*}(BP_*, BP_*)$, $Ext_{BP_*BP}^{2,*}(BP_*, BP_*)$, and $Ext_{BP_*BP}^{3,*}(BP_*, BP_*)$, respectively.

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Furthermore, using the Adams spectral sequence, Cohen [2] discovered another family of nontrivial elements $\zeta_n \in \pi_*(S)$ with $n \geq 1$. The representation of ζ_n in $Ext_{BP_*BP}^{3,*}(BP_*, BP_*)$ has also been studied in [2] (also see [3]).

Nontrivial products on $\pi_*(S)$. There exists a natural ring structure on $\pi_*(S)$ in which multiplication is defined by the composition of representing maps. In order to gain a deeper understanding of the ring structure of $\pi_*(S)$, it is necessary to determine whether the product of certain given elements is trivial. The main purpose of this paper is to find nontrivial products formed by the elements in $\{\alpha_s, \beta_s, \gamma_s, \zeta_s | s \geq 1\}$. To ensure these elements are well-defined, we assume $p \geq 7$ for the remainder of the paper, unless otherwise specified.

Numerous results have been obtained in this direction. Just to mention a few:

- (a) Aubry [1] shows that $\alpha_1\beta_2\gamma_2, \beta_1^r\beta_2\gamma_2 \neq 0$ if $r \leq p-1$.
- (b) Lee-Ravenel [8] show $\beta_1^{p^2-p-1} \neq 0$ for $p \geq 7$.
- (c) Lee [7] shows: (1) $\beta_1^r\beta_s, \beta_1^{r-1}\beta_2\beta_{kp-1} \neq 0$ for $p \geq 5$, if $r, k \leq p-1$, $s < p^2 - p - 1$, and $s \not\equiv 0 \pmod p$; (2) $\beta_1^r\gamma_t, \beta_1^{r-1}\beta_2\gamma_t \neq 0$, if $r, t \leq p-1$; (3) $\alpha_1\beta_1^r\gamma_t \neq 0$, if $r \leq p-2$, $2 \leq t \leq p-1$; (4) $\beta_1^{p-1}\zeta_n \neq 0$.
- (d) Liu-Liu [9] show that $\alpha_1\beta_1^2\beta_2\gamma_s \neq 0$ if $4 < s < p$.
- (e) Zhao-Wang-Zhong [23] show that $\gamma_{p-1}\zeta_n \neq 0$ if $n \neq 4$.

In recent years, Kato-Shimomura [5] have developed a method for detecting nontrivial products on $\pi_*(S)$ through the use of $S(3)$, where $S(3)$ denotes the third Morava stabilizer algebra [16]. This new approach offers an advantage when studying products involving γ_s for arbitrarily large values of s . We can briefly recall their strategy as follows.

There exists a natural map $\phi : Ext_{BP_*BP}^{*,*}(BP_*, BP_*) \rightarrow Ext_{S(3)}^{*,*}(\mathbb{F}_p, \mathbb{F}_p) =: H^{*,*}(S(3))$. The cohomology $H^{*,*}(S(3))$ is studied in [3, 17, 22]. Given a product $x = x_1x_2 \cdots x_n \in \pi_*(S)$, we let $y = y_1y_2 \cdots y_n \in Ext_{BP_*BP}^{*,*}(BP_*, BP_*)$ represent x on the E_2 -page of the ANSS. If $\phi(y) \neq 0$, then $y \neq 0 \in Ext_{BP_*BP}^{*,*}(BP_*, BP_*)$. For the examples of interest, y will not be eliminated by any Adams-Novikov differential due to degree considerations. Thus, we can conclude that $x \neq 0 \in \pi_*(S)$ in this case.

Using this strategy, Kato-Shimomura [5] demonstrate the following: (1) $\alpha_1\gamma_s \neq 0$, if $s \not\equiv 0, \pm 1 \pmod p$; (2) $\beta_1\gamma_s \neq 0$, if $s \not\equiv 0, 1 \pmod p$; (3) $\beta_2\gamma_s \neq 0$, if $s \not\equiv 0, \pm 1 \pmod p$.

Similarly, Gu-Wang-Wu [3] show that $\zeta_n\gamma_s \neq 0$ if $n \not\equiv 1 \pmod 3$ and $s \not\equiv 0, \pm 1 \pmod p$.

Our main results. In this paper, we employ the “Detection via $H^{*,*}(S(3))$ ” method, which was developed in [3, 5], to detect nontrivial products on $\pi_*(S)$. However, instead of focusing on specific examples, we fully utilize the potential of this method and enumerate all detectable products. The main results of our study are as follows:

Theorem 1.1. *Let $p \geq 7$ be a prime. Let $n \equiv 2 \pmod 3$, and $s \not\equiv 0, \pm 1 \pmod p$. Then $\beta_1\gamma_s\zeta_n \neq 0 \in \pi_*(S)$.*

Remark 1.2. Utilizing the Adams spectral sequence, Kato-Shimomura [6] demonstrated that $\beta_1\gamma_s\zeta_n \neq 0 \in \pi_*(S)$ holds true when $3 \leq s < p-2$. The findings presented in [6] and Theorem 1.1 address distinct ranges of (n, s) , with neither

being a subset of the other. The method of “Detection via $H^{*,*}(S(3))$ ” possesses the advantage of accommodating products involving γ_s for arbitrarily large s .

Theorem 1.3. *Let $p \geq 7$ be a prime. We consider the products in $\pi_*(S)$ where each factor belongs to $\{\alpha_s, \beta_s, \gamma_s, \zeta_s : s \geq 1\}$. Among all such products, only the following ones can be detected as nontrivial products using the comparison with $H^{*,*}S(3)$.*

- (1) $\alpha_1\beta_1$,
- (2) $\alpha_1\beta_2$,
- (3) $\alpha_1\gamma_s$, if $s \not\equiv 0, \pm 1 \pmod p$,
- (4) β_1^2 ,
- (5) $\beta_1\gamma_s$, if $s \not\equiv 0, 1 \pmod p$,
- (6) $\beta_1\zeta_n$,
- (7) $\beta_2\gamma_s$, if $s \not\equiv 0, \pm 1 \pmod p$,
- (8) $\gamma_s\zeta_n$, if $n \not\equiv 1 \pmod 3$, $s \not\equiv 0, \pm 1 \pmod p$,
- (9) $\alpha_1\beta_1^2$,
- (10) $\alpha_1\beta_1\gamma_s$, if $s \not\equiv 0, \pm 1 \pmod p$,
- (11) $\beta_1^2\zeta_n$, if $n \equiv 1 \pmod 3$
- (12) $\beta_1\gamma_s\zeta_n$, if $n \equiv 2 \pmod 3$, $s \not\equiv 0, \pm 1 \pmod p$.

The nontriviality of (1) $\sim$ (11) has been determined by earlier works in [3, 5, 7, 11]. We single out the new result (12) as Theorem 1.1. We have exhausted the potential of the “Detection via $H^{*,*}(S(3))$ ” strategy in Theorem 1.3. To detect other nontrivial products in $\pi_*(S)$, different methods would need to be employed.

Organization of the paper. In Section 2, we review the basic structures of the Hopf algebroid (BP_*, BP_*BP) and the third Morava stabilizer algebra $S(3)$. In Section 3, we analyze the $\mathbb{F}_p$ -algebra structure of $H^{*,*}S(3)$. We also discuss some typos in the previous literature [3, 22]. In Section 4, we determine the images of $\{\alpha_s, \beta_s, \gamma_s, \zeta_s | s \geq 1\}$ under the comparison map $\phi : Ext_{BP_*BP}^{*,*}(BP_*, BP_*) \rightarrow H^{*,*}(S(3))$. In Section 5, we prove Theorem 1.1 and Theorem 1.3.

2. HOPF ALGEBROIDS

This section recalls the basic definitions and constructions related to Hopf algebroids. In particular, we review the basic structures of the Hopf algebroid (BP_*, BP_*BP) and the third Morava stabilizer algebra $S(3)$.

2.1. The Hopf algebroid (BP_*, BP_*BP) .

Definition 2.2. A Hopf algebroid over a commutative ring K is a pair (A, Γ) of commutative K -algebras with structure maps

$$\begin{aligned} \text{left unit map } \eta_L : A &\rightarrow \Gamma, \\ \text{right unit map } \eta_R : A &\rightarrow \Gamma, \\ \text{coproduct map } \Delta : \Gamma &\rightarrow \Gamma \otimes_A \Gamma, \\ \text{counit map } \varepsilon : \Gamma &\rightarrow A, \\ \text{conjugation map } c : \Gamma &\rightarrow \Gamma \end{aligned}$$

such that for any other commutative K -algebra B , the two sets $\text{Hom}(A, B)$ and $\text{Hom}(\Gamma, B)$ are the objects and morphisms of a groupoid.

An important example of Hopf algebroids is (BP_*, BP_*BP) . Recall that we have

$$(2.1) \quad BP_* := \pi_*(BP) = \mathbb{Z}_{(p)}[v_1, v_2, \dots], \quad BP_*BP = BP_*[t_1, t_2, \dots],$$

where the inner degrees are $|v_n| = |t_n| = 2(p^n - 1)$. Throughout this paper, we denote $v_0 = p$, and $t_0 = 1$. The structure maps of the Hopf algebroid (BP_*, BP_*BP) are described in [4, 11, 18]. In practice, the following formulas [5] are useful.

$$(2.2) \quad \eta_R(v_1) = v_1 + pt_1,$$

$$(2.3) \quad \eta_R(v_2) \equiv v_2 + v_1 t_1^p + pt_2 \pmod{(p^2, v_1^p)},$$

$$(2.4) \quad \Delta(t_1) = t_1 \otimes 1 + 1 \otimes t_1,$$

$$(2.5) \quad \Delta(t_2) = t_2 \otimes 1 + t_1 \otimes t_1^p + 1 \otimes t_2 - v_1 b_{1,0}.$$

Notation 2.3. We denote $b_{i,j} = \frac{1}{p}[(\sum_{k=0}^i t_{i-k} \otimes t_k^{p^{i-k}})^{p^{j+1}} - \sum_{k=0}^i t_{i-k}^{p^{j+1}} \otimes t_k^{p^{i-k+j+1}}]$ for $i \geq 1, j \geq 0$. See [21] for related discussions.

2.4. Morava stabilizer algebras. We recall the basic properties of the Morava stabilizer algebras, which are studied in detail in [10, 16].

Let $K(n)_*$ denote $\mathbb{F}_p[v_n, v_n^{-1}]$. We can equip $K(n)_*$ with a BP_* -algebra structure via the ring homomorphism which sends all v_i with $i \neq n$ to 0. Then we define $\Sigma(n) := K(n)_* \otimes_{BP_*} BP_*BP \otimes_{BP_*} K(n)_*$. As an algebra, one has $\Sigma(n) \cong K(n)_*[t_1, t_2, \dots]/(v_n t_i^{p^n} - v_n^p t_i | i > 0)$. The coproduct structure of $\Sigma(n)$ is inherited from that of BP_*BP .

Moreover, one can prove $Ext_{BP_*BP}^{*,*}(BP_*, v_n^{-1}BP_*/I_n) \cong Ext_{\Sigma(n)}^{*,*}(K(n)_*, K(n)_*)$, where we let I_n denote the ideal $(p, v_1, v_2, \dots, v_{n-1}) \subset BP_*$.

We define the Hopf algebra $S(n) := \Sigma(n) \otimes_{K(n)_*} \mathbb{F}_p$, where $K(n)_*$ and $\Sigma(n)$ are here regarded as graded over $\mathbb{Z}/2(p^n - 1)$ and $\mathbb{F}_p$ is a $K(n)_*$ -algebra via the map sending v_n to 1. We call $S(n)$ the n -th Morava stabilizer algebra. One can show

$$(2.6) \quad Ext_{\Sigma(n)}^{*,*}(K(n)_*, K(n)_*) \otimes_{K(n)_*} \mathbb{F}_p \cong Ext_{S(n)}^{*,*}(\mathbb{F}_p, \mathbb{F}_p) =: H^{*,*}(S(n)).$$

For the purpose of this paper, from now on, we will only consider the case when $n = 3$. We have the following results.

Proposition 2.5 ([17]). *As an algebra, $S(3) \cong \mathbb{F}_p[t_1, t_2, \dots]/(t_i^{p^3} - t_i)$ and the inner degrees are $|t_s| \equiv 2(p^s - 1) \pmod{2(p^3 - 1)}$. The coproduct structure of $S(3)$ is that inherited from BP_*BP . In particular, $\Delta(t_s) = \sum_{k=0}^s t_k \otimes t_{s-k}^{p^k}$ for $s \leq 3$, and $\Delta(t_s) = \sum_{k=0}^s t_k \otimes t_{s-k}^{p^k} - \tilde{b}_{s-3,2}$ for $s > 3$.*

Notation 2.6. We let $\tilde{b}_{i,j}$ denote the mod p reduction of $b_{i,j}$ in Notation 2.3.

2.7. Cobar complexes. Cobar complexes are helpful in computing certain Ext groups, such as

$$Ext_{BP_*BP}^{*,*}(BP_*, BP_*), Ext_{BP_*BP}^{*,*}(BP_*, v_n^{-1}BP_*/I_n), \text{ and } Ext_{S(n)}^{*,*}(\mathbb{F}_p, \mathbb{F}_p).$$

We now recall the relevant definitions and constructions.

Definition 2.8. Let (A, Γ) be a Hopf algebroid. A *right Γ -comodule* M is a right A -module M together with a right A -linear map $\psi : M \rightarrow M \otimes_A \Gamma$ which is counitary and coassociative. Left Γ -comodules are defined similarly.

Definition 2.9. Let (A, Γ) be a Hopf algebroid. Let M be a right Γ -comodule. The cobar complex $\Omega_\Gamma^{*,*}(M)$ is a cochain complex with $\Omega_\Gamma^{s,*}(M) = M \otimes_A \bar{\Gamma}^{\otimes s}$, where $\bar{\Gamma}$ is the augmentation ideal of $\varepsilon : \Gamma \rightarrow A$. The differentials $d : \Omega_\Gamma^{s,*}(M) \rightarrow \Omega_\Gamma^{s+1,*}(M)$ are given by

$$d(m \otimes x_1 \otimes x_2 \otimes \cdots \otimes x_s) = -(\psi(m) - m \otimes 1) \otimes x_1 \otimes x_2 \otimes \cdots \otimes x_s \\ - \sum_{i=1}^s (-1)^{\lambda_{i,j_i}} m \otimes x_1 \otimes \cdots \otimes x_{i-1} \otimes \left(\sum_{j_i} x'_{i,j_i} \otimes x''_{i,j_i} \right) \otimes x_{i+1} \otimes \cdots \otimes x_s,$$

where we denote

$$(2.7) \quad \sum_{j_i} x'_{i,j_i} \otimes x''_{i,j_i} = \Delta(x_i) - 1 \otimes x_i - x_i \otimes 1,$$

$$(2.8) \quad \lambda_{i,j_i} = i + |x_1| + \cdots + |x_{i-1}| + |x'_{i,j_i}|.$$

Proposition 2.10 ([18, Section A1.2]). *The cohomology of $\Omega_\Gamma^{s,*}(M)$ is $Ext_\Gamma^{s,*}(A, M)$. Moreover, if M is also a commutative associative A -algebra such that the structure map ψ is an algebra map, then $Ext_\Gamma^{s,*}(A, M)$ has a naturally induced product structure.*

3. THE COHOMOLOGY OF $S(3)$

In this section, we discuss the cohomology $H^{*,*}S(3) := Ext_{S(3)}^{*,*}(\mathbb{F}_p, \mathbb{F}_p)$ of the Hopf algebra $S(3)$. Ravenel [17] computed the $\mathbb{F}_p$ -module structure of $H^{*,*}S(3)$. The $\mathbb{F}_p$ -algebra structure of $H^{*,*}S(3)$ was subsequently computed by Yamaguchi in [22], and revisited by Gu-Wang-Wu in [3]. Unfortunately, both [22] and [3] contain typos. We will say more about these typos in this section.

Theorem 3.1 ([3, 22]). *As an $\mathbb{F}_p$ -algebra, $H^*(S(3))$ is isomorphic to the cohomology $H^*(E; d_1)$ of a certain differential graded algebra E , where*

$$(3.1) \quad E := E[h_{i,j} | i = 1, 2, 3, j \in \mathbb{Z}/3],$$

is the exterior algebra with generators $h_{i,j}$, and the differential d_1 is given by

$$(3.2) \quad d_1(h_{i,j}) = - \sum_{1 \leq k \leq i-1} h_{k,j} h_{i-k,j+k}.$$

Moreover,

$$(3.3) \quad d_1(xy) = d_1(x)y + (-1)^s x d_1(y)$$

for all monomials $x, y \in E$ and s denotes the homological degree of x . The generator $h_{i,j}$ corresponds to $t_i^{p^j} \in S(3)$ under the isomorphism $H^*(E; d_1) \cong H^*(S(3))$. The generator $h_{i,j}$ has induced inner degree $|h_{i,j}| = 2(p^i - 1)p^j \bmod 2(p^3 - 1)$.

Remark 3.2. Recall from Proposition 2.5 that $S(3) \cong \mathbb{F}_p[t_1, t_2, \dots] / (t_i^{p^3} - t_i)$. This implies $t_i^{p^j} = t_i^{p^{j+3}} \in S(3)$. Corresponding to this, we have $j \in \mathbb{Z}/3$ in Theorem 3.1.

Proposition 3.3 ([3, 22]). *Let $p \geq 7$ be a prime number. As an $\mathbb{F}_p$ -module, $H^{*,*}S(3)$ is isomorphic to $E[\rho] \otimes M$, where $\rho := h_{3,0} + h_{3,1} + h_{3,2} \in H^{1,*}S(3)$, M is an $\mathbb{F}_p$ -module with the following generators ($i \in \mathbb{Z}/3$):*

- dim0:* 1;
- dim1:* $h_{1,i}$;
- dim2:* $e_{4,i}, \quad g_i, \quad k_i$;

$$\begin{aligned}
\dim 3: & e_{4,i}h_{1,i}, \quad e_{4,i}h_{1,i+1}, \quad g_ih_{1,i+1}, \quad \mu_i, \quad \nu_i, \quad \xi; \\
\dim 4: & e_{4,i}^2, \quad e_{4,i}e_{4,i+1}, \quad e_{4,i}g_{i+1}, \quad e_{4,i}g_{i+2}, \quad e_{4,i}k_i, \quad \theta_i; \\
\dim 5: & e_{4,i}^2h_{1,i+1}, \quad e_{4,i}^2h_{1,i+2}, \quad e_{4,i}e_{4,i+1}h_{1,i+2} \quad (e_{4,i}e_{4,i+1}h_{1,i+2} = e_{4,i+1}e_{4,i+2}h_{1,i}), \\
& e_{4,i}\mu_{i+2}, \quad e_{4,i}\nu_i, \quad \eta_i; \\
\dim 6: & e_{4,i}^2e_{4,i+1}, \quad e_{4,i}^2e_{4,i+2}, \quad e_{4,i}e_{4,i+1}g_{i+2}; \\
\dim 7: & e_{4,i}e_{4,i+1}\mu_{i+2}; \\
\dim 8: & e_{4,i}^2e_{4,i+2}g_{i+1} \quad (e_{4,i}^2e_{4,i+2}g_{i+1} = e_{4,i+1}^2e_{4,i}g_{i+2}).
\end{aligned}$$

Here, the generators are defined as follows:

$$\begin{aligned}
e_{4,i} &:= h_{1,i}h_{3,i+1} + h_{2,i}h_{2,i+2} + h_{3,i}h_{1,i} & g_i &:= h_{2,i}h_{1,i}, \\
k_i &:= h_{2,i}h_{1,i+1} & \mu_i &:= h_{3,i}h_{2,i}h_{1,i}, \\
\nu_i &:= h_{3,i}h_{2,i+1}h_{1,i+2} & \xi &= \sum_{i=0}^2 h_{3,i}e_{3,i+1} + h_{2,0}h_{2,1}h_{2,2}, \\
\theta_i &:= h_{3,i}h_{2,i+2}h_{2,i}h_{1,i} & \eta_i &:= h_{3,i}h_{3,i+1}h_{2,i+2}h_{2,i}h_{1,i}.
\end{aligned}$$

Here we denote $e_{3,i} := h_{1,i}h_{2,i+1} + h_{2,i}h_{1,i+2}$ for $i \in \mathbb{Z}/3$.

Remark 3.4. The original formula for ξ in [3] was $\xi = \sum h_{3,i+1}e_{3,i} + \sum h_{2,i}h_{2,i+1}h_{2,i+2}$. However, that doesn't represent a cocycle. We have corrected the formula for ξ in Proposition 3.3. It corresponds to Yamaguchi's generator c in [22] under the relation $c - \xi = -d_1(h_{3,0}h_{3,2})$.

Using Theorem 3.1, one can compute the product relations of these additive generators by hand.

Example 3.5. Direct computation shows

- (1) $h_{1,i}k_i = h_{1,i}h_{2,i}h_{1,i+1} = -h_{2,i}h_{1,i}h_{1,i+1} = -g_ih_{1,i+1}$.
- (2) $e_{4,i+1}k_i = h_{1,i+1}h_{3,i+2}h_{2,i}h_{1,i+1} + h_{2,i+1}h_{2,i}h_{2,i}h_{1,i+1} + h_{3,i+1}h_{1,i+1}h_{2,i}h_{1,i+1} = 0$.
- (3) $k_i^2 = h_{2,i}h_{1,i+1}h_{2,i}h_{1,i+1} = 0$.

It is also useful to notice that, for $x \in H^{i,*}S(3)$, $y \in H^{j,*}S(3)$, we have $x \cdot y = (-1)^{ij}y \cdot x$.

Computing the entire $\mathbb{F}_p$ -algebra structure of $H^{*,*}S(3)$ is straightforward but quite tedious. Yamaguchi [22] and Gu-Wang-Wu [3] both listed the product relations without providing proofs. Unfortunately, both papers contain typos. As pointed out in [3, Remark A.1], the formula $a_0g'_0 = h_0b'_0 - h_1b_0$ in [22, Theorem 4.4] should be corrected to $a_0g'_0 = h_0b'_0 - 2h_1b_0$ under Yamaguchi's notation. On the other hand, [3, Appendix A] claimed $e_{4,i}e_{4,i+1}v_i = -e_{4,i}e_{4,i+1}\mu_{i+2} + \frac{2}{3}\rho e_{4,i+2}e_{4,i}g_{i+1}$. However, one can tell this is wrong since $e_{4,i}e_{4,i+1}\mu_{i+2}$ and $\rho e_{4,i+2}e_{4,i}g_{i+1}$ have different inner degrees. As another example, [3, Appendix A] claimed that $v_ih_{1,i} = \frac{1}{3}e_{4,i+1}g_{i+2}$ and $h_{1,i}e_{4,i+1}v_i = 0$. However, we have $h_{1,i}e_{4,i+1}v_i = h_{1,i}v_ie_{4,i+1} = -\frac{1}{3}e_{4,i+1}g_{i+2}e_{4,i+1} = -\frac{1}{3}e_{4,i+1}^2g_{i+2} \neq 0$, since $e_{4,i+1}^2e_{4,i}g_{i+2} \neq 0$ is a generator in dimension 8. This brings to a contradiction.

We have not reproduced all the product relations in $H^{*,*}S(3)$. We do not claim we have found all typos in [3, 22].

For the purpose of this paper, we do not attempt to determine the entire $\mathbb{F}_p$ -algebra structure of $H^{*,*}S(3)$. In Proposition 3.6, we will recompute only the products that we actually need in this paper. Therefore, the result of this paper does not depend on the computation of the full $\mathbb{F}_p$ -algebra structure of $H^{*,*}S(3)$ in [3, 22].

Proposition 3.6. *Let $p \geq 7$ be a prime number. We have the following nontrivial products among generators of $H^{*,*}S(3)$:*

- $v_0 h_{1,0} = \frac{1}{3} e_{4,1} g_2 \neq 0$.
- $e_{4,1} v_0 h_{1,0} e_{4,0} = \frac{1}{3} e_{4,1}^2 e_{4,0} g_2 \neq 0$.
- $v_0 e_{4,1} = -e_{4,2} \mu_1 + \frac{1}{3} \rho e_{4,2} g_1 + \frac{1}{3} \rho e_{4,1} k_1 \neq 0$.
- $e_{4,i+1} h_{1,i} = e_{4,i} h_{1,i+1} \neq 0$.
- $k_0 v_0 = \frac{1}{2} e_{4,1}^2 h_{1,2} \neq 0$.

Meanwhile, the following products are trivial:

$$h_{1,0} k_1 = 0, \quad k_0 k_1 = 0, \quad k_0 h_{1,0} e_{4,i} = 0, \quad h_{1,0} k_0 v_0 = 0.$$

Proof. We pick several typical examples to illustrate the method of computation. The rest of the products can be computed similarly.

(1) $v_0 h_{1,0} = \frac{1}{3} e_{4,1} g_2 \neq 0$. Recall that by definition, we have

$$(3.4) \quad v_0 h_{1,0} = [h_{3,0} h_{2,1} h_{1,2} h_{1,0}],$$

$$(3.5) \quad e_{4,1} g_2 = [h_{1,1} h_{3,2} h_{2,2} h_{1,2} + h_{2,1} h_{2,0} h_{2,2} h_{1,2} + h_{3,1} h_{1,1} h_{2,2} h_{1,2}].$$

Here, we use the bracket $[]$ to emphasize these are cohomology classes. To simplify notations, we denote $A := h_{3,0} h_{2,1} h_{1,2} h_{1,0}$, $B := h_{1,1} h_{3,2} h_{2,2} h_{1,2}$, $C := h_{2,1} h_{2,0} h_{2,2} h_{1,2}$, and $D := h_{3,1} h_{1,1} h_{2,2} h_{1,2}$. We want to show $3[A] = [B+C+D]$. We also denote $E := h_{1,0} h_{2,1} h_{3,1} h_{1,2}$, $F := h_{3,0} h_{1,1} h_{2,2} h_{1,2}$, and $G := h_{2,1} h_{1,0} h_{3,2} h_{1,2}$.

Next, we consider elements in homological degree 3, which has the same inner degree as A, B, C, D . The point is, the differential of such elements might provide relations among A, B, C, D . Direct computation shows:

$$(3.6) \quad d_1(h_{3,0} h_{3,1} h_{1,2}) = E - F + A \Rightarrow [A] = [F] - [E].$$

$$(3.7) \quad d_1(h_{3,0} h_{2,1} h_{2,2}) = -C + F + A \Rightarrow [A] = [C] - [F].$$

$$(3.8) \quad d_1(h_{3,1} h_{3,2} h_{1,2}) = -B + G + D \Rightarrow [G] = [B] - [D].$$

$$(3.9) \quad d_1(h_{3,0} h_{3,2} h_{1,2}) = -G + F \Rightarrow [F] = [G].$$

$$(3.10) \quad d_1(h_{3,1} h_{2,1} h_{2,2}) = D + E \Rightarrow [E] = -[D].$$

Then we have $3[A] = 2([F] - [E]) + ([C] - [F]) = [C] - 2[E] + [F] = [C] + 2[D] + [G] = [C] + 2[D] + [B] - [D] = [B] + [C] + [D]$. This shows $v_0 h_{1,0} = \frac{1}{3} e_{4,1} g_2$. Moreover, $e_{4,1} g_2 \neq 0$ since it is a generator in dimension 4 by Proposition 3.3.

(2) $e_{4,1} v_0 h_{1,0} e_{4,0} = \frac{1}{3} e_{4,1}^2 e_{4,0} g_2 \neq 0$. This follows directly from (1). Moreover, $e_{4,1}^2 e_{4,0} g_2 \neq 0$ since it is a generator in dimension 8 by Proposition 3.3.

(3) $h_{1,0} k_1 = [h_{1,0} h_{2,1} h_{1,2}] = 0$. This is because $d_1(h_{3,0} h_{1,2}) = h_{1,0} h_{2,1} h_{1,2} + h_{2,0} h_{1,2} h_{1,2} = h_{1,0} h_{2,1} h_{1,2}$.

(4) $k_0 k_1 = [h_{2,0} h_{1,1} h_{2,1} h_{1,2}] = 0$. This is because $d_1(h_{3,0} h_{1,1} h_{2,1}) = h_{1,0} h_{2,1} h_{1,1} h_{2,1} + h_{2,0} h_{1,2} h_{1,1} h_{2,1} + h_{3,0} h_{1,1} h_{1,1} h_{1,2} = h_{2,0} h_{1,2} h_{1,1} h_{2,1}$.

The rest of the products can be computed similarly. $\square$

4. IMAGES OF $\alpha, \beta, \gamma, \zeta$ -FAMILY ELEMENTS

In this section, we recall the constructions of the Greek letter family elements in the E_2 -page $Ext_{BP_*BP}^{*,*}(BP_*, BP_*)$ of the Adams-Novikov spectral sequence. Then we determine the images of $\{\alpha_s, \beta_s, \gamma_s, \zeta_s | s \geq 1\}$ under the comparison map $\phi : Ext_{BP_*BP}^{*,*}(BP_*, BP_*) \rightarrow H^{*,*}(S(3))$.

Note we can write ϕ as the composition of several maps. We have

$$(4.1) \quad \phi = \text{Ext}_{BP_*BP}^{*,*}(BP_*, BP_*) \xrightarrow{\eta} \text{Ext}_{BP_*BP}^{*,*}(BP_*, v_3^{-1}BP_*/I_3) \xrightarrow{\psi} H^{*,*}(S(3))$$

with $I_3 = (p, v_1, v_2) \subset BP_*$ and $\psi = \psi_3\psi_2\psi_1$, where

$$(4.2) \quad \psi_1 : \text{Ext}_{BP_*BP}^{*,*}(BP_*, v_3^{-1}BP_*/I_3) \xrightarrow{\cong} \text{Ext}_{\Sigma(3)}^{*,*}(K(3)_*, K(3)_*),$$

$$(4.3) \quad \psi_2 : \text{Ext}_{\Sigma(3)}^{*,*}(K(3)_*, K(3)_*) \rightarrow \text{Ext}_{\Sigma(3)}^{*,*}(K(3)_*, K(3)_*) \otimes_{K(3)_*} \mathbb{F}_p,$$

$$(4.4) \quad \psi_3 : \text{Ext}_{\Sigma(3)}^{*,*}(K(3)_*, K(3)_*) \otimes_{K(3)_*} \mathbb{F}_p \xrightarrow{\cong} \text{Ext}_{S(3)}^{*,*}(\mathbb{F}_p, \mathbb{F}_p) = H^{*,*}(S(3)).$$

4.1. α -Family elements. Let $n \geq 0$, $p \nmid s \geq 1$. Then

$$v_1^{sp^n} \in \text{Ext}_{BP_*BP}^{0,*}(BP_*, BP_*/p^{n+1}).$$

We define $\alpha_{sp^n/n+1} := \delta_0(v_1^{sp^n}) \in \text{Ext}_{BP_*BP}^{1,*}(BP_*, BP_*)$, where δ_0 is the boundary-homomorphism associated to the short exact sequence

$$(4.5) \quad 0 \rightarrow \Omega_{BP_*BP}(BP_*) \xrightarrow{p^{n+1}} \Omega_{BP_*BP}(BP_*) \rightarrow \Omega_{BP_*BP}(BP_*/p^{n+1}) \rightarrow 0$$

of cobar complexes (Definition 2.9). We often abbreviate $\alpha_{s/1}$ to α_s .

In order to determine the image of η , we introduce the following notion.

Definition 4.2. Let $n \geq 1$. We define $I[n]$ as the ideal of BP_* generated by monomials $p^i v_1^j v_2^k$ such that $i + j + k = n$. In particular, $I[1] = (p, v_1, v_2) = I_3 \subset BP_*$.

Lemma 4.3. Let d denote the differential of the cobar complex $\Omega_{BP_*BP}^{*,*}(BP_*)$. Let $x \in I[n] \subset BP_* = \Omega_{BP_*BP}^{0,*}(BP_*)$ for some $n \geq 1$. Then $d(x) \in I[n] \cdot \Omega_{BP_*BP}^{1,*}(BP_*)$.

Proof. BP_* can be regarded as a right BP_*BP -comodule with $\eta_R : BP_* \rightarrow BP_*BP$ as the structure map. According to Definition 2.9, for $x \in BP_*$, we have

$$(4.6) \quad d(x) = -\psi(x) + x \otimes 1 = -\eta_R(x) + x \otimes 1.$$

Note that if $x \in I[n]$, then $x \otimes 1 \in I[n] \cdot \Omega_{BP_*BP}^{1,*}(BP_*)$. Therefore, it is sufficient to show that $\eta_R(x) \in I[n] \cdot \Omega_{BP_*BP}^{1,*}(BP_*)$. Furthermore, by considering each summand separately, we can assume that x is a monomial in $BP_* = \mathbb{Z}_{(p)}[v_1, v_2, \dots]$. Write $x = p^i v_1^j v_2^k y$, where $i + j + k \geq n$. Using (2.2) and (2.3), we have

$$(4.7) \quad \begin{aligned} \eta_R(p^i v_1^j v_2^k y) &= \eta_R(p^i) \eta_R(v_1^j) \eta_R(v_2^k) \eta_R(y) \\ &= p^i (v_1 + pt_1)^j (v_2 + v_1 t_1^p + pt_2 + L)^k \eta_R(y), \end{aligned}$$

where $L \in (p^2, v_1^p) \cdot \Omega_{BP_*BP}^{1,*}(BP_*)$. By counting the exponents, we can see that $\eta_R(x) \in I[n] \cdot \Omega_{BP_*BP}^{1,*}(BP_*)$. $\square$

Proposition 4.4. Concerning the image of the α -family elements under the map ϕ specified in (4.1), we have

- (1) $\phi(\alpha_1) = -h_{1,0}$.
- (2) $\phi(\alpha_s) = 0$, for $s > 1$.

Proof. (1) The image of α_1 is computed in [5, Lemma 3.4]. Here, we still provide a detailed computation to illustrate the method.

We have $\alpha_1 = \delta_0(v_1)$. By definition of the connecting homomorphism δ_0 , we have

$$(4.8) \quad \delta_0(v_1) = \frac{d(v_1)}{p} = -\frac{(\eta_R(v_1) - v_1 \otimes 1)}{p} = -t_1,$$

where we let v_1 also denote the preimage of v_1 with respect to the map

$$\Omega_{BP_*BP}(BP_*) \rightarrow \Omega_{BP_*BP}(BP_*/p^{n+1})$$

and let d denote the differential of the cobar complex $\Omega_{BP_*BP}(BP_*)$. Therefore, upon reduction modulo I_3 , we find that $\eta(\alpha_1) = -t_1$.

On the level of cobar complexes, the effect of ψ is sending v_3 to 1. By Proposition 3.3, $-t_1$ represents $-h_{1,0}$ in $H^{*,*}(S(3))$. Therefore, $\phi(\alpha_1) = \psi(-t_1) = -h_{1,0}$.

(2) For $s \geq 2$, we have

$$(4.9) \quad \alpha_s = \delta_0(v_1^s) = \frac{d(v_1^s)}{p}.$$

Note that $v_1^s \in I[s]$. By Lemma 4.3, $d(v_1^s) \in I[s] \cdot \Omega_{BP_*BP}(BP_*)$. Then

$$(4.10) \quad \frac{d(v_1^s)}{p} \in I[s-1] \cdot \Omega_{BP_*BP}(BP_*).$$

Note $s-1 \geq 1$, we have

$$(4.11) \quad \alpha_s \in I[s-1] \cdot \Omega_{BP_*BP}(BP_*) \subset I[1] \cdot \Omega_{BP_*BP}(BP_*) = I_3 \cdot \Omega_{BP_*BP}(BP_*).$$

Upon reduction modulo I_3 , we have $\eta(\alpha_s) = 0$. Therefore, $\phi(\alpha_s) = 0$ for $s > 1$. $\square$

Notation 4.5. In this paper, we often abuse the notation and refer to the elements in $Ext_{\Gamma}^{s,*}(A, M)$ by their representatives in the associated cobar complex $\Omega_{\Gamma}^{s,*}(M)$ when no confusion arises. For example, here we let $-t_1$ denote the element in $Ext_{BP_*BP}^{1,*}(BP_*, v_3^{-1}BP_*/I_3)$ represented by $-t_1 \in \Omega_{BP_*BP}^{1,*}(v_3^{-1}BP_*/I_3)$.

Remark 4.6. Here, the result for $\phi(\alpha_1)$ differs from the formula in [5, Lemma 3.4] by a negative sign, as our definitions of the differential in the cobar complex (Definition 2.9) differ by a negative sign.

4.7. β -Family elements. Let $a_0 = 1$, $a_n = p^n + p^{n-1} - 1$ for $n \geq 1$. Define $x_n \in v_2^{-1}BP_*$ as

$$(4.12) \quad x_0 = v_2,$$

$$(4.13) \quad x_1 = x_0^p - v_1^p v_2^{-1} v_3,$$

$$(4.14) \quad x_2 = x_1^p - v_1^{p^2-1} v_2^{p^2-p+1} - v_1^{p^2+p-1} v_2^{p^2-2p} v_3,$$

$$(4.15) \quad x_n = x_{n-1}^p - 2v_1^{b_n} v_2^{p^n - p^{n-1} + 1}, n \geq 3$$

with $b_n = (p+1)(p^{n-1} - 1)$ for $n > 1$. Now, if $s \geq 1$ and $p^i | j \leq a_{n-i}$ with $j \leq p^n$ if $s = 1$, then $x_n^s \in Ext_{BP_*BP}^{0,*}(BP_*, BP_*/(p^{i+1}, v_1^j))$. Define

$$(4.16) \quad \beta_{sp^n/j, i+1} := \delta' \delta''(x_n^s) \in Ext_{BP_*BP}^{2,*}(BP_*, BP_*),$$

where δ' (resp. δ'') is the boundary-homomorphism associated to E' (resp. E'')

$$(4.17) \quad E' : 0 \rightarrow \Omega(BP_*) \xrightarrow{p^{i+1}} \Omega(BP_*) \rightarrow \Omega(BP_*/p^{i+1}) \rightarrow 0,$$

$$(4.18) \quad E'' : 0 \rightarrow \Omega(BP_*/p^{i+1}) \xrightarrow{v_1^j} \Omega(BP_*/p^{i+1}) \rightarrow \Omega(BP_*/(p^{i+1}, v_1^j)) \rightarrow 0,$$

where we let $\Omega(-)$ denote $\Omega_{BP_*BP}(-)$. We often abbreviate $\beta_{sp^n/j,1}$ to $\beta_{sp^n/j}$ and $\beta_{sp^n/1}$ to β_{sp^n} . When we work with β -family elements in practice, we require the indexes (s, n, j, i) to satisfy certain relations as specified in Theorem 4.8.

Theorem 4.8 ([11, 12]). *Let p be an odd prime. $\text{Ext}_{BP_*BP}^{2,*}(BP_*, BP_*)$ is the direct sum of cyclic subgroups generated by $\beta_{sp^n/j, i+1}$ for $n \geq 0$, $p \nmid s \geq 1$, $j \geq 1$, $i \geq 0$, subject to: (1) $j \leq p^n$, if $s = 1$, (2) $p^i | j \leq a_{n-i}$, and (3) $a_{n-i-1} < j$, if $p^{i+1} | j$.*

Proposition 4.9. *Concerning the image of the β -family elements under the map ϕ specified in (4.1), we have*

- (1) $\phi(\beta_1) = -e_{4,1}$.
- (2) $\phi(\beta_2) = 2k_0$.
- (3) $\phi(\beta_s) = 0$, for $s > 2$.
- (4) $\phi(\beta_{p^n/p^n}) = -e_{4,n+1}$, for $n \geq 1$.
- (5) $\phi(\beta_{sp^n/p^n}) = 0$, for $n \geq 1$, $s \geq 2$.

Proof. (1) and (2) are computed in [5, Lemma 3.4]. Note the elements $b_0 := h_{1,1}h_{3,2} + h_{2,1}h_{2,0} + h_{3,1}h_{1,1}$, $k_0 := h_{2,0}h_{1,1}$ defined in [5, Theorem 2.7] correspond to $e_{4,1}$ and k_0 respectively in our notation, see Proposition 3.3.

Before proving (3), (4), (5), we first introduce some notations.

Consider $\beta_{sp^n/p^n} = \delta' \delta''(x_n^s)$, we denote

$$(4.19) \quad y_{sp^n/p^n} := \delta''(x_n^s) = \frac{d'(x_n^s)}{v_1^{p^n}} \in \Omega(BP_*/p),$$

where we let x_n^s also denote the preimage of x_n^s with respect to the map $\Omega(BP_*/p) \rightarrow \Omega(BP_*/(p, v_1^{p^n}))$ and let d' denote the differential map of the cobar complex $\Omega(BP_*/p)$.

Similarly, using the definition of the connecting homomorphism δ' , we have

$$(4.20) \quad \beta_{sp^n/p^n} = \delta'(y_{sp^n/p^n}) = \frac{d(y_{sp^n/p^n})}{p} \in \Omega(BP_*),$$

where we let y_{sp^n/p^n} also denote the preimage of y_{sp^n/p^n} with respect to the map $\Omega(BP_*) \rightarrow \Omega(BP_*/p)$ and let d denote the differential map of the cobar complex $\Omega(BP_*)$.

(3) Let $n = 0$, $s \geq 3$. Then we have:

$$(4.21) \quad y_s = \delta''(v_2^s) = \frac{d'(v_2^s)}{v_1} \in I[s-1] \cdot \Omega(BP_*/p).$$

$$(4.22) \quad \beta_s = \delta'(y_s) = \frac{d(y_s)}{p} \in I[s-2] \cdot \Omega(BP_*) \subset I_3 \cdot \Omega(BP_*).$$

Upon reduction modulo I_3 , we have $\eta(\beta_s) = 0$. Then $\phi(\beta_s) = 0$ for $s \geq 3$.

(4) Let $n \geq 1$. We claim that in $\Omega(BP_*/(p, v_1^{p^n}))$, we can express x_n as $v_2^{p^n} + L_n$, where $L_n \in I[2p^n - p^{n-1}] \cdot \Omega(BP_*/(p, v_1^{p^n}))$.

If $n = 1$, we have $x_1 = v_2^p - v_1^p v_2^{-1} v_3 = v_2^p \in \Omega(BP_*/(p, v_1^p))$ since $v_1^p = 0 \in \Omega(BP_*/(p, v_1^p))$. If $n = 2$, we have $x_2 = x_1^p - v_1^{p^2-1} v_2^{p^2-p+1} - v_1^{p^2+p-1} v_2^{p^2-2p} v_3 = v_2^{p^2} - v_1^{p^2-1} v_2^{p^2-p+1} \in \Omega(BP_*/(p, v_1^{p^2}))$ since $p, v_1^{p^2} = 0 \in \Omega(BP_*/(p, v_1^{p^2}))$. The case for general $n \geq 3$ can be proved analogously.

Consequently, we have

$$(4.23) \quad y_{p^n/p^n} = \delta''(x_n) = \frac{d'(x_n)}{v_1^{p^n}} = \frac{d'(v_2^{p^n})}{v_1^{p^n}} + \frac{d'(L_n)}{v_1^{p^n}}.$$

$$(4.24) \quad \beta_{p^n/p^n} = \delta'(y_{p^n/p^n}) = \frac{d(y_{p^n/p^n})}{p} = \frac{1}{p}d\left(\frac{d'(v_2^{p^n})}{v_1^{p^n}}\right) + \frac{1}{p}d\left(\frac{d'(L_n)}{v_1^{p^n}}\right).$$

Note that $L_n \in I[2p^n - p^{n-1}] \cdot \Omega(BP_*/(p, v_1^{p^n}))$. Analogous to Lemma 4.3, we have

$$(4.25) \quad \frac{d'(L_n)}{v_1^{p^n}} \in I[p^n - p^{n-1}] \cdot \Omega(BP_*/p),$$

$$(4.26) \quad \frac{1}{p}d\left(\frac{d'(L_n)}{v_1^{p^n}}\right) \in I[p^n - p^{n-1} - 1] \cdot \Omega(BP_*) \subset I_3 \cdot \Omega(BP_*).$$

Using (2.3), we can write $\eta_R(v_2) = v_2 + v_1 t_1^p + p t_2 + L$, where $L \in (p^2, v_1^p) \cdot BP_* BP$. Since $p = 0$ in $\Omega(BP_*/p)$, we can write:

$$(4.27) \quad \frac{d'(v_2^{p^n})}{v_1^{p^n}} = -t_1^{p^{n+1}} + L_{p^n},$$

where $L_{p^n} \in I[p^{n+1} - p^n] \cdot \Omega(BP_*/p)$. This implies

$$(4.28) \quad \frac{1}{p}d(L_{p^n}) \in I[p^{n+1} - p^n - 1] \cdot \Omega(BP_*) \subset I_3 \cdot \Omega(BP_*).$$

Moreover, we have

$$(4.29) \quad \begin{aligned} \frac{1}{p}d(-t_1^{p^{n+1}}) &= -\frac{1}{p}[\Delta(t_1^{p^{n+1}}) - 1 \otimes t_1^{p^{n+1}} - t_1^{p^{n+1}} \otimes 1] \\ &= -\frac{1}{p}[(1 \otimes t_1 + t_1 \otimes 1)^{p^{n+1}} - 1 \otimes t_1^{p^{n+1}} - t_1^{p^{n+1}} \otimes 1] \\ &= -b_{1,n}, \end{aligned}$$

as defined in Notation 2.3.

Combining (4.24) $\sim$ (4.29), we have $\eta(\beta_{p^n/p^n}) = -b_{1,n}$ using Notation 2.6.

Next, we will show that $\psi(b_{1,n}) = e_{4,n+1} \in H^{*,*}S(3)$ for $n \geq 0$. Then, we can conclude that $\phi(\beta_{p^n/p^n}) = -e_{4,n+1}$, for $n \geq 1$.

Following Notation 2.6, we have $\psi(b_{1,n}) = \tilde{b}_{1,n}$. By Proposition 2.5, in the cobar complex $\Omega_{S(3)}^{*,*}(\mathbb{F}_p)$, we have $d(t_4) = t_1 \otimes t_3^p + t_2 \otimes t_2^{p^2} + t_3 \otimes t_1^{p^3} - \tilde{b}_{1,2}$. Hence, we have equivalent cohomology classes $[\tilde{b}_{1,2}] = [t_1 \otimes t_3^p + t_2 \otimes t_2^{p^2} + t_3 \otimes t_1^{p^3}] = e_{4,3}$. This implies $\psi(b_{1,2}) = e_{4,3}$.

Note that if a is not a multiple of p , then $a^p \equiv a$ modulo p . Hence, working over $\mathbb{F}_p$, we have $\tilde{b}_{1,n+1} = \tilde{b}_{1,n}^p$. Moreover, note that $t_1^{p^3} = t_1$ in $S(3)$, so we have $\tilde{b}_{1,n+3} = \tilde{b}_{1,n}$. Similarly, one can show that $e_{4,n+1} = e_{4,n}^p$ and $e_{4,n+3} = e_{4,n}$. Hence, we conclude that $\psi(b_{1,n}) = e_{4,n+1}$ for each $n \geq 0$.

This finishes the proof for statement (4).

(5) Let $n \geq 1, s \geq 2$. Direct observation shows $x_n \in I[p^n - p^{n-1}] \cdot \Omega(BP_*/(p, v_1^{p^n}))$. From this, we can conclude

$$(4.30) \quad x_n^s \in I[sp^n - sp^{n-1}] \cdot \Omega(BP_*/(p, v_1^{p^n})),$$

$$(4.31) \quad y_{sp^n/p^n} \in I[sp^n - sp^{n-1} - p^n] \cdot \Omega(BP_*/p),$$

$$(4.32) \quad \beta_{sp^n/p^n} \in I[sp^n - sp^{n-1} - p^n - 1] \cdot \Omega(BP_*).$$

Note $sp^n - sp^{n-1} - p^n - 1 = (sp - s - p)p^{n-1} - 1 \geq sp - s - p - 1 \geq p - 3 > 1$. We have $\eta(\beta_{sp^n/p^n}) = 0$. Then $\phi(\beta_{sp^n/p^n}) = 0$, for $n \geq 1, s \geq 2$. $\square$

4.10. γ -Family elements. Analogous to α -family and β -family elements, one can construct γ -family elements in $Ext_{BP_*BP}^{3,*}(BP_*, BP_*)$. For the purpose of this paper, we only need to consider γ_s for $s \geq 1$. Recall the following result concerning $\phi(\gamma_s)$ (see [5, Lemma 3.4], also [3, Lemma 4.1]).

Proposition 4.11. *Concerning the image of the γ -family elements under the map ϕ specified in (4.1), we have*

$$\phi(\gamma_s) = -s(s^2 - 1)\nu_0 + s(s - 1)\rho k_1, \quad s \geq 1.$$

Remark 4.12. Here, the result in Proposition 4.11 differs from the formula in [3, 5] by a negative sign, as our definitions of the differential in the cobar complex (Definition 2.9) differ by a negative sign.

4.13. ζ -Family elements. Let $p \geq 7, s \geq 1$. It is proved in [11, 15, 19, 20] that $\alpha_s, \beta_s, \gamma_s$ all represent nontrivial elements in $\pi_*(S)$. Using the Adams spectral sequence, Cohen [2] also found another family of nontrivial elements $\zeta_n \in \pi_*(S)$, for $n \geq 1$. We also denote the representative of ζ_n in the Adams-Novikov E_2 -page $Ext_{BP_*BP}^{*,*}(BP_*, BP_*)$ by ζ_n .

Cohen [2] shows $\zeta_n = \alpha_1 \beta_{p^n/p^n} + \alpha_1 x \in Ext_{BP_*BP}^{3,*}(BP_*, BP_*)$, where $x = \sum_{s,k,j} a_{s,k,j} \beta_{sp^k/j}$, $0 \leq a_{s,k,j} \leq p - 1$, and $a_{1,n,p^n} = 0$. Moreover, by comparing the inner degrees, one can show [3] $x = \sum_{s,k,p^k} a_{s,k,p^k} \beta_{sp^k/p^k}$, where $k \leq n, s = \frac{p^{n-k+1}+1}{p+1} \neq 1$. Then, simple calculation shows that $s > 2$.

Proposition 4.14. *Concerning the image of the ζ -family elements under the map ϕ specified in (4.1), we have*

$$\phi(\zeta_n) = h_{1,0}e_{4,n+1}, \quad n \geq 1.$$

Proof. We have $\zeta_n = \alpha_1 \beta_{p^n/p^n} + \alpha_1 x$. By Proposition 4.4 and Proposition 4.9, we have $\phi(\alpha_1) = -h_{1,0}$, $\phi(\beta_{p^n/p^n}) = -e_{4,n+1}$, and $\phi(x) = 0$. Therefore, $\phi(\zeta_n) = \phi(\alpha_1)\phi(\beta_{p^n/p^n}) = h_{1,0}e_{4,n+1}$. $\square$

Gathering the analysis of the α -family, β -family, γ -family, and ζ -family elements, we have the following result.

Proposition 4.15. *Under the comparison map*

$$\phi : Ext_{BP_*BP}^{*,*}(BP_*, BP_*) \rightarrow H^{*,*}(S(3)),$$

all nonzero images of $\{\alpha_s, \beta_s, \gamma_s, \zeta_s | s \geq 1\}$ are listed as follows:

- (1) $\phi(\alpha_1) = -h_{1,0}$,
- (2) $\phi(\beta_1) = -e_{4,1}$
- (3) $\phi(\beta_2) = 2k_0$,

- (4) $\phi(\gamma_s) = -s(s^2 - 1)\nu_0 + s(s - 1)\rho k_1$, for $s \not\equiv 0, 1 \pmod p$.
 (5) $\phi(\zeta_n) = h_{1,0}e_{4,n+1}$, for $n \geq 1$.

Proof. This follows directly from Propositions 4.4, 4.9, 4.11, and 4.14. Note $h_{1,0}e_{4,0} = e_{4,0}h_{1,0} \neq 0$ since $e_{4,0}h_{1,0}$ is a generator of $H^{3,*}(S(3))$ (Proposition 3.3). Similarly, $h_{1,0}e_{4,2} = e_{4,2}h_{1,0} \neq 0$. Finally, $h_{1,0}e_{4,1} \neq 0$ since $e_{4,1}e_{4,2}h_{1,0} \neq 0$ is a generator in dimension 5. Therefore, $\phi(\zeta_n) \neq 0$ for $n \geq 1$. $\square$

5. DETECTION OF NONTRIVIAL PRODUCTS IN $\pi_*(S)$

In this section, we prove Theorems 1.1 and 1.3.

Proof of Theorem 1.1. We consider the representation of $\beta_1\gamma_s\zeta_n$ on the E_2 -page of the ANSS. According to Propositions 4.15 and 3.6, we have

$$\begin{aligned}\phi(\beta_1\gamma_s\zeta_n) &= e_{4,1}(s(s^2 - 1)\nu_0 - s(s - 1)\rho k_1)h_{1,0}e_{4,n+1} \\ &= s(s^2 - 1)e_{4,1}\nu_0h_{1,0}e_{4,n+1} - s(s - 1)\rho e_{4,1}k_1h_{1,0}e_{4,n+1} \\ &= s(s^2 - 1)e_{4,1}\nu_0h_{1,0}e_{4,0} \quad (n \equiv 2 \pmod 3, h_{1,0}k_1 = 0) \\ &= \frac{s(s^2 - 1)}{3}e_{4,1}^2e_{4,0}g_2 \\ &\neq 0\end{aligned}$$

when $n \equiv 2 \pmod 3$, and $s \not\equiv 0, \pm 1 \pmod p$.

Hence, we conclude $\beta_1\gamma_s\zeta_n \neq 0 \in Ext_{BP_*BP}^{8,*}(BP_*, BP_*)$. Since β_1 , γ_s , and ζ_n are all permanent cycles in the ANSS, their product is also a permanent cycle.

Note the differentials of the ANSS have the form $d_r : E_r^{s,t} \rightarrow E_r^{s+r,t+r-1}$, where $r \geq 2$. Additionally, the inner degrees of the elements in the ANSS are all multiples of $q = 2p - 2$. Thus, the first potentially nontrivial differentials in the ANSS occur at d_{2p-1} . Suppose $\beta_1\gamma_s\zeta_n$ is in the target of a differential d_r . Then we have $8 = s + r \geq 2p - 1 \geq 13$. This is a contradiction. Hence, $\beta_1\gamma_s\zeta_n$ is not in the target of any differential in the ANSS. This proves $\beta_1\gamma_s\zeta_n$ survives to the nontrivial product $\beta_1\gamma_s\zeta_n \neq 0 \in \pi_*(S)$. $\square$

Proof of Theorem 1.3. Let $X = X_1X_2 \cdots X_m$ be a product in $\pi_*(S)$ where each factor belongs to the set $\{\alpha_s, \beta_s, \gamma_s, \zeta_s \mid s \geq 1\}$. Let

$$x = x_1x_2 \cdots x_m \in Ext_{BP_*BP}^{*,*}(BP_*, BP_*)$$

represent X on the Adams-Novikov E_2 -page. If X can be detected as nontrivial by comparing with $H^{*,*}(S(3))$, then we have $\phi(x) \neq 0 \in H^{*,*}(S(3))$.

On the other hand, if $0 \neq \phi(x) \in H^{a,*}(S(3))$, then it follows that $a \leq 9$. Similar to the arguments in the proof of Theorem 1.1, we can show that x cannot be in the target of any differential in the ANSS by degree reasons. Hence, the product X is nontrivial in $\pi_*(S)$.

For the rest of the proof, our task is to find all products X such that $\phi(x) \neq 0$. Note $\phi(x) \neq 0$ implies $\phi(x_i) \neq 0$ for $1 \leq i \leq m$. Then $X_i \in \{\alpha_1, \beta_1, \beta_2, \gamma_s, \zeta_n \mid s \not\equiv 0, 1 \pmod p, n \geq 1\}$ by Proposition 4.15.

We first consider binary products. By Propositions 3.3, 3.6, and 4.15, we have:

- (1) $\phi(\alpha_1\alpha_1) = h_{1,0}^2 = 0$,
- (2) $\phi(\alpha_1\beta_1) = h_{1,0}e_{4,1} \neq 0$.
- (3) $\phi(\alpha_1\beta_2) = -2h_{1,0}k_0 \neq 0$, by Example 3.5.

- (4) $\phi(\alpha_1\gamma_s) = s(s^2 - 1)h_{1,0}\nu_0 + s(s - 1)\rho h_{1,0}k_1 = s(s^2 - 1)h_{1,0}\nu_0 \neq 0$, if and only if $s \not\equiv 0, \pm 1 \pmod p$, since $h_{1,0}\nu_0 \neq 0$, $h_{1,0}k_1 = 0$.
- (5) $\phi(\alpha_1\zeta_n) = -h_{1,0}^2e_{4,n+1} = 0$,
- (6) $\phi(\beta_1^2) = e_{4,1}^2 \neq 0$, since $e_{4,1}^2 \in H^{*,*}(S(3))$ is a generator.
- (7) $\phi(\beta_1\beta_2) = -2e_{4,1}k_0 = 0$, by Example 3.5.
- (8) $\phi(\beta_1\gamma_s) = s(s^2 - 1)e_{4,1}\nu_0 - s(s - 1)\rho e_{4,1}k_1 \neq 0$, if and only if $s \not\equiv 0, 1 \pmod p$ by Proposition 3.6.
- (9) $\phi(\beta_1\zeta_n) = -e_{4,1}h_{1,0}e_{4,n+1} \neq 0$, since $h_{1,0}e_{4,1}^2$, $h_{1,0}e_{4,1}e_{4,2}$, and $h_{1,0}e_{4,1}e_{4,0} = e_{4,0}^2h_{1,1}$ are all generators in $H^{*,*}(S(3))$.
- (10) $\phi(\beta_2^2) = 4k_0^2 = 0$, by Example 3.5.
- (11) $\phi(\beta_2\gamma_s) = -2s(s^2 - 1)k_0\nu_0 + 2s(s - 1)\rho k_0k_1 = -2s(s^2 - 1)k_0\nu_0 \neq 0$, if and only if $s \not\equiv 0, \pm 1 \pmod p$,
- (12) $\phi(\beta_2\zeta_n) = 2k_0h_{1,0}e_{4,n+1} = 0$,
- (13) $\phi(\gamma_s\gamma_t) = 0$, since $\rho^2 = 0$, $\nu_0^2 = 0$, and $\nu_0k_1 = 0$ by direct computation similar to Example 3.5.
- (14) $\phi(\gamma_s\zeta_n) = -s(s^2 - 1)\nu_0h_{1,0}e_{4,n+1} + s(s - 1)\rho k_1h_{1,0}e_{4,n+1} = -s(s^2 - 1)\nu_0h_{1,0}e_{4,n+1} \neq 0$, if and only if $n \not\equiv 1 \pmod 3$, $s \not\equiv 0, \pm 1 \pmod p$. Note $\nu_0h_{1,0}e_{4,2} = 0$ by direct computation similar to Example 3.5. Besides, we note that $\nu_0h_{1,0}e_{4,0} \neq 0$ and $\nu_0h_{1,0}e_{4,1} \neq 0$. These assertions follow from the result that $e_{4,1}\nu_0h_{1,0}e_{4,0} \neq 0$ by Proposition 3.6.
- (15) $\phi(\zeta_m\zeta_n) = 0$, since $h_{1,0}^2 = 0$.

For triple products, if $X = X_1X_2X_3 \neq 0$, then X_1X_2 , X_2X_3 , and X_1X_3 are all nontrivial. By the above analysis, we only need to consider the following products.

- (1) $\phi(\alpha_1\beta_1^2) = -h_{1,0}e_{4,1}^2 \neq 0$.
- (2) $\phi(\alpha_1\beta_1\gamma_s) = -s(s^2 - 1)h_{1,0}e_{4,1}\nu_0 - s(s - 1)\rho h_{1,0}e_{4,1}k_1 = -s(s^2 - 1)h_{1,0}e_{4,1}\nu_0 \neq 0$, if and only if $s \not\equiv 0, \pm 1 \pmod p$. Note $h_{1,0}e_{4,1}\nu_0 \neq 0$ since $e_{4,1}\nu_0h_{1,0}e_{4,0} \neq 0$ by Proposition 3.6.
- (3) $\phi(\alpha_1\beta_2\gamma_s) = 2s(s^2 - 1)h_{1,0}k_0\nu_0 + 2s(s - 1)\rho h_{1,0}k_0k_1 = 0$, since $h_{1,0}k_0\nu_0$, $h_{1,0}k_1 = 0$ by Proposition 3.6.
- (4) $\phi(\beta_1^3) = -e_{4,1}^3 = 0$ by direct computation similar to Example 3.5.
- (5) $\phi(\beta_1^2\gamma_s) = -s(s^2 - 1)e_{4,1}^2\nu_0 + s(s - 1)\rho e_{4,1}^2k_1 = 0$, since $e_{4,1}^2\nu_0$, $e_{4,1}^2k_1 = 0$ by direct computation similar to Example 3.5.
- (6) $\phi(\beta_1^2\zeta_n) = h_{1,0}e_{4,1}^2e_{4,n+1} \neq 0$, if and only if $n \equiv 1 \pmod 3$. Note $h_{1,0}e_{4,1}^2e_{4,0}$, $h_{1,0}e_{4,1}^3 = 0$ by direct computation. Using the formula $e_{4,i+1}h_{1,i} = e_{4,i}h_{1,i+1}$ from Proposition 3.6, we have $h_{1,0}e_{4,1}^2e_{4,2} = e_{4,1}^2e_{4,2}h_{1,0} = e_{4,1}^2e_{4,0}h_{1,2} \neq 0$ since $e_{4,1}^2e_{4,0}h_{2,2}h_{1,2} = e_{4,1}^2e_{4,0}g_2 \neq 0$ is a generator.
- (7) $\phi(\beta_1\gamma_s\zeta_n) = s(s^2 - 1)e_{4,1}\nu_0h_{1,0}e_{4,n+1} \neq 0$, if and only if $n \equiv 2 \pmod 3$, $s \not\equiv 0, \pm 1 \pmod p$. Note we have $e_{4,1}\nu_0h_{1,0}e_{4,0} \neq 0$ by Proposition 3.6. On the other hand, $e_{4,1}\nu_0h_{1,0}e_{4,1} = 0$, and $e_{4,1}\nu_0h_{1,0}e_{4,2} = 0$ by direct computation similar to Example 3.5.

For four-fold products, if $X = X_1X_2X_3X_4 \neq 0$, then $X_1X_2X_3$, $X_2X_3X_4$, $X_1X_3X_4$, and $X_1X_2X_4$ are all nontrivial. By the above analysis, there are no nontrivial four-fold products.

In this way, we have found all nontrivial products of the desired form. The result is summarized as Theorem 1.3. $\square$

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