Regularities and Exponential Ergodicity in Entropy for SDEs Driven by Distribution Dependent Noise*

Xing Huang [a)], **Feng-Yu Wang** [a),b)]

a)Center for Applied Mathematics, Tianjin University, Tianjin 300072, China

xinghuang@tju.edu.cn

b)Department of Mathematics, Swansea University, Bay Campus, SA1 8EN, United Kingdom

wangfy@tju.edu.cn

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Abstract

As two crucial tools characterizing regularity properties of stochastic systems, the log-Harnack inequality and Bismut formula have been intensively studied for distribution dependent (McKean-Vlasov) SDEs. However, due to technical difficulties, existing results mainly focus on the case with distribution free noise. In this paper, we introduce a noise decomposition argument to establish the log-Harnack inequality and Bismut formula for SDEs with distribution dependent noise, in both non-degenerate and degenerate situations. As application, the exponential ergodicity in entropy is investigated.

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1 Introduction

Let $\mathscr{P}(\mathbb{R}^d)$ be the space of all probability measures on $\mathbb{R}^d$ equipped with the weak topology. Consider the following distribution dependent SDE on $\mathbb{R}^d$:

$$(1.1)\qquad \mathrm{d}X_t = b_t(X_t, \mathscr{L}_{X_t})\mathrm{d}t + \sigma_t(X_t, \mathscr{L}_{X_t})\mathrm{d}B_t, \quad t \in [0,T],$$

where $T > 0$ is a fixed time, $\mathscr{L}_{X_t}$ is the distribution of X_t,

$$b : [0,T] \times \mathbb{R}^d \times \mathscr{P}(\mathbb{R}^d) \to \mathbb{R}^d, \quad \sigma : [0,T] \times \mathbb{R}^d \times \mathscr{P}(\mathbb{R}^d) \to \mathbb{R}^d \otimes \mathbb{R}^d$$

are measurable, and B_t is a d-dimensional Brownian motion on a complete filtration probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\in[0,T]}, \mathbb{P})$.

We investigate the regularity in initial distributions for solutions to (1.1). More precisely, for $k > 1$ let

$$\mathscr{P}_k(\mathbb{R}^d) := \big\{\mu \in \mathscr{P}(\mathbb{R}^d) : \ \|\mu\|_k := \mu(|\cdot|^k)^{\frac{1}{k}} < \infty\big\},$$

which is a Polish space under the L^k-Wasserstein distance

$$\mathbb{W}_k(\mu,\nu) = \inf_{\pi\in\mathscr{C}(\mu,\nu)} \bigg(\int_{\mathbb{R}^d\times\mathbb{R}^d} |x-y|^k \pi(\mathrm{d}x, \mathrm{d}y)\bigg)^{\frac{1}{k}}, \quad \mu,\nu \in \mathscr{P}_k(\mathbb{R}^d),$$

where $\mathscr{C}(\mu,\nu)$ is the set of all couplings of μ and ν. When (1.1) is well-posed for distributions in $\mathscr{P}_k(\mathbb{R}^d)$, i.e. for any $\mathscr{F}_0$-measurable initial value X_0 with $\mathscr{L}_{X_0} \in \mathscr{P}_k(\mathbb{R}^d)$ (correspondingly, any initial distribution $\mu \in \mathscr{P}_k(\mathbb{R}^d)$), the SDE (1.1) has a unique solution (correspondingly, a unique weak solution) with $\mathscr{L}_{X_\cdot} \in C([0,T], \mathscr{P}_k(\mathbb{R}^d))$, we consider the regularity of the maps

$$\mathscr{P}_k(\mathbb{R}^d) \ni \mu \mapsto P_t^*\mu := \mathscr{L}_{X_t} \text{ for } \mathscr{L}_{X_0} = \mu, \quad t \in (0,T].$$

Since $P_t^*\mu$ is uniquely determined by

$$(1.2)\qquad P_t f(\mu) := \int_{\mathbb{R}^d} f\mathrm{d}(P_t^*\mu), \quad f \in \mathscr{B}_b(\mathbb{R}^d),$$

where $\mathscr{B}_b(\mathbb{R}^d)$ is the space of bounded measurable functions on $\mathbb{R}^d$, we study the regularity of functionals

$$\mathscr{P}_k(\mathbb{R}^d) \ni \mu \mapsto P_t f(\mu), \quad t \in (0,T], f \in \mathscr{B}_b(\mathbb{R}^d).$$

When the noise is distribution free, i.e. $\sigma_t(x,\mu) = \sigma_t(x)$ does not depend on the distribution argument μ, the log-Harnack inequality

$$(1.3)\quad P_t \log f(\mu) \le \log P_t f(\nu) + \frac{c}{t}\mathbb{W}_2(\mu,\nu)^2, \quad 0 < f \in \mathscr{B}_b(\mathbb{R}^d), t \in (0,T], \mu,\nu \in \mathscr{P}_2(\mathbb{R}^d),$$

for some constant $c > 0$ has been established in [9, 10, 14, 21, 22] under different conditions, see also [6, 7] for extensions to the infinite-dimensional case. A crucial application of this inequality is that it is equivalent to the entropy-cost estimate

$$\mathrm{Ent}(P_t^*\nu|P_t^*\mu) \le \frac{c}{t}\mathbb{W}_2(\mu,\nu)^2, \quad t \in (0,T], \mu,\nu \in \mathscr{P}_2(\mathbb{R}^d),$$

where $\mathrm{Ent}(\nu|\mu)$ is the relative entropy of ν with respect to μ. With this estimate, the exponential ergodicity of P_t^* in entropy is proved in [14] for a class of time-homogeneous distribution dependent SDEs. The study of (1.3) goes back to [17, 18] where the family of dimension-free Harnack inequalities is introduced, see [19] for various applications of this type inequalities.

Another crucial tool characterizing the regularity of $\mu \mapsto P_t^*\mu$ is the following Bismut type formula for the intrinsic derivative D^I in $\mu \in \mathscr{P}_k(\mathbb{R}^d)$ (see Definition 2.1 below):

$$\text{(1.4)} \qquad \begin{aligned} D_\phi^I P_t f(\mu) = \mathbb{E}\bigg[f(X_t^\mu)\int_0^t \langle M_s^{\mu,\phi}, \mathrm{d}B_s\rangle\bigg],& \\ t \in (0,T],\ f \in \mathscr{B}_b(\mathbb{R}^d), \phi \in L^k(\mathbb{R}^d \to \mathbb{R}^d;\mu),& \end{aligned}$$

where $\int_0^t \langle M_s^{\mu,\phi}, \mathrm{d}B_s\rangle$ is a martingale depending on μ and ϕ. Again, when $\sigma_t(x,\mu) = \sigma_t(x)$ is distribution free, this type formula has been established in [3, 8, 13, 22] under different conditions, but it is open for distribution dependent noise.

However, arguments used in the above mentioned references do not apply to distribution dependent noise. The only known log-Harnack inequality for distribution dependent noise is established in [2] for Ornstein-Ulenbeck type SDEs whose solutions are Gaussian processes and thus easy to manage. This again does not apply to more general case. On the other hand, intrinsic derivative estimates have been presented for a class of SDEs with distribution dependent noise, see [11] and references. This convinces us of establishing the log-Harnack inequality and Bimsut formula for SDEs with distribution dependent noise.

In this paper, we propose a noise decomposition argument which reduces the study of distribution dependent noise to distribution free noise. For simplicity, we only explain here the idea on establishing the log-Harnack inequality for the following distribution dependent SDE:

$$\text{(1.5)} \qquad \mathrm{d}X_t = b_t(X_t, \mathscr{L}_{X_t})\mathrm{d}t + \sigma_t(\mathscr{L}_{X_t})\mathrm{d}B_t, \quad t \in [0,T].$$

When the equation is well-posed for distributions on $\mathscr{P}_2(\mathbb{R}^d)$, let $P_t^*\mu = \mathscr{L}_{X_t}$ for the solution with $\mathscr{L}_{X_0} = \mu \in \mathscr{P}_2(\mathbb{R}^d)$. Assume that σ_t is bounded and Lipschitz continuous on $\mathscr{P}_2(\mathbb{R}^d)$, such that

$$(\sigma_t\sigma_t^*)(\gamma) \ge 2\lambda^2 I_d, \quad \gamma \in \mathscr{P}_2(\mathbb{R}^d)$$

holds for some constant $\lambda > 0$, where I_d is the $d \times d$ identity matrix. We take

$$\tilde\sigma_t(\gamma) := \sqrt{(\sigma_t\sigma_t^*)(\gamma) - \lambda^2 I_d}.$$

Then $\tilde{\sigma}_t(\gamma) \geq \lambda I_d$, and [12, Lemma 3.3] implies that $\tilde{\sigma}_t(\gamma)$ is Lipschitz continuous in $\gamma \in \mathscr{P}_2(\mathbb{R}^d)$ as well. Moreover, for two independent d-dimensional Brownian motions W_t and $\tilde{W}_t$,

$$\mathrm{d}B_t := \sigma_t(\mathscr{L}_{X_t})^{-1}\big\{\lambda \mathrm{d}W_t + \tilde{\sigma}_t(\mathscr{L}_{X_t})\mathrm{d}\tilde{W}_t\big\}$$

is a d-dimensional Brownian motion, so that (1.5) is reduced to

$$\mathrm{d}X_t = b_t(X_t, \mathscr{L}_{X_t})\mathrm{d}t + \lambda \mathrm{d}W_t + \tilde{\sigma}_t(\mathscr{L}_{X_t})\mathrm{d}\tilde{W}_t, \quad t \in [0,T]. \tag{1.6}$$

Thus, by the well-posedness, (1.5) and (1.6) provide the same operator P_t. Now, consider the conditional probability $\mathbb{P}^{\tilde{W}}$ given $\tilde{W}$, under which $\int_0^t \tilde{\sigma}_s(\mathscr{L}_{X_s})\mathrm{d}\tilde{W}_s$ is deterministic so that (1.6) becomes an SDE with constant noise $\lambda \mathrm{d}W_t$, and hence its log-Harnack inequality follows from exiting arguments developed for distribution free noise.

However, this noise decomposition argument is hard to extend to spatial-distribution dependent noise. So, in the following we only consider (1.5) or (1.6), rather than (1.1).

Closely related to the log-Harnack inequality, a very nice entropy estimate has been derived in [4] for two SDEs with different noise coefficients. Consider, for instance, the following SDEs on $\mathbb{R}^d$ for $i = 1, 2$:

$$\mathrm{d}X_t^i = b_i(t, X_t^i)\mathrm{d}t + \sqrt{a_i(t)}\mathrm{d}B_t, \quad X_0^i = x \in \mathbb{R}^d, t \geq 0,$$

where $a_i(t)$ is positive definite, and for some constant $K > 1$,

$$|b_i(t,x) - b_i(t,y)| \leq K|x-y|, \quad K^{-1}I_d \leq a_i(t) \leq KI_d, \quad x, y \in \mathbb{R}^d, t \geq 0.$$

Then [4, Theorem 1.1] gives the entropy estimate

$$\begin{aligned}&\mathrm{Ent}(\mathscr{L}_{X_t^2}|\mathscr{L}_{X_t^1}) \leq \frac{1}{2}\int_0^t \mathrm{d}s \int_{\mathbb{R}^d} \big|a_1(s)^{-\frac{1}{2}}\Phi(s,y)\big|^2 \rho_2(s,y)\mathrm{d}y,\\ &\Phi(s,y) := (a_1(s) - a_2(s))\nabla \log \rho_2(s,y) + b_2(s,y) - b_1(s,y), \quad s > 0, y \in \mathbb{R}^d,\end{aligned}$$

where $\rho_2(s,y) := \frac{\mathscr{L}_{X_s^2}(\mathrm{d}y)}{\mathrm{d}y}$ is the distribution density function of X_s^2. Since for elliptic diffusion processes

$$\int_{\mathbb{R}^d} \big|\nabla \log \rho_2(s,y)\big|^2 \rho_2(s,y)\mathrm{d}y$$

behaves like $\frac{c}{s}$ for some constant $c > 0$ and small $s > 0$, to derive finite entropy upper bound from this estimate one my assume

$$\int_0^1 \frac{\|a_1(s) - a_2(s)\|^2}{s}\mathrm{d}s < \infty, \tag{1.7}$$

where $\|\cdot\|$ is the operator norm of matrices. To bound $\mathrm{Ent}(P_t^*\nu|P_t^*\mu)$ for (1.5), we take

$$a_1(s) := (\sigma_s\sigma_s^*)(P_s^*\mu), \quad a_2(s) := (\sigma_s\sigma_s^*)(P_s^*\nu).$$

But (1.7) fails when $\|(\sigma_s\sigma_s^*)(P_s^*\mu) - (\sigma_s\sigma_s^*)(P_s^*\nu)\|$ is uniformly positive for small s.

The remainder of the paper is organized as follows. In Section 2 and Section 3, we establish the log-Harnack inequality and Bismut formula for the non-degenerate case and degenerate cases respectively. In Section 4 we apply the log-Harnack inequality to study the exponential ergodicity in entropy.

2 Non-degenerate case

In this part, we establish the log-Harnack inequality and Bismut formula for $P_t f$ defined in (1.2), where $P_t^*\mu := \mathscr{L}_{X_t^\mu}$ for X_t^μ solving (1.6) with initial distribution μ.

2.1 Log-Harnack inequality

To establish the log-Harnack inequality, we make the following assumption.

(A) $\lambda > 0$ is a constant, and there exists $0 \le K \in L^1([0,T])$ such that

$$|b_t(x,\mu) - b_t(y,\nu)|^2 + \|\tilde\sigma_t(\mu) - \tilde\sigma_t(\nu)\|^2 \le K_t(|x-y|^2 + \mathbb{W}_2(\mu,\nu)^2),$$
$$|b_t(0,\delta_0)| + \|\tilde\sigma_t(\delta_0)\|^2 \le K_t, \quad t \in [0,T],\ x,y \in \mathbb{R}^d,\ \mu,\nu \in \mathscr{P}_2(\mathbb{R}^d).$$

By [5, Theorems 2.1 and 3.3] or [21, Theorem 2.1], assumption **(A)** implies that the SDE (1.6) is well-posed for distributions in $\mathscr{P}_2(\mathbb{R}^d)$, and there exists a constant $c > 0$ such that

$$\mathbb{W}_2(P_t^*\nu, P_t^*\mu) \le c\mathbb{W}_2(\nu,\mu), \quad \mu,\nu \in \mathscr{P}_2(\mathbb{R}^d), t \in [0,T]. \tag{2.1}$$

Theorem 2.1. *Assume* **(A)** *and let* P_t *be defined in* (1.2) *for the SDE* (1.6)*. Then there exists a constant* $c > 0$ *such that*

$$P_t \log f(\nu) \le \log P_t f(\mu) + \frac{c}{t}\mathbb{W}_2(\mu,\nu)^2, \quad 0 < f \in \mathscr{B}_b(\mathbb{R}^d), \mu,\nu \in \mathscr{P}_2(\mathbb{R}^d), t \in (0,T].$$

Equivalently,

$$\mathrm{Ent}(P_t^*\nu | P_t^*\mu) \le \frac{c}{t}\mathbb{W}_2(\mu,\nu)^2, \quad \mu,\nu \in \mathscr{P}_2(\mathbb{R}^d), t \in (0,T].$$

Proof. As explained in Introduction, we will use coupling by change of measure under the conditional expectation given $\tilde W$, which will be enough for the proof of the log-Harnack inequality. But for the study of Bismut formula later on, we will use the conditional probability and the conditional expectation given both $\tilde W$ and $\mathscr{F}_0$:

$$\mathbb{P}^{\tilde W,0} := \mathbb{P}(\,\cdot\,|\tilde W, \mathscr{F}_0), \quad \mathbb{E}^{\tilde W,0} := \mathbb{E}(\,\cdot\,|\tilde W, \mathscr{F}_0).$$

(a) For any $t \in [0,T], \mu \in \mathscr{P}_2(\mathbb{R}^d)$ and $f \in \mathscr{B}_b(\mathbb{R}^d)$, let

$$P_t^{\tilde W,0} f(X_0^\mu) := \mathbb{E}^{\tilde W,0}[f(X_t^\mu)] = \mathbb{E}\big[f(X_t^\mu)\big|\tilde W, \mathscr{F}_0\big],$$

where X_t^μ solves (1.6) with $\mathscr{L}_{X_0^\mu} = \mu$. By (1.2),

$$P_t f(\mu) = \mathbb{E}\big[P_t^{\tilde W,0} f(X_0^\mu)\big], \quad t \in [0,T], \mu \in \mathscr{P}_2(\mathbb{R}^d), f \in \mathscr{B}_b(\mathbb{R}^d). \tag{2.2}$$

Next, let

$$\xi_t^\mu := \int_0^t \tilde\sigma_s(P_s^*\mu)\mathrm{d}\tilde W_s, \quad t \in [0,T], \mu \in \mathscr{P}_2(\mathbb{R}^d). \tag{2.3}$$

By **(A)**, BDG's inequality and (2.1), we find constants $C_1, C_2 > 0$ such that

$$\mathbb{E}\Big[\sup_{t\in[0,T]} |\xi_t^\mu - \xi_t^\nu|^2\Big] \le C_1 \mathbb{W}_2(\mu,\nu)^2 \int_0^T K_s \mathrm{d}s \le C_2 \mathbb{W}_2(\mu,\nu)^2, \quad \mu,\nu \in \mathscr{P}_2(\mathbb{R}^d). \tag{2.4}$$

(b) For fixed $\mu,\nu \in \mathscr{P}_2(\mathbb{R}^d)$, we take $\mathscr{F}_0$-measurable X_0^μ and X_0^ν such that

$$\mathscr{L}_{X_0^\mu} = \mu, \quad \mathscr{L}_{X_0^\nu} = \nu, \quad \mathbb{E}[|X_0^\mu - X_0^\nu|^2] = \mathbb{W}_2(\mu,\nu)^2. \tag{2.5}$$

Since X_t^μ solves (1.6) with $\mathscr{L}_{X_0^\mu} = \mu$, we have $\mathscr{L}_{X_t^\mu} = P_t^*\mu$ and the SDE becomes

$$\mathrm{d}X_t^\mu = b_t(X_t^\mu, P_t^*\mu)\mathrm{d}t + \lambda \mathrm{d}W_t + \tilde\sigma_t(P_t^*\mu)\mathrm{d}\tilde W_t, \quad t \in [0,T]. \tag{2.6}$$

For fixed $t_0 \in (0,T]$, consider the following SDE:

$$\begin{aligned} &\mathrm{d}Y_t = \Big\{b_t(X_t^\mu, P_t^*\mu) + \frac{1}{t_0}\big[\xi_{t_0}^\mu - \xi_{t_0}^\nu + X_0^\mu - X_0^\nu\big]\Big\}\mathrm{d}t + \lambda \mathrm{d}W_t + \tilde\sigma_t(P_t^*\nu)\mathrm{d}\tilde W_t,\\ &\quad t \in [0,t_0],\ Y_0 = X_0^\nu. \end{aligned} \tag{2.7}$$

By (2.3), (2.6) and (2.7), we obtain

$$Y_t - X_t^\mu = \frac{t_0 - t}{t_0}(X_0^\nu - X_0^\mu) + \frac{t}{t_0}\big(\xi_{t_0}^\mu - \xi_{t_0}^\nu\big) + \xi_t^\nu - \xi_t^\mu, \quad t \in [0,t_0]. \tag{2.8}$$

To formulate $P_{t_0} f(\nu)$ using Y_{t_0}, we make Girsanov's transform as follows. Let

$$\eta_t := b_t(Y_t, P_t^*\nu) - b_t(X_t^\mu, P_t^*\mu) + \frac{1}{t_0}\big[\xi_{t_0}^\nu - \xi_{t_0}^\mu + X_0^\nu - X_0^\mu\big], \quad t \in [0,t_0]. \tag{2.9}$$

By **(A)** and (2.1), we find a constant $c_1 > 0$ such that

$$\begin{aligned} |\eta_t|^2 \le\, & c_1 K_t\big(\mathbb{W}_2(\mu,\nu)^2 + |\xi_t^\nu - \xi_t^\mu|^2\big)\\ & + c_1\Big(\frac{t^2 K_t + 1}{t_0^2}|\xi_{t_0}^\mu - \xi_{t_0}^\nu|^2 + \frac{1}{t_0^2}|X_0^\mu - X_0^\nu|^2\Big), \quad t \in [0,t_0]. \end{aligned}$$

Since $\int_0^T K_t \mathrm{d}t < \infty$, we find a constant $c_2 > 0$ uniform in $t_0 \in (0, T]$, such that

$$\frac{1}{2\lambda^2} \int_0^{t_0} |\eta_t|^2 \mathrm{d}t \le c_2 \mathbb{W}_2(\mu, \nu)^2 + \frac{c_2}{t_0} \Big(|X_0^\mu - X_0^\nu|^2 + \sup_{t \in [0, t_0]} |\xi_t^\mu - \xi_t^\nu|^2 \Big). \tag{2.10}$$

Let $\mathrm{d}\mathbb{Q}^{\tilde{W},0} := R^{\tilde{W},0} \mathrm{d}\mathbb{P}^{\tilde{W},0}$, where

$$R^{\tilde{W},0} := \mathrm{e}^{\int_0^{t_0} \langle \frac{1}{\lambda} \eta_s, \mathrm{d}W_s \rangle - \frac{1}{2} \int_0^{t_0} |\frac{1}{\lambda} \eta_s|^2 \mathrm{d}s}. \tag{2.11}$$

By Girsanov's theorem, under the weighted conditional probability $\mathbb{Q}^{\tilde{W},0}$,

$$\hat{W}_t := W_t - \int_0^t \frac{1}{\lambda} \eta_s \mathrm{d}s, \quad t \in [0, t_0]$$

is a d-dimensional Brownian motion. By (2.7), $\hat{Y}_t := Y_t - \xi_t^\nu$ solves the SDE

$$\mathrm{d}\hat{Y}_t = b_t(\hat{Y}_t + \xi_t^\nu, P_t^* \nu) \mathrm{d}t + \lambda \mathrm{d}\hat{W}_t, \quad t \in [0, t_0], \hat{Y}_0 = X_0^\nu.$$

On the other hand, let X_t^ν solve (1.6) with initial value X_0^ν. Then

$$\hat{X}_t^\nu := X_t^\nu - \xi_t^\nu, \quad t \in [0, t_0]$$

solves the same SDE as $\hat{Y}_t$ for W replacing $\hat{W}$. Then the weak uniqueness of this equation ensured by **(A)** implies

$$\mathscr{L}_{\hat{Y}_{t_0} | \mathbb{Q}^{\tilde{W},0}} = \mathscr{L}_{\hat{X}_{t_0}^\nu | \mathbb{P}^{\tilde{W},0}},$$

where $\mathscr{L}_{\hat{Y}_{t_0} | \mathbb{Q}^{\tilde{W},0}}$ is the law of $\hat{Y}_{t_0}$ under $\mathbb{Q}^{\tilde{W},0}$, while $\mathscr{L}_{\hat{X}_{t_0}^\nu | \mathbb{P}^{\tilde{W},0}}$ is the law of $\hat{X}_{t_0}^\nu$ under $\mathbb{P}^{\tilde{W},0}$. Since $\xi_{t_0}^\nu$ is deterministic given $\tilde{W}$, it follows that

$$\mathscr{L}_{Y_{t_0} | \mathbb{Q}^{\tilde{W},0}} = \mathscr{L}_{\hat{Y}_{t_0} + \xi_{t_0}^\nu | \mathbb{Q}^{\tilde{W},0}} = \mathscr{L}_{\hat{X}_{t_0}^\nu + \xi_{t_0}^\nu | \mathbb{P}^{\tilde{W},0}} = \mathscr{L}_{X_{t_0}^\nu | \mathbb{P}^{\tilde{W},0}}.$$

Combining this with $X_{t_0}^\mu = Y_{t_0}$ due to (2.8), we obtain

$$P_{t_0}^{\tilde{W},0} f(X_0^\nu) := \mathbb{E}^{\tilde{W},0}[f(X_{t_0}^\nu)] = \mathbb{E}_{\mathbb{Q}^{\tilde{W},0}}[f(Y_{t_0})] = \mathbb{E}^{\tilde{W},0}[R^{\tilde{W},0} f(X_{t_0}^\mu)], \quad f \in \mathscr{B}_b(\mathbb{R}^d). \tag{2.12}$$

By Young's inequality [1, Lemma 2.4], we derive

$$\begin{aligned} &P_{t_0}^{\tilde{W},0} \log f(X_0^\nu) := \mathbb{E}^{\tilde{W},0}[\log f(X_{t_0}^\nu)] = \mathbb{E}_{\mathbb{Q}^{\tilde{W},0}}[\log f(Y_{t_0})] \\ &= \mathbb{E}^{\tilde{W},0}[R^{\tilde{W},0} \log f(X_{t_0}^\mu)] \le \log \mathbb{E}^{\tilde{W},0}[f(X_{t_0}^\mu)] + \mathbb{E}^{\tilde{W},0}[R^{\tilde{W},0} \log R^{\tilde{W},0}] \\ &= \log P_{t_0}^{\tilde{W},0} f(X_0^\mu) + \frac{1}{2} \int_0^{t_0} \frac{1}{\lambda^2} \mathbb{E}_{\mathbb{Q}^{\tilde{W},0}}[|\eta_t|^2] \mathrm{d}t, \quad 0 < f \in \mathscr{B}_b(\mathbb{R}^d). \end{aligned}$$

This together with (2.10) gives

$$P_{t_0}^{\tilde W,0}\log f(X_0^\nu)\le \log P_{t_0}^{\tilde W,0} f(X_0^\mu)+c_2\mathbb{W}_2(\mu,\nu)^2+\frac{c_2}{t_0}\Big(|X_0^\mu-X_0^\nu|^2+\sup_{t\in[0,t_0]}|\xi_t^\mu-\xi_t^\nu|^2\Big). \tag{2.13}$$

Taking expectation for both sides, by (2.2), (2.4), (2.5) and Jensen's inequality, we find a constant $c>0$ such that

$$\begin{aligned}P_{t_0}\log f(\nu)&=\mathbb{E}\big[P_{t_0}^{\tilde W,0}\log f(X_0^\nu)\big]\le \mathbb{E}\big[\log P_{t_0}^{\tilde W,0} f(X_0^\mu)\big]+\frac{c}{t_0}\mathbb{W}_2(\mu,\nu)^2\\&\le \log P_{t_0}f(\mu)+\frac{c}{t_0}\mathbb{W}_2(\mu,\nu)^2,\quad t_0\in(0,T],\mu,\nu\in\mathscr{P}_2(\mathbb{R}^d).\end{aligned}$$

□

2.2 Bismut formula

We aim to establish the Bismut type formula (1.4) for the intrinsic derivative of P_tf. To this end, we first recall the definition of intrinsic derivative, see [15] for historical remarks on this derivative and links to other derivatives for functions of measures.

Definition 2.1. Let $k\in(1,\infty)$.

(1) A continuous function f on $\mathscr{P}_k(\mathbb{R}^d)$ is called intrinsically differentiable, if for any $\mu\in\mathscr{P}_k(\mathbb{R}^d)$,

$$T_{\mu,k}(\mathbb{R}^d):=L^k(\mathbb{R}^d\to\mathbb{R}^d;\mu)\ni\phi\mapsto D_\phi^If(\mu):=\lim_{\varepsilon\downarrow 0}\frac{f(\mu\circ(id+\varepsilon\phi)^{-1})-f(\mu)}{\varepsilon}\in\mathbb{R}$$

is a well defined bounded linear operator. In this case, the norm of the intrinsic derivative $D^If(\mu)$ is given by

$$\|D^If(\mu)\|_{L^{k^*}(\mu)}:=\sup_{\|\phi\|_{L^k(\mu)}\le 1}|D_\phi^If(\mu)|.$$

(2) f is called L-differentiable on $\mathscr{P}_k(\mathbb{R}^d)$, if it is intrinsically differentiable and

$$\lim_{\|\phi\|_{T_{\mu,k}(\mathbb{R}^d)}\downarrow 0}\frac{|f(\mu\circ(id+\phi)^{-1})-f(\mu)-D_\phi^If(\mu)|}{\|\phi\|_{T_{\mu,k}(\mathbb{R}^d)}}=0,\quad \mu\in\mathscr{P}_k(\mathbb{R}^d).$$

We denote $f\in C^1(\mathscr{P}_k(\mathbb{R}^d))$, if it is L-differentiable such that $D^If(\mu)(x)$ has a jointly continuous version in $(x,\mu)\in\mathbb{R}^d\times\mathscr{P}_k(\mathbb{R}^d)$.

(3) We denote $g\in C^{1,1}(\mathbb{R}^d\times\mathscr{P}_k(\mathbb{R}^d))$, if $g:\mathbb{R}^d\times\mathscr{P}_k(\mathbb{R}^d)\to\mathbb{R}$ is C^1 in $x\in\mathbb{R}^d$ and $\mu\in\mathscr{P}_k(\mathbb{R}^d)$ respectively, such that

$$\nabla g(x,\mu):=\nabla\{g(\cdot,\mu)\}(x),\quad D^Ig(x,\mu)(y):=D^I\{g(x,\cdot)\}(\mu)(y)$$

are jointly continuous in $(x,y,\mu)\in\mathbb{R}^d\times\mathbb{R}^d\times\mathscr{P}_k(\mathbb{R}^d)$.

In this part, we consider (1.6) with coefficients

$$\tilde\sigma : [0,T]\times \mathscr{P}_k(\mathbb{R}^d)\to \mathbb{R}^d\otimes\mathbb{R}^d,\quad b:[0,T]\times\mathbb{R}^d\times\mathscr{P}_k(\mathbb{R}^d)\to\mathbb{R}^d$$

satisfying the following assumption.

(B) $\lambda>0$ and $k\in(1,\infty)$ are constants, denote $k^*:=\frac{k}{k-1}$. For any $t\in[0,T]$, $b_t\in C^{1,1}(\mathbb{R}^d\times\mathscr{P}_k(\mathbb{R}^d))$, $\tilde\sigma_t\in C^1(\mathscr{P}_k(\mathbb{R}^d))$, and there exists $0\le K\in L^1([0,T])$ such that

$$\begin{aligned}&|D^I b_t(x,\cdot)(\mu)(y)|+\|D^I\tilde\sigma_t(\mu)(y)\|\le\sqrt{K_t}(1+|y|^{k-1}),\\ &|b_t(0,\delta_0)|+|\nabla b_t(\cdot,\mu)(x)|\le\sqrt{K_t},\quad (t,x,\mu)\in[0,T]\times\mathbb{R}^d\times\mathscr{P}_k(\mathbb{R}^d),\ y\in\mathbb{R}^d.\end{aligned}$$

By [22, Lemma 3.1], **(B)** implies **(A)** for $(\mathscr{P}_k(\mathbb{R}^d),\mathbb{W}_k)$ replacing $(\mathscr{P}_2(\mathbb{R}^d),\mathbb{W}_2)$. So, according to [5, Theorem 3.3], the SDE (1.6) is well-posed for distributions in $\mathscr{P}_k(\mathbb{R}^d)$, and there exists a constant $c>0$ such that

$$\mathbb{W}_k(P_t^*\mu,P_t^*\nu)\le c\mathbb{W}_k(\mu,\nu),\quad \mu,\nu\in\mathscr{P}_k(\mathbb{R}^d),t\in[0,T]. \tag{2.14}$$

By this estimate and **(A)** for $(\mathscr{P}_k(\mathbb{R}^d),\mathbb{W}_k)$ replacing $(\mathscr{P}_2(\mathbb{R}^d),\mathbb{W}_2)$, the argument leading to (2.13) yields that there exists a constant $c>0$ such that for any $t\in(0,T],0<f\in\mathscr{B}_b(\mathbb{R}^d)$,

$$P_t^{\tilde W,0}\log f(X_0^\nu)\le\log P_t^{\tilde W,0}f(X_0^\mu)+c\mathbb{W}_k(\mu,\nu)^2+\frac{c}{t}\Big(|X_0^\mu-X_0^\nu|^2+\sup_{s\in[0,t]}|\xi_s^\mu-\xi_s^\nu|^2\Big). \tag{2.15}$$

To calculate $D_\phi^I P_t f(\mu)$ for $\mu\in\mathscr{P}_k(\mathbb{R}^d)$ and $\phi\in T_{\mu,k}(\mathbb{R}^d)$, let X_0^μ be $\mathscr{F}_0$-measurable such that $\mathscr{L}_{X_0^\mu}=\mu$. Then

$$\mathscr{L}_{X_0^\mu+\varepsilon\phi(X_0^\mu)}=\mu^\varepsilon:=\mu\circ(id+\varepsilon\phi)^{-1},\quad \varepsilon\in[0,1].$$

For any $\varepsilon\in[0,1]$, let $X_t^{\mu^\varepsilon}$ solve (1.6) with $X_0^{\mu^\varepsilon}=X_0^\mu+\varepsilon\phi(X_0^\mu)$, i.e.

$$\begin{aligned}\mathrm{d}X_t^{\mu^\varepsilon}&=b_t(X_t^{\mu^\varepsilon},P_t^*\mu^\varepsilon)\mathrm{d}t+\lambda\mathrm{d}W_t+\tilde\sigma_t(P_t^*\mu^\varepsilon)\mathrm{d}\tilde W_t,\\ X_0^{\mu^\varepsilon}&=X_0^\mu+\varepsilon\phi(X_0^\mu),t\in[0,T],\varepsilon\in[0,1].\end{aligned}$$

Consider the spatial derivative of X_t^μ along ϕ:

$$\nabla_\phi X_t^\mu:=\lim_{\varepsilon\downarrow0}\frac{X_t^{\mu^\varepsilon}-X_t^\mu}{\varepsilon},\quad t\in[0,T],\phi\in T_{\mu,k}(\mathbb{R}^d).$$

For any $0\le s<t\le T$, define

$$N_{s,t}^{\mu,\phi}:=\frac{t-s}{t}\phi(X_0^\mu)+\int_0^s\Big\langle\mathbb{E}\big[\langle D^I\tilde\sigma_r(P_r^*\mu)(X_r^\mu),\nabla_\phi X_r^\mu\rangle\big],\ \mathrm{d}\tilde W_r\Big\rangle$$

$$
\begin{aligned}
&\quad - \frac{s}{t}\int_0^t \Big\langle \mathbb{E}\big[\langle D^I\tilde\sigma_r(P_r^*\mu)(X_r^\mu), \nabla_\phi X_r^\mu\rangle\big],\ \mathrm{d}\tilde W_r\Big\rangle,\\
M_{s,t}^{\mu,\phi} &:= \mathbb{E}\big[\big\langle \{D^I b_s(y,\cdot)\}(P_s^*\mu)(X_s^\mu), \nabla_\phi X_s^\mu\big\rangle\big]_{y=X_s^\mu} + \frac{1}{t}\phi(X_0^\mu)\\
&\quad + \frac{1}{t}\int_0^t \Big\langle \mathbb{E}\big[\langle D^I\tilde\sigma_r(P_r^*\mu)(X_r^\mu), \nabla_\phi X_r^\mu\rangle\big],\ \mathrm{d}\tilde W_r\Big\rangle.
\end{aligned}
$$

The main result in this part is the following.

Theorem 2.2. *Assume* **(B)**.

(1) *For any $\mu\in\mathscr{P}_k(\mathbb{R}^d)$ and $\phi\in T_{\mu,k}(\mathbb{R}^d)$, $(\nabla_\phi X_\cdot^\mu)$ exists in $L^k(\Omega\to C([0,T],\mathbb{R}^d),\mathbb{P})$ such that for some constant $c>0$,*

$$
\mathbb{E}\Big[\sup_{t\in[0,T]}|\nabla_\phi X_t^\mu|^k\Big] \le c\|\phi\|_{L^k(\mu)}^k,\quad \mu\in\mathscr{P}_k(\mathbb{R}^d), \phi\in T_{\mu,k}(\mathbb{R}^d).
$$

(2) *For any $f\in\mathscr{B}_b(\mathbb{R}^d), t\in(0,T]$, $\mu\in\mathscr{P}_k(\mathbb{R}^d)$ and $\phi\in T_{\mu,k}(\mathbb{R}^d)$, $D_\phi^I P_t f(\mu)$ exists and satisfies*

$$
D_\phi^I P_t f(\mu) = \frac{1}{\lambda}\mathbb{E}\Big[f(X_t^\mu)\int_0^t \Big\langle \nabla_{N_{s,t}^{\mu,\phi}} b_s(\cdot, P_s^*\mu)(X_s^\mu) + M_{s,t}^{\mu,\phi},\ \mathrm{d}W_s\Big\rangle\Big]. \tag{2.16}
$$

Consequently, $P_t f$ is intrinsically differentiable and for some constant $c>0$,

$$
\begin{aligned}
&\|D^I P_t f(\mu)\|_{L^{k^*}(\mu)} \le \frac{c}{\sqrt{t}}\big(P_t|f|^{k^*}(\mu)\big)^{\frac{1}{k^*}},\\
&\qquad f\in\mathscr{B}_b(\mathbb{R}^d), \mu\in\mathscr{P}_k(\mathbb{R}^d), t\in(0,T].
\end{aligned} \tag{2.17}
$$

Proof. The first assertion follows from [3, Lemma 5.2]. By the first assertion, **(B)** and the definition of $(N_{s,t}^{\mu,\phi}, M_{s,t}^{\mu,\phi})$, we deduce (2.17) from (2.16). So, it remains to prove (2.16).

(a) Since **(B)** implies **(A)** for $\mathscr{P}_k(\mathbb{R}^d)$ replacing $\mathscr{P}_2(\mathbb{R}^d)$, the argument in the proof of Theorem 2.1 up to (2.12) still applies. For fixed $t_0\in(0,T], \mu\in\mathscr{P}_k(\mathbb{R}^d)$ and $\phi\in T_{\mu,k}(\mathbb{R}^d)$, let X_t^μ solve (2.6). Next, for any $\varepsilon\in(0,1]$, let Y_t^ε solve (2.7) for

$$
\nu=\mu^\varepsilon,\quad Y_0=Y_0^\varepsilon := X_0^\mu+\varepsilon\phi(X_0^\mu).
$$

Then (2.8) with $(Y_t,\nu)=(Y_t^\varepsilon,\mu^\varepsilon)$ becomes

$$
Y_t^\varepsilon - X_t^\mu = \frac{t_0-t}{t_0}\varepsilon\phi(X_0^\mu) + \frac{t}{t_0}(\xi_{t_0}^\mu - \xi_{t_0}^{\mu^\varepsilon}) + \xi_t^{\mu^\varepsilon} - \xi_t^\mu,\quad t\in[0,t_0]. \tag{2.18}
$$

Let

$$
H_t := \int_0^t \Big\langle \mathbb{E}\big[\big\langle D^I\tilde\sigma_s(P_s^*\mu)(X_s^\mu), \nabla_\phi X_s^\mu\big\rangle\big],\ \mathrm{d}\tilde W_s\Big\rangle,\quad t\in[0,T].
$$

By **(B)** and (2.14), we obtain

$$\big\|\tilde{\sigma}_s(P_s^*\mu^\varepsilon) - \tilde{\sigma}_s(P_s^*\mu)\big\|^2 \le \varepsilon^2 c^2 K_s \|\phi\|_{L^k(\mu)}^2, \quad \varepsilon \in [0,1], s \in [0,T]. \tag{2.19}$$

So, by **(B)**, the chain rule in [3, Theorem 2.1(1)], (2.3), BDG's inequality and the dominated convergence theorem, we obtain

$$\lim_{\varepsilon \downarrow 0} \mathbb{E}\bigg[\sup_{t\in[0,T]} \Big|\frac{\xi_t^{\mu^\varepsilon} - \xi_t^\mu}{\varepsilon} - H_t\Big|^2\bigg] = 0. \tag{2.20}$$

Let $(\eta_t^\varepsilon, R^\varepsilon) = (\eta_t, R^{\tilde{W},0})$ be defined in (2.9) and (2.11) for $(Y_t, \nu) = (Y_t^\varepsilon, \mu^\varepsilon)$. By **(B)** and (2.18), we find a constant $\kappa > 0$ such that

$$\begin{aligned}
&\frac{|\eta_s^\varepsilon|^2}{\varepsilon^2} \le \kappa K_s \bigg(\|\phi\|_{L^k(\mu)}^2 + |\phi(X_0^\mu)|^2 + \sup_{t\in[0,t_0]} \frac{|\xi_t^{\mu^\varepsilon} - \xi_t^\mu|^2}{\varepsilon^2}\bigg) =: \Lambda_s,\\
&\lim_{\varepsilon\downarrow 0} \frac{\eta_s^\varepsilon}{\varepsilon} = \nabla_{N_{s,t_0}^{\mu,\phi}} b_s(\cdot, P_s^*\mu)(X_s^\mu) + M_{s,t_0}^{\mu,\phi}, \quad s \in [0,t_0].
\end{aligned}$$

Since Λ_s is deterministic given $\tilde{W}$ and $\mathscr{F}_0$, this together with (2.12) and the dominated convergence theorem yields

$$\begin{aligned}
&\lim_{\varepsilon\downarrow 0} \frac{P_{t_0}^{\tilde{W},0} f(X_0^{\mu^\varepsilon}) - P_{t_0}^{\tilde{W},0} f(X_0^\mu)}{\varepsilon} = \lim_{\varepsilon\downarrow 0} \mathbb{E}^{\tilde{W},0}\Big[f(X_{t_0}^\mu)\frac{R^\varepsilon - 1}{\varepsilon}\Big]\\
&= \frac{1}{\lambda}\mathbb{E}^{\tilde{W},0}\bigg[f(X_{t_0}^\mu)\int_0^{t_0} \Big\langle \nabla_{N_{s,t_0}^{\mu,\phi}} b_s(\cdot, P_s^*\mu)(X_s^\mu) + M_{s,t_0}^{\mu,\phi},\ \mathrm{d}W_s\Big\rangle\bigg].
\end{aligned} \tag{2.21}$$

(b) Let $\mathscr{L}_{\xi|\mathbb{P}^{\tilde{W},0}}$ be the conditional distribution of a random variable ξ under $\mathbb{P}^{\tilde{W},0}$. By Pinsker's inequality and (2.15), we have

$$\begin{aligned}
&\sup_{\|f\|_\infty \le 1} \big|P_{t_0}^{\tilde{W},0} f(X_0^{\mu^\varepsilon}) - P_{t_0}^{\tilde{W},0} f(X_0^\mu)\big|^2 \le 2\,\mathrm{Ent}\big(\mathscr{L}_{X_{t_0}^{\mu^\varepsilon}|\mathbb{P}^{\tilde{W},0}}\big|\mathscr{L}_{X_{t_0}^\mu|\mathbb{P}^{\tilde{W},0}}\big)\\
&\le c\mathbb{W}_k(\mu^\varepsilon, \mu)^2 + \frac{c}{t_0}\Big(\varepsilon^2|\phi(X_0^\mu)|^2 + \sup_{t\in[0,t_0]} |\xi_t^\mu - \xi_t^{\mu^\varepsilon}|^2\Big).
\end{aligned}$$

This together with $\mathbb{W}_k(\mu^\varepsilon, \mu) \le \varepsilon\|\phi\|_{L^k(\mu)}$ implies that for some constant $c(t_0) > 0$,

$$\begin{aligned}
&\frac{|P_{t_0}^{\tilde{W},0} f(X_0^{\mu^\varepsilon}) - P_{t_0}^{\tilde{W},0} f(X_0^\mu)|}{\varepsilon}\\
&\le \|f\|_\infty c(t_0)\Big(\|\phi\|_{L^k(\mu)} + |\phi(X_0^\mu)| + \sup_{t\in[0,t_0]} \frac{|\xi_t^{\mu^\varepsilon} - \xi_t^\mu|}{\varepsilon}\Big), \quad \varepsilon \in (0,1].
\end{aligned}$$

Combining this with (2.4) and (2.19), we may apply the dominated convergence theorem to (2.21) to derive

$$D^I_\phi P_{t_0} f(\mu) := \lim_{\varepsilon\downarrow 0} \mathbb{E}\bigg[\frac{P^{\tilde W,0}_{t_0} f(X_0^{\mu^\varepsilon}) - P^{\tilde W,0}_{t_0} f(X_0^\mu)}{\varepsilon}\bigg] = \mathbb{E}\bigg[\lim_{\varepsilon\downarrow 0} \frac{P^{\tilde W,0}_{t_0} f(X_0^{\mu^\varepsilon}) - P^{\tilde W,0}_{t_0} f(X_0^\mu)}{\varepsilon}\bigg]$$
$$= \frac{1}{\lambda}\mathbb{E}\bigg[f(X^\mu_{t_0}) \int_0^{t_0} \Big\langle \nabla_{N^{\mu,\phi}_{s,t_0}} b_s(\cdot, P_s^*\mu)(X_s^\mu) + M^{\mu,\phi}_{s,t_0},\ \mathrm{d}W_s\Big\rangle\bigg].$$

□

3 Degenerate case

Consider the following distribution dependent stochastic Hamiltonian system for $X_t = (X_t^{(1)}, X_t^{(2)}) \in \mathbb{R}^{m+d}$:

$$\begin{cases} \mathrm{d}X_t^{(1)} = \{AX_t^{(1)} + MX_t^{(2)}\}\mathrm{d}t, \\ \mathrm{d}X_t^{(2)} = b_t(X_t, \mathscr{L}_{X_t})\mathrm{d}t + \sigma_t(\mathscr{L}_{X_t})\mathrm{d}B_t, \quad t\in[0,T], \end{cases} \tag{3.1}$$

where $B = (B_t)_{t\in[0,T]}$ is a d-dimensional standard Brownian motion, A is an $m\times m$ and M is an $m\times d$ matrix, and

$$\sigma : [0,T]\times\mathscr{P}(\mathbb{R}^{m+d}) \to \mathbb{R}^d\otimes\mathbb{R}^d, \quad b : [0,T]\times\mathbb{R}^{m+d}\times\mathscr{P}(\mathbb{R}^{m+d}) \to \mathbb{R}^d$$

are measurable, where $\mathscr{P}(\mathbb{R}^{m+d})$ is the space of probability measures on $\mathbb{R}^{m+d}$ equipped with the weak topology. For any $k\ge 1$, let

$$\mathscr{P}_k(\mathbb{R}^{m+d}) := \big\{\mu\in\mathscr{P}(\mathbb{R}^{m+d}) :\ \|\mu\|_k := \mu(|\cdot|^k)^{\frac{1}{k}} < \infty\big\},$$

which is a Polish space under the L^k-Wasserstein distance $\mathbb{W}_k$. When (3.1) is well-posed for distributions in $\mathscr{P}_k(\mathbb{R}^{m+d})$, let $P_t^*\mu = \mathscr{L}_{X_t}$ for the solution with initial distribution $\mu\in\mathscr{P}_k(\mathbb{R}^{m+d})$. We aim to establish the log-Harnack inequality and Bismut formula for

$$P_t f(\mu) := \int_{\mathbb{R}^{m+d}} f\mathrm{d}(P_t^*\mu), \quad f\in\mathscr{B}_b(\mathbb{R}^{m+d}).$$

By the same reason reformulating (1.5) as (1.6), instead of (3.1) we consider

$$\begin{cases} \mathrm{d}X_t^{(1)} = \{AX_t^{(1)} + MX_t^{(2)}\}\mathrm{d}t, \\ \mathrm{d}X_t^{(2)} = b_t(X_t, \mathscr{L}_{X_t})\mathrm{d}t + \lambda\mathrm{d}W_t + \tilde\sigma_t(\mathscr{L}_{X_t})\mathrm{d}\tilde W_t, \quad t\in[0,T], \end{cases} \tag{3.2}$$

where $W_t, \tilde W_t$ are two independent d-dimensional Brownian motions, and

$$\tilde\sigma : [0,T]\times\mathscr{P}(\mathbb{R}^{m+d}) \to \mathbb{R}^d\otimes\mathbb{R}^d$$

are measurable.

3.1 Log-Harnack inequality

To establish the log-Harnack inequality, we make the following assumption.

(C) $\lambda > 0$ is a constant, $(\tilde{\sigma}, b)$ satisfies conditions in **(A)** for $(x, \mu) \in \mathbb{R}^{m+d} \times \mathscr{P}_2(\mathbb{R}^{m+d})$, and the following Kalman's rank condition holds for some integer $1 \le l \le m$:

$$\text{Rank}[A^i M, 0 \le i \le l-1] = m, \tag{3.3}$$

where $A^0 := I_m$ is the $m \times m$-identity matrix.

By [21, Theorem 2.1], **(C)** implies that (3.2) is well-posed for distributions in $\mathscr{P}_2(\mathbb{R}^{m+d})$, and there exists a constant $c > 0$ such that

$$\mathbb{W}_2(P_t^*\mu, P_t^*\nu) \le c\mathbb{W}_2(\mu, \nu), \quad \mu, \nu \in \mathscr{P}_2(\mathbb{R}^{m+d}), t \in [0, T].$$

So, as in (2.4), we find a constant $C > 0$ such that

$$\mathbb{E}\left[\sup_{t\in[0,T]} |\xi_t^\mu - \xi_t^\nu|^2\right] \le C\mathbb{W}_2(\mu, \nu)^2, \quad \mu, \nu \in \mathscr{P}_2(\mathbb{R}^{m+d}). \tag{3.4}$$

To distinguish the singularity of P_t in the degenerate component $x^{(1)}$ and the non-degenerate one $x^{(2)}$, for any $t > 0$ we consider the modified distance

$$\rho_t(x, y) := \sqrt{t^{-2}|x^{(1)} - y^{(1)}|^2 + |x^{(2)} - y^{(2)}|^2}, \quad x, y \in \mathbb{R}^{m+d},$$

and define the associated L^2-Wasserstein distance

$$\mathbb{W}_{2,t}(\mu, \nu) := \inf_{\pi\in\mathscr{C}(\mu,\nu)} \left(\int_{\mathbb{R}^{m+d}\times\mathbb{R}^{m+d}} \rho_t(x, y)^2 \pi(\mathrm{d}x, \mathrm{d}y)\right)^{\frac{1}{2}}.$$

It is clear that

$$\frac{1}{T^2 \vee 1}\mathbb{W}_2^2 \le \mathbb{W}_{2,t}^2 \le \frac{1 \vee T^2}{t^2}\mathbb{W}_2^2, \quad t \in (0, T]. \tag{3.5}$$

For $t \in (0, T]$, let

$$Q_t := \int_0^t \frac{s(t-s)}{t^2} \mathrm{e}^{-sA} MM^* \mathrm{e}^{-sA^*} \mathrm{d}s.$$

According to [16], see also [23, Proof of Theorem 4.2(1)], the rank condition (3.3) implies

$$\|Q_t^{-1}\| \le c_0 t^{1-2l}, \quad t \in (0, T] \tag{3.6}$$

for some constant $c_0 > 0$.

Theorem 3.1. *Assume* **(C)** *and let* P_t^* *be associated with the degenerate SDE* (3.2). *Then there exists a constant* $c > 0$ *such that*

$$\begin{aligned}&P_t \log f(\nu) - \log P_t f(\mu) \le \frac{c}{t^{4l-3}} \mathbb{W}_{2,t}(\mu,\nu)^2 \le \frac{c(1\vee T^2)}{t^{4l-1}} \mathbb{W}_2(\mu,\nu)^2,\\ &t\in(0,T],\ \ \mu,\nu\in\mathscr{P}_2(\mathbb{R}^{m+d}),\quad 0<f\in\mathscr{B}_b(\mathbb{R}^{m+d}).\end{aligned} \tag{3.7}$$

Equivalently, for any $t \in (0, T]$ *and* $\mu, \nu \in \mathscr{P}_2(\mathbb{R}^{m+d})$,

$$\mathrm{Ent}(P_t^*\nu | P_t^*\mu) \le \frac{c}{t^{4l-3}} \mathbb{W}_{2,t}(\mu,\nu)^2 \le \frac{c(1\vee T^2)}{t^{4l-1}} \mathbb{W}_2(\mu,\nu)^2.$$

Proof. For any $t_0 \in (0, T]$ and $\mu, \nu \in \mathscr{P}_2(\mathbb{R}^{m+d})$, let X_0, Y_0 be $\mathscr{F}_0$-measurable such that

$$\mathscr{L}_{X_0} = \mu,\ \ \mathscr{L}_{Y_0} = \nu,\ \ \mathbb{E}[\rho_{t_0}(X_0, Y_0)^2] = \mathbb{W}_{2,t_0}(\mu,\nu)^2. \tag{3.8}$$

Let X_t solve (3.2) with initial value X_0, we have $P_t^*\mu = \mathscr{L}_{X_t}$. Let

$$v = (v^{(1)}, v^{(2)}) := (Y_0^{(1)} - X_0^{(1)}, Y_0^{(2)} - X_0^{(2)}) = Y_0 - X_0. \tag{3.9}$$

For fixed $t_0 \in (0, T]$, let

$$\begin{aligned}&\alpha_{t_0}(s) := \frac{s}{t_0}\big(\xi_{t_0}^\mu - \xi_{t_0}^\nu - v^{(2)}\big) - \frac{s(t_0-s)}{t_0^2} M^* \mathrm{e}^{-sA^*} Q_{t_0}^{-1}\big(v^{(1)} + V_{t_0}^{\mu,\nu}\big),\\ &V_{t_0}^{\mu,\nu} := \int_0^{t_0} \mathrm{e}^{-rA} M\Big\{\frac{t_0-r}{t_0} v^{(2)} + \frac{r}{t_0}\big(\xi_{t_0}^\mu - \xi_{t_0}^\nu\big) + \xi_r^\nu - \xi_r^\mu\Big\}\mathrm{d}r.\end{aligned} \tag{3.10}$$

By (3.6), we find a constant $c_1 > 0$ independent of $t_0 \in (0, T]$ such that

$$\sup_{t\in[0,t_0]} \big\{t_0|\alpha_{t_0}'(t)| + |\alpha_{t_0}(t)|\big\} \le \frac{c_1}{t_0^{2(l-1)}}\Big(t_0^{-1}|v^{(1)}| + |v^{(2)}| + \sup_{t\in[0,t_0]}|\xi_t^\mu - \xi_t^\nu|\Big). \tag{3.11}$$

Let Y_t solve the SDE with initial value Y_0:

$$\begin{cases}\mathrm{d}Y_t^{(1)} = \big\{AY_t^{(1)} + MY_t^{(2)}\big\}\mathrm{d}t,\\ \mathrm{d}Y_t^{(2)} = \big\{b_t(X_t, P_t^*\mu) + \alpha_{t_0}'(t)\big\}\mathrm{d}t + \lambda \mathrm{d}W_t + \tilde\sigma_t(P_t^*\nu)\mathrm{d}\tilde W_t,\quad t\in[0,t_0].\end{cases} \tag{3.12}$$

This and (3.2) imply

$$\begin{aligned}&Y_t^{(2)} - X_t^{(2)} = \alpha_{t_0}(t) + v^{(2)} + \xi_t^\nu - \xi_t^\mu,\\ &Y_t^{(1)} - X_t^{(1)} = \mathrm{e}^{tA} v^{(1)} + \int_0^t \mathrm{e}^{(t-s)A} M\big\{\alpha_{t_0}(s) + v^{(2)} + \xi_s^\nu - \xi_s^\mu\big\}\mathrm{d}s,\quad t\in[0,t_0].\end{aligned} \tag{3.13}$$

Consequently,

$$Y_{t_0}^{(2)} - X_{t_0}^{(2)} = \xi_{t_0}^\mu - \xi_{t_0}^\nu - v^{(2)} + v^{(2)} + \xi_{t_0}^\nu - \xi_{t_0}^\mu = 0,$$

$$
\begin{aligned}
Y^{(1)}_{t_0}-X^{(1)}_{t_0}&=\mathrm{e}^{t_0A}v^{(1)}+\int_0^{t_0}\mathrm{e}^{(t_0-s)A}M\Big\{\frac{s}{t_0}\big(\xi^\mu_{t_0}-\xi^\nu_{t_0}-v^{(2)}\big)+v^{(2)}+\xi^\nu_s-\xi^\mu_s\Big\}\mathrm{d}s\\
&\quad-\mathrm{e}^{t_0A}Q_{t_0}Q_{t_0}^{-1}\bigg(v^{(1)}+\int_0^{t_0}\mathrm{e}^{-rA}M\Big\{\frac{t_0-r}{t_0}v^{(2)}+\frac{r}{t_0}\big(\xi^\mu_{t_0}-\xi^\nu_{t_0}\big)+\xi^\nu_r-\xi^\mu_r\Big\}\mathrm{d}r\bigg)\\
&=0,
\end{aligned}
$$

so that

$$
Y_{t_0}=X_{t_0}. \tag{3.14}
$$

On the other hand, by (3.13) and (3.11) we find a constant $c_2>0$ uniform in $t_0\in(0,T]$ such that

$$
\begin{aligned}
\sup_{t\in[0,t_0]}|Y_t-X_t|^2&\le\frac{c_2}{t_0^{4(l-1)}}\Big\{t_0^{-2}|v^{(1)}|^2+|v^{(2)}|^2+\sup_{t\in[0,t_0]}|\xi^\mu_t-\xi^\nu_t|^2\Big\}\\
&=\frac{c_2}{t_0^{4(l-1)}}\Big\{\rho_{t_0}(X_0,Y_0)^2+\sup_{t\in[0,t_0]}|\xi^\mu_t-\xi^\nu_t|^2\Big\}.
\end{aligned}\tag{3.15}
$$

To formulate the equation of Y_t as (3.2), let

$$
\eta_s:=\frac{1}{\lambda}\big\{b_s(Y_s,P_s^*\nu)-b_s(X_s,P_s^*\mu)-\alpha'_{t_0}(s)\big\},\quad s\in[0,t_0]. \tag{3.16}
$$

By **(C)**, (3.11) and (3.15), we find a constant $c_3>0$ uniformly in $t_0\in(0,T]$ such that

$$
\begin{aligned}
|\eta_s|^2\le&\, c_3K_s\Big\{\mathbb{W}_2(\mu,\nu)^2+t_0^{4(1-l)}\rho_{t_0}(X_0,Y_0)^2+t_0^{4(1-l)}\sup_{t\in[0,t_0]}|\xi^\mu_t-\xi^\nu_t|^2\Big\}\\
&+c_3t_0^{2-4l}\Big(\rho_{t_0}(X_0,Y_0)^2+\sup_{t\in[0,t_0]}|\xi^\nu_t-\xi^\mu_t|^2\Big).
\end{aligned}\tag{3.17}
$$

By Girsanov's theorem,

$$
\hat W_t:=W_t-\int_0^t\eta_s\mathrm{d}s,\quad t\in[0,t_0]
$$

is a d-dimensional Brownian motion under the weighted conditional probability measure $\mathrm{d}\mathbb{Q}^{\tilde W,0}:=R^{\tilde W,0}\mathrm{d}\mathbb{P}^{\tilde W,0}$, where

$$
R^{\tilde W,0}:=\mathrm{e}^{\int_0^{t_0}\langle\eta_s,\mathrm{d}W_s\rangle-\frac12\int_0^{t_0}|\eta_s|^2\mathrm{d}s}.
$$

Let $\tilde\xi^\nu_t=(0,\xi^\nu_t)$. By (3.12), $\hat Y_t:=Y_t-\tilde\xi^\nu_t$ solves the SDE

$$
\begin{cases}
\mathrm{d}\hat Y^{(1)}_t=\big\{A\hat Y^{(1)}_t+M\hat Y^{(2)}_t+M\xi^\nu_t\big\}\mathrm{d}t,\\
\mathrm{d}\hat Y^{(2)}_t=b_t(\hat Y_t+\tilde\xi^\nu_t,P_t^*\nu)\mathrm{d}t+\lambda\mathrm{d}\hat W_t,\quad t\in[0,t_0],\ \hat Y_0=Y_0.
\end{cases}
$$

Letting X_t^ν solve (3.2) with $X_0^\nu = Y_0$, we see that $\hat{X}_t^\nu := X_t^\nu - \tilde{\xi}_t^\nu$ solves the same equation as $\hat{Y}_t$ for W_t replacing $\hat{W}_t$. By the weak uniqueness and (3.14), (2.12) holds for $\mathbb{R}^{m+d}$ replacing $\mathbb{R}^d$, i.e. for any $f \in \mathscr{B}_b(\mathbb{R}^{m+d})$,

$$P_{t_0}^{\tilde{W},0} f(X_{t_0}^\nu) := \mathbb{E}^{\tilde{W},0}[f(X_{t_0}^\nu)] = \mathbb{E}^{\tilde{W},0}[R^{\tilde{W},0} f(Y_{t_0})] = \mathbb{E}^{\tilde{W},0}[R^{\tilde{W},0} f(X_{t_0})]. \tag{3.18}$$

Combining this with Young's inequality and (3.17), we find constants $c_4 > 0$ such that

$$\begin{aligned} &P_{t_0}^{\tilde{W},0} \log f(X_0^\nu) - \log P_{t_0}^{\tilde{W},0} f(X_0^\mu) \le \mathbb{E}^{\tilde{W},0}[R^{\tilde{W},0} \log R^{\tilde{W},0}] = \frac{1}{2}\mathbb{E}_{\mathbb{Q}^{\tilde{W},0}} \int_0^{t_0} |\eta_t|^2 \mathrm{d}t \\ &\le c_4 \Big\{ \mathbb{W}_2(\mu,\nu)^2 + t_0^{3-4l} \rho_{t_0}(X_0, Y_0)^2 + t_0^{3-4l} \sup_{t\in[0,t_0]} |\xi_t^\nu - \xi_t^\mu|^2 \Big\}. \end{aligned} \tag{3.19}$$

By taking expectation, using Jensen's inequality, (3.4), (3.5) and (3.8), we prove (3.7). □

3.2 Bismut formula

We will use Definition 2.1 for $\mathbb{R}^{m+d}$ replacing $\mathbb{R}^d$. The following assumption is parallel to **(B)** with an additional rank condition.

- **(D)** $(\tilde{\sigma}, b)$ satisfies **(B)** for $\mathbb{R}^{m+d}$ replacing $\mathbb{R}^d$, and the rank condition (3.3) holds for some $1 \le l \le m$.

Let X_0^μ be $\mathscr{F}_0$-measurable such that $\mathscr{L}_{X_0^\mu} = \mu \in \mathscr{P}_k(\mathbb{R}^{m+d})$, and let X_t^μ solve (3.2) with initial value X_0^μ. For any $\varepsilon \ge 0$, denote

$$\mu^\varepsilon := \mu \circ (id + \varepsilon\phi)^{-1}, \quad X_0^{\mu^\varepsilon} := X_0^\mu + \varepsilon\phi(X_0^\mu).$$

Let $X_t^{\mu^\varepsilon}$ solve (3.2) with initial value $X_0^{\mu^\varepsilon}$. So,

$$X_t^\mu = X_t^{\mu^0}, \quad P_t^*\mu^\varepsilon = \mathscr{L}_{X_t^{\mu^\varepsilon}}, \quad t \in [0,T], \varepsilon \ge 0.$$

By [3, Lemma 5.2], for any $\phi = (\phi^{(1)}, \phi^{(2)}) \in T_{\mu,k}(\mathbb{R}^{m+d})$, **(D)** implies that

$$\nabla_\phi X_\cdot^\mu := \lim_{\varepsilon \downarrow 0} \frac{X_\cdot^{\mu^\varepsilon} - X_\cdot^\mu}{\varepsilon}$$

exists in $L^k(\Omega \to C([0,T];\mathbb{R}^{m+d});\mathbb{P})$, and there exists a constant $c > 0$ such that

$$\mathbb{E}\bigg[\sup_{t\in[0,T]} |\nabla_\phi X_t^\mu|^k\bigg] \le c\|\phi\|_{L^k(\mu)}^k, \quad \mu \in \mathscr{P}_k(\mathbb{R}^{m+d}), \phi \in T_{\mu,k}(\mathbb{R}^{m+d}).$$

Finally, for any $t \in (0,T]$ and $s \in [0,t]$, let

$$\gamma_t^{\mu,\phi} := \int_0^t \Big\langle \mathbb{E}\big[\big\langle D^I \tilde{\sigma}_r(P_r^*\mu)(X_r^\mu),\ \nabla_\phi X_r^\mu \big\rangle\big],\ \mathrm{d}\tilde{W}_r \Big\rangle$$

$$V_t^{\mu,\phi} := \int_0^t \mathrm{e}^{-rA} M\Big\{\frac{t-r}{t}\phi^{(2)}(X_0^\mu) - \frac{r}{t}\gamma_t^{\mu,\phi} + \gamma_r^{\mu,\phi}\Big\}\mathrm{d}r,$$
$$\alpha_t^{\mu,\phi}(s) := -\frac{s}{t}\big\{\phi^{(2)}(X_0^\mu) + \gamma_t^{\mu,\phi}\big\} - \frac{s(t-s)}{t^2} M^* \mathrm{e}^{-sA^*} Q_t^{-1}\big\{\phi^{(1)}(X_0^\mu) + V_t^{\mu,\phi}\big\},$$

and define

$$N_{s,t}^{(1)} := \mathrm{e}^{sA}\phi^{(1)}(X_0^\mu) + \int_0^s \mathrm{e}^{(s-r)A} M\big\{\alpha_t^{\mu,\phi}(r) + \phi^{(2)}(X_0^\mu) + \gamma_r^{\mu,\phi}\big\}\mathrm{d}r$$
$$N_{s,t}^{(2)} := \alpha_t^{\mu,\phi}(s) + \phi^{(2)}(X_0^\mu) + \gamma_s^{\mu,\phi},$$
$$M_{s,t}^{\mu,\phi} := \mathbb{E}\big[\langle D_\phi^I b_s(z,\cdot)(P_s^*\mu)(X_s^\mu), \nabla_\phi X_s^\mu\rangle\big]_{z=X_s^\mu} - (\alpha_t^{\mu,\phi})'(s).$$

Then we have the following result.

Theorem 3.2. *Assume* **(D)** *and let* $N_{s,t}^{\mu,\phi} := \big(N_{s,t}^{(1)}, N_{s,t}^{(2)}\big) \in \mathbb{R}^{m+d}, 0 \le s \le t$. *For any* $t \in (0,T]$, $\mu \in \mathscr{P}_k(\mathbb{R}^{m+d})$, $\phi \in T_{\mu,k}(\mathbb{R}^{m+d})$ *and* $f \in \mathscr{B}_b(\mathbb{R}^{m+d})$,

$$D_\phi^I P_t f(\mu) = \frac{1}{\lambda}\mathbb{E}\bigg[f(X_t^\mu)\int_0^t \Big\langle \nabla_{N_{s,t}^{\mu,\phi}} b_s(\cdot, P_s^*\mu)(X_s^\mu) + M_{s,t}^{\mu,\phi},\ \mathrm{d}W_s\Big\rangle\bigg]. \tag{3.20}$$

Consequently, $P_t f$ *is intrinsically differentiable, and there exists a constant* $c > 0$ *such that*

$$\|D^I P_t f(\mu)\|_{L^{k^*}(\mu)} \le \frac{c}{t^{2l-\frac{1}{2}}}\big(P_t|f|^{k^*}(\mu)\big)^{\frac{1}{k^*}}, \quad t \in (0,T], f \in \mathscr{B}_b(\mathbb{R}^{m+d}). \tag{3.21}$$

Proof. It is easy to see that under **(D)**, (3.21) follows from (3.20). So, it suffices to prove (3.20).

Let X_t^μ solve (3.2) with initial value X_0^μ, and for any $\varepsilon \in (0,1]$, let Y_t^ε solve (3.12) for $Y_0 = Y_0^\varepsilon := X_0^\mu + \varepsilon\phi(X_0^\mu)$ and $\nu = \mu^\varepsilon$. Then

$$\mathscr{L}_{Y_0} = \mathscr{L}_{Y_0^\varepsilon} = \mu^\varepsilon.$$

Let $\alpha_{t_0}^\varepsilon(s)$ be defined in (3.10) for $\nu = \mu^\varepsilon$. By (2.20) and (3.9), we have

$$\lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\alpha_{t_0}^\varepsilon(s) = \alpha_{t_0}^{\mu,\phi}(s), \quad s \in [0,t_0], \tag{3.22}$$

while (3.13) and (3.16) reduces to

$$(Y_t^\varepsilon)^{(2)} - (X_t^\mu)^{(2)} = \alpha_{t_0}^\varepsilon(t) + \varepsilon\phi^{(2)}(X_0^\mu) + \xi_t^{\mu^\varepsilon} - \xi_t^\mu,$$
$$(Y_t^\varepsilon)^{(1)} - (X_t^\mu)^{(1)} = \varepsilon \mathrm{e}^{tA}\phi^{(1)}(X_0^\mu) + \int_0^t \mathrm{e}^{(t-s)A} M\big\{\alpha_{t_0}^\varepsilon(s) + \varepsilon\phi^{(2)}(X_0^\mu) + \xi_s^{\mu^\varepsilon} - \xi_s^\mu\big\}\mathrm{d}s,$$

and

$$\eta_t^\varepsilon = \frac{1}{\lambda}\Big\{b_t(Y_t^\varepsilon, P_t^*\mu^\varepsilon) - b_t(X_t^\mu, P_t^*\mu) - \{\alpha_{t_0}^\varepsilon\}'(t)\Big\}, \quad t \in [0,t_0].$$

Then by (2.20) and (3.22), we have

$$\lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}(Y_t^\varepsilon - X_t^\mu) = N_{t,t_0}^{\mu,\phi}. \tag{3.23}$$

Let

$$R^\varepsilon := \mathrm{e}^{\int_0^{t_0}\langle \eta_t^\varepsilon, \mathrm{d}W_t\rangle - \frac{1}{2}\int_0^{t_0}|\eta_t^\varepsilon|^2\mathrm{d}t}.$$

By (3.18), we obtain

$$P_{t_0}^{\tilde W,0} f(X_0^{\mu^\varepsilon}) := \mathbb{E}^{\tilde W,0}[f(X_{t_0}^{\mu^\varepsilon})] = \mathbb{E}^{\tilde W,0}[R^\varepsilon f(X_{t_0}^\mu)],\quad f\in\mathscr{B}_b(\mathbb{R}^{m+d}).$$

As in (2.21), by **(D)**, (3.23) and (2.20), we derive

$$\begin{aligned}
&\lim_{\varepsilon\downarrow 0}\frac{P_{t_0}^{\tilde W,0} f(X_0^{\mu^\varepsilon}) - P_{t_0}^{\tilde W,0} f(X_0^\mu)}{\varepsilon} = \lim_{\varepsilon\downarrow 0}\mathbb{E}^{\tilde W,0}\Big[f(X_{t_0}^\mu)\frac{R^\varepsilon-1}{\varepsilon}\Big]\\
&= \frac{1}{\lambda}\mathbb{E}^{\tilde W,0}\bigg[f(X_{t_0}^\mu)\int_0^{t_0}\Big\langle \nabla_{N_{s,t_0}^{\mu,\phi}} b_s(\cdot, P_s^*\mu)(X_s^\mu) + M_{s,t_0}^{\mu,\phi},\ \mathrm{d}W_s\Big\rangle\bigg].
\end{aligned} \tag{3.24}$$

Finally, similarly to the proof of (2.15), since **(D)** implies **(C)** for $(\mathscr{P}_k(\mathbb{R}^{m+d}), \mathbb{W}_k)$ replacing $(\mathscr{P}_2(\mathbb{R}^{m+d}), \mathbb{W}_2)$, the argument leading to (3.19) implies

$$P_{t_0}^{\tilde W,0}\log f(X_0^\nu) - \log P_{t_0}^{\tilde W,0} f(X_0^\mu) \le c(t_0)\Big\{\mathbb{W}_k(\mu,\nu)^2 + \rho_{t_0}(X_0^\mu, X_0^\nu)^2 + \sup_{t\in[0,t_0]}|\xi_t^\nu - \xi_t^\mu|^2\Big\}$$

for some constant $c(t_0) > 0$. Therefore, as shown in step (b) of the proof of Theorem 2.2, this enables us to apply the dominated convergence theorem with (3.24) to derive

$$\begin{aligned}
&D_\phi^I P_{t_0} f(\mu) = \lim_{\varepsilon\downarrow 0}\frac{\mathbb{E}[P_{t_0}^{\tilde W,0} f(X_0^{\mu^\varepsilon}) - P_{t_0}^{\tilde W,0} f(X_0^\mu)]}{\varepsilon} = \mathbb{E}\Big\{\lim_{\varepsilon\downarrow 0}\mathbb{E}^{\tilde W,0}\Big[f(X_{t_0}^\mu)\frac{R^\varepsilon - 1}{\varepsilon}\Big]\Big\}\\
&= \frac{1}{\lambda}\mathbb{E}\bigg[f(X_{t_0}^\mu)\int_0^{t_0}\Big\langle \nabla_{N_{s,t_0}^{\mu,\phi}} b_s(\cdot, P_s^*\mu)(X_s^\mu) + M_{s,t_0}^{\mu,\phi},\ \mathrm{d}W_s\Big\rangle\bigg].
\end{aligned}$$

□

4 Exponential ergodicity in entropy

Following the line of (2.11), we may use the log-Harnack inequality to study the exponential ergodicity in entropy. To this end, we consider the time homogeneous equation on $\mathbb{R}^d$

$$\mathrm{d}X_t = b(X_t, \mathscr{L}_{X_t})\mathrm{d}t + \lambda\mathrm{d}W_t + \tilde\sigma(\mathscr{L}_{X_t})\mathrm{d}\tilde W_t,\quad t\ge 0, \tag{4.1}$$

and the degenerate model on $\mathbb{R}^{m+d}$

$$\begin{cases}\mathrm{d}X_t^{(1)} = \big\{AX_t^{(1)} + MX_t^{(2)}\big\}\mathrm{d}t,\\ \mathrm{d}X_t^{(2)} = b(X_t, \mathscr{L}_{X_t})\mathrm{d}t + \lambda\mathrm{d}W_t + \tilde\sigma(\mathscr{L}_{X_t})\mathrm{d}\tilde W_t,\quad t\ge 0,\end{cases} \tag{4.2}$$

where $\lambda > 0$ is a constant.

4.1 Non-degenerate case

(E) There exist constants $K, \theta_1, \theta_2 > 0$ with $\theta := \theta_2 - \theta_1 > 0$, such that for any $\mu, \nu \in \mathscr{P}_2(\mathbb{R}^d)$ and $x, y \in \mathbb{R}^d$,

$$|b(x,\mu) - b(y,\nu)| + |\tilde\sigma(\mu) - \tilde\sigma(\nu)| \le K(|x-y| + \mathbb{W}_2(\mu,\nu)),$$
$$2\langle b(x,\mu) - b(y,\nu), x-y\rangle + \|\sigma(\mu) - \sigma(\nu)\|_{HS}^2 \le -\theta_2|x-y|^2 + \theta_1 \mathbb{W}_2(\mu,\nu)^2,$$

where $\|\cdot\|_{HS}$ is the Hilbert-Schmidt norm.

By [21, Theorem 2.1], this assumption implies that (4.1) is well-posed for distributions in $\mathscr{P}_2$, and P_t^* has a unique invariant probability measure $\bar\mu \in \mathscr{P}_2(\mathbb{R}^d)$ such that

$$\mathbb{W}_2(P_t^*\mu, \bar\mu)^2 \le \mathrm{e}^{-\theta t}\mathbb{W}_2(\mu,\bar\mu)^2, \quad t \ge 0. \tag{4.3}$$

The following result ensures the exponential convergence in entropy.

Theorem 4.1. *Assume* **(E)** *and let* P_t^* *be associated with* (4.1). *Then there exists a constant* $c > 0$ *such that*

$$\max\left\{\mathbb{W}_2(P_t^*\mu,\bar\mu)^2, \mathrm{Ent}(P_t^*\mu|\bar\mu)\right\}$$
$$\le c\mathrm{e}^{-\theta t}\min\left\{\mathbb{W}_2(\mu,\bar\mu)^2, \mathrm{Ent}(\mu|\bar\mu)\right\}, \quad \mu \in \mathscr{P}_2(\mathbb{R}^d), t \ge 1.$$

Proof. According to the proof of [14, Theorem 2.3], **(E)** implies the Talagrand inequality

$$\mathbb{W}_2(\mu,\bar\mu)^2 \le c_1 \mathrm{Ent}(\mu|\bar\mu), \quad \mu \in \mathscr{P}_2(\mathbb{R}^d)$$

for some constant $c_1 > 0$. According to [14, Theorem 2.1], this together with (4.3) and Theorem 2.1 implies the desired assertion.

□

4.2 Degenerate case

To study the exponential ergodicity for the degenerate model (4.2), we extend the assumption $(A1)$-$(A3)$ in [20] to the present distribution dependent case.

(F) $\tilde\sigma$ and b are Lipschitz continuous on $\mathscr{P}_2(\mathbb{R}^{m+d})$ and $\mathbb{R}^{m+d} \times \mathscr{P}_2(\mathbb{R}^{m+d})$ respectively. (A, M) satisfies the rank condition (3.3) for some $1 \le l \le m$, and there exist constants $r > 0, \theta_2 > \theta_1 \ge 0$ and $r_0 \in (-\|M\|^{-1}, \|M\|^{-1})$ such that

$$\frac{1}{2}\|\tilde\sigma(\mu) - \tilde\sigma(\nu)\|_{HS}^2 + \left\langle b(x,\mu) - b(y,\nu),\ x^{(2)} - y^{(2)} + rr_0M^*(x^{(1)} - y^{(1)})\right\rangle$$
$$+ \left\langle r^2(x^{(1)} - y^{(1)}) + rr_0M(x^{(2)} - y^{(2)}),\ A(x^{(1)} - y^{(1)}) + M(x^{(2)} - y^{(2)})\right\rangle$$
$$\le \theta_1\mathbb{W}_2(\mu,\nu)^2 - \theta_2|x-y|^2, \quad x, y \in \mathbb{R}^{m+d},\ \mu,\nu \in \mathscr{P}_2(\mathbb{R}^{m+d}).$$

In the distribution free case, some examples are presented in [20, Section 5], which can be extended to the present setting if the Lipschitz constant of $\tilde\sigma(\mu)$ and $b(x,\mu)$ in $\mu \in \mathscr{P}_2(\mathbb{R}^{m+d})$ is small enough.

Theorem 4.2. *Assume* **(F)***. Then* P_t^* *associated with* (4.2) *has a unique invariant probability measure* $\bar\mu$*, and there exist constants* $c,\lambda>0$ *such that*

$$\max\big\{\mathrm{Ent}(P_t^*\mu|\bar\mu), \mathbb{W}_2(P_t^*\mu,\bar\mu)^2\big\} \le c\mathrm{e}^{-\lambda t}\mathbb{W}_2(\mu,\bar\mu)^2, \quad t\ge 1, \mu\in\mathscr{P}_2(\mathbb{R}^{m+d}).$$

Proof. Let

$$\rho(x) := \frac{r^2}{2}|x^{(1)}|^2 + \frac{1}{2}|x^{(2)}|^2 + rr_0\langle x^{(1)}, Mx^{(2)}\rangle, \quad x=(x^{(1)},x^{(2)})\in\mathbb{R}^{m+d}.$$

By $r_0\|M\|<1$ and $r>0$, we find a constant $c_0\in(0,1)$ such that

$$c_0|x|^2 \le \rho(x) \le c_0^{-1}|x|^2, \quad x\in\mathbb{R}^{m+d}. \tag{4.4}$$

Let X_t and Y_t solve (4.2) with initial values

$$\mathscr{L}_{X_0}=\mu, \quad \mathscr{L}_{Y_0}=\nu, \quad \mathbb{W}_2(\mu,\nu)^2=\mathbb{E}[|X_0-Y_0|^2]. \tag{4.5}$$

By **(F)** and Itô's formula, we obtain

$$\mathrm{d}\rho(X_t-Y_t) \le \Big\{\theta_1\mathbb{W}_2(P_t^*\mu,P_t^*\nu)^2 - \theta_2|X_t-Y_t|^2\Big\}\mathrm{d}t + \mathrm{d}M_t \tag{4.6}$$

for some martingale M_t, and

$$\mathrm{d}\rho(X_t) \le \Big\{\theta_1\mathbb{E}[|X_t|^2] - \theta_2|X_t|^2 + C + C|X_t|\Big\}\mathrm{d}t + \mathrm{d}\tilde M_t$$

for some martingale $\tilde M_t$ and constant $C>0$. In particular, by (4.4), the latter implies

$$\sup_{t\ge 0}\mathbb{E}[|X_t|^2] < \infty. \tag{4.7}$$

Since

$$\mathbb{W}_2(P_t^*\mu,P_t^*\nu)^2 \le \mathbb{E}[|X_t-Y_t|^2], \tag{4.8}$$

(4.6) and (4.4) imply

$$\mathbb{E}[\rho(X_t-Y_t)] - \mathbb{E}[\rho(X_s-Y_s)] \le -c_0(\theta_2-\theta_1)\int_s^t \mathbb{E}[\rho(X_r-Y_r)]\mathrm{d}r, \quad t\ge s\ge 0.$$

By Gronwall"s inequality, we derive

$$\mathbb{E}[\rho(X_t-Y_t)] \le \mathrm{e}^{-c_0(\theta_2-\theta_1)t}\mathbb{E}[\rho(X_0-Y_0)], \quad t\ge 0.$$

This together with (4.4), (4.5) and (4.8) yields

$$\begin{aligned}&\mathbb{W}_2(P_t^*\mu, P_t^*\nu)^2 \le \mathbb{E}[|X_t - Y_t|^2] \le c_0^{-1}\mathbb{E}[\rho(X_t - Y_t)] \le c_0^{-1}\mathrm{e}^{-c_0(\theta_2-\theta_1)t}\mathbb{E}[\rho(X_0 - Y_0)]\\ &\le c_0^{-2}\mathrm{e}^{-c_0(\theta_2-\theta_1)t}\mathbb{E}[|X_0 - Y_0|^2] = c_0^{-2}\mathrm{e}^{-c_0(\theta_2-\theta_1)t}\mathbb{W}_2(\mu,\nu)^2, \quad t \ge 0, \mu, \nu \in \mathscr{P}_2(\mathbb{R}^{m+d}).\end{aligned}$$

As shown in [21, Proof of Theorem 3.1(2)], this together with (4.7) implies that P_t^* has a unique invariant probability measure $\bar{\mu} \in \mathscr{P}_2(\mathbb{R}^d)$, and

$$\mathbb{W}_2(P_t^*\mu, \bar{\mu})^2 \le c_0^{-2}\mathrm{e}^{-c_0(\theta_2-\theta_1)t}\mathbb{W}_2(\mu,\bar{\mu})^2, \quad t \ge 0, \mu \in \mathscr{P}_2(\mathbb{R}^{m+d}). \tag{4.9}$$

Finally, by the log-Harnack inequality (3.7), there exists a constant $c_1 > 0$ such that

$$\mathrm{Ent}(P_1^*\mu|\bar{\mu}) \le c_1\mathbb{W}_2(\mu,\bar{\mu})^2.$$

Combining this with (4.9) and using the semigroup property $P_t^* = P_{t-1}^*P_1^*$ for $t \ge 1$, we finish the proof. □

When b is of a gradient type (induced by σ) as in [14, (2.21)] such that the invariant probability measure $\bar{\mu}$ is explicitly given and satisfies the Talagrand inequality, we may also derive the stronger upper bound as in Theorem 4.1. We skip the details.

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