

WEAK TYPE A_p ESTIMATE FOR BILINEAR CALDERÓN-ZYGMUND OPERATORS

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ABSTRACT. In this paper, we investigate the boundedness of bilinear Calderón-Zygmund operators T from $L^{p_1}(w_1) \times L^{p_2}(w_2)$ to $L^{p,\infty}(v_{\vec{w}})$ with the stopping time method, where $1/p = 1/p_1 + 1/p_2$, $1 < p_1, p_2 < \infty$ and $\vec{w}$ is a multiple $A_{\vec{p}}$ weight. Specifically, we study the exponent α of $A_{\vec{p}}$ constant in formula

$$\|T(\vec{f})\|_{L^{p,\infty}(v_{\vec{w}})} \leq C_{m,n,\vec{p},T} [\vec{w}]_{A_{\vec{p}}}^\alpha \|f_1\|_{L^{p_1}(w_1)} \|f_2\|_{L^{p_2}(w_2)}.$$

Surprisingly, we show that when $p \geq \frac{3+\sqrt{5}}{2}$ or $\min\{p_1, p_2\} > 4$, the exponent α in the above estimate can be less than 1, which is different from the linear scenario.

1. INTRODUCTION AND MAIN RESULTS

In recent years, the theory of Calderón-Zygmund operators has attracted widespread attention. There have been many advances in the optimal control of weighted operator norms with A_p weights.

In the linear case, in 2012, Hytönen proved the A_2 conjecture in [3] and obtained

$$\|T(f)\|_{L^p(w)} \lesssim [w]_{A_p}^{\max\{1, \frac{p'}{p}\}} \|f\|_{L^p(w)}.$$

Just one year later, in [6], Lerner proved that Calderón-Zygmund operators can be controlled by sparse operators and provided an alternative proof of the A_2 theorem. We recommend interested readers to learn about the history of A_2 theorem in the above two papers and the references therein.

For the weak weighted operator norms of Calderón-Zygmund operators, in [4], Hytönen, Lacey, Martikainen, Orponen, Reguera, Sawyer, and Uriarte-Tuero obtained

$$\|T(f)\|_{L^{p,\infty}(w)} \lesssim [w]_{A_p} \|f\|_{L^p(w)}.$$

In the multilinear case, in [10], Li, Moen, and Sun proved that when $1 < p, p_1, p_2 < \infty$,

$$(1.1) \quad \|T(\vec{f})\|_{L^p(v_{\vec{w}})} \lesssim [\vec{w}]_{A_{\vec{p}}}^{\max\{1, \frac{p'_1}{p}, \frac{p'_2}{p}\}} \|f_1\|_{L^{p_1}(w_1)} \|f_2\|_{L^{p_2}(w_2)},$$

and they provided a beautiful example to show that their result is optimal. When $p < 1$, the estimate (1.1) still holds since the Calderón-Zygmund operators can be controlled by sparse operators pointwise, as shown in [1, 7, 2]. For weak norms, it is generally believed that the optimal exponent in $A_{\vec{p}}$ estimate is 1, which is the same as the linear case. Li and Sun gave a mixed $A_p - A_\infty$ estimate in [11], just in terms of the $A_{\vec{p}}$ constant, with the exponent larger than 1. In Li's master's thesis [9], he established a Coifman-Fefferman

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type inequality to prove that the exponent of the $A_{\vec{P}}$ constant for the weak type estimate can be $1 + 1/p$.

In this paper, we show that the exponent in weak $A_{\vec{P}}$ estimate for bilinear Calderón-Zygmund operators can be less than 1 for certain $\vec{P}$. Specifically, we prove the following theorem.

Theorem 1.1. *Let T be a bilinear Calderón-Zygmund operator, $\vec{P} = (p_1, p_2)$ with $1/p_1 + 1/p_2 = 1/p$ and $1 < p, p_1, p_2 < \infty$. Suppose $\vec{w} = (w_1, w_2) \in A_{\vec{P}}$, then*

$$(1.2) \quad \|T(\vec{f})\|_{L^{p,\infty}(v_{\vec{w}})} \leq C_{m,n,\vec{P},T} [\vec{w}]_{A_{\vec{P}}}^\alpha \|f_1\|_{L^{p_1}(w_1)} \|f_2\|_{L^{p_2}(w_2)},$$

where

$$\alpha = \min\{\beta, \gamma\}, \beta = \frac{1}{p} + \max\left\{\min\left\{\frac{1}{p_1}, \frac{1}{p_1} \frac{p_2'}{p}\right\}, \min\left\{\frac{1}{p_2'}, \frac{1}{p_2'} \frac{p_1'}{p}\right\}\right\}, \gamma = \max\left\{1, \frac{p_1'}{p}, \frac{p_2'}{p}\right\}.$$

It should be noted that the exponent γ in the theorem comes from the strong type estimate (1.1), so we only need to prove estimate (1.2) with exponent $\alpha = \beta$.

Remark 1.2. Note that $\beta \leq \max\left\{\frac{1}{p} + \frac{1}{p_1}, \frac{1}{p} + \frac{1}{p_2}\right\} < 1 + \frac{1}{p}$, so our results improve the one from [9].

Remark 1.3. We can apply the extrapolation techniques demonstrated in [12, Theorem 4.1], to generalize Theorem 1.1 to the case of $p < 1$. In particular, if we use $\vec{P} = (p_1, p_2)$ with $2 < p_1 = p_2 < \sqrt{2} + 1$ as the starting point, we can obtain better results than the strong type estimate (1.1), and further details are left for interested readers.

2. PRELIMINARIES

2.1. Bilinear Calderón-Zygmund operators. We call T a *bilinear Calderón-Zygmund operator* if it is originally defined on the product of Schwartz spaces and takes values in tempered distributions, meanwhile, for some $1 < q_1, q_2 < \infty$, it can be extended to a bounded bilinear operator from $L^{q_1}(\mathbb{R}^n) \times L^{q_2}(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$, where $1/q_1 + 1/q_2 = 1/q$, and if there exists a function $K(y_0, y_1, y_2)$, defined off the diagonal $y_0 = y_1 = y_2$ in $(\mathbb{R}^n)^3$ that satisfies

$$T(f_1, f_2)(y_0) = \int_{(\mathbb{R}^n)^2} K(y_0, y_1, y_2) f_1(y_1) f_2(y_2) dy_1 dy_2, \quad \forall y_0 \notin \text{supp } f_1 \cap \text{supp } f_2;$$

$$|K(y_0, y_1, y_2)| \leq \frac{C}{(|y_0 - y_1| + |y_0 - y_2|)^{2n}};$$

and for some $A, \varepsilon > 0$, whenever $|h| \leq \frac{1}{2} \max\{|y_0 - y_1|, |y_0 - y_2|\}$,

$$\begin{aligned} & |K(y_0 + h, y_1, y_2) - K(y_0, y_1, y_2)| + |K(y_0, y_1 + h, y_2) - K(y_0, y_1, y_2)| \\ & \quad + |K(y_0, y_1, y_2 + h) - K(y_0, y_1, y_2)| \\ & \leq \frac{1}{(|y_0 - y_1| + |y_0 - y_2|)^{2n}} \omega\left(\frac{|h|}{|y_0 - y_1| + |y_0 - y_2|}\right), \end{aligned}$$

where ω is the modulus of Dini-continuity, i.e., an increasing function satisfies $\omega(0) = 0$, $\omega(t + s) \leq \omega(t) + \omega(s)$, and

$$\|\omega\|_{\text{Dini}} := \int_0^1 \omega(t) \frac{dt}{t} < \infty.$$

In [2], Damián, Hormozi, and Li proved that bilinear Calderón-Zygmund operators can be pointwise controlled by sparse operators, which will be introduced in section 2.3.

2.2. Multiple $A_{\vec{p}}$ weight. Recall that in the linear case, a *weight* is a non-negative locally integrable function. When $1 < p < \infty$, The set A_p is composed of weights that satisfy

$$[w]_{A_p} := \sup_{Q: \text{cube in } \mathbb{R}^n} \langle w \rangle_Q \langle w^{1-p'} \rangle_Q^{p-1} < \infty,$$

where $\langle w \rangle_Q := w(Q)/|Q|$. When $p = \infty$, $A_\infty := \bigcup_{1 < p < \infty} A_p$ and the A_∞ constant $[w]_{A_\infty}$ is defined by

$$[w]_{A_\infty} := \sup_Q \frac{1}{w(Q)} \int_Q M(w\chi_Q),$$

where M denotes the Hardy-Littlewood maximal function. Meanwhile, for any $w \in A_p$, we have $[w]_{A_\infty} \leq [w]_{A_p}$.

As is well known, in [8], Lerner, Ombrosi, Pérez, Torres, and Trujillo-González extended the above definition to the multilinear case, and defined *multiple $A_{\vec{p}}$ weights* as follows. Let $\vec{P} = (p_1, \dots, p_m)$ with $1 < p_1, \dots, p_m < \infty$ and $1/p_1 + \dots + 1/p_m = 1/p$. Given $\vec{w} = (w_1, \dots, w_m)$, set

$$v_{\vec{w}} = \prod_{i=1}^m w_i^{p/p_i}.$$

The $A_{\vec{p}}$ constant is defined by

$$[\vec{w}]_{A_{\vec{p}}} := \sup_Q \langle v_{\vec{w}} \rangle_Q \prod_{i=1}^m \langle \sigma_i \rangle_Q^{p/p'_i},$$

where $\sigma_i = w_i^{1-p'_i}$. We say that $\vec{w}$ satisfies the multilinear $A_{\vec{p}}$ condition if $[\vec{w}]_{A_{\vec{p}}} < \infty$. Particularly, in Theorem 3.6 of the aforementioned paper, they proved that

$$(2.1) \quad [v_{\vec{w}}]_{A_{mp}} \leq [\vec{w}]_{A_{\vec{p}}}, [\sigma_i]_{A_{mp'_i}} \leq [\vec{w}]_{A_{\vec{p}}}^{p'_i/p}, \quad \forall \vec{w} \in A_{\vec{p}}.$$

2.3. Dyadic cubes system, sparse operators and stopping time argument. The *dyadic cubes system* $\mathcal{D}$ is a family of cubes with the following properties:

(1) for any $Q \in \mathcal{D}$, its sides are parallel to the coordinate axes and its sidelength is of the form of 2^k ;

(2) $Q \cap R \in \{Q, R, \emptyset\}$, for any $Q, R \in \mathcal{D}$;

(3) the cubes of fixed sidelength 2^k form a partition of $\mathbb{R}^n$.

A collection $\mathcal{S} \subset \mathcal{D}$ is called *sparse* if for each $Q \in \mathcal{S}$, there exists a subset $E_Q \subset Q$ such that $|E_Q| \geq \frac{1}{2}|Q|$ and the sets $\{E_Q\}_{Q \in \mathcal{S}}$ are pairwise disjoint. For a sparse family $\mathcal{S}$, we can define the *sparse operator* $A_{\mathcal{D}, \mathcal{S}}$ as follows:

$$A_{\mathcal{D}, \mathcal{S}}(\vec{f}) = \sum_{Q \in \mathcal{S}} \langle f_1 \rangle_Q \langle f_2 \rangle_Q \chi_Q,$$

where $\vec{f} = (f_1, f_2)$.

Below, we will introduce the main technique of this paper, stopping time argument, which was introduced by Li and Sun in [11], and further improved by Damián, Hormozi, and Li in [2].

Let w be a weight and $f \in L^p(w)$ for some $0 < p < \infty$. Suppose that the sparse family $\mathcal{S}$ has a collection of maximal cubes, in other words, there exists a collection of disjoint cubes $\{Q_i\}_{i \in \Lambda} \subset \mathcal{S}$, such that for any cube $Q \in \mathcal{S}$, there exists $i \in \Lambda$ such that $Q \subset Q_i$. Now we construct the *stopping time family* $\mathcal{F}$ from the pair (f, w) . Let $\mathcal{F}_0 := \{Q_i\}_{i \in \Lambda}$ and

$$\mathcal{F}_k := \bigcup_{F \in \mathcal{F}_{k-1}} \{F' \subset F : F' \text{ is the maximal cube in } \mathcal{S} \text{ that satisfies } \langle f \rangle_{F'}^w > 2 \langle f \rangle_F^w\},$$

where $\langle f \rangle_Q^w := \int_Q f w dx / w(Q)$, then the stopping time family is $\mathcal{F} := \bigcup_{k=0}^{\infty} \mathcal{F}_k$. It is easy to deduce from the construction above that

$$(2.2) \quad \sum_{F \in \mathcal{F}} (\langle f \rangle_F^w)^p w(F) \lesssim \|M_{\mathcal{D}}^w(f)\|_{L^p(w)}^p \lesssim \|f\|_{L^p(w)}^p,$$

where $M_{\mathcal{D}}^w(f)(x) := \sup_{x \in Q, Q \in \mathcal{D}} \langle f \rangle_Q^w$. We use $\pi_{\mathcal{F}}(Q)$ to represent the *stopping parents* of Q , that is, the minimal cube containing Q in $\mathcal{F}$. According to the definition, we have $\langle f \rangle_Q^w \leq 2 \langle f \rangle_{\pi_{\mathcal{F}}(Q)}^w$.

3. PROOF OF THE MAIN RESULTS

In order to prove the main theorem, we need the following lemmas.

Lemma 3.1. ([10, lemma 3.1]) *Let $\vec{P} = (p_1, p_2)$ with $1/p_1 + 1/p_2 = 1/p$ and $1 < p, p_1, p_2 < \infty$, $\vec{w} = (w_1, w_2) \in A_{\vec{P}}$. Then $\vec{w}^1 := (v_{\vec{w}}^{1-p'}, w_2) \in A_{\vec{P}^1}$, with $\vec{P}^1 = (p', p_2)$ and*

$$[\vec{w}^1]_{A_{\vec{P}^1}} = [\vec{w}]_{A_{\vec{P}}}^{p'_1/p}.$$

Lemma 3.2. ([11, lemma 4.5]) *Let $\vec{P} = (p_1, p_2)$ with $1/p_1 + 1/p_2 = 1/p$ and $1 < p, p_1, p_2 < \infty$, $\vec{w} = (w_1, w_2) \in A_{\vec{P}}$. Suppose that $\mathcal{D}$ is a dyadic cubes system and $\mathcal{S}$ is a sparse family in $\mathcal{D}$. Then the following assertions are equivalent.*

- (1) $\|A_{\mathcal{D}, \mathcal{S}}(|f_1| \sigma_1, |f_2| \sigma_2)\|_{L^{p, \infty}(v_{\vec{w}})} \leq C \prod_{i=1}^2 \|f_i\|_{L^{p_i}(\sigma_i)}.$
- (2) $\int_Q A_{\mathcal{D}, \mathcal{S}}(|f_1| \sigma_1 \chi_Q, |f_2| \sigma_2 \chi_Q) v_{\vec{w}} dx \leq C \prod_{i=1}^2 \|f_i\|_{L^{p_i}(\sigma_i)} v_{\vec{w}}(Q)^{1/p'}$ for all dyadic cubes $Q \in \mathcal{S}$ and all functions $f_i \in L^{p_i}(\sigma_i)$, $i = 1, 2$.

A careful read of [11, lemma 4.5] reveals that the constants C appearing in Lemma 3.2 are comparable.

Lemma 3.3. *Let $\vec{P} = (p_1, p_2)$ with $1/p_1 + 1/p_2 = 1/p$ and $1 < p, p_1, p_2 < \infty$, $\vec{w} = (w_1, w_2) \in A_{\vec{P}}$. Suppose that $\tilde{Q}$ is a dyadic cube and $\text{supp } f_2 \subset \tilde{Q}$, then*

$$\begin{aligned} \|\chi_{\tilde{Q}} A_{\mathcal{D}, \mathcal{S}}(\sigma_1 \chi_{\tilde{Q}}, |f_2| \sigma_2)\|_{L^p(v_{\vec{w}})} &\lesssim \max \left\{ \min \{[\sigma_1]_{A_{\infty}}, [\sigma_2]_{A_{\infty}}\}^{1/p}, \min \{[\sigma_1]_{A_{\infty}}, [v_{\vec{w}}]_{A_{\infty}}\}^{1/p'_2} \right\} \\ &\quad \times [\vec{w}]_{A_{\vec{P}}}^{1/p} \|f_2\|_{L^{p_2}(\sigma_2)} \sigma_1(\tilde{Q})^{1/p_1}. \end{aligned}$$

Suppose Lemma 3.3 is proven, referring to the method in [11], we can directly prove Theorem 1.1 as follows.

Proof of Theorem 1.1. Using Lemma 3.1 and Lemma 3.3, for each $Q \in \mathcal{S}$, we have

$$v_{\vec{w}}(Q)^{-1/p'} \int_Q A_{\mathcal{D}, \mathcal{S}}(|f_1| \sigma_1 \chi_Q, |f_2| \sigma_2 \chi_Q) v_{\vec{w}} dx$$

$$\begin{aligned}
&= v_{\vec{w}}(Q)^{-1/p'} \int_Q A_{\mathcal{Q},S}(v_{\vec{w}}\chi_Q, |f_2|\sigma_2\chi_Q) |f_1|\sigma_1 dx \\
&\leq v_{\vec{w}}(Q)^{-1/p'} \left(\int_Q (A_{\mathcal{Q},S}(v_{\vec{w}}\chi_Q, |f_2|\sigma_2\chi_Q))^{p'_1} \sigma_1 dx \right)^{1/p'_1} \left(\int_Q |f_1|^{p_1} \sigma_1 dx \right)^{1/p_1} \\
&\lesssim \max \left\{ \min \{ [v_{\vec{w}}]_{A_\infty}, [\sigma_2]_{A_\infty} \}^{\frac{1}{p'_1}}, \min \{ [v_{\vec{w}}]_{A_\infty}, [\sigma_1]_{A_\infty} \}^{\frac{1}{p'_2}} \right\} [\vec{w}^1]_{A_{\vec{P}}^1}^{\frac{1}{p'_1}} \|f_1\|_{L^{p_1}(\sigma_1)} \|f_2\|_{L^{p_2}(\sigma_2)} \\
(2.1) \quad &\leq [\vec{w}]_{A_{\vec{P}}}^{\frac{1}{p} + \max \left\{ \min \left\{ \frac{1}{p'_1}, \frac{1}{p'_1} \frac{p'_2}{p} \right\}, \min \left\{ \frac{1}{p'_2}, \frac{1}{p'_2} \frac{p'_1}{p} \right\} \right\}} \|f_1\|_{L^{p_1}(\sigma_1)} \|f_2\|_{L^{p_2}(\sigma_2)}.
\end{aligned}$$

Finally, according to Lemma 3.2, we get the desired result. This finishes the proof. $\square$

To prove Lemma 3.3, we need the following lemma.

Lemma 3.4. ([2, lemma 4.15]) *Let $\vec{P} = (p_1, p_2)$ with $1/p_1 + 1/p_2 = 1/p$ and $1 < p, p_1, p_2 < \infty$, $\vec{w} = (w_1, w_2) \in A_{\vec{P}}$. Then for any sparse family $\mathcal{S}$, we have*

$$(3.1) \quad \left\| \sum_{Q \in \mathcal{S}} \langle \sigma_1 \rangle_Q \langle \sigma_2 \rangle_Q \chi_Q \right\|_{L^p(v_{\vec{w}})} \lesssim [\vec{w}]_{A_{\vec{P}}}^{1/p} \left(\sum_{Q \in \mathcal{S}} \langle \sigma_1 \rangle_Q^{p/p_1} \langle \sigma_2 \rangle_Q^{p/p_2} |Q| \right)^{1/p},$$

$$(3.2) \quad \left\| \sum_{Q \in \mathcal{S}} \langle \sigma_1 \rangle_Q \langle v_{\vec{w}} \rangle_Q \chi_Q \right\|_{L^{p'_2}(\sigma_2)} \lesssim [\vec{w}]_{A_{\vec{P}}}^{1/p} \left(\sum_{Q \in \mathcal{S}} \langle \sigma_1 \rangle_Q^{p'_2/p_1} \langle v_{\vec{w}} \rangle_Q^{p'_2/p'_1} |Q| \right)^{1/p'_2}.$$

Proof of Lemma 3.3. In the first half of the proof, we will use a method similar to that in [2] and [11]. Since $\text{supp } f_2 \subset \tilde{Q}$, we have

$$\begin{aligned}
A_{\mathcal{Q},S}(\sigma_1\chi_{\tilde{Q}}, |f_2|\sigma_2) &= \sum_{\substack{Q \in \mathcal{S} \\ Q \cap \tilde{Q} \neq \emptyset}} \langle \sigma_1\chi_{\tilde{Q}} \rangle_Q \langle |f_2|\sigma_2 \rangle_Q \chi_Q \\
&= \sum_{\substack{Q \in \mathcal{S} \\ \tilde{Q} \subset Q}} \langle \sigma_1\chi_{\tilde{Q}} \rangle_Q \langle |f_2|\sigma_2 \rangle_Q \chi_Q + \sum_{\substack{Q \in \mathcal{S} \\ Q \subset \tilde{Q}}} \langle \sigma_1 \rangle_Q \langle |f_2|\sigma_2 \rangle_Q \chi_Q \\
&:= A_{\mathcal{Q},S}^1(\sigma_1\chi_{\tilde{Q}}, |f_2|\sigma_2) + A_{\mathcal{Q},S}^2(\sigma_1\chi_{\tilde{Q}}, |f_2|\sigma_2).
\end{aligned}$$

For $A_{\mathcal{Q},S}^1(\sigma_1\chi_{\tilde{Q}}, |f_2|\sigma_2)$, the calculation is not difficult,

$$\begin{aligned}
\left\| \chi_{\tilde{Q}} A_{\mathcal{Q},S}^1(\sigma_1\chi_{\tilde{Q}}, |f_2|\sigma_2) \right\|_{L^p(v_{\vec{w}})} &= \left\| \sum_{\tilde{Q} \subset Q} \frac{\sigma_1(Q \cap \tilde{Q}) \int_{\tilde{Q}} f_2(y_2) \sigma_2 dy_2}{|Q|^2} \chi_{\tilde{Q}} \right\|_{L^p(v_{\vec{w}})} \\
&\lesssim \left\| \frac{\sigma_1(\tilde{Q}) \int_{\tilde{Q}} f_2(y_2) \sigma_2 dy_2}{|\tilde{Q}|^2} \chi_{\tilde{Q}} \right\|_{L^p(v_{\vec{w}})} \\
&\leq \frac{\sigma_1(\tilde{Q}) \|f_2\|_{L^{p_2}(\sigma_2)} \sigma_2(\tilde{Q})^{1/p'_2}}{|\tilde{Q}|^2} v_{\vec{w}}(\tilde{Q})^{1/p} \\
&\leq [\vec{w}]_{A_{\vec{P}}}^{1/p} \|f_2\|_{L^{p_2}(\sigma_2)} \sigma_1(\tilde{Q})^{1/p_1}.
\end{aligned}$$

It remains to estimate $A_{\mathcal{D},\mathcal{S}}^2(\sigma_1\chi_{\tilde{Q}}, |f_2|\sigma_2)$. By duality, we have

$$\begin{aligned} \left\| A_{\mathcal{D},\mathcal{S}}^2(\sigma_1\chi_{\tilde{Q}}, |f_2|\sigma_2) \right\|_{L^p(v_{\tilde{w}})} &= \left\| \sum_{Q \subset \tilde{Q}} \langle f_2 \rangle_Q^{\sigma_2} \langle \sigma_1 \rangle_Q \langle \sigma_2 \rangle_Q \chi_Q \right\|_{L^p(v_{\tilde{w}})} \\ &= \sup_{\|h\|_{L^{p'}(v_{\tilde{w}})}=1} \sum_{Q \subset \tilde{Q}} \langle f_2 \rangle_Q^{\sigma_2} \langle \sigma_1 \rangle_Q \langle \sigma_2 \rangle_Q \int_Q h \, dv_{\tilde{w}} \\ &= \sup_{\|h\|_{L^{p'}(v_{\tilde{w}})}=1} \sum_{Q \subset \tilde{Q}} \langle f_2 \rangle_Q^{\sigma_2} \langle h \rangle_Q^{v_{\tilde{w}}} \langle \sigma_1 \rangle_Q \langle \sigma_2 \rangle_Q v_{\tilde{w}}(Q). \end{aligned}$$

Let $\mathcal{S}' = \mathcal{S} \cap \tilde{Q}$, then $\tilde{Q}$ is the maximal cube in the sparse family $\mathcal{S}'$ and we can use the stopping time argument mentioned above. Let $\mathcal{F}_2$ and $\mathcal{H}$ represent the stopping time family constructed by (f_2, σ_2) and $(h, v_{\tilde{w}})$ respectively, and write $\pi_{\mathcal{F}_2}(Q) = F_2$, and $\pi_{\mathcal{H}}(Q) = H$ together as $\pi(Q) = (F_2, H)$. Then,

$$\begin{aligned} \sum_{Q \in \mathcal{S}'} \langle f_2 \rangle_Q^{\sigma_2} \langle h \rangle_Q^{v_{\tilde{w}}} \langle \sigma_1 \rangle_Q \langle \sigma_2 \rangle_Q v_{\tilde{w}}(Q) &= \sum_{F_2 \in \mathcal{F}_2} \sum_{\substack{H \in \mathcal{H} \\ H \subset F_2}} \sum_{\substack{Q \in \mathcal{S}' \\ \pi(Q)=(F_2, H)}} \langle f_2 \rangle_Q^{\sigma_2} \langle h \rangle_Q^{v_{\tilde{w}}} \lambda_Q \\ &\quad + \sum_{H \in \mathcal{H}} \sum_{\substack{F_2 \in \mathcal{F}_2 \\ F_2 \subset H}} \sum_{\substack{Q \in \mathcal{S}' \\ \pi(Q)=(F_2, H)}} \langle f_2 \rangle_Q^{\sigma_2} \langle h \rangle_Q^{v_{\tilde{w}}} \lambda_Q \\ &:= I_1 + I_2, \end{aligned}$$

where $\lambda_Q = \langle \sigma_1 \rangle_Q \langle \sigma_2 \rangle_Q v_{\tilde{w}}(Q)$.

For I_1 , we have

$$\begin{aligned} I_1 &\leq 4 \sum_{F_2 \in \mathcal{F}_2} \langle f_2 \rangle_{F_2}^{\sigma_2} \sum_{\substack{H \in \mathcal{H} \\ H \subset F_2}} \langle h \rangle_H^{v_{\tilde{w}}} \sum_{\substack{Q \in \mathcal{S}' \\ \pi(Q)=(F_2, H)}} \lambda_Q \\ &\lesssim \sum_{F_2 \in \mathcal{F}_2} \langle f_2 \rangle_{F_2}^{\sigma_2} \int_{F_2} \sum_{\substack{H \in \mathcal{H} \\ H \subset F_2}} \langle h \rangle_H^{v_{\tilde{w}}} \sum_{\substack{Q \in \mathcal{S}' \\ \pi(Q)=(F_2, H)}} \frac{\lambda_Q \chi_Q}{v_{\tilde{w}}(Q)} \, dv_{\tilde{w}} \\ &\lesssim \sum_{F_2 \in \mathcal{F}_2} \langle f_2 \rangle_{F_2}^{\sigma_2} \int_{F_2} \left(\sup_{\substack{H' \in \mathcal{H} \\ \pi_{\mathcal{F}_2}(H')=F_2}} \langle h \rangle_{H'}^{v_{\tilde{w}}} \chi_{H'} \right) \sum_{\substack{H \in \mathcal{H} \\ H \subset F_2}} \sum_{\substack{Q \in \mathcal{S}' \\ \pi(Q)=(F_2, H)}} \frac{\lambda_Q \chi_Q}{v_{\tilde{w}}(Q)} \, dv_{\tilde{w}} \\ &\lesssim \sum_{F_2 \in \mathcal{F}_2} \langle f_2 \rangle_{F_2}^{\sigma_2} \left\| \sum_{\substack{H \in \mathcal{H} \\ H \subset F_2}} \sum_{\substack{Q \in \mathcal{S}' \\ \pi(Q)=(F_2, H)}} \frac{\lambda_Q \chi_Q}{v_{\tilde{w}}(Q)} \right\|_{L^p(v_{\tilde{w}})} \left\| \sup_{\substack{H' \in \mathcal{H} \\ \pi_{\mathcal{F}_2}(H')=F_2}} \langle h \rangle_{H'}^{v_{\tilde{w}}} \chi_{H'} \right\|_{L^{p'}(v_{\tilde{w}})} \\ &\leq \left(\sum_{F_2 \in \mathcal{F}_2} \left(\langle f_2 \rangle_{F_2}^{\sigma_2} \right)^p \left\| \sum_{\substack{H \in \mathcal{H} \\ H \subset F_2}} \sum_{\substack{Q \in \mathcal{S}' \\ \pi(Q)=(F_2, H)}} \frac{\lambda_Q \chi_Q}{v_{\tilde{w}}(Q)} \right\|_{L^p(v_{\tilde{w}})}^p \right)^{\frac{1}{p}} \\ &\quad \times \left(\sum_{F_2 \in \mathcal{F}_2} \sum_{\substack{H' \in \mathcal{H} \\ \pi_{\mathcal{F}_2}(H')=F_2}} \left(\langle h \rangle_{H'}^{v_{\tilde{w}}} \right)^{p'} v_{\tilde{w}}(H') \right)^{\frac{1}{p'}} \end{aligned}$$

$$\lesssim \left(\sum_{F_2 \in \mathcal{F}_2} \left(\langle f_2 \rangle_{F_2}^{\sigma_2} \right)^p \left\| \sum_{\substack{H \in \mathcal{H} \\ H \subset F_2}} \sum_{\substack{Q \in \mathcal{S}' \\ \pi(Q) = (F_2, H)}} \frac{\lambda_Q \chi_Q}{v_{\vec{w}}(Q)} \right\|_{L^p(v_{\vec{w}})}^p \right)^{\frac{1}{p}}.$$

The last inequality is due to (2.2). By (3.1), we have

$$\begin{aligned} \left\| \sum_{\substack{H \in \mathcal{H} \\ H \subset F_2}} \sum_{\substack{Q \in \mathcal{S}' \\ \pi(Q) = (F_2, H)}} \frac{\lambda_Q \chi_Q}{v_{\vec{w}}(Q)} \right\|_{L^p(v_{\vec{w}})} &= \left\| \sum_{\substack{Q \in \mathcal{S}' \\ \pi_{\mathcal{F}_2}(Q) = F_2}} \frac{\lambda_Q \chi_Q}{v_{\vec{w}}(Q)} \right\|_{L^p(v_{\vec{w}})} \\ &\lesssim [\vec{w}]_{A_{\vec{P}}}^{\frac{1}{p}} \left(\sum_{\substack{Q \in \mathcal{S}' \\ \pi_{\mathcal{F}_2}(Q) = F_2}} \langle \sigma_1 \rangle_Q^{\frac{p}{p_1}} \langle \sigma_2 \rangle_Q^{\frac{p}{p_2}} |Q| \right)^{\frac{1}{p}}. \end{aligned}$$

Let $\varepsilon = \frac{1}{2^{11+d}[\sigma_1]_{A_\infty}}$. Hytönen and Pérez [5] proved the reverse Hölder inequality

$$\langle \sigma_1^{1+\varepsilon} \rangle_Q \lesssim \langle \sigma_1 \rangle_Q^{1+\varepsilon}, \quad \forall Q \subset \mathbb{R}^n.$$

Let $\gamma := \frac{p}{p_1} \frac{1}{1+\varepsilon}$, $\eta := \frac{p}{p_2}$, $\frac{1}{r} := \gamma + \eta$, $\frac{1}{s} := \gamma + \frac{1}{2}(1 - \frac{1}{r})$, $\frac{1}{s'} := 1 - \frac{1}{s}$. We have

$$\begin{aligned} I_1 &\lesssim [\vec{w}]_{A_{\vec{P}}}^{\frac{1}{p}} \left(\sum_{F_2 \in \mathcal{F}_2} \left(\langle f_2 \rangle_{F_2}^{\sigma_2} \right)^p \sum_{\substack{Q \in \mathcal{S}' \\ \pi_{\mathcal{F}_2}(Q) = F_2}} \langle \sigma_1 \rangle_Q^{\frac{p}{p_1}} \langle \sigma_2 \rangle_Q^{\frac{p}{p_2}} |Q| \right)^{\frac{1}{p}} \\ &\leq [\vec{w}]_{A_{\vec{P}}}^{\frac{1}{p}} \left(\sum_{F_2 \in \mathcal{F}_2} \left(\langle f_2 \rangle_{F_2}^{\sigma_2} \right)^p \sum_{\substack{Q \in \mathcal{S}' \\ \pi_{\mathcal{F}_2}(Q) = F_2}} \langle \sigma_1^{1+\varepsilon} \rangle_Q^\gamma \langle \sigma_2 \rangle_Q^\eta |Q| \right)^{\frac{1}{p}} \\ &\leq [\vec{w}]_{A_{\vec{P}}}^{\frac{1}{p}} \left(\sum_{F_2 \in \mathcal{F}_2} \left(\langle f_2 \rangle_{F_2}^{\sigma_2} \right)^p \left(\sum_{\substack{Q \in \mathcal{S}' \\ \pi_{\mathcal{F}_2}(Q) = F_2}} \langle \sigma_1^{1+\varepsilon} \rangle_Q^{s\gamma} |Q| \right)^{\frac{1}{s}} \left(\sum_{\substack{Q \in \mathcal{S}' \\ \pi_{\mathcal{F}_2}(Q) = F_2}} \langle \sigma_2 \rangle_Q^{s'\eta} |Q| \right)^{\frac{1}{s'}} \right)^{\frac{1}{p}} \\ &\leq [\vec{w}]_{A_{\vec{P}}}^{\frac{1}{p}} \left(\sum_{F_2 \in \mathcal{F}_2} \sum_{\substack{Q \in \mathcal{S}' \\ \pi_{\mathcal{F}_2}(Q) = F_2}} \langle \sigma_1^{1+\varepsilon} \rangle_Q^{s\gamma} |Q| \right)^{\frac{1}{sp}} \times \left(\sum_{F_2 \in \mathcal{F}_2} \left(\langle f_2 \rangle_{F_2}^{\sigma_2} \right)^{s'p} \sum_{\substack{Q \in \mathcal{S}' \\ \pi_{\mathcal{F}_2}(Q) = F_2}} \langle \sigma_2 \rangle_Q^{s'\eta} |Q| \right)^{\frac{1}{s'p}} \\ &:= [\vec{w}]_{A_{\vec{P}}}^{\frac{1}{p}} J_1 \times J_2. \end{aligned}$$

Since $\mathcal{S}'$ is sparse, for J_1 , we have

$$\begin{aligned} J_1 &\lesssim \left(\sum_{F_2 \in \mathcal{F}_2} \sum_{\substack{Q \in \mathcal{S}' \\ \pi_{\mathcal{F}_2}(Q) = F_2}} \langle \sigma_1^{1+\varepsilon} \rangle_Q^{s\gamma} |E_Q| \right)^{\frac{1}{sp}} \\ &\leq \left(\int_{\tilde{Q}} (M(\sigma_1^{1+\varepsilon} \chi_{\tilde{Q}}))^{s\gamma} dx \right)^{\frac{1}{sp}} \\ &= \|M(\sigma_1^{1+\varepsilon} \chi_{\tilde{Q}})\|_{L^{s\gamma}(\frac{dx}{|\tilde{Q}|})}^{\frac{\gamma}{p}} |\tilde{Q}|^{\frac{1}{sp}} \end{aligned}$$

$$\begin{aligned}
&\leq [\sigma_1]_{A_\infty}^{\frac{1}{sp}} \left\| M(\sigma_1^{1+\varepsilon} \chi_{\tilde{Q}}) \right\|_{L^{1,\infty}(\frac{dx}{|\tilde{Q}|})}^{\frac{\gamma}{p}} |\tilde{Q}|^{\frac{1}{sp}} \\
&\lesssim [\sigma_1]_{A_\infty}^{\frac{1}{sp}} \langle \sigma_1^{1+\varepsilon} \rangle_{\tilde{Q}}^{\frac{\gamma}{p}} |\tilde{Q}|^{\frac{1}{sp}} \lesssim [\sigma_1]_{A_\infty}^{\frac{1}{sp}} \langle \sigma_1 \rangle_{\tilde{Q}}^{\frac{p_1}{p}} |\tilde{Q}|^{\frac{1}{sp}}.
\end{aligned}$$

The third inequality in the estimate above is due to Kolmogorov's inequality, that is, for any cube Q in $\mathbb{R}^n$ and $f \in L^{1,\infty}(Q)$,

$$\|f\|_{L^p(\frac{dx}{|\tilde{Q}|})} \leq \left(\frac{1}{p} + \frac{1}{1-p} \right)^{\frac{1}{p}} \|f\|_{L^{1,\infty}(\frac{dx}{|\tilde{Q}|})}, \quad 0 < p < 1.$$

Specifically,

$$\left(\frac{1}{s\gamma} + \frac{1}{1-s\gamma} \right)^{\frac{1}{sp}} = \left(\frac{1}{1 - \frac{s}{2} \frac{\varepsilon}{1+\varepsilon} \frac{p}{p_1}} + \frac{2}{s} \frac{1+\varepsilon}{\varepsilon} \frac{p_1}{p} \right)^{\frac{1}{sp}} \lesssim [\sigma_1]_{A_\infty}^{\frac{1}{sp}}.$$

For J_2 , using the same method as for J_1 and (2.2), we obtain

$$\begin{aligned}
J_2 &\lesssim \left(\sum_{F_2 \in \mathcal{F}_2} \left(\langle f_2 \rangle_{F_2}^{\sigma_2} \right)^{s'p} [\sigma_1]_{A_\infty} \langle \sigma_2 \rangle_{F_2}^{s'\eta} |F_2| \right)^{\frac{1}{s'p}} \\
&\leq [\sigma_1]_{A_\infty}^{\frac{1}{s'p}} \left(\sum_{F_2 \in \mathcal{F}_2} \left(\langle f_2 \rangle_{F_2}^{\sigma_2} \right)^{p_2} \langle \sigma_2 \rangle_{F_2} |F_2| \right)^{\frac{1}{p_2}} \left(\sum_{F_2 \in \mathcal{F}_2} |F_2| \right)^{\frac{1}{s'p} - \frac{1}{p_2}} \\
&\lesssim [\sigma_1]_{A_\infty}^{\frac{1}{s'p}} \|f_2\|_{L^{p_2}(\sigma_2)} |\tilde{Q}|^{\frac{1}{s'p} - \frac{1}{p_2}}.
\end{aligned}$$

If we apply the reverse Hölder inequality for σ_2 , we can obtain another bound similarly. Therefore, we get

$$I_1 \lesssim [\tilde{w}]_{A_{\tilde{P}}}^{1/p} \min\{[\sigma_1]_{A_\infty}, [\sigma_2]_{A_\infty}\}^{1/p} \sigma_1(\tilde{Q})^{1/p_1} \|f_2\|_{L^{p_2}(\sigma_2)}.$$

The estimation of I_2 is similar to I_1 , only by replacing formula (3.1) with (3.2). By combining the above estimates of I_1 and I_2 , we conclude the proof of the theorem. $\square$

At the end of this section, we use Python to draw a graph to compare the weak type estimate we obtained with the sharp strong type estimate (1.1). In particular, we show that when $p \geq \frac{3+\sqrt{5}}{2}$ or $\min\{p_1, p_2\} > 4$, the exponent we obtained is smaller than 1.

Without loss of generality, we assume $p_1 \leq p_2$ in the following calculations.

- When $p \geq \frac{3+\sqrt{5}}{2}$, it is obvious that $p'_1 \leq p$. In this case, the exponent in Theorem 1.1 is $\frac{1}{p} + \frac{1}{p'_2} \frac{p'_1}{p}$. If it is greater than or equal to 1, we obtain

$$\frac{1}{p} + \frac{1}{p'_2} \frac{p'_1}{p} \geq 1 \Rightarrow \frac{p'_1}{p'_2} \geq p-1 \Rightarrow p'_1 \geq p-1 \Rightarrow \frac{1}{p_1} \geq \frac{p-2}{p-1}.$$

Since $p \geq \frac{3+\sqrt{5}}{2}$, we have $\frac{p-2}{p-1} \geq \frac{1}{p}$, which leads to a contradiction.

- When $\min\{p_1, p_2\} > 4$, we can also obtain $p'_1 \leq p$, thus,

$$\frac{1}{p} + \frac{1}{p'_2} \frac{p'_1}{p} = \frac{p'_1}{p} \left(2 - \frac{1}{p} \right) < 1 \Leftrightarrow \frac{2}{p} - \frac{1}{p^2} < \frac{1}{p'_1},$$

and this holds automatically since the left-hand side is always less than $\frac{3}{4}$, while the right-hand side is greater than it.

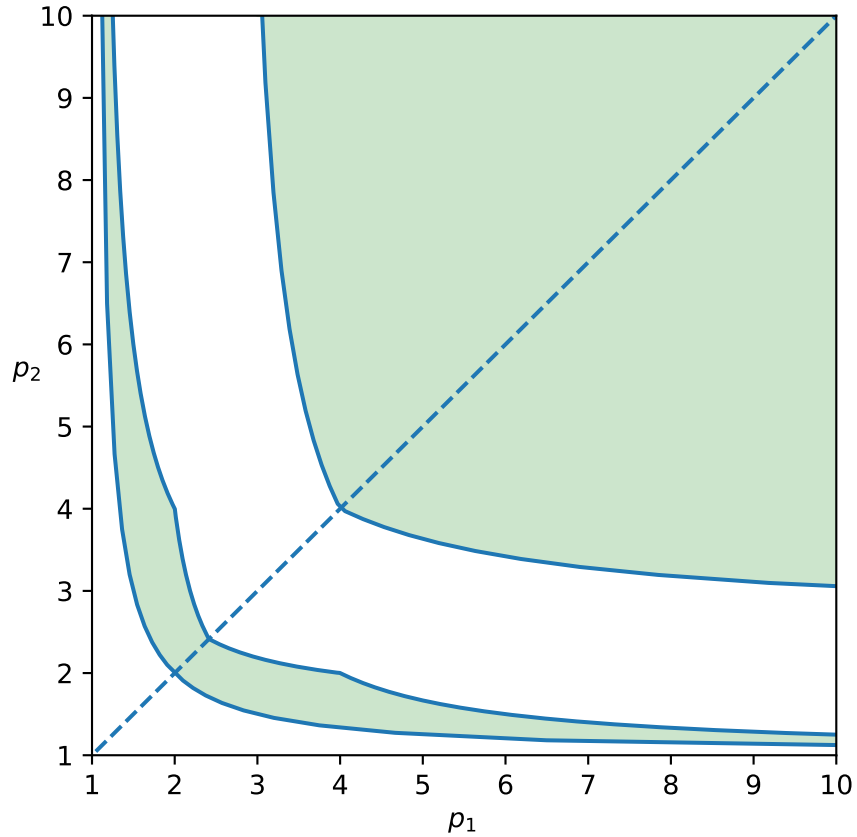


FIGURE 1. Compared to the sharp strong type estimate, our results are better in shaded areas.

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