

Path Dependent McKean-Vlasov SDEs with Hölder Continuous Diffusion*

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Abstract

In this paper, the well-posedness for one-dimensional path dependent McKean-Vlasov SDEs with $\alpha(\alpha \geq \frac{1}{2})$ -Hölder continuous diffusion is investigated. Moreover, the associated quantitative propagation of chaos in the sense of Wasserstein distance, total variation distance as well as relative entropy is studied.

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1 Introduction

Distribution dependent SDEs can be used to characterize the nonlinear Fokker-Planck-Kolmogorov equations. They are also called McKean-Vlasov SDEs due to the pioneer work in [18]. On the other hand, McKean-Vlasov SDE can be viewed as the limit equation of a single particle in the mean field interacting particle system, which is related to the propagation of chaos [24], so it is also called mean field SDE. Recently, there are plentiful results on McKean-Vlasov SDEs. With respect to the well-posedness, one can refer to [1, 4, 5, 12, 13, 19, 23, 25] and references therein, see also [14] for the path dependent case with singular drifts. In [4, 5, 12, 13, 23], the diffusion is assumed to be uniformly elliptic. For the propagation of chaos, see [2, 3, 6, 8, 9, 11, 15, 17, 24, 27]. One can also refer to [10, 22, 26] for the long time behavior of mean field interacting particle system and McKean-Vlasov SDEs.

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The aim of this paper is to investigate the well-posedness and propagation of chaos of one-dimensional path dependent McKean-Vlasov SDEs with $\alpha(\alpha \geq \frac{1}{2})$ -Hölder continuous diffusion. With respect to the well-posedness, we do not assume that the diffusion is elliptic.

Throughout the paper, fix a constant $r > 0$. Let $\mathcal{C} = C([-r, 0]; \mathbb{R})$, the continuous map from $[-r, 0]$ to $\mathbb{R}$. For any $f \in C([-r, \infty); \mathbb{R})$, $t \geq 0$, define $f_t \in \mathcal{C}$ as $f_t(s) = f(t + s)$, $s \in [-r, 0]$, which is called the segment process. Let $\mathcal{P}(\mathbb{R})$ be the set of all probability measures in $\mathbb{R}$ equipped with the weak topology. Define

$$\mathcal{P}_1(\mathbb{R}) = \{\mu \in \mathcal{P}(\mathbb{R}) : \mu(|\cdot|) < \infty\}.$$

It is well known that $\mathcal{P}_1(\mathbb{R})$ is a Polish space under the L^1 -Wasserstein distance

$$\mathbb{W}_1(\mu, \nu) := \inf_{\pi \in \mathbf{C}(\mu, \nu)} \left(\int_{\mathbb{R} \times \mathbb{R}} |x - y| \pi(dx, dy) \right), \quad \mu, \nu \in \mathcal{P}_1(\mathbb{R}),$$

where $\mathbf{C}(\mu, \nu)$ is the set of all couplings of μ and ν . By the adjoint formula, it holds

$$\mathbb{W}_1(\mu, \nu) = \sup_{\|f\|_{\text{Lip}} \leq 1} |\mu(f) - \nu(f)|,$$

where

$$\|f\|_{\text{Lip}} := \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|}.$$

Recall that for two probability measures μ, ν on some measurable space $(E, \mathcal{E})$, the entropy and total variation distance are defined as follows:

$$\text{Ent}(\nu|\mu) := \begin{cases} \int_E (\log \frac{d\nu}{d\mu}) d\nu, & \text{if } \nu \text{ is absolutely continuous with respect to } \mu, \\ \infty, & \text{otherwise,} \end{cases}$$

and

$$\|\mu - \nu\|_{\text{var}} := \sup_{\{f: |f(x)| \leq 1, x \in E\}} |\mu(f) - \nu(f)|.$$

By Pinsker's inequality (see [20]),

$$(1.1) \quad \|\mu - \nu\|_{\text{var}}^2 \leq 2\text{Ent}(\nu|\mu), \quad \mu, \nu \in \mathcal{P}(E),$$

here $\mathcal{P}(E)$ denotes all probability measures on $(E, \mathcal{E})$.

Let $W = (W(t))_{t \geq 0}$ be a one-dimensional standard Brownian motion on a complete filtration probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$. Consider the following one-dimensional path dependent McKean-Vlasov SDE:

$$(1.2) \quad dX(t) = b(t, X(t), \mathcal{L}_{X(t)})dt + B(t, X_t, \mathcal{L}_{X(t)})dt + \sigma(t, X(t))dW(t),$$

where $b : [0, \infty) \times \mathbb{R} \times \mathcal{P}(\mathbb{R}) \rightarrow \mathbb{R}$, $B : [0, \infty) \times \mathcal{C} \times \mathcal{P}(\mathbb{R}) \rightarrow \mathbb{R}$, $\sigma : [0, \infty) \times \mathbb{R} \rightarrow \mathbb{R}$ are measurable and X_0 is an $\mathcal{F}_0$ -measurable $\mathcal{C}$ -valued random variable. Define the uniform norm $\|\xi\|_\infty := \sup_{s \in [-r, 0]} |\xi(s)|$, $\xi \in \mathcal{C}$.

Definition 1.1. A continuous process $(X(t))_{t \geq -r}$ on $\mathbb{R}$ is called a strong solution of (1.2) in $\mathcal{P}_1(\mathbb{R})$, if for any $t \geq 0$, $X(t)$ is $\mathcal{F}_t$ -measurable, $\mathbb{E}\|X_t\|_\infty < \infty$, and $\mathbb{P}$ -a.s.

$$X(t) = X(0) + \int_0^t (b(s, X(s), \mathcal{L}_{X(s)}) + B(s, X_s, \mathcal{L}_{X(s)}))ds + \int_0^t \sigma(s, X(s))dW(s), \quad t \geq 0.$$

Throughout the paper, we fix $T > 0$, and consider the solution on $[0, T]$.

The remainder of this paper is organized as follows: In Section 2, the strong well-posedness of path dependent classical SDEs is addressed by Yamada-Watanabe's approximation; In Section 3, the well-posedness and quantitative propagation of chaos for path dependent McKean-Vlasov SDEs are investigated.

2 Multi-dimensional Path Dependent Classical SDEs with Hölder Continuous Diffusion

For any $n \geq 1$, let $\mathcal{C}^n = C([-r, 0]; \mathbb{R}^n)$, the continuous map from $[-r, 0]$ to $\mathbb{R}^n$, equipped with the uniform norm. In this section, we study the well-posedness of multi-dimensional path dependent classical SDEs with Hölder continuous diffusion. The main tool is the Yamada-Watanabe approximation, see [16]. Before going on, we state a time nonhomogeneous version of [21, Theorem 2.3]. More precisely, consider path dependent SDE on $\mathbb{R}^m$:

$$(2.1) \quad dX(t) = f(t, X_t)dt + g(t, X_t)d\bar{W}(t), \quad X_0 = \xi \in \mathcal{C}^m,$$

here $f : [0, T] \times \mathcal{C}^m \rightarrow \mathbb{R}^m$, $g : [0, T] \times \mathcal{C}^m \rightarrow \mathbb{R}^m \otimes \mathbb{R}^d$ and $\bar{W}(t)$ is a d -dimensional standard Brownian motion on a complete filtration probability space $(\bar{\Omega}, \bar{\mathcal{F}}, \{\bar{\mathcal{F}}_t\}_{t \in [0, T]}, \bar{\mathbb{P}})$.

Proposition 2.1. *Assume that f and g are bounded on bounded sets. Suppose that for any $t \in [0, T]$, $f(t, \cdot)$ and $g(t, \cdot)$ are continuous in $(\mathcal{C}^m, \|\cdot\|_\infty)$ and there exists a constant $K \in \mathbb{R}$ such that*

$$2\langle f(t, \xi) - f(t, \eta), \xi(0) - \eta(0) \rangle + \|g(t, \xi) - g(t, \eta)\|_{HS}^2 \leq K\|\xi - \eta\|_\infty^2, \quad t \in [0, T], \xi, \eta \in \mathcal{C}^m.$$

Then (2.1) has a unique non-explosive strong solution on $[0, T]$.

Since the proof of Proposition 2.1 is completely the same with that of [21, Theorem 2.3], we omit it here.

The following stochastic Grönwall lemma comes from [21, Lemma 5.2], which is crucial in the proof of the main result of this section.

Lemma 2.2. *Let Z be a continuous adapted non-negative stochastic process which satisfies the inequality*

$$Z(t) \leq K \int_0^t \sup_{u \in [0, s]} Z(u)ds + M(t) + C, \quad t \geq 0,$$

where $C \geq 0$, $K > 0$ and M is a continuous local martingale with $M(0) = 0$. Then for any $p \in (0, 1)$, there exist finite constants $c_1(p), c_2(p)$ (not depending on K, C, T and M) such that

$$\mathbb{E}(\sup_{s \in [0, t]} |Z(s)|^p) \leq C^p c_1(p) e^{c_2(p) K t}, \quad t \geq 0.$$

For $x \in \mathbb{R}^d$, we denote x_i as the i -th component of x , that is $x = (x_1, x_2, \dots, x_d)$. Consider

$$(2.2) \quad dX(t) = F(t, X(t))dt + H(t, X_t)dt + G(t, X(t))d\bar{W}(t), \quad X_0 = \xi \in \mathcal{C}^d,$$

where $F = (F_1, F_2, \dots, F_d) : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$, $H = (H_1, H_2, \dots, H_d) : [0, T] \times \mathcal{C}^d \rightarrow \mathbb{R}^d$, $G : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^d \otimes \mathbb{R}^d$. We make the following assumptions.

(A1) F is locally bounded in $[0, T] \times \mathbb{R}^d$. For any $t \in [0, T]$, $F(t, \cdot)$ is continuous, and there exists a constant $K_1 \geq 0$ such that

$$(F_i(t, x) - F_i(t, y)) \operatorname{sgn}(x_i - y_i) \leq K_1 |x - y|, \quad t \in [0, T], x, y \in \mathbb{R}^d, 1 \leq i \leq d,$$

where $\operatorname{sgn}(\cdot)$ means the sign function.

(A2) There exist m real valued functions $(G_i)_{1 \leq i \leq d}$ on $[0, T] \times \mathbb{R}$ such that

$$G(t, x) = \operatorname{diag}(G_1(t, x_1), G_2(t, x_2), \dots, G_d(t, x_d)), \quad x = (x_1, x_2, \dots, x_d) \in \mathbb{R}^d, t \in [0, T].$$

Moreover, there exist constants $(\alpha_i)_{1 \leq i \leq d} \subset [\frac{1}{2}, 1]$ and $K_2 \geq 0$ such that

$$|G_i(t, z) - G_i(t, \bar{z})| \leq K_2 |z - \bar{z}|^{\alpha_i}, \quad |G_i(t, 0)| \leq K_2, \quad z, \bar{z} \in \mathbb{R}, t \in [0, T], 1 \leq i \leq d.$$

(A3) There exists a constant $K_3 \geq 0$ such that

$$|H(t, \xi) - H(t, \eta)| \leq K_3 \|\xi - \eta\|_\infty, \quad |H(t, 0)| \leq K_3, \quad \xi, \eta \in \mathcal{C}^d, t \in [0, T].$$

Now, we provide the main result in this section.

Theorem 2.3. *Assume (A1)-(A3). Then for any $\xi \in \mathcal{C}^d$, (2.2) has a unique strong solution $(X^\xi(t))_{t \in [-r, T]}$ with initial value $X_0^\xi = \xi$ and for any $q \in (0, 2]$, there exists a constant $C(T, q) > 0$ such that*

$$(2.3) \quad \mathbb{E} \sup_{t \in [-r, T]} |X^\xi(t)|^q \leq C(T, q)(1 + \|\xi\|_\infty^q).$$

Moreover, for any $p \in (0, 1)$,

$$(2.4) \quad \mathbb{E} \sup_{t \in [-r, T]} |X^\xi(t) - X^\eta(t)|^p \leq C(p, T) \|\xi - \eta\|_\infty^p$$

for some constant $C(p, T) > 0$.

Proof. Step 1. Existence of the strong solution.

For $\varepsilon \in (0, 1)$, note $\int_{\varepsilon/e^{\frac{1}{\varepsilon}}}^{\varepsilon} \frac{\varepsilon}{x} dx = 1$, so there exists a continuous function $\psi_\varepsilon : [0, \infty) \rightarrow [0, \infty)$ with support $[\varepsilon/e^{\frac{1}{\varepsilon}}, \varepsilon]$ such that

$$0 \leq \psi_\varepsilon(x) \leq \frac{2\varepsilon}{x}, \quad x \in [\varepsilon/e^{\frac{1}{\varepsilon}}, \varepsilon], \quad \int_{\varepsilon/e^{\frac{1}{\varepsilon}}}^{\varepsilon} \psi_\varepsilon(u) du = 1.$$

Let

$$V_\varepsilon(x) := \int_0^{|x|} \int_0^y \psi_\varepsilon(z) dz dy, \quad x \in \mathbb{R}.$$

Then $V_\varepsilon \in C^2$,

$$(2.5) \quad |x| - \varepsilon \leq V_\varepsilon(x) \leq |x|, \quad 0 \leq \operatorname{sgn}(x) V'_\varepsilon(x) \leq 1, \quad x \in \mathbb{R},$$

and

$$(2.6) \quad 0 \leq V''_\varepsilon(x) \leq \frac{2\varepsilon}{|x|} \mathbf{1}_{[\varepsilon/e^{\frac{1}{\varepsilon}}, \varepsilon]}(|x|), \quad x \in \mathbb{R}.$$

Let $\rho \in C_0^\infty(\mathbb{R})$ with $\rho \geq 0$ and $\int_{\mathbb{R}} \rho(x) dx = 1$ be supported in $[-1, 1]$. For any $n \geq 1$, define $\rho^n(x) = n\rho(nx)$, $x \in \mathbb{R}$ and let

$$G_i^n(t, \cdot) = G_i(t, \cdot) * \rho^n, \quad 1 \leq i \leq d, t \in [0, T]$$

and

$$G^n(t, x) = \operatorname{diag}(G_1^n(t, x_1), G_2^n(t, x_2), \dots, G_d^n(t, x_d)), \quad x = (x_1, x_2, \dots, x_d) \in \mathbb{R}^d, t \in [0, T].$$

(A2) implies that

$$(2.7) \quad \begin{aligned} & \lim_{n \rightarrow \infty} \sup_{t \in [0, T], x \in \mathbb{R}} |G_i^n(t, x) - G_i(t, x)| \\ & \leq \lim_{n \rightarrow \infty} \sup_{t \in [0, T], x \in \mathbb{R}} \int_{\mathbb{R}} \rho^n(y) |G_i(t, x - y) - G_i(t, x)| dy \\ & \leq K_2 \lim_{n \rightarrow \infty} \int_{\mathbb{R}} |y|^{\alpha_i} \rho^n(y) dy \\ & = K_2 \lim_{n \rightarrow \infty} \int_{\mathbb{R}} \frac{1}{n^{\alpha_i}} |x|^{\alpha_i} \rho(x) dx = 0, \quad 1 \leq i \leq d \end{aligned}$$

and for any $n \geq 1$, there exists a constant $L_n \geq 0$ such that

$$(2.8) \quad |G^n(t, x) - G^n(t, y)| \leq L_n |x - y|, \quad |G^n(t, 0)| \leq 2\sqrt{d}K_2, \quad t \in [0, T], x, y \in \mathbb{R}^d,$$

where for the second inequality, it is sufficient to note that

$$|G_i^n(t, 0)| \leq \int_{\mathbb{R}} |G_i(t, -y)| n\rho(ny) dy \leq K_2 \int_{\mathbb{R}} |y|^{\alpha_i} n\rho(ny) dy + K_2 \leq 2K_2, \quad 1 \leq i \leq d.$$

According to Proposition 2.1, it follows from **(A1)**, **(A3)** and (2.8) that for any $n \geq 1$, the SDE

$$(2.9) \quad dX^n(t) = F(t, X^n(t))dt + H(t, X_t^n)dt + G^n(t, X^n(t))d\bar{W}(t), \quad X_0^n = \xi$$

has a unique non-explosive strong solution. Moreover, in view of the second inequality of (2.8) and

$$(2.10) \quad |G_i^n(t, z) - G_i^n(t, \bar{z})| \leq K_2|z - \bar{z}|^{\alpha_i}, \quad t \in [0, T], z, \bar{z} \in \mathbb{R}, 1 \leq i \leq d, n \geq 1,$$

this together with **(A1)** and **(A3)** implies that there exists a constant $C(T) > 0$ such that

$$(2.11) \quad \sup_{n \geq 1} \mathbb{E} \sup_{t \in [-r, T]} |X^n(t)|^2 \leq C(T)(1 + \|\xi\|_\infty^2).$$

In fact, by **(A1)**, **(A3)**, (2.10) and the second inequality in (2.8), it holds

$$(2.12) \quad \begin{aligned} \langle F(t, x), x \rangle &\leq C(T)(1 + |x|^2), \quad |H(t, \xi)| \leq C(T)(1 + \|\xi\|_\infty), \\ |G^n(t, x)| &\leq C(T)(1 + |x|), \quad x \in \mathbb{R}^d, t \in [0, T], \xi \in \mathcal{C}^d, n \geq 1. \end{aligned}$$

For each integer $N \geq 1$, define $\tau_N^n = \inf \{t \in [0, T] : |X^n(t)| \geq N\}$ and $\inf \emptyset = \infty$ by convention. By Itô's formula, we derive from (2.12) that

$$\begin{aligned} |X^n(t \wedge \tau_N^n)|^2 &\leq \xi(0)^2 + C + C \int_0^{t \wedge \tau_N^n} \sup_{u \in [-r, s]} |X^n(u)|^2 ds \\ &\quad + 2 \int_0^{t \wedge \tau_N^n} \langle X^n(s), G^n(s, X^n(s)) d\bar{W}(s) \rangle \end{aligned}$$

for some constant $C > 0$. Applying BDG's inequality, (2.12) and Grönwall's inequality, it is standard to derive (2.11). For any $1 \leq i \leq d$, let $X^{n,i}$ be the i -th component of X^n . For any $m, n \geq 1$, it follows from Itô's formula that

$$\begin{aligned} V_\varepsilon(X^{m,i}(t) - X^{n,i}(t)) &= \int_0^t V'_\varepsilon(X^{m,i}(s) - X^{n,i}(s))(F_i(s, X^m(s)) - F_i(s, X^n(s)))ds \\ &\quad + \int_0^t V'_\varepsilon(X^{m,i}(s) - X^{n,i}(s))(H_i(s, X_s^m) - H_i(s, X_s^n))ds \\ &\quad + \frac{1}{2} \int_0^t V''_\varepsilon(X^{m,i}(s) - X^{n,i}(s))(G_i^m(s, X^{m,i}(s)) - G_i^n(s, X^{n,i}(s)))^2 ds \\ &\quad + \int_0^t V'_\varepsilon(X^{m,i}(s) - X^{n,i}(s))(G_i^m(s, X^{m,i}(s)) - G_i^n(s, X^{n,i}(s)))d\bar{W}^i(s). \end{aligned}$$

This combined with (2.5) implies that for any $1 \leq i \leq d$,

$$\sum_{i=1}^d |X^{m,i}(t) - X^{n,i}(t)| - d\varepsilon$$

$$\begin{aligned}
(2.13) \quad & \leq \int_0^t \sum_{i=1}^d V'_\varepsilon(X^{m,i}(s) - X^{n,i}(s))(F_i(s, X^m(s)) - F_i(s, X^n(s)))ds \\
& + \int_0^t \sum_{i=1}^d V'_\varepsilon(X^{m,i}(s) - X^{n,i}(s))(H_i(s, X_s^m) - H_i(s, X_s^n))ds \\
& + \frac{1}{2} \int_0^t \sum_{i=1}^d V''_\varepsilon(X^{m,i}(s) - X^{n,i}(s))(G_i^m(s, X^{m,i}(s)) - G_i^n(s, X^{n,i}(s)))^2 ds \\
& + \int_0^t \sum_{i=1}^d V'_\varepsilon(X^{m,i}(s) - X^{n,i}(s))(G_i^m(s, X^{m,i}(s)) - G_i^n(s, X^{n,i}(s)))d\bar{W}^i(s) \\
& =: J_1^{m,n} + J_2^{m,n} + J_3^{m,n} + J_4^{m,n}, \quad t \in [0, T].
\end{aligned}$$

By (2.5) and **(A1)**, we conclude that

$$\begin{aligned}
J_1^{m,n} &= \int_0^t \sum_{i=1}^d \left[V'_\varepsilon(X^{m,i}(s) - X^{n,i}(s)) \operatorname{sgn}(X^{m,i}(s) - X^{n,i}(s)) \right. \\
& \quad \left. \times (F_i(s, X^m(s)) - F_i(s, X^n(s))) \operatorname{sgn}(X^{m,i}(s) - X^{n,i}(s)) \right] ds \\
&\leq K_1 d \int_0^t |X^m(s) - X^n(s)| ds, \quad t \in [0, T].
\end{aligned}$$

(2.5) and **(A3)** yield that

$$J_2^{m,n} \leq K_3 d \int_0^t \|X_s^m - X_s^n\|_\infty ds, \quad t \in [0, T].$$

Note that (2.6) and (2.10) derive

$$\begin{aligned}
J_3^{m,n} &\leq K_2^2 \int_0^t \sum_{i=1}^d \varepsilon^{2\alpha_i} ds + \int_0^t \sum_{i=1}^d e^{\frac{1}{\varepsilon}} (G_i^m(s, X^{n,i}(s)) - G_i^n(s, X^{n,i}(s)))^2 ds \\
&\leq t K_2^2 \sum_{i=1}^d \varepsilon^{2\alpha_i} + t e^{\frac{1}{\varepsilon}} \sum_{i=1}^d \sup_{s \in [0, t], z \in \mathbb{R}} |G_i^m(s, z) - G_i^n(s, z)|^2 \\
&\leq t K_2^2 \sum_{i=1}^d \varepsilon^{2\alpha_i} + 2t e^{\frac{1}{\varepsilon}} \sum_{i=1}^d \sup_{s \in [0, t], z \in \mathbb{R}} |G_i^m(s, z) - G_i(s, z)|^2 \\
&\quad + 2t e^{\frac{1}{\varepsilon}} \sum_{i=1}^d \sup_{s \in [0, t], z \in \mathbb{R}} |G_i^m(s, z) - G_i(s, z)|^2 \\
&=: H^{m,n,\varepsilon}(t), \quad t \in [0, T].
\end{aligned}$$

So, (2.13) and Lemma 2.2 imply that for any $p \in (0, 1)$, there exist constants $c_1(p), c_2(p) > 0$ such that

$$\mathbb{E} \sup_{t \in [0, T]} |X^m(t) - X^n(t)|^p \leq c_1(p) e^{c_2(p)(K_1 + K_3)dT} (H^{m,n,\varepsilon}(T) + d\varepsilon)^p.$$

Thanks to (2.7), we derive

$$\limsup_{m,n \rightarrow \infty} |H^{m,n,\varepsilon}(T)| \leq TK_2^2 \sum_{i=1}^d \varepsilon^{2\alpha_i}.$$

So, for $p \in (0, 1)$, it holds

$$\limsup_{m,n \rightarrow \infty} \mathbb{E} \sup_{t \in [0, T]} |X^m(t) - X^n(t)|^p \leq c_1(p) e^{c_2(p)(K_1 + K_3)dT} (TK_2^2 \sum_{i=1}^d \varepsilon^{2\alpha_i} + d\varepsilon)^p.$$

Letting $\varepsilon \rightarrow 0$, we conclude that for any $p \in (0, 1)$,

$$\limsup_{m,n \rightarrow \infty} \mathbb{E} \sup_{t \in [0, T]} |X^m(t) - X^n(t)|^p = 0.$$

So, there exists a continuous stochastic process $\{\bar{X}(t)\}_{t \in [-r, T]}$ satisfying $\bar{X}_0 = \xi$ and

$$(2.14) \quad \limsup_{n \rightarrow \infty} \mathbb{E} \sup_{t \in [0, T]} |X^n(t) - \bar{X}(t)|^p = 0.$$

This yields that there exists a subsequence $\{n_k\}_{k \geq 1}$ such that $\mathbb{P}$ -a.s.

$$(2.15) \quad \lim_{k \rightarrow \infty} \sup_{t \in [0, T]} |X^{n_k}(t) - \bar{X}(t)| = 0, \quad \sup_{k \geq 1} \sup_{t \in [-r, T]} (|X^{n_k}(t)| + |\bar{X}(t)|) < \infty.$$

Moreover, (2.11) and Fatou's Lemma imply

$$\mathbb{E} \sup_{t \in [-r, T]} |\bar{X}(t)|^2 \leq C(T)(1 + \|\xi\|_\infty^2).$$

So, by the local boundedness of F, H , the second inequality in (2.15), we conclude that $\mathbb{P}$ -a.s.

$$\sup_{k \geq 1} \sup_{s \in [0, T]} (|F(s, X^{n_k}(s))| + |H(s, X_s^{n_k})| + |F(s, \bar{X}(s))| + |H(s, \bar{X}_s)|) < \infty.$$

which together with the continuity of $F(s, \cdot), H(s, \cdot)$, the first equality in (2.15) and the dominated convergence theorem implies that $\mathbb{P}$ -a.s.

$$\lim_{k \rightarrow \infty} \sup_{t \in [0, T]} \left| \int_0^t (F(s, X^{n_k}(s)) + H(s, X_s^{n_k})) ds - \int_0^t (F(s, \bar{X}(s)) + H(s, \bar{X}_s)) ds \right| = 0.$$

Moreover, by Markov's inequality, BDG's inequality, (2.7), (2.10) and (2.14), for any $\varepsilon > 0$ and $p \in (0, 1)$, we have

$$\begin{aligned} & \limsup_{k \rightarrow \infty} \mathbb{P} \left(\sup_{t \in [0, T]} \left| \int_0^t [G^{n_k}(s, X^{n_k}(s)) - G(s, \bar{X}(s))] d\bar{W}(s) \right| \geq \varepsilon \right) \\ & \leq \limsup_{k \rightarrow \infty} \frac{1}{\varepsilon^p} \mathbb{E} \sup_{t \in [0, T]} \left| \int_0^t [G^{n_k}(s, X^{n_k}(s)) - G(s, \bar{X}(s))] d\bar{W}(s) \right|^p \end{aligned}$$

$$\begin{aligned}
&\leq c(p) \limsup_{k \rightarrow \infty} \frac{1}{\varepsilon^p} \mathbb{E} \left(\int_0^T \sum_{i=1}^d [G_i^{n_k}(s, X^{n_k,i}(s)) - G_i(s, \bar{X}^i(s))]^2 ds \right)^{\frac{p}{2}} \\
&\leq c(p) T^{\frac{p}{2}} \frac{1}{\varepsilon^p} \limsup_{k \rightarrow \infty} \left(\sum_{i=1}^d \sup_{s \in [0, T], x \in \mathbb{R}} |G_i^{n_k}(s, x) - G_i(s, x)|^2 \right)^{\frac{p}{2}} \\
&\quad + c(p) T^{\frac{p}{2}} K_2^p \frac{1}{\varepsilon^p} \limsup_{k \rightarrow \infty} \mathbb{E} \left(\sup_{s \in [0, T]} \sum_{i=1}^d |X^{n_k,i}(s) - \bar{X}^i(s)|^{2\alpha_i} \right)^{\frac{p}{2}} = 0.
\end{aligned}$$

Therefore, replacing n by n_k in (2.9) and letting $k \rightarrow \infty$, it holds $\mathbb{P}$ -a.s.

$$\bar{X}(t) = \int_0^t F(s, \bar{X}(s)) ds + \int_0^t H(s, \bar{X}_s) ds + \int_0^t G(s, \bar{X}(s)) d\bar{W}(s), \quad t \in [0, T].$$

This means that $\{\bar{X}(t)\}_{t \in [-r, T]}$ is a strong solution to (2.2).

Step 2. Uniqueness of the strong solution.

Let $X^\xi(t)$ be the solution to (2.2) with initial value $\xi \in \mathcal{C}^d$. By the same argument to derive (2.11), we obtain

$$\mathbb{E} \sup_{t \in [-r, T]} |X^\xi(t)|^2 \leq C(T)(1 + \|\xi\|_\infty^2).$$

So, Jensen's inequality implies (2.3). For any $1 \leq i \leq d$, let $X^{\xi,i}$ be the i -th component of X^ξ . Applying Ito's formula, for any $1 \leq i \leq d$, we have

$$\begin{aligned}
V_\varepsilon(X^{\xi,i}(t) - X^{\eta,i}(t)) &= V_\varepsilon(\xi^i(0) - \eta^i(0)) \\
&\quad + \int_0^t V'_\varepsilon(X^{\xi,i}(s) - X^{\eta,i}(s)) \{F_i(s, X^\xi(s)) - F_i(s, X^\eta(s))\} ds \\
&\quad + \int_0^t V'_\varepsilon(X^{\xi,i}(s) - X^{\eta,i}(s)) \{H_i(s, X_s^\xi) - H_i(s, X_s^\eta)\} ds \\
&\quad + \frac{1}{2} \int_0^t V''_\varepsilon(X^{\xi,i}(s) - X^{\eta,i}(s)) \{G_i(s, X^{\xi,i}(s)) - G_i(s, X^{\eta,i}(s))\}^2 ds \\
&\quad + \int_0^t V'_\varepsilon(X^{\xi,i}(s) - X^{\eta,i}(s)) \{G_i(s, X^{\xi,i}(s)) - G_i(s, X^{\eta,i}(s))\} d\bar{W}^i(s).
\end{aligned}$$

By **(A1)**-**(A3)**, (2.5) and (2.6), it holds

$$\begin{aligned}
|X^\xi(t) - X^\eta(t)| &\leq d\varepsilon + K_2^2 T \sum_{i=1}^d \varepsilon^{2\alpha_i} + C(T) \|\xi - \eta\|_\infty \\
&\quad + C \int_0^t \sup_{u \in [0, s]} |X^\xi(u) - X^\eta(u)| ds + M_t, \quad t \in [0, T]
\end{aligned}$$

for a martingale M_t and some constants $C, C(T) > 0$. Then for any $p \in (0, 1)$, applying Lemma 2.2, we get

$$\mathbb{E} \sup_{t \in [0, T]} |X^\xi(t) - X^\eta(t)|^p \leq c_1(p) e^{c_2(p)T} \left(d\varepsilon + K_2^2 T \sum_{i=1}^d \varepsilon^{2\alpha_i} + C(T) \|\xi - \eta\|_\infty \right)^p.$$

Letting $\varepsilon \rightarrow 0$, we derive (2.4), which yields the uniqueness of the strong solution of (2.2). $\square$

3 Path Dependent McKean-Vlasov SDEs with Hölder Continuous Diffusion

Throughout this section, we make the following assumption.

(H) Assume that the following conditions hold.

(Hb) b is locally bounded in $[0, T] \times \mathbb{R} \times (\mathcal{P}_1(\mathbb{R}), \mathbb{W}_1)$. For any $t \in [0, T]$, $b(t, \cdot, \cdot)$ is continuous in $\mathbb{R} \times (\mathcal{P}_1(\mathbb{R}), \mathbb{W}_1)$, and there exists a constant $K_b \geq 0$ such that for $x, y \in \mathbb{R}$ and $\mu, \nu \in \mathcal{P}_1(\mathbb{R})$,

$$[b(t, x, \mu) - b(t, y, \nu)] \operatorname{sgn}(x - y) \leq K_b (\mathbb{W}_1(\mu, \nu) + |x - y|), \quad t \in [0, T].$$

(H σ) There exist constants $K_\sigma \geq 0$ and $\alpha \in [\frac{1}{2}, 1]$ such that

$$|\sigma(t, x) - \sigma(t, y)| \leq K_\sigma |x - y|^\alpha, \quad |\sigma(t, 0)| \leq K_\sigma, \quad x, y \in \mathbb{R}, t \in [0, T].$$

(HB) There exists a constant $K_B \geq 0$ and a probability measure m on $[-r, 0]$ such that for any $\xi, \eta \in \mathcal{C}$, $\mu, \nu \in \mathcal{P}_1(\mathbb{R})$, $t \in [0, T]$,

$$|B(t, \xi, \mu) - B(t, \eta, \nu)| \leq K_B \{ \|\xi - \eta\|_{L^1(m)} + \mathbb{W}_1(\mu, \nu) \}, \quad |B(t, 0, \delta_0)| \leq K_B,$$

here δ_0 is the Dirac measure at the point 0.

3.1 Well-posedness

Theorem 3.1. *Assume (H). Then for any $X_0 \in L^1(\Omega \rightarrow (\mathcal{C}, \|\cdot\|_\infty); \mathcal{F}_0, \mathbb{P})$, (1.2) has a unique strong solution $(X(t))_{t \in [-r, T]}$ with initial value X_0 and there exists a constant $C(T) > 0$ such that*

$$(3.1) \quad \mathbb{E} \sup_{t \in [0, T]} \|X_t\|_\infty \leq C(T) (1 + \mathbb{E} \|X_0\|_\infty).$$

Moreover, for two solutions $X(t)$ and $\tilde{X}(t)$,

$$(3.2) \quad \begin{aligned} & \sup_{t \in [0, T]} \mathbb{E} |X(t) - \tilde{X}(t)| \\ & \leq C(T) \mathbb{E} \left\{ |X(0) - \tilde{X}(0)| + K_B \int_{-r}^0 m([-r, u]) |X(u) - \tilde{X}(u)| du \right\}. \end{aligned}$$

Proof. For $\mu \in C([0, T]; \mathcal{P}_1(\mathbb{R}))$, $x \in \mathbb{R}$ and $\xi \in \mathcal{C}$, let $b^\mu(t, x) = b(t, x, \mu_t)$, $B^\mu(t, \xi) = B(t, \xi, \mu_t)$. Consider

$$(3.3) \quad dX^\mu(t) = b^\mu(t, X^\mu(t))dt + B^\mu(t, X_t^\mu)dt + \sigma(t, X^\mu(t))dW(t), \quad t \in [0, T].$$

By **(H)** and Theorem 2.3, (3.3) is strongly well-posed and let $\Phi_t(\mu) = \mathcal{L}_{X^\mu(t)}$, $t \in [0, T]$, where $(X^\mu(t))_{t \in [-r, T]}$ solves (3.3) with $X_0^\mu \in L^1(\Omega \rightarrow (\mathcal{C}, \|\cdot\|_\infty); \mathcal{F}_0, \mathbb{P})$. In view of **(Hb)**,

$$(3.4) \quad b(t, x, \mu) \operatorname{sgn}(x) \leq C_0(T)(1 + |x| + \mu(|\cdot|)), \quad t \in [0, T]$$

holds for some constant $C_0(T) > 0$. So, by the similar argument to derive (2.11), we get

$$(3.5) \quad \mathbb{E}(\sup_{s \in [-r, t]} |X^\mu(s)|^2 | \mathcal{F}_0) \leq C(T)^2 \left(1 + \|X_0^\mu\|_\infty^2 + \int_0^t \mu_s(|\cdot|)^2 ds \right), \quad t \in [0, T],$$

which yields

$$(3.6) \quad \mathbb{E}(\sup_{s \in [-r, t]} |X^\mu(s)|) \leq C(T) \left(1 + \mathbb{E}\|X_0^\mu\|_\infty + \left(\int_0^t \mu_s(|\cdot|)^2 ds \right)^{\frac{1}{2}} \right), \quad t \in [0, T]$$

for some constant $C(T) \geq 0$. By Itô's formula, it follows that

$$(3.7) \quad \begin{aligned} V_\varepsilon(X^\mu(t) - X^\nu(t)) &= V_\varepsilon(X^\mu(0) - X^\nu(0)) \\ &+ \int_0^t V'_\varepsilon(X^\mu(s) - X^\nu(s)) \{b(s, X^\mu(s), \mu_s) - b(s, X^\nu(s), \nu_s)\} ds \\ &+ \int_0^t V'_\varepsilon(X^\mu(s) - X^\nu(s)) \{B(s, X_s^\mu, \mu_s) - B(s, X_s^\nu, \nu_s)\} ds \\ &+ \frac{1}{2} \int_0^t V''_\varepsilon(X^\mu(s) - X^\nu(s)) \{\sigma(s, X^\mu(s)) - \sigma(s, X^\nu(s))\}^2 ds \\ &+ \int_0^t V'_\varepsilon(X^\mu(s) - X^\nu(s)) \{\sigma(s, X^\mu(s)) - \sigma(s, X^\nu(s))\} dW(s) \\ &=: I_{1,\varepsilon} + I_{2,\varepsilon}(t) + I_{3,\varepsilon}(t) + I_{4,\varepsilon}(t) + I_{5,\varepsilon}(t). \end{aligned}$$

Using (2.5), we get

$$I_{1,\varepsilon} \leq |X^\mu(0) - X^\nu(0)|.$$

Moreover, it follows from (2.5) and **(Hb)** that

$$I_{2,\varepsilon}(t) \leq K_b \int_0^t \{|X^\mu(s) - X^\nu(s)| + \mathbb{W}_1(\mu_s, \nu_s)\} ds, \quad t \in [0, T].$$

By (2.5), **(HB)**, Fubini's theorem and $t + \theta \leq t, \theta \in [-r, 0]$, we arrive at

$$I_{3,\varepsilon}(t) \leq K_B \int_0^t \{\|X_s^\mu - X_s^\nu\|_{L^1(m)} + \mathbb{W}_1(\mu_s, \nu_s)\} ds$$

$$\begin{aligned}
&= K_B \int_{-r}^0 \left(\int_{\theta}^{t+\theta} |X^\mu(u) - X^\nu(u)| du \right) m(d\theta) + K_B \int_0^t \mathbb{W}_1(\mu_s, \nu_s) ds \\
&\leq K_B \int_{-r}^0 \left(\int_0^t |X^\mu(u) - X^\nu(u)| du \right) m(d\theta) \\
&+ K_B \int_{-r}^0 \left(\int_{\theta}^0 |X^\mu(u) - X^\nu(u)| du \right) m(d\theta) + K_B \int_0^t \mathbb{W}_1(\mu_s, \nu_s) ds \\
&\leq K_B \left(\int_0^t |X^\mu(u) - X^\nu(u)| du \right) \\
&+ K_B \int_{-r}^0 m([-r, u]) |X^\mu(u) - X^\nu(u)| du + K_B \int_0^t \mathbb{W}_1(\mu_s, \nu_s) ds, \quad t \in [0, T].
\end{aligned}$$

Furthermore, by **(H σ)**, (2.6) and using $\alpha \in [1/2, 1]$, we deduce

$$I_{4,\varepsilon}(t) \leq K_\sigma^2 T \varepsilon^{2\alpha}, \quad t \in [0, T].$$

In addition, by (2.5), **(H σ)** and (3.6), we have $\mathbb{E}I_{5,\varepsilon}(t) = 0$. Taking expectation in (3.7), using (2.5) and letting $\varepsilon \downarrow 0$, there exists a constant $C > 0$ such that

$$\begin{aligned}
(3.8) \quad \mathbb{E}|X^\mu(t) - X^\nu(t)| &\leq \mathbb{E}|X^\mu(0) - X^\nu(0)| + K_B \int_{-r}^0 m([-r, u]) \mathbb{E}|X^\mu(u) - X^\nu(u)| du \\
&+ C \int_0^t \mathbb{E}|X^\mu(s) - X^\nu(s)| ds + (K_b + K_B) \int_0^t \mathbb{W}_1(\mu_s, \nu_s) ds.
\end{aligned}$$

It follows from Grönwall's inequality that

$$\begin{aligned}
\mathbb{E}|X^\mu(t) - X^\nu(t)| &\leq e^{Ct} \left\{ \mathbb{E}|X^\mu(0) - X^\nu(0)| + K_B \int_{-r}^0 m([-r, u]) \mathbb{E}|X^\mu(u) - X^\nu(u)| du \right\} \\
&+ C(T) \int_0^t \mathbb{W}_1(\mu_s, \nu_s) ds.
\end{aligned}$$

So, when $X_0^\mu = X_0^\nu$, for $\lambda = 2C(T)$, we get

$$\sup_{t \in [0, T]} e^{-\lambda t} \mathbb{W}_1(\Phi_t(\mu), \Phi_t(\nu)) \leq \frac{1}{2} \sup_{t \in [0, T]} e^{-\lambda t} \mathbb{W}_1(\mu_t, \nu_t).$$

Set

$$E_\lambda := \{ \mu \in C([0, T]; \mathcal{P}_1(\mathbb{R})) : \mu_0 = \mathcal{L}_{X^\mu(0)} \}$$

and equip it with the complete metric

$$\rho(\mu, \nu) := \sup_{t \in [0, T]} e^{-\lambda t} \mathbb{W}_1(\mu_t, \nu_t), \quad \mu, \nu \in E_\lambda.$$

Then Φ is strictly contractive in E_λ . Consequently, the Banach fixed point theorem together with the definition of Φ implies that there exists a unique $\mu \in E_\lambda$ such that

$$\Phi_t(\mu) = \mu_t = \mathcal{L}_{X^\mu(t)}, \quad t \in [0, T].$$

Finally, taking $\mu_t = \mathcal{L}_{X^\mu(t)}$ in (3.6), (3.1) follows from Grönwall's inequality. Similarly, taking $\mu_t = \mathcal{L}_{X(t)}$, $\nu_t = \mathcal{L}_{\tilde{X}(t)}$, $X^\mu(t) = X(t)$, $X^\nu(t) = \tilde{X}(t)$ in (3.8), (3.2) holds by Grönwall's inequality. $\square$

3.2 Propagation of Chaos

Let $N \geq 1$ be an integer and $(X_0^i, W^i(t))_{1 \leq i \leq N}$ be i.i.d. copies of $(X_0, W(t))$ with $\mathcal{F}_0$ -measurable $\mathcal{C}$ -valued random variable X_0 . Consider

$$dX^i(t) = b(t, X^i(t), \mathcal{L}_{X^i(t)})dt + B(t, X_t^i, \mathcal{L}_{X^i(t)})dt + \sigma(t, X^i(t))dW^i(t), \quad 1 \leq i \leq N.$$

Let

$$(3.9) \quad \tilde{\mu}_t^N = \frac{1}{N} \sum_{j=1}^N \delta_{X^j(t)}.$$

Consider the stochastic N -interacting particle system:

$$(3.10) \quad dX^{i,N}(t) = b(t, X^{i,N}(t), \hat{\mu}_t^N)dt + B(t, X_t^i, \hat{\mu}_t^N)dt + \sigma(t, X^{i,N}(t))dW^i(t), \quad X_0^{i,N} = X_0^i,$$

where $\hat{\mu}_t^N$ is the empirical distribution corresponding to $X^{1,N}(t), \dots, X^{N,N}(t)$, i.e.

$$\hat{\mu}_t^N := \frac{1}{N} \sum_{j=1}^N \delta_{X^{j,N}(t)}.$$

Applying Theorem 2.3, the well-posedness of the stochastic N -interacting particle system (3.10) can be proved in the following lemma.

Lemma 3.2. *Assume (H) and $X_0^i \in L^1(\Omega \rightarrow (\mathcal{C}, \|\cdot\|_\infty); \mathcal{F}_0, \mathbb{P})$, $1 \leq i \leq N$. Then, for each $N \geq 1$, (3.10) admits a unique strong solution $\{(X^{i,N}(t))_{t \in [-r, T]}\}_{1 \leq i \leq N}$ and*

$$(3.11) \quad \mathbb{E} \sup_{t \in [-r, T]} |X^{i,N}(t)| \leq C(T) \mathbb{E} \sqrt{\frac{1}{N} \sum_{i=1}^N (1 + \|X_0^i\|_\infty^2)}, \quad 1 \leq i \leq N$$

holds for some constant $C(T) > 0$.

Proof. For $x := (x_1, x_2, \dots, x_N)^* \in \mathbb{R}^N$, $\xi := (\xi_1, \xi_2, \dots, \xi_N)^* \in \mathcal{C}^N$, set $\tilde{\mu}_x^N = \frac{1}{N} \sum_{i=1}^N \delta_{x_i}$ and

$$\begin{aligned} \hat{b}(t, x) &:= (b(t, x_1, \tilde{\mu}_x^N), \dots, b(t, x_N, \tilde{\mu}_x^N))^*, \quad \hat{B}(t, \xi) := (B(t, \xi_1, \tilde{\mu}_{\xi(0)}^N), \dots, B(t, \xi_N, \tilde{\mu}_{\xi(0)}^N))^*, \\ \hat{\sigma}(t, x) &:= \text{diag}(\sigma(t, x_1), \dots, \sigma(t, x_N)), \quad \hat{W}(t) := (W^1(t), \dots, W^N(t))^*, \quad t \in [0, T]. \end{aligned}$$

Then it is clear that $(\hat{W}(t))_{t \in [0, T]}$ is an N -dimensional Brownian motion and (3.10) can be reformulated as

$$(3.12) \quad d\hat{X}(t) = \hat{b}(t, \hat{X}(t))dt + \hat{B}(t, \hat{X}_t)dt + \hat{\sigma}(t, \hat{X}(t))d\hat{W}(t), \quad \hat{X}_0 = (X_0^1, X_0^2, \dots, X_0^N)^*.$$

Note that

$$(3.13) \quad \mathbb{W}_1 \left(\frac{1}{N} \sum_{i=1}^N \delta_{x_i}, \frac{1}{N} \sum_{i=1}^N \delta_{\tilde{x}_i} \right) \leq \frac{1}{N} \sum_{i=1}^N |x_i - \tilde{x}_i|, \quad x_i, \tilde{x}_i \in \mathbb{R}, 1 \leq i \leq N.$$

It is not difficult to see from **(Hb)**, **(HB)** and (3.13) that $\hat{b}$ is locally bounded in $[0, T] \times \mathbb{R}^N$, for any $t \in [0, T]$, $\hat{b}(t, \cdot)$ is continuous,

$$(3.14) \quad \begin{aligned} (\hat{b}_i(t, x) - \hat{b}_i(t, y)) \operatorname{sgn}(x_i - y_i) &= (b(t, x_i, \tilde{\mu}_x^N) - b(t, y_i, \tilde{\mu}_y^N)) \operatorname{sgn}(x_i - y_i) \\ &\leq K_b(|x_i - y_i| + \mathbb{W}_1(\tilde{\mu}_x^N, \tilde{\mu}_y^N)) \\ &\leq K_b(|x_i - y_i| + \frac{1}{N} \sum_{i=1}^N |x_i - y_i|) \\ &\leq K_b(1 + N^{-\frac{1}{2}})|x - y|, \quad x, y \in \mathbb{R}^N, 1 \leq i \leq N, \end{aligned}$$

and

$$(3.15) \quad \begin{aligned} |\hat{B}(t, \xi) - \hat{B}(t, \eta)|^2 &\leq \sum_{i=1}^N |B(t, \xi_i, \tilde{\mu}_{\xi(0)}^N) - B(t, \eta_i, \tilde{\mu}_{\eta(0)}^N)|^2 \\ &\leq 2K_B^2 \sum_{i=1}^N (\|\xi_i - \eta_i\|_\infty^2 + \mathbb{W}_1(\tilde{\mu}_{\xi(0)}^N, \tilde{\mu}_{\eta(0)}^N)^2) \\ &\leq 2K_B^2 \sum_{i=1}^N (\|\xi_i - \eta_i\|_\infty^2 + |\xi_i(0) - \eta_i(0)|^2) \\ &\leq 4K_B^2 \sum_{i=1}^N \|\xi_i - \eta_i\|_\infty^2, \quad \xi, \eta \in \mathcal{C}^N. \end{aligned}$$

So, (3.14), (3.15) and **(H σ)** yield that **(A1)**-**(A3)** hold for $\hat{b}, \hat{B}, \hat{\sigma}, N$ replacing F, H, G, d respectively. Therefore, according to Theorem 2.3, for each $N \geq 1$, (3.12) and consequently (3.10) admits a unique strong solution $\{(X^{i,N}(t))_{t \in [-r, T]}\}_{1 \leq i \leq N}$. Finally, by Itô's formula, (3.4), **(HB)** and **(H σ)**, there exists a constant $C > 0$ such that

$$\begin{aligned} |X^{i,N}(t)|^2 &\leq |X^i(0)|^2 + C \int_0^t \left[1 + |X^{i,N}(s)|^2 + \frac{1}{N} \sum_{j=1}^N |X^{j,N}(s)|^2 \right] ds \\ &\quad + C \int_0^t \|X_s^{i,N}\|_{L^1(m)}^2 ds + \int_0^t 2X^{i,N}(s) \sigma(s, X^{i,N}(s)) dW^i(s). \end{aligned}$$

Using the same argument to derive (3.5), we arrive at

$$\sum_{i=1}^N \mathbb{E} \left(\sup_{t \in [-r, T]} |X^{i,N}(t)|^2 | \mathcal{F}_0 \right) \leq C_0(T) \sum_{i=1}^N (1 + \|X_0^i\|_\infty^2)$$

for some constant $C_0(T) > 0$. This together with Jensen's inequality with respect to conditional expectation implies that

$$\begin{aligned}
& \mathbb{E} \sup_{t \in [-r, T]} |X^{i, N}(t)| \\
&= \frac{1}{N} \mathbb{E} \left\{ \mathbb{E} \left(\sum_{i=1}^N \sup_{t \in [-r, T]} |X^{i, N}(t)| \middle| \mathcal{F}_0 \right) \right\} \\
&\leq \frac{1}{N} \mathbb{E} \left\{ \left[\mathbb{E} \left(N \sum_{i=1}^N \sup_{t \in [-r, T]} |X^{i, N}(t)|^2 \middle| \mathcal{F}_0 \right) \right]^{\frac{1}{2}} \right\} \\
&\leq \frac{1}{\sqrt{N}} \mathbb{E} \left[C(T) \sum_{i=1}^N (1 + \|X_0^i\|_\infty^2) \right]^{\frac{1}{2}} \\
&= \sqrt{C(T)} \mathbb{E} \sqrt{\frac{1}{N} \sum_{i=1}^N (1 + \|X_0^i\|_\infty^2)}.
\end{aligned}$$

So, we complete the proof. $\square$

Finally, we give the quantitative propagation of chaos.

Theorem 3.3. *Assume that $\mathbb{E}\|X_0^i\|_\infty^p < \infty$ for some $p > 1$ and $p \neq 2$. Let $\mu_t = \mathcal{L}_{X^i(t)}$.*

(1) *Then there exists a constant $C(p, T) > 0$ depending only on p, T such that*

$$(3.16) \quad \sup_{t \in [0, T]} \mathbb{E} |X^i(t) - X^{i, N}(t)| \leq C(p, T) (1 + (\mathbb{E}\|X_0^i\|_\infty^p)^{\frac{1}{p}}) (N^{-1/2} + N^{-\frac{p-1}{p}}),$$

and consequently,

$$(3.17) \quad \sup_{t \in [0, T]} \mathbb{E} \mathbb{W}_1(\hat{\mu}_t^N, \mu_t) \leq C(p, T) (1 + (\mathbb{E}\|X_0^i\|_\infty^p)^{\frac{1}{p}}) (N^{-1/2} + N^{-\frac{p-1}{p}}).$$

(2) *If in addition, $\sigma^2 \geq \delta$ for some $\delta > 0$ and there exists a constant $K \geq 0$ such that*

$$(3.18) \quad \begin{aligned} & |b(t, x, \mu) - b(t, x, \nu)| + |B(t, \xi, \mu) - B(t, \xi, \nu)| \\ & \leq K(1 \wedge \mathbb{W}_1(\mu, \nu)), \quad \mu, \nu \in \mathcal{P}_1(\mathbb{R}), t \in [0, T], x \in \mathbb{R}, \xi \in \mathcal{C}. \end{aligned}$$

then there exists a constant $C(p, T) > 0$ depending only on p, T such that for any $1 \leq k \leq N$,

$$\begin{aligned}
& \sup_{t \in [0, T]} \|\mathcal{L}_{(X^{1, N}(t), X^{2, N}(t), \dots, X^{k, N}(t))} - \mu_t^{\otimes k}\|_{var}^2 \\
& \leq 2 \sup_{t \in [0, T]} \text{Ent}(\mu_t^{\otimes k} | \mathcal{L}_{(X^{1, N}(t), X^{2, N}(t), \dots, X^{k, N}(t))}) \\
& \leq kC(p, T) (1 + (\mathbb{E}\|X_0^i\|_\infty^p)^{\frac{1}{p}}) (N^{-\frac{1}{2}} + N^{-\frac{(p-1)}{p}}),
\end{aligned}$$

where $\mu_t^{\otimes k} = \prod_{i=1}^k \mu_t$, the k -independent product of μ_t .

Proof. Applying Itô's formula, it holds

$$\begin{aligned}
& V_\varepsilon(X^i(t) - X^{i,N}(t)) \\
&= \int_0^t V'_\varepsilon(X^i(s) - X^{i,N}(s)) (b(s, X^i(s), \hat{\mu}_s^N) - b(s, X^{i,N}(s), \mu_s)) ds \\
&\quad + \int_0^t V'_\varepsilon(X^i(s) - X^{i,N}(s)) (B(s, X_s^i, \hat{\mu}_s^N) - B(s, X_s^{i,N}, \mu_s)) ds \\
&\quad + \frac{1}{2} \int_0^t V''_\varepsilon(X^i(s) - X^{i,N}(s)) (\sigma(s, X^i(s)) - \sigma(s, X^{i,N}(s)))^2 ds \\
&\quad + \int_0^t V'_\varepsilon(X^i(s) - X^{i,N}(s)) (\sigma(s, X^i(s)) - \sigma(s, X^{i,N}(s))) dW^i(s).
\end{aligned}$$

By the same argument to derive (3.8) and adopting the triangle inequality for $\mathbb{W}_1$, we arrive at

$$\mathbb{E}|X^i(t) - X^{i,N}(t)| \leq C \int_0^t \{ \mathbb{E}|X^i(s) - X^{i,N}(s)| + \mathbb{E}\mathbb{W}_1(\mu_s, \tilde{\mu}_s^N) + \mathbb{E}\mathbb{W}_1(\tilde{\mu}_s^N, \hat{\mu}_s^N) \} ds,$$

where $\tilde{\mu}^N$ was introduced in (3.9). By [7, Theorem 1], there exists a constant $C(p, T) > 0$ such that

$$(3.19) \quad \mathbb{E}\mathbb{W}_1(\mu_t, \tilde{\mu}_t^N) \leq C(p, T)(1 + (\mathbb{E}\|X_0^i\|_\infty^p)^{\frac{1}{p}})(N^{-1/2} + N^{-\frac{p-1}{p}}).$$

As a result, it follows from (3.13) and (3.19) that

$$\begin{aligned}
& \mathbb{E}|X^i(t) - X^{i,N}(t)| \\
& \leq C_1 \int_0^t \left\{ \mathbb{E}|X^i(s) - X^{i,N}(s)| + C(p, T)(1 + (\mathbb{E}\|X_0^i\|_\infty^p)^{\frac{1}{p}})(N^{-1/2} + N^{-\frac{p-1}{p}}) \right\} ds
\end{aligned}$$

for some constant $C_1 > 0$. Consequently, we derive (3.16) by (3.1), (3.11) and Grönwall's inequality. Finally, note that

$$\mathbb{W}_1(\hat{\mu}_s^N, \mu_s) \leq \mathbb{W}_1(\hat{\mu}_s^N, \tilde{\mu}_s^N) + \mathbb{W}_1(\tilde{\mu}_s^N, \mu_s) \leq \frac{1}{N} \sum_{i=1}^N |X^{i,N}(s) - X^i(s)| + \mathbb{W}_1(\tilde{\mu}_s^N, \mu_s),$$

which together with (3.16) and (3.19) yields (3.17).

(2) Rewrite (3.10) as

$$dX^{i,N}(t) = b(t, X^{i,N}(t), \mu_t)dt + B(t, X_t^{i,N}, \mu_t)dt + \sigma(t, X^{i,N}(t))d\tilde{W}^i(t), \quad 1 \leq i \leq k$$

with

$$d\tilde{W}^i(t) = \tilde{\Gamma}^i(t)dt + dW^i(t), \quad 1 \leq i \leq k$$

and

$$\tilde{\Gamma}^i(t) = \sigma(t, X^{i,N}(t))^{-1} [b(t, X^{i,N}(t), \hat{\mu}_t^N) - b(t, X^{i,N}(t), \mu_t) + B(t, X_t^{i,N}, \hat{\mu}_t^N) - B(t, X_t^{i,N}, \mu_t)].$$

It follows from (3.18) and $\sigma^2 \geq \delta$ that there exists a constant $C > 0$ such that

$$(3.20) \quad |\tilde{\Gamma}^i(t)| \leq C(\mathbb{W}_1(\hat{\mu}_t^N, \mu_t) \wedge 1), \quad t \in [0, T], 1 \leq i \leq k.$$

Let

$$R_t^k = \exp \left\{ - \sum_{i=1}^k \int_0^t \langle \tilde{\Gamma}^i(s), dW^i(s) \rangle - \frac{1}{2} \sum_{i=1}^k \int_0^t |\tilde{\Gamma}^i(s)|^2 ds \right\}, \quad t \in [0, T].$$

(3.20) and Girsanov's theorem imply that $\{R_t^k\}_{t \in [0, T]}$ is a martingale and $((\tilde{W}^i(t))_{1 \leq i \leq k})_{t \in [0, T]}$ is a k -dimensional Brownian motion under $\mathbb{Q}_T^k = R_T^k \mathbb{P}$ and

$$(3.21) \quad \mathcal{L}_{(X^{1,N}(t), X^{2,N}(t), \dots, X^{k,N}(t))} | \mathbb{Q}_T^k = \mu_t^{\otimes k}, \quad t \in [0, T].$$

This implies that

$$\begin{aligned} \mu_t^{\otimes k}(f) &= \mathbb{E}[R_T^k f(X^{1,N}(t), X^{2,N}(t), \dots, X^{k,N}(t))] \\ &= \mathbb{E}[R_t^k f(X^{1,N}(t), X^{2,N}(t), \dots, X^{k,N}(t))], \quad f \in \mathcal{B}_b(\mathbb{R}^k), t \in [0, T]. \end{aligned}$$

So, there exists a constant $C > 0$ such that

$$\begin{aligned} &\text{Ent}(\mu_t^{\otimes k} | \mathcal{L}_{(X^{1,N}(t), X^{2,N}(t), \dots, X^{k,N}(t))}) \\ &= \mathbb{E}(R_t^k \log R_t^k) = \frac{1}{2} \sum_{i=1}^k \int_0^t \mathbb{E}^{\mathbb{Q}_T^k} |\tilde{\Gamma}^i(s)|^2 ds \leq C^2 k \int_0^t \mathbb{E}^{\mathbb{Q}_T^k} (\mathbb{W}_1(\hat{\mu}_s^N, \mu_s) \wedge 1)^2 ds, \quad t \in [0, T]. \end{aligned}$$

This together with Pinsker's inequality (1.1) yields

$$\begin{aligned} &\|\mu_t^{\otimes k} - \mathcal{L}_{(X^{1,N}(t), X^{2,N}(t), \dots, X^{k,N}(t))}\|_{var}^2 \\ &\leq 2 \text{Ent}(\mu_t^{\otimes k} | \mathcal{L}_{(X^{1,N}(t), X^{2,N}(t), \dots, X^{k,N}(t))}) \\ &\leq 2C^2 k \int_0^t \mathbb{E}^{\mathbb{Q}_T^k} (\mathbb{W}_1(\hat{\mu}_s^N, \mu_s)) ds. \end{aligned}$$

The proof is finished by (3.21) and (3.19). □

Remark 3.4. For quantitative propagation of chaos, one can refer to [11] and references therein for the convolution type distribution dependent SDEs. Since we only assume that the drift is Lipschitz continuous under L^1 -Wasserstein distance and the estimate in [7, Theorem 1] for the convergence rate of empirical distribution of i.i.d. random variables plays crucial role, the order of the quantitative propagation of chaos may be not optimal.

References

- [1] M. Bauer, T. M-Brandis, F. Proske, *Strong Solutions of Mean-Field Stochastic Differential Equations with irregular drift*, Electron. J. Probab. 23(2018), 1-35.

- [2] R. J. Berman, M. Önnheim, *Propagation of Chaos for a Class of First Order Models with Singular Mean Field Interactions*, SIAM J. Math. Anal. 51(2019), 159-196.
- [3] R. Carmona, F. Delarue, *Probabilistic theory of mean field games with applications. I. Mean field FBSDEs, control, and games*. Probability Theory and Stochastic Modelling, 83. Springer, Cham, 2018.
- [4] P.-E. Chaudru de Raynal, *Strong well-posedness of McKean-Vlasov stochastic differential equation with Hölder drift*, Stochastic Process. Appl. 130(2020), 79-107.
- [5] P.-E. Chaudru de Raynal, N. Frikha, *Well-posedness for some non-linear diffusion processes and related pde on the Wasserstein space*, J. Math. Pures Appl. 159(2022), 1-167.
- [6] G. dos Reis, S. Engelhardt, G. Smith, *Simulation of McKean-Vlasov SDEs with super linear growth*, IMA J. Numer. Anal. 42(2022), 874-922.
- [7] N. Fournier, A. Guillin, *On the rate of convergence in Wasserstein distance of the empirical measure*, Probab. Theory Related Fields 162(2015), 707-738.
- [8] N. Fournier, M. Hauray, *Propagation of Chaos for the Landau equation with moderately soft potentials*, Ann. Probab. 44(2016), 3581-3660.
- [9] O. Guéant O, J.-M. Lasry, P.-L. Lions, *Mean Field Games and Applications, Paris-Princeton Lectures on Mathematical Finance 2010. Lecture Notes in Math.* 2003, Springer, Berlin, 205-266.
- [10] A. Guillin, W. Liu, L. Wu, C. Zhang, *The kinetic Fokker-Planck equation with mean field interaction*, J. Math. Pures Appl. 150(2021), 1-23.
- [11] Z. Hao, M. Röckner, X. Zhang, *Strong convergence of propagation of chaos for McKean-Vlasov SDEs with singular interactions*, arXiv:2204.07952.
- [12] X. Huang, F.-Y. Wang, *Distribution dependent SDEs with singular coefficients*, Stochastic Process. Appl. 129(2019), 4747-4770.
- [13] X. Huang, F.-Y. Wang, *McKean-Vlasov SDEs with drifts discontinuous under Wasserstein distance*, Discrete Contin. Dyn. Syst. 41(2021), 1667-1679.
- [14] X. Huang, *Path-distribution dependent SDEs with singular coefficients*, Electron. J. Probab. 26(2021), 1-21.
- [15] B. Jourdain, J. Reygner, *Propagation of chaos for rank-based interacting diffusions and long time behaviour of a scalar quasilinear parabolic equation*, Stoch. PDE: Anal. Comp. 1(2013), 455-506.
- [16] N. Ikeda, S. Watanabe, *Stochastic differential equations and diffusion processes*, North-Holland Mathematical Library, 24. North-Holland Publishing Co., Amsterdam-New York; Kodansha, Ltd., Tokyo, 1981.

- [17] D. Lacker, *On a strong form of propagation of chaos for McKean-Vlasov equations*, Electron. Commun. Probab. 23(2018), 1-11.
- [18] H. P. McKean, *A class of Markov processes associated with nonlinear parabolic equations*, Proc Natl Acad Sci U S A, 56(1966), 1907-1911.
- [19] Yu. S. Mishura, A. Yu. Veretennikov, *Existence and uniqueness theorems for solutions of McKean-Vlasov stochastic equations*, Theor. Probability and Math. Statist. 103(2020), 59-101.
- [20] M. S. Pinsker, *Information and Information Stability of Random Variables and Processes*, Holden-Day, San Francisco, 1964.
- [21] M. von Renesse, M. Scheutzow, *Existence and uniqueness of solutions of stochastic functional differential equations*, Random Oper. Stoch. Equ. 18(2010), 267-284.
- [22] P. Ren, F.-Y. Wang, *Exponential convergence in entropy and Wasserstein for McKean-Vlasov SDEs*, Nonlinear Anal. 206(2021), 112259.
- [23] M. Röckner, X. Zhang, *Well-posedness of distribution dependent SDEs with singular drifts*, Bernoulli 27(2021), 1131-1158.
- [24] A.-S. Sznitman, *Topics in propagation of chaos*, École d' Été de Prob. de Saint-Flour XIX-1989, 165-251, Lecture Notes in Math., 1464, Springer, Berlin, 1991.
- [25] F.-Y. Wang, *Distribution-dependent SDEs for Landau type equations*, Stochastic Process. Appl. 128(2018), 595-621.
- [26] F.-Y. Wang, *Exponential Ergodicity for Singular Reflecting McKean-Vlasov SDEs*, arXiv:2108.03908.
- [27] X. Zhang, *A discretized version of Krylov's estimate and its applications*, Electron. J. Probab. (24)2019, 1-17.