Singular Density Dependent Stochastic Differential Equations *

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Abstract

The (strong and weak) well-posedness is proved for singular SDEs depending on the distribution density point-wisely and globally, where the drift satisfies a local integrability condition in time-spatial variables, and is Lipschitz continuous in the distribution density with respect to a local L^k-norm. Density dependent reflecting SDEs are also studied.

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1 Introduction

The study of distribution dependent SDEs goes back to McKean's pioneering work [9] where an expectation dependent SDE is proposed to characterize Maxwellian gas. Comparing with the dependence on the global distribution, the point-wise dependence on the density function is more singular for SDEs. The point-wisely density dependent SDE is called Nemytskii-type McKean-Vlasov SDE, see [1, 2] for the correspondence of this type SDEs and nonlinear PDEs.

In recent years, distribution dependent SDEs have been intensively investigated and a plenty of results have been derived. However, much less is known for density dependent SDEs, see Remark 1.1 below for existing results.

Let $\ell_\xi : \mathbb{R}^d \to [0, \infty)$ be the distribution density function of an absolutely continuous random variable ξ on $\mathbb{R}^d$. We investigate the following SDE depending on the distribution

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density point-wisely and globally:

$$\mathrm{d}X_t = b_t(X_t, \ell_{X_t}(X_t), \ell_{X_t})\mathrm{d}t + \sigma_t(X_t, \ell_{X_t})\mathrm{d}W_t, \quad t \in [0,T], \tag{1.1}$$

where $T > 0$ is fixed, $\{W_t\}_{t\in[0,T]}$ is an m-dimensional Brownian motion on a complete filtration probability space $(\Omega, \{\mathscr{F}_t\}_{t\in[0,T]}, \mathbb{P})$, and

$$b : [0,T] \times \mathbb{R}^d \times [0,\infty) \times \mathscr{D}^1_+ \to \mathbb{R}^d, \quad \sigma : [0,T] \times \mathbb{R}^d \times \mathscr{D}^1_+ \to \mathbb{R}^d \otimes \mathbb{R}^m$$

are measurable, where

$$\mathscr{D}^1_+ := \left\{ f \in L^1(\mathbb{R}^d) : f \geq 0, \int_{\mathbb{R}^d} f(x)\mathrm{d}x \leq 1 \right\}$$

is a closed subspace of $L^1(\mathbb{R}^d)$.

Definition 1.1. A continuous adapted process $(X_t)_{t\in[0,T]}$ on $\mathbb{R}^d$ is called a (strong) solution of (1.1), if

$$\int_0^T \mathbb{E}\big[|b_s(X_s, \ell_{X_s}(X_s), \ell_{X_s})| + \|\sigma_s(X_s, \ell_{X_s})\|^2\big]\mathrm{d}s < \infty$$

and $\mathbb{P}$-a.s.

$$X_t = X_0 + \int_0^t b_s(X_s, \ell_{X_s}(X_s), \ell_{X_s})\mathrm{d}s + \int_0^t \sigma_s(X_s, \ell_{X_s})\mathrm{d}W_s, \quad t \in [0,T].$$

A pair $(X_t, W_t)_{t\in[0,T]}$ is called a weak solution of (1.1), if $(W_t)_{t\in[0,T]}$ is an m-dimensional Brownian under a complete filtration probability space $(\Omega, \{\mathscr{F}_t\}_{t\in[0,T]}, \mathbb{P})$ such that $(X_t)_{t\in[0,T]}$ solves (1.1). We identify any two weak solutions $(X_t, W_t)_{t\in[0,T]}$ and $(\bar{X}_t, \bar{W}_t)_{t\in[0,T]}$ if $(X_t)_{t\in[0,T]}$ and $(\bar{X}_t)_{t\in[0,T]}$ have the same distribution under the corresponding probability spaces.

Remark 1.1. We introduce below some existing results concerning the well-posdedness of (1.1) in two special situations.

(1) When $m = d, \sigma = I_d$ (the $d \times d$ identity matrix), and $b_t(x, r, \rho) = b_t(x, r)$ does not depend on ρ, the weak solutions are studied in [5, 6]. In [5], the weak existence is proved for $b_t(x,r)$ bounded and continuous in (t,r) locally uniformly in x, and the weak and strong uniqueness holds when $b_t(x,r)$ is furthermore Lipschitz continuous in r uniformly in (t,x). In [6] the initial density is in $C^{\beta+} := \cup_{p>\beta} C^p$ for some $\beta \in (0, \frac{1}{2})$, the weak well-posedness is proved for $b_t(x,r) := F(r)\tilde{b}_t(x)$, where $\tilde{b} \in C([0,T]; C^{-\beta})$ and F is bounded and Lipschitz continuous such that $rF(r)$ is Lipschitz continuous in $r \geq 0$. See [7] and references within for the case with better drift.

(2) In (1.1) the noise does not point-wisely depend on the density. It seems that to solve SDEs with point-wisely density dependent noise, one needs stronger regularity for the initial density and the coefficients. For instance, [8] proved the well-posedness and studied the propagation of chaos for the following SDE with point-wisely density dependent noise:

$$\mathrm{d}X_t = b(\ell_{X_t}(X_t))\mathrm{d}t + \sigma(\ell_{X_t}(X_t))\mathrm{d}W_t,$$

where the initial distribution density is C^{2+}-smooth, b is C^2-smooth, and σ is uniformly elliptic and C^3-smooth.

In this paper, we only consider density dependent SDEs with singular (non-continuous in spatial) drifts, and leave to a forthcoming paper for the study of regular SDEs with point-wisely density dependent noise.

In Section 2, we state two main results of the paper which provide the well-posedness of (1.1) for density free noise and density dependent noise respectively, and explain the main idea of the proof. To realize the idea, in Section 3 we recall some heat kernel estimates based on [11] and present new estimates. With these estimates we prove the main results in Sections 4 and 5 respectively, and finally make an extension to the reflecting setting in Section 6.

2 Main results and idea of proof

To characterize the time-spatial singularity, we recall some spaces of locally integrable functions introduced in [14].

For $p \in [1,\infty]$ and $f \in \mathscr{B}(\mathbb{R}^d)$, the space of measurable functions on $\mathbb{R}^d$, let

$$\|f\|_{L^p} := \left(\int_{\mathbb{R}^d} |f|^p(x)\mathrm{d}x\right)^{\frac{1}{p}}, \quad \|f\|_{\tilde{L}^p} := \sup_{z\in\mathbb{R}^d} \|1_{B(z,1)} f\|_{L^p} < \infty,$$

where $B(z,r) := \{x \in \mathbb{R}^d : |x-z| \le r\}, r > 0, z \in \mathbb{R}^d$. We write $f \in L^p$ ($f \in \tilde{L}^p$) if $\|f\|_{L^p} < \infty$ ($\|f\|_{\tilde{L}^p} < \infty$).

For any $p, q \in [1,\infty]$ and $f \in \mathscr{B}([0,T]\times\mathbb{R}^d)$, the space of measurable functions on $[0,T]\times\mathbb{R}^d$, let

$$\|f\|_{L^p_q} := \left(\int_0^T \|f_t\|_{L^p}^q \mathrm{d}t\right)^{\frac{1}{q}}, \quad \|f\|_{\tilde{L}^p_q} := \sup_{z\in\mathbb{R}^d} \left(\int_0^T \|1_{B(z,1)} f_t\|_{L^p}^q \mathrm{d}t\right)^{\frac{1}{q}}.$$

We denote $f \in L^p_q (f \in \tilde{L}^p_q)$ if $\|f\|_{L^p_q} < \infty$ ($\|f\|_{\tilde{L}^p_q} < \infty$).

In the following the parameter (p,q) will be taken from the class

$$\mathscr{K} := \left\{(p,q) \in (2,\infty] : \frac{d}{p} + \frac{2}{q} < 1\right\}.$$

For simplicity, we identify $L^\infty = \tilde{L}^\infty$ with $\mathscr{B}_b(\mathbb{R}^d)$, the space of bounded measurable functions on $\mathbb{R}^d$, equipped with the uniform norm

$$\|f\|_{L^\infty} = \|f\|_{\tilde{L}^\infty} = \|f\|_\infty := \sup_{\mathbb{R}^d} |f|.$$

This uniform norm is defined for real functions on an abstract space. Similarly, $\tilde{L}^\infty_\infty = L^\infty_\infty = \mathscr{B}_b([0,T]\times\mathbb{R}^d)$ is the space of bounded measurable functions on $[0,T]\times\mathbb{R}^d$ equipped with the uniform norm, and L^k_∞ ($\tilde{L}^k_\infty$) is the space of functions $f \in \mathscr{B}([0,T]\times\mathbb{R}^d)$ such that

$$\|f\|_{L^k_\infty} := \sup_{t\in[0,T]} \|f_t\|_{L^k} < \infty \ \big(\|f\|_{\tilde{L}^k_\infty} := \sup_{t\in[0,T]} \|f_t\|_{\tilde{L}^k} < \infty\big).$$

Finally, let ∇ be the gradient in $\mathbb{R}^d$, and let $\|\nabla f\|_\infty$ denote the Lipschitz constant of a real function f on $\mathbb{R}^d$.

In the following, we state our main results for density free noise and density dependent noise respectively, and briefly explain the main idea of proof.

2.1 Density free noise

In this part, we let $\sigma_t(x,\rho) = \sigma_t(x)$ do not depend on ρ. For $k > 1$ and a signed measure μ with density function $\ell_\mu(x) := \frac{\mu(\mathrm{d}x)}{\mathrm{d}x}$, let

$$\|\mu\|_{L^k} := \|\ell_\mu\|_{L^k}, \quad \|\mu\|_{\tilde{L}^k} := \|\ell_\mu\|_{\tilde{L}^k}.$$

When $k = 1$, we define

$$\|\mu\|_{L^1} := \sup_{\|f\|_\infty \le 1} |\mu(f)|, \quad \|\mu\|_{\tilde{L}^1} := \sup_{z\in\mathbb{R}^d} \sup_{\|f\|_\infty\le 1} |\mu(1_{B(z,1)} f)|,$$

where $\mu(f) := \int_{\mathbb{R}^d} f \mathrm{d}\mu$. Note that $\|\cdot\|_{L^1}$ is the total variation norm.

Let $\mathscr{P}$ be the set of all probability measures on $\mathbb{R}^d$. We will solve (1.1) with initial distributions in the classes

$$\mathscr{P}^k := \Big\{\nu \in \mathscr{P} : \|\nu\|_{L^k} < \infty\Big\}, \quad \tilde{\mathscr{P}}^k := \Big\{\nu \in \mathscr{P} : \|\nu\|_{\tilde{L}^k} < \infty\Big\}, \quad k \in [1,\infty],$$

which are complete metric spaces under distances $\|\nu_1 - \nu_2\|_{L^k}$ and $\|\nu_1 - \nu_2\|_{\tilde{L}^k}$ respectively.

(A) $a_t(x) := (\sigma_t\sigma_t^*)(x)$ and $b_t(x,r,\rho) = b_t^{(1)}(x) + b_t^{(0)}(x,r,\rho)$ satisfy the following conditions for some $k \in [1,\infty]$.

(A_1) $a_t(x)$ is invertible with $\|a\|_\infty + \|a^{-1}\|_\infty < \infty$, and there exist constants $\alpha \in (0,1)$ and $C > 0$ such that

$$\sup_{t\in[0,T]} \|a_t(x) - a_t(y)\| \le C|x-y|^\alpha, \quad x,y \in \mathbb{R}^d.$$

(A_2) There exist $(p_0,q_0) \in \mathscr{K}$, $\theta > \frac{2}{q_0} + \frac{d}{p_0} - 1$, and $1 \le f_0 \in \tilde{L}^{p_0}_{q_0}$ such that

$$|b_t^{(0)}(x,r,\rho) - b_t^{(0)}(x,\tilde{r},\tilde{\rho})| \le f_0(t,x)t^\theta\big(|r-\tilde{r}| + \|\rho-\tilde{\rho}\|_{\tilde{L}^k}\big),$$
$$|b_t^{(0)}(x,r,\rho)| \le f_0(t,x), \quad (t,x) \in (0,T]\times\mathbb{R}^d, r,\tilde{r} \in [0,\infty), \rho,\tilde{\rho} \in \tilde{L}^k \cap \mathscr{D}^1_+.$$

(A_3) $b_t^{(1)}(0)$ is bounded in $t \in [0,T]$ and

$$\|\nabla b^{(1)}\|_\infty := \sup_{t\in[0,T]} \sup_{x\ne y} \frac{|b_t^{(1)}(x) - b_t^{(1)}(y)|}{|x-y|} < \infty.$$

To ensure $\ell_{X_\cdot} \in L^k$ for $\ell_{X_0} \in L^k$, we replace (A_2) by the following (A_2').

(A_2') There exist $C \in (0,\infty)$, $(p_0,q_0)\in\mathscr{K}$, $\theta > \frac{2}{q_0}+\frac{d}{p_0}-1$, and $0\le f_0\in L^{p_0}_{q_0}$ such that

$$|b_t^{(0)}(x,r,\rho)-b_t^{(0)}(x,s,\tilde\rho)| \le t^\theta\big(C+f_0(t,x)\big)\big(|r-s|+\|\rho-\tilde\rho\|_{L^k}\big),$$
$$|b_t^{(0)}(x,r,\rho)|\le f_0(t,x),\quad (t,x)\in(0,T]\times\mathbb{R}^d, r,s\in[0,\infty),\rho,\tilde\rho\in L^k\cap\mathscr{D}^1_+.$$

Under the above assumptions, the following result ensures the well-posedness of (1.1) for initial distributions in $\tilde{\mathscr{P}}^k$ or $\mathscr{P}^k$ for

$$k\in\Big[\frac{p_0}{p_0-1},\infty\Big]\cap(k_0,\infty],\quad k_0:=\frac{d}{2\theta+1-2q_0^{-1}-dp_0^{-1}}.$$

This explains the role played by the quantity θ in (A_2) and (A_2'): for bigger θ, (A_2) and (A_2') provide stronger upper bound condition on $|b_t^{(0)}(x,r,\rho)-b_t^{(0)}(x,\tilde r,\tilde\rho)|$ for small t, so that the SDE is solvable for initial distributions in larger classes $\mathscr{P}^k$ and $\tilde{\mathscr{P}}^k$. In particular, when $p_0=\infty$ and θ is large enough such that $k_0<1$, we may take $k=1$ so that the SDE is well-posed for any initial distribution $\nu\in\mathscr{P}$.

Theorem 2.1. *Let* $k\in[\frac{p_0}{p_0-1},\infty]$ *with* $k>k_0:=\frac{d}{2\theta+1-2q_0^{-1}-dp_0^{-1}}$.

(1) *Under* **(A)**, *for any* $\nu\in\tilde{\mathscr{P}}^k$, (1.1) *has a unique weak solution with* $\mathscr{L}_{X_0}=\nu$ *satisfying* $\ell_{X_\cdot}\in\tilde L^k_\infty$, *and there exist an increasing function* $\Lambda:[0,\infty)\to(0,\infty)$ *such that for any two weak solutions* $\{X_t^i\}_{i=1,2}$ *of* (1.1) *with* $\ell_{X^i_\cdot}\in\tilde L^k_\infty$,

$$\sup_{t\in[0,T]}\|\ell_{X_t^1}-\ell_{X_t^2}\|_{\tilde L^k}\le\Lambda\big(\|\mathscr{L}_{X_0^1}\|_{\tilde L^k}\wedge\|\mathscr{L}_{X_0^1}\|_{\tilde L^k}\big)\|\mathscr{L}_{X_0^1}-\mathscr{L}_{X_0^2}\|_{\tilde L^k}. \tag{2.1}$$

If moreover σ_t *is weakly differentiable with*

$$\|\nabla\sigma\|\le\sum_{i=1}^l f_i \text{ for some } l\in\mathbb{N}, 0\le f_i\in\tilde L^{p_i}_{q_i}, (p_i,q_i)\in\mathscr{K}, 1\le i\le l, \tag{2.2}$$

then for any X_0 *with* $\mathscr{L}_{X_0}\in\tilde L^k$, (1.1) *has a unique strong solution with* $\ell_{X_\cdot}\in\tilde L^k_\infty$.

(2) *Under* **(A)** *with* (A_2') *replacing* (A_2), *assertions in* (1) *hold for* $(\mathscr{P}^k, L^k_\infty, L^k)$ *replacing* $(\tilde{\mathscr{P}}^k,\tilde L^k_\infty,\tilde L^k)$.

2.2 Density dependent noise

In this part, we allow σ to be density dependent but make stronger assumptions on the initial density and the coefficients in the spatial variable.

For any $n\in\mathbb{Z}^+$, let $C_b^n(\mathbb{R}^d)$ be the class of real functions f on $\mathbb{R}^d$ with continuous derivatives $\{\nabla^i f\}_{0\le i\le n}$ such that

$$\|f\|_{C_b^n}:=\sum_{i=0}^n\|\nabla^i f\|_\infty<\infty.$$

For any $n\in\mathbb{Z}^+$ and $\alpha\in(0,1)$, $C_b^{n+\alpha}(\mathbb{R}^d)$ is the space of functions $f\in C_b^n(\mathbb{R}^d)$ such that

$$\|f\|_{C_b^{n+\alpha}}:=\|f\|_{C_b^n}+\sup_{x\ne y}\frac{|\nabla^n f(x)-\nabla^n f(y)|}{|x-y|^\alpha}<\infty.$$

(B) There exist $1 \le f_0 \in \tilde{L}^{p_0}_{q_0}$, $C \in (0,\infty)$ and $\alpha \in (0,1)$, such that the following conditions hold for all $t \in (0,T]$, $x, y \in \mathbb{R}^d$, $r, \tilde{r} \in [0,\infty)$ and $\rho, \tilde{\rho} \in L^\infty \cap \mathscr{D}^1_+$:

$$|b_t(x,r,\rho)| \le f_0(t,x),$$

$$\begin{aligned}
|b_t(x,r,\rho) - b_t(x,\tilde{r},\tilde{\rho})| &\le C(|r-\tilde{r}| + \|\rho-\tilde{\rho}\|_\infty),\\
\|\sigma\|_\infty + \|\nabla\sigma\|_\infty + \|(\sigma\sigma^*)^{-1}\|_\infty &\le C,\\
\|\nabla\sigma_t(\cdot,\rho)(x) - \nabla\sigma_t(\cdot,\rho)(y)\| &\le C|x-y|^\alpha,\\
\|\sigma_t(\cdot,\rho) - \sigma_t(\cdot,\tilde{\rho})\|_{C_b^\alpha} &\le C\|\rho-\tilde{\rho}\|_\infty.
\end{aligned}$$

Theorem 2.2. *Assume* **(B)** *and let $\beta \in (0, 1 - \frac{d}{p_0} - \frac{2}{q_0})$. For any initial value (initial density) with $\ell_{X_0} \in C_b^\beta(\mathbb{R}^d)$, (1.1) has a unique strong (weak) solution satisfying $\ell_{X_\cdot} \in L^\infty_\infty$, and there exists a constant $c > 0$ such that*

$$\sup_{t\in[0,T]} \|\ell_{X_t}\|_{C_b^\beta} \le c\|\ell_{X_0}\|_{C_b^\beta}. \tag{2.3}$$

Moreover, there exists an increasing function $\Lambda : (0,\infty) \to (0,\infty)$ such that for any two solutions $\{X_t^i\}_{i=1,2}$ with $\ell_{X_0^i} \in C_b^\beta(\mathbb{R}^d)$ and $\ell_{X_\cdot^i} \in L^\infty_\infty$,

$$\sup_{t\in[0,T]} \|\ell_{X_t^1} - \ell_{X_t^2}\|_\infty \le \Lambda\big(\|\ell_{X_0^1}\|_{C_b^\beta} \wedge \|\ell_{X_0^1}\|_{C_b^\beta}\big)\|\ell_{X_0^1} - \ell_{X_0^2}\|_\infty. \tag{2.4}$$

2.3 Idea of proof

For fixed $k \ge 1$ and $\nu \in \tilde{\mathscr{P}}^k$, let $\tilde{\mathscr{P}}^k_{\nu,T}$ be the set of all bounded measurable maps

$$\gamma : (0,T] \to \tilde{L}^k \cap \mathscr{D}^1_+, \quad \gamma_0 = \nu.$$

When $k = 1$, the initial value γ_0 may be singular, and if it is absolutely continuous we regard it as its density function.

Then $\tilde{\mathscr{P}}^k_{\nu,T}$ is complete under the metric

$$\tilde{d}_{k,\lambda}(\gamma^1,\gamma^2) := \sup_{t\in[0,T]} e^{-\lambda t}\|\gamma_t^1 - \gamma_t^2\|_{\tilde{L}^k}, \quad \gamma^1,\gamma^2 \in \tilde{\mathscr{P}}^k_{\nu,T}$$

for $\lambda > 0$. We define $(\mathscr{P}^k_{\nu,T}, d_{k,\lambda})$ in the same way with $(L^k, \mathscr{P}^k)$ replacing $(\tilde{L}^k, \tilde{\mathscr{P}}^k)$.

For any $\gamma \in \tilde{\mathscr{P}}^k_{\nu,T}$, let

$$b_t^\gamma(x) := b_t(x,\gamma_t(x),\gamma_t), \quad \sigma_t^\gamma(x) := \sigma_t(x,\gamma_t), \quad t \in (0,T], x \in \mathbb{R}^d.$$

Then for $\nu := \mathscr{L}_{X_0} \in \tilde{L}^k$, (1.1) has a unique (weak or strong) solution with $\ell_{X_\cdot} \in \tilde{L}^k_\infty$ if we could verify the following two things:

1) For any $\gamma \in \tilde{\mathscr{P}}^k_{\nu,T}$, the SDE

$$\mathrm{d}X_t^\gamma = b_t^\gamma(X_t^\gamma)\mathrm{d}t + \sigma_t^\gamma(X_t^\gamma)\mathrm{d}W_t, \quad t \in [0,T],\ X_0^\gamma = X_0 \tag{2.5}$$

is (weakly or strongly) well-posed, and

$$\gamma \mapsto \Phi_t^\nu \gamma := \ell_{X_t^\gamma}, \quad t \in (0,T],\ \Phi_0^\nu \gamma := \gamma_0 = \nu$$

provides a map $\Phi^\nu : \tilde{\mathscr{P}}^k_{\nu,T} \to \tilde{\mathscr{P}}^k_{\nu,T}$.

2) Φ^ν has a unique fixed point $\bar{\gamma}$ in $\tilde{\mathscr{P}}^k_{\nu,T}$.

Indeed, from these we see that $X_t := X_t^{\bar{\gamma}}$ is the unique (weak or strong) solution of (1.1) with $\mathscr{L}_{X_\cdot} \in \tilde{L}^k_\infty$.

To verify 1) and 2), in Section 2 we recall some heat kernel upper bounds of [11], and estimate the $\tilde{L}^p_q$-$\tilde{L}^{p'}_q$ norm for time inhomogeneous semigroups.

3 Heat kernel estimates

We first recall a result of [11]. Let

$$a : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^d, \quad b : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d.$$

We consider heat kernel estimates for the time dependent second order differential operator

$$L_t^{a,b} := \frac{1}{2}\mathrm{tr}\{a_t \nabla^2\} + \nabla_{b_t}$$

satisfying the following conditions.

$(H^{a,b})$ $a_t(x)$ is invertible and there exist constants $C > 0$ and $\alpha \in (0,1)$ such that

$$\begin{aligned} &\|b_\cdot(0)\|_\infty + \|a\|_\infty + \|a^{-1}\|_\infty \le C,\\ &\sup_{t\in[0,T]} \|a_t(x) - a_t(y)\| \le C|x-y|^\alpha,\\ &\sup_{t\in[0,T]} |b_t(x) - b_t(y)| \le C\big(|x-y| + |x-y|^\alpha\big), \quad x,y \in \mathbb{R}^d. \end{aligned}$$

(H^a) $a_t(x)$ is differentiable in x, and there exist constants $C \in (0,\infty)$ and $\alpha \in (0,1)$ such that

$$\|\nabla a\|_\infty \le C, \quad \sup_{t\in[0,T]} \|\nabla a_t(x) - \nabla a_t(y)\| \le C|x-y|^\alpha, \quad x,y \in \mathbb{R}^d.$$

Under $(H^{a,b})$, for any $s \in [0,T)$, the SDE

$$\mathrm{d}X_{s,t}^x = b_s(X_{s,t}^x)\mathrm{d}s + \sqrt{a_s}(X_{s,t}^x)\mathrm{d}W_s, \quad t \in [s,T], X_{s,s}^x = x \in \mathbb{R}^d$$

is weakly well-posed with semigroup $\{P^{a,b}_{s,t}\}_{0\le s<t\le T}$ and transition density $\{p^{a,b}_{s,t}\}_{0\le s<t\le T}$ given by

$$P^{a,b}_{s,t}f(x)=\int_{\mathbb{R}^d}p^{a,b}_{s,t}(x,y)f(y)\mathrm{d}y=\mathbb{E}[f(X^x_{s,t})],\quad f\in\mathscr{B}_b(\mathbb{R}^d),$$

and we have the following Kolmogorov backward equation (see Remark 2.2 in [11])

$$\partial_s P^{a,b}_{s,t}f=-L_sP^{a,b}_{s,t}f,\quad f\in C_b^\infty(\mathbb{R}^d), s\in[0,t], t\in(0,T]. \tag{3.1}$$

Next, we denote $\psi_{s,t}=\theta^{(1)}_{t,s}$ presented in [11]. Then $(\psi_{s,t})_{0\le s\le t\le T}$ is a family of diffeomorphisms on $\mathbb{R}^d$ satisfying

$$\sup_{0\le s\le t\le T}\big\{\|\nabla\psi_{s,t}\|_\infty+\|\nabla\psi_{s,t}^{-1}\|_\infty\big\}\le\delta \tag{3.2}$$

for some constant $\delta>0$ depending on α, C. For any $\kappa>0$, consider the Gaussian heat kernel

$$p^\kappa_t(x):=(\kappa\pi t)^{-\frac{d}{2}}\mathrm{e}^{-\frac{|x|^2}{\kappa t}},\quad t>0,\ x\in\mathbb{R}^d.$$

The following result is taken from [11, Theorem 1.2].

Theorem 3.1 ([11]). *Assume $(H^{a,b})$. Then there exist constants $c,\kappa>0$ depending on C,α such that*

$$\begin{aligned}|\nabla^i p^{a,b}_{s,t}(\cdot,y)(x)|\le c(t-s)^{-\frac{i}{2}}p^\kappa_{t-s}(\psi_{s,t}(x)-y),\\ i=0,1,2,\ 0\le s<t\le T,\ x,y\in\mathbb{R}^d.\end{aligned} \tag{3.3}$$

If moreover (H^a) holds, then

$$|\nabla p^{a,b}_{s,t}(x,\cdot)(y)|\le c(t-s)^{-\frac{1}{2}}p^\kappa_{t-s}(\psi_{s,t}(x)-y),\quad 0\le s<t\le T, x,y\in\mathbb{R}^d, \tag{3.4}$$

and for any $\beta\in(0,1)$ there exists a constant $c'>0$ depending on C,α,β such that

$$\begin{aligned}&|\nabla p^{a,b}_{s,t}(\cdot,y)(x)-\nabla p^{a,b}_{s,t}(\cdot,y')(x)|+|\nabla p^{a,b}_{s,t}(x,\cdot)(y)-\nabla p^{a,b}_{s,t}(x,\cdot)(y')|\\ &\le c'|y-y'|^\beta(t-s)^{-\frac{1+\beta}{2}}\big\{p^\kappa_{t-s}(\psi_{s,t}(x)-y)+p^\kappa_{t-s}(\psi_{s,t}(x)-y')\big\},\\ &\quad 0\le s<t\le T,\ x,x',y\in\mathbb{R}^d.\end{aligned} \tag{3.5}$$

For any $f\in\mathscr{B}_b(\mathbb{R}^d)\cup\mathscr{B}^+(\mathbb{R}^d)$, let

$$\begin{aligned}&P^\kappa_t f(x):=\int_{\mathbb{R}^d}p^\kappa_t(x-y)f(y)\mathrm{d}y,\\ &\hat P^\kappa_{s,t}f(x):=\int_{\mathbb{R}^d}p^\kappa_{t-s}(\psi_{s,t}(x)-y)f(y)\mathrm{d}y,\\ &\tilde P^\kappa_{s,t}f(x):=\int_{\mathbb{R}^d}p^\kappa_{t-s}(\psi_{s,t}(y)-x)f(y)\mathrm{d}y,\quad 0\le s<t\le T,\ x\in\mathbb{R}^d.\end{aligned} \tag{3.6}$$

It is well known that for some constant $c>0$,

$$\|P^\kappa_t\|_{L^p\to L^{p'}}:=\sup_{\|f\|_p\le1}\|P^\kappa_t f\|_{L^{p'}}\le ct^{-\frac{d(p'-p)}{2pp'}},\quad t>0, 1\le p\le p'\le\infty. \tag{3.7}$$

Combining this with (3.2) we obtain

$$(3.8)\qquad \|\hat P^\kappa_{s,t}\|_{L^p\to L^{p'}}+\|\tilde P^\kappa_{s,t}\|_{L^p\to L^{p'}}\le c(t-s)^{-\frac{d(p'-p)}{2pp'}},\ \ 0\le s<t\le T, 1\le p\le p'\le\infty.$$

for some different constant $c>0$. Below we extend this estimate to the $\tilde L^p_q$-$\tilde L^{p'}_q$ norm. For any $t\in(0,T]$, let

$$\|f\|_{\tilde L^p_q(t)}:=\sup_{z\in\mathbb{R}^d}\bigg(\int_0^t\|1_{B(z,1)}f_s\|^q_{\tilde L^p}\mathrm{d}s\bigg)^{\frac1q},\ \ p,q\in[1,\infty].$$

Lemma 3.2. *There exists a constant $c>0$ such that for any $0\le s<t\le T,\ 1\le p\le p'\le\infty$ and $q\in[1,\infty]$,*

$$(3.9)\qquad \|\hat P^\kappa_{\cdot,t}f\|_{\tilde L^{p'}_q(t)}+\|\tilde P^\kappa_{\cdot,t}f\|_{\tilde L^{p'}_q(t)}\le c\big\|(t-\cdot)^{-\frac{d(p'-p)}{2pp'}}f\big\|_{\tilde L^p_q(t)},\quad f\in\mathscr{B}^+(\mathbb{R}^d),$$

where and in the sequel, $(t-\cdot)(s):=t-s$ is a function on $[0,t]$, and

$$(3.10)\qquad \sup_{z\in\mathbb{R}^d}\|g\hat P^\kappa_{s,t}(1_{B(z,1)}f)\|_{L^1}\le c(t-s)^{-\frac{d(p'-p)}{2pp'}}\|g\|_{\tilde L^{\frac{p'}{p'-1}}}\|f\|_{\tilde L^p},\quad f,g\in\mathscr{B}^+(\mathbb{R}^d).$$

Proof. Let $\mathbf{B}_n:=\{v\in\mathbb{Z}^d: |v|_1:=\sum_{i=1}^d|v_i|=n\}, n\ge0$. By (3.2), we find a constant $\varepsilon\in(0,1)$ such that for any $n\ge0$ and $0\le s<t\le T$,

$$|\psi_{s,t}(x)-y|^2\ge\varepsilon n^2,\ \ x\in B(\psi_{s,t}^{-1}(z),\varepsilon),\ \ y\in\cup_{v\in\mathbf{B}_n}B(z+v,d),\ z\in\mathbb{R}^d.$$

Combining this with (3.7), we find constants $c_2,c_3,c_4>0$ such that for any $z\in\mathbb{R}^d$, $0\le s<t\le T$, and $f,g\in\mathscr{B}^+_b(\mathbb{R}^d)$,

$$\begin{aligned}
\|1_{B(\psi^{-1}_{s,t}(z),\varepsilon)}g\hat P^\kappa_{s,t}f\|_{L^1}&\le\sum_{n=0}^\infty\sum_{v\in\mathbb{Z}^d:|v|_1=n}\|1_{B(\psi^{-1}_{s,t}(z),1)}g\hat P^\kappa_{s,t}(1_{B(z+v,d)}f)\|_{L^1}\\
&\le\sum_{n=0}^\infty\sum_{v\in\mathbf{B}_n}\int_{\mathbb{R}^d\times\mathbb{R}^d}|1_{B(\psi^{-1}_{s,t}(z),\varepsilon)}g|(x)p^\kappa_{t-s}(\psi_{s,t}(x)-y)|1_{B(z+v,d)}f|(y)\mathrm{d}x\mathrm{d}y\\
&\le c_2\sum_{n=0}^\infty\sum_{v\in\mathbf{B}_n}\mathrm{e}^{-\frac{n^2}{c_3(t-s)}}\int_{\mathbb{R}^d\times\mathbb{R}^d}|1_{B(\psi^{-1}_{s,t}(z),\varepsilon)}g|(x)p^\kappa_{2(t-s)}(\psi_{s,t}(x)-y)|1_{B(z+v,d)}f|(y)\mathrm{d}x\mathrm{d}y\\
&\le c_3\sum_{n=0}^\infty\sum_{v\in\mathbf{B}_n}\mathrm{e}^{-\frac{n^2}{c_3(t-s)}}\|\{P^\kappa_{2(t-s)}(1_{B(\psi^{-1}_{s,t}(z),\varepsilon)}g)\}1_{B(z+v,d)}f\|_{L^1}\\
&\le c_3\sum_{n=0}^\infty\sum_{v\in\mathbf{B}_n}\mathrm{e}^{-\frac{n^2}{c_3(t-s)}}\|P^\kappa_{2(t-s)}(1_{B(\psi^{-1}_{s,t}(z),\varepsilon)}g)\|_{L^{\frac{p}{p-1}}}\|1_{B(z+v,d)}f\|_{L^p}\\
&\le c_4(t-s)^{-\frac{d(p'-p)}{2pp'}}\|g\|_{L^{\frac{p'}{p'-1}}}\sum_{n=0}^\infty\sum_{v\in\mathbf{B}_n}\mathrm{e}^{-\frac{n^2}{c_3(t-s)}}\|1_{B(z+v,d)}f\|_{L^p}.
\end{aligned}$$

Since

$$\sup_{z\in\mathbb{R}^d}\left(\int_0^t \|1_{B(z,d)}f\|_{L^p}^q \mathrm{d}s\right)^{\frac{1}{q}} \le c_5\|f\|_{\tilde L_q^p(t)} \tag{3.11}$$

holds for some constant $c_5>0$, we find a constant $c_6>0$ such that this and Hölder's inequality imply

$$\begin{aligned}
&\sup_{z\in\mathbb{R}^d}\left(\int_0^t \|1_{B(z,\varepsilon)}\hat P^\kappa_{s,t}f_s\|_{L^{p'}}^q \mathrm{d}s\right)^{\frac{1}{q}} = \sup_{z\in\mathbb{R}^d}\left(\int_0^t \|1_{B(\psi_{s,t}^{-1}(z),\varepsilon)}\hat P^\kappa_{s,t}f_s\|_{L^{p'}}^q \mathrm{d}s\right)^{\frac{1}{q}}\\
&\le \sup_{z\in\mathbb{R}^d}\left(\int_0^t \left\{c_4\|1_{B(z,d)}f_s\|_{L^p}(t-s)^{-\frac{d(p'-p)}{2pp'}}\right\}^q \mathrm{d}s\right)^{\frac{1}{q}} \sup_{r\in(0,T]}\sum_{n=0}^\infty\sum_{v\in\mathbf{B}_n}\mathrm{e}^{-\frac{n^2}{c_3 r}}\\
&\le c_6\left\|(t-\cdot)^{-\frac{d(p'-p)}{2pp'}}f\right\|_{\tilde L_q^p(t)}\sum_{n=0}^\infty\sum_{v\in\mathbf{B}_n}\mathrm{e}^{-\frac{n^2}{c_3 T}}.
\end{aligned}$$

This implies the upper bound for $\hat P^\kappa$ in (3.9), by noting that for some constant $K>0$

$$\sum_{n=0}^\infty\sum_{v\in\mathbf{B}_n}\mathrm{e}^{-\frac{n^2}{c_3T}} \le \sum_{n=0}^\infty K(1+n^{d-1})\mathrm{e}^{-\frac{n^2}{c_3T}}<\infty. \tag{3.12}$$

By (3.2) and integral transforms, the estimate on $\tilde P_{s,t}$ follows from that of $\hat P^\kappa_{s,t}$.

Similarly, we find a constant $K>1$ such that

$$\begin{aligned}
&\|g\hat P^\kappa_{s,t}(1_{B(\psi_{s,t}(z),1)}f)\|_{L^1} \le \sum_{n=0}^\infty\sum_{v\in\mathbb{Z}^d:|v|_1=n}\|1_{B(z+v,d)}g\hat P^\kappa_{s,t}(1_{B(\psi_{s,t}(z),\varepsilon)}f)\|_{L^1}\\
&\le \sum_{n=0}^\infty\sum_{v\in\mathbf{B}_n}\int_{\mathbb{R}^d\times\mathbb{R}^d}|1_{B(z+v,d)}g|(x)p^\kappa_{t-s}(\psi_{s,t}(x)-y)|1_{B(\psi_{s,t}(z),\varepsilon)}f|(y)\mathrm{d}x\mathrm{d}y\\
&\le K(t-s)^{-\frac{d(p'-p)}{2pp'}}\sum_{n=0}^\infty\sum_{v\in\mathbf{B}_n}\mathrm{e}^{-\frac{n^2}{K(t-s)}}\|g\|_{\tilde L^{\frac{p'}{p'-1}}}\|1_{B(z,\varepsilon)}f\|_{L^p}.
\end{aligned}$$

This together with (3.11) and (3.12) implies (3.10) for some $c>0$. □

4 Proof of Theorem 2.1

We first prove assertion 1), i.e. the well-posedness of (2.5). For $\gamma\in\tilde{\mathscr{P}}^k_{\nu,T}$, we denote

$$\begin{aligned}
&\sigma_t^\gamma(x):=\sigma_t(x,\gamma_t),\quad b_t^{\gamma,0}(x):=b_t^{(0)}(x,\gamma_t(x),\gamma_t),\\
&b_t^\gamma(x):=b_t(x,\gamma_t(x),\gamma_t)=b_t^{(1)}(x)+b_t^{\gamma,0}(x),\quad t\in[0,T],x\in\mathbb{R}^d.
\end{aligned}$$

Lemma 4.1. *Assume* **(A)** *with* (A_1) *holding for* σ^γ *replacing* σ *uniformly in* $\gamma\in\tilde{\mathscr P}^k_{\nu,T}$, *where* $k\in[\frac{p_0}{p_0-1},\infty]$. *Then* (2.5) *is weakly well-posed for any* $\mathscr L_{X_0}\in\tilde L^k$ *and* $\gamma\in\tilde{\mathscr P}^k_{\nu,T}$, *and for any* $\beta\in(0,1)$ *there exists a constant* $c>1$ *independent of* ν *and* γ *such that* $\Phi^\nu_t\gamma:=\ell_{X_t^\gamma}$ *for* $\mathscr L_{X_0}=\nu$ *satisfies*

$$\|\Phi^\nu\gamma\|_{\tilde L^k_\infty}\le c\|\nu\|_{\tilde L^k}. \tag{4.1}$$

Moreover, under the assumption with (A_2') *replacing* (A_2), *the assertion holds for* $(\mathscr P^k_{\nu,T},L^k)$ *replacing* $(\tilde{\mathscr P}^k_{\nu,T},\tilde L^k)$.

Proof. (a) By (A_2), we have

$$\sup_{\gamma\in\tilde{\mathscr P}^k_{\nu,T}}\|b^{\gamma,0}\|\le f_0,\quad \|f_0\|_{\tilde L^{p_0}_{q_0}}<\infty. \tag{4.2}$$

According to [17], see also [13, Theorem 1.1(1)], this together with (A_1) and (A_3) imply the well-posedness of (2.5). Moreover, by Theorem 6.2.7(ii)-(iii) in [3], the distribution density function $\ell_{X_t^\gamma}$ exists.

(b) To estimate $\Phi^\nu_t\gamma$ for $\gamma\in\tilde{\mathscr P}^k_{\nu,T}$, consider the SDE

$$\mathrm d\bar X^\gamma_s=b^{(1)}_s(\bar X^\gamma_s)\mathrm ds+\sigma^\gamma_s(\bar X^\gamma_s)\mathrm dW_s,\quad s\in[0,t],\ \bar X^\gamma_0=X^\gamma_0=X_0 \text{ with } \mathscr L_{X_0}=\nu. \tag{4.3}$$

Let $a^\gamma:=\sigma^\gamma(\sigma^\gamma)^*$. Then

$$\mathbb E[f(\bar X^\gamma_t)]=\mathbb E[(P^{a^\gamma,b^{(1)}}_{0,t}f)(X_0)]=\int_{\mathbb R^d\times\mathbb R^d}p^{a^\gamma,b^{(1)}}_{0,t}(x,y)f(y)\nu(\mathrm dx)\mathrm dy,\quad f\in\mathscr B^+(\mathbb R^d),$$

and (3.3) holds for $p^{a^\gamma,b^{(1)}}_{s,t}$ with constants $c,\kappa>0$ uniformly in γ. So, we find a constant $c_1>0$ such that

$$\mathbb E[f(\bar X^\gamma_t)]\le c_1\int_{\mathbb R^d}(\hat P^\kappa_{0,t}f)(x)\nu(\mathrm dx)=c_1(\hat P^{\kappa*}_{0,t}\nu)(f),\quad f\in\mathscr B^+(\mathbb R^d), \tag{4.4}$$

where

$$(\hat P^{\kappa*}_{0,t}\nu)(\mathrm dy):=\bigg(\int_{\mathbb R^d}\hat p^\kappa_{0,t}(x,y)\nu(\mathrm dx)\bigg)\mathrm dy,\quad t\in(0,T],\nu\in\mathscr P. \tag{4.5}$$

On the other hand, let

$$R_t:=\mathrm e^{\int_0^t\langle\xi_s,\mathrm dW_s\rangle-\frac12\int_0^t|\xi_s|^2\mathrm ds},\quad \xi_s:=\big\{\sigma^\gamma_s(\sigma_s(\sigma^\gamma_s)^*)^{-1}b^{\gamma,0}_s\big\}(\bar X_s).$$

By (4.2), the uniform boundedness of $\|\sigma^\gamma(\sigma^\gamma(\sigma^\gamma)^*)^{-1}\|_\infty$, and Khasminskii's estimate implied by the Krylov's estimate in [17, Theorem 3.1] (see the proof of [14, Lemma 4.1(ii)]), we find a map $K_\gamma:[1,\infty)\to(0,\infty)$ such that

$$K_\gamma(p):=(\mathbb E[R_t^p])^{\frac1p}<\infty,\quad p\ge1. \tag{4.6}$$

By Girsanov's theorem,

$$\tilde W_s := W_s - \int_0^s \xi_r \mathrm{d}r, \quad s \in [0,t]$$

is an m-dimensional Brownian motion under the probability measure $\mathbb{Q}_t := R_t \mathbb{P}$, with which the SDE (4.3) reduces to

$$\mathrm{d}\bar X_s = b_s^\gamma(\bar X_s)\mathrm{d}s + \sigma_s^\gamma(\bar X_s)\mathrm{d}\tilde W_s, \quad s \in [0,t], \bar X_0 = X_0^\gamma.$$

By the weak uniqueness, the law of X_t^γ under $\mathbb{P}$ coincides with that of $\bar X_t$ under $\mathbb{Q}_t$. Combining this with (4.4), (4.6) and (3.10), for any $p>1$ and $k' \ge k$ we find constants $c_1(p), c_2(p) > 0$ such that

$$\begin{aligned}
&\int_{\mathbb{R}^d} \big\{(\Phi_t^\nu \gamma) 1_{B(z,1)} f\big\}(y)\mathrm{d}y = \mathbb{E}\big[(1_{B(z,1)}f)(X_t^\gamma)\big] = \mathbb{E}\big[R_t(1_{B(z,1)}f)(\bar X_t^\gamma)\big] \\
&\le \big(\mathbb{E}[R_t^{\frac{p}{p-1}}]\big)^{\frac{p-1}{p}} \big(\mathbb{E}\big[(1_{B(z,1)}f^p)(\bar X_t^\gamma)\big]\big)^{\frac{1}{p}} \le c_1(p)\bigg(\int_{\mathbb{R}^d} \big\{(\hat P_{0,t}^\kappa(1_{B(z,1)}f^p)\big\}(x)\nu(\mathrm{d}x)\bigg)^{\frac{1}{p}} \\
&\le c_2(p)\|\nu\|_{\tilde L^k}^{\frac{1}{p}} t^{-\frac{d(k'-k)}{2kk'p}} \|f\|_{\tilde L^{\frac{pk'}{k'-1}}}, \quad t \in (0,T], f \in \mathscr{B}^+(\mathbb{R}^d).
\end{aligned}$$

Therefore, for any $\nu \in \tilde{\mathscr{P}}^k$,

$$\|\Phi_t^\nu \gamma\|_{\tilde L^{\frac{pk'}{pk'-k'+1}}} \le c_2(p)\|\nu\|_{\tilde L^k}^{\frac{1}{p}} t^{-\frac{d(k'-k)}{2kk'p}}, \quad p>1, k' \ge k, \gamma \in \tilde{\mathscr{P}}_{\nu,T}^k, t \in (0,T], \tag{4.7}$$

where for $k' = k = \infty$ we set $\frac{pk'}{pk'-k'+1} := \frac{p}{p-1}$, $\frac{d(k'-k)}{2kk'p} := 0$. Using (3.8) replacing the estimate in Lemma 3.2, we find a ma $c : (1,\infty) \to (0,\infty)$ such that

$$\|\Phi_t^\nu \gamma\|_{L^{\frac{pk'}{pk'-k'+1}}} \le c(p)\|\nu\|_{L^k}^{\frac{1}{p}} t^{-\frac{d(k'-k)}{2kk'p}}, \quad p>1, k' \ge k, \gamma \in L_\infty^k \cap \mathscr{D}_+^1, t \in (0,T]. \tag{4.8}$$

(c) By the backward Kolmogorov equation (3.1) and Itô's formula, for any $f \in C_0^\infty(\mathbb{R}^d)$ we have

$$\begin{aligned}
\mathrm{d}\big\{(P_{s,t}^{a^\gamma,b^{(1)}} f)(X_s^\gamma)\big\} &= \big\{\big(\partial_s + L_s^{a^\gamma,b^{(1)}} + \nabla_{b_s^{\gamma,0}}\big) P_{s,t}^{a^\gamma,b^{(1)}} f\big\}(X_s^\gamma)\mathrm{d}s + \mathrm{d}M_s \\
&= \big\{\nabla_{b_s^{\gamma,0}} P_{s,t} f\big\}(X_s^\gamma)\mathrm{d}s + \mathrm{d}M_s, \quad s \in [0,t]
\end{aligned}$$

for some martingale M_s. Then

$$\begin{aligned}
&\mathbb{E}[f(X_t^\gamma)] = \mathbb{E}[P_{t,t}^{a^\gamma,b^{(1)}} f(X_t^\gamma)] \\
&= \mathbb{E}[P_{0,t}^{a^\gamma,b^{(1)}} f(X_0)] + \int_0^t \mathbb{E}\big[(\nabla_{b_s^{\gamma,0}} P_{s,t}^{a^\gamma,b^{(1)}} f)(X_s^\gamma)\big]\mathrm{d}s, \quad s \in [0,t].
\end{aligned} \tag{4.9}$$

We explain that the last term in (4.9) exists. Indeed, by [13, Theorem 1.1(2)], there exists a constant $c_2 > 0$ such that

$$\|\nabla P_{s,t}^{a^\gamma,b^{(1)}} f\|_\infty \le c_2 \|\nabla f\|_\infty, \quad 0 \le s \le t, f \in C_b^1(\mathbb{R}^d),$$

so that (4.2) and Krylov's estimate (see Theorem 3.1 in [17]) yield

$$\mathbb{E}\bigg(\int_0^t \big|\big(\nabla_{b_s^{\gamma,0}} P_{s,t}^{a^\gamma,b^{(1)}} f\big)(X_s^\gamma)\big|\mathrm{d}s\bigg)^n \le \mathbb{E}\bigg(\int_0^t c_2\|\nabla f\|_\infty |b_s^{\gamma,0}|(X_s^\gamma)\mathrm{d}s\bigg)^n < \infty, \quad n \ge 1.$$

Noting that $\Phi_s^\nu \gamma := \ell_{X_s^\gamma}$ and $P_{s,t}^{a^\gamma,b^{(1)}} f(x) = \int_{\mathbb{R}^d} p_{s,t}^{a^\gamma,b^{(1)}}(x,y)f(y)\mathrm{d}y$, (4.9) is equivalent to

$$\int_{\mathbb{R}^d} \big\{\Phi_s^\nu f\big\}(y)\mathrm{d}y = \int_{\mathbb{R}^d\times\mathbb{R}^d} \nu(x) p_{0,t}^{a^\gamma,b^{(1)}}(x,y)f(y)\mathrm{d}x\mathrm{d}y$$
$$+ \int_0^t \mathrm{d}s \int_{\mathbb{R}^d\times\mathbb{R}^d} (\Phi_s^\nu\gamma)(x)\big\{\nabla_{b_s^{\gamma,0}} p_{s,t}^{a^\gamma,b^{(1)}}(\cdot,y)(x)\big\}f(y)\mathrm{d}y, \quad f \in C_0^\infty(\mathbb{R}^d), s \in [0,t].$$

Thus,

$$\begin{aligned}(\Phi_t^\nu\gamma)(y) = &\int_{\mathbb{R}^d} p_{0,t}^{a^\gamma,b^{(1)}}(x,y)\nu(\mathrm{d}x)\\ &+ \int_0^t \mathrm{d}s \int_{\mathbb{R}^d} (\Phi_s^\nu\gamma)(x)\big\{\nabla_{b_s^{\gamma,0}} p_{s,t}^{a^\gamma,b^{(1)}}(\cdot,y)(x)\big\}\mathrm{d}x, \quad t \in [0,T].\end{aligned} \tag{4.10}$$

By (3.8) for $p = p'$, $\|\hat{P}_t^{\kappa *}\nu\|_{\tilde{L}^l} \le K\|\nu\|_{\tilde{L}^l}$ holds for some constant $K > 0$. Combining this with (3.3), (4.2) and (4.10), we find a constant $c_3 > 0$ such that

$$\|\Phi_t^\nu\gamma\|_{\tilde{L}^l} \le c_3\|\nu\|_{\tilde{L}^l} + c_3 \sup_{z\in\mathbb{R}^d} \int_0^t (t-s)^{-\frac{1}{2}} \big\|1_{B(z,1)} \hat{P}_{t-s}^\kappa \big\{(\Phi_s^\nu\gamma) f_0(s,\cdot)\big\}\big\|_{L^l}\mathrm{d}s, \quad l \in [1,\infty]. \tag{4.11}$$

By $k > k_0$ and $k \ge \frac{p_0}{p_0-1}$, for any $l \in (k_0,k] \cap [\frac{p_0}{p_0-1},k]$ we have

$$q_l := \frac{p_0 l}{p_0 + l} \in (1,l], \qquad \frac{1}{q_l} = \frac{1}{p_0} + \frac{1}{l}, \tag{4.12}$$

and $(p_0,q_0) \in \mathscr{K}$ implies

$$\frac{1}{2} + \frac{d(l-q_l)}{2lq_l} = \frac{1}{2} + \frac{d}{2p_0} =: \delta' < \frac{q_0-1}{q_0}. \tag{4.13}$$

Combining these with (3.9) for $(p',p) = (l,q_l)$ and applying Hölder's inequality, we find a constant $c_4 > 0$ such that

$$\begin{aligned}&\int_0^t (t-s)^{-\frac{1}{2}} \big\|\hat{P}_{t-s}^\kappa\big\{(\Phi_s^\nu\gamma)f_0(s,\cdot)\big\}\big\|_{\tilde{L}^l}\mathrm{d}s \le c_4\big\|(t-\cdot)^{-\delta'} f_0 \Phi_\cdot^\nu\gamma\big\|_{\tilde{L}_1^{q_l}(t)}\\ &\le c_4\|f_0\|_{\tilde{L}_{q_0}^{p_0}} \big\|(t-\cdot)^{-\delta'}\Phi_\cdot^\nu\gamma\big\|_{\tilde{L}^l_{\frac{q_0}{q_0-1}}(t)}, \quad l \in (k_0,k] \cap \Big[\frac{p_0}{p_0-1},k\Big],\end{aligned}$$

where $\{(t-\cdot)^{-\delta'}\Phi_\cdot^\nu\}(s,x) := (t-s)^{-\delta'}\Phi_s^\nu(x)$. This together with (4.11) implies that for some constant $c_5 > 0$,

$$\begin{aligned}&\|\Phi_t^\nu\gamma\|_{\tilde{L}^l} \le c_5\|\nu\|_{\tilde{L}^l} + c_5\|f_0\|_{\tilde{L}_{q_0}^{p_0}} \bigg(\int_0^t \Big\{(t-s)^{-\delta'}\|\Phi_s^\nu\gamma\|_{\tilde{L}^l}\Big\}^{\frac{q_0}{q_0-1}}\mathrm{d}s\bigg)^{\frac{q_0-1}{q_0}},\\ &\qquad t \in [0,T], l \in (k_0,k] \cap \Big[\frac{p_0}{p_0-1},k\Big].\end{aligned} \tag{4.14}$$

Similarly, using (3.8) replacing Lemma 3.2, we derive the same estimate for L^l replacing $\tilde{L}^l$ and $\|f_0\|_{L^{p_0}_{q_0}}$ replacing $\|f_0\|_{\hat{L}^{p_0}_{q_0}}$:

$$\begin{aligned}(4.15)\qquad &\|\Phi_t^\nu\gamma\|_{L^l} \le c_5\|\nu\|_{L^l} + c_5\|f_0\|_{L^{p_0}_{q_0}}\bigg(\int_0^t \Big\{(t-s)^{-\delta'}\|\Phi_s^\nu\gamma\|_{\tilde{L}^l}\Big\}^{\frac{q_0}{q_0-1}}\mathrm{d}s\bigg)^{\frac{q_0-1}{q_0}},\\ &t\in[0,T], l\in(k_0,k]\cap\Big[\frac{p_0}{p_0-1},k\Big].\end{aligned}$$

Below we prove (4.1) by considering two different situations.

(c_1) $k<\infty$. For any $k'\in(k,\infty)$ we have

$$p_{k,k'} := \frac{k(k'-1)}{k'(k-1)} > 1,\quad \frac{p_{k,k'}k'}{p_{k,k'}k'-k'+1} = k.$$

Noting that

$$\lim_{k'\downarrow k}\frac{d(k'-k)}{2kk'p_{k,k'}} = 0,$$

by (4.13) we find $k'>k$ such that

$$\varepsilon_{k,k'} := \frac{d(k'-k)}{2kk'p_{k,k'}} \in \Big(0, 1-\frac{\delta' q_0}{q_0-1}\Big).$$

Combining this with (4.7) and (4.14) for $l=k$, we find a constant $K>0$ such that

$$\sup_{t\in[0,T]}\|\Phi_t^\nu\gamma\|_{\tilde{L}^k} \le K\|\nu\|_{\tilde{L}^k} + K\sup_{t\in[0,T]}\bigg(\int_0^t (t-s)^{-\frac{q_0\delta'}{q_0-1}}s^{-\varepsilon_{k,k'}}\|\nu\|_{\tilde{L}^k}^{\frac{q_0}{p_{k,k'}(q_0-1)}}\mathrm{d}s\bigg)^{\frac{q_0-1}{q_0}} < \infty.$$

Therefore, by the generalized Gronwall inequality (see [16]), (4.13) and (4.14) implies (4.1).

When $f_0\in L^{p_0}_{q_0}$, by using (4.8) and (4.15) replacing (4.7) and (4.14), we obtain this estimate for L replacing $\tilde{L}$.

(c_2) $k=\infty$. We take $k'=k=\infty$, so that by (4.7), for any $p>1$ we find a constant $c(p)>0$ such that

$$\|\Phi_t^\nu\gamma\|_{\tilde{L}^{\frac{p}{p-1}}} \le c(p)\|\nu\|_{\tilde{L}^k}^{\frac{1}{p}}.$$

Combining this with (4.14) for $l\in(\frac{p_0}{p_0-1}\vee k_0,\infty)$ and $p:=\frac{l}{l-1}>1$, we obtain

$$\sup_{t\in[0,T]}\|\Phi_t^\nu\gamma\|_{\tilde{L}^l}<\infty,$$

so that by the generalized Gronwall inequality, (4.14) implies (4.1) for $l\in(\frac{p_0}{p_0-1}\vee k_0,\infty)$ replacing $k=\infty$ with a uniform constant $c>0$. By letting $l\uparrow k=\infty$, we prove (4.1).

Noting that a probability density function $\rho\in L^\infty$ implies $\rho\in L^l$ for any $l\ge1$, when $f_0\in L^{p_0}_{q_0}$ we prove (4.1) for L replacing $\tilde{L}$ by using (4.8) and (4.15) replacing (4.7) and (4.14). □

Proof of Theorem 2.1(1). By Lemma 3.2, (2.5) is weakly well-posed. By [17] or [13], it is also strongly well-posed provided (2.2) holds. Thus, as explained in the end of Section 1 that for the weak or strong well-posedness of (1.1), it suffices to prove that Φ^ν has a unique fixed point in $\tilde{\mathscr{P}}^k_{\nu,T}$. In general, for any $\nu_1,\nu_2\in\tilde{\mathscr{P}}^k$ and $\gamma^1,\gamma^2\in\tilde{\mathscr{P}}^k_{\nu,T}$, we estimate

$$\tilde d_{k,\lambda}(\Phi^{\nu_1}\gamma^1,\Phi^{\nu_2}\gamma^2):=\sup_{t\in[0,T]}\mathrm{e}^{-\lambda t}\|\Phi^{\nu_1}_t\gamma^1-\Phi^{\nu_2}_t\gamma^2\|_{\tilde L^k},\quad \lambda>0.$$

By (4.10), (A_2) and (3.3), we find a constant $c_1>0$ such that

$$\begin{aligned}&\|\Phi^{\nu_1}_t\gamma^1-\Phi^{\nu_2}_t\gamma^2\|_{\tilde L^k}-c_1\|\nu_1-\nu_2\|_{\tilde L^k}\\ &\le c_1\int_0^t(t-s)^{-\frac12}\Big\|\hat P^\kappa_{s,t}\Big\{f_0(s,\cdot)\big[|\Phi^{\nu_1}_s\gamma^1-\Phi^{\nu_2}_s\gamma^2|\\ &\qquad+s^\theta(\Phi^{\nu_1}_s\gamma^1)\big(|\gamma^1_s-\gamma^2_s|+\|\gamma^1_s-\gamma^2_s\|_{\tilde L^k}\big)\big]\Big\}\Big\|_{\tilde L^k}\mathrm{d}s.\end{aligned}\tag{4.16}$$

Letting

$$F_l(s,x):=(t-s)^{-\frac{d(k-l)}{2kl}-\frac12}\big[|\Phi^{\nu_1}_s\gamma^1-\Phi^{\nu_2}_s\gamma^2|+s^\theta(\Phi^{\nu_1}_s\gamma^1)\big(|\gamma^1_s-\gamma^2_s|+\|\gamma^1_s-\gamma^2_s\|_{\tilde L^k}\big)\big](x)$$

for $l\in[1,\frac{kp_0}{k+p_0}]$, by (3.9) for $q=1$ and $(p',p)=(k,l)$, and applying Hölder's inequality, we find a constant $c_2>0$ such that

$$\begin{aligned}&\int_0^t(t-s)^{-\frac12}\Big\|\hat P^\kappa_{s,t}\Big\{f_0(s,\cdot)\big[|\Phi^{\nu_1}_s\gamma^1-\Phi^{\nu_2}_s\gamma^2|+s^\theta(\Phi^{\nu_1}_s\gamma^1)\big(|\gamma^1_s-\gamma^2_s|+\|\gamma^1_s-\gamma^2_s\|_{\tilde L^k}\big)\big]\Big\}\Big\|_{\tilde L^k}\mathrm{d}s\\ &\le c_2\|f_0F_l\|_{\tilde L^l_1}\le c\|f_0\|_{\tilde L^{p_0}_{q_0}}\|F_l\|_{\tilde L^{\frac{p_0l}{p_0-l}}_{\frac{q_0}{q_0-1}}}\\ &\le c_2\|f_0\|_{\tilde L^{p_0}_{q_0}}\bigg(\int_0^t\Big\{(t-\cdot)^{-\frac{d(k-l)}{2kl}-\frac12}\Big[\big\|\Phi^{\nu_1}\gamma^1-\Phi^{\nu_2}\gamma^2\big\|_{\tilde L^{\frac{p_0l}{p_0-l}}}\\ &\qquad+s^\theta\big\|\big(\Phi^{\nu_1}_s\gamma^1\big)\big(|\gamma^1_s-\gamma^2_s|+\|\gamma^1_s-\gamma^2_s\|_{\tilde L^k}\big)\big\|_{\tilde L^{\frac{p_0k}{p_0-k}}}\Big]\Big\}^{\frac{q_0}{q_0-1}}\mathrm{d}s\bigg)^{\frac{q_0-1}{q_0}}.\end{aligned}$$

Since $l\in[1,\frac{kp_0}{k+p_0}]$ implies $\frac{p_0l}{p_0-l}\le k$, combining this with (4.16) and applying Hölder's inequality, we find a constant $c_3>0$ such that

$$\begin{aligned}&\|\Phi^{\nu_1}_t\gamma^1-\Phi^{\nu_2}_t\gamma^2\|_{\tilde L^k}-c_1\|\nu_1-\nu_2\|_{\tilde L^k}\\ &\le c_3\bigg(\int_0^t\Big\{(t-s)^{-\frac{d(k-l)}{2kl}-\frac12}\Big[\big\|\Phi^{\nu_1}_s\gamma^1-\Phi^{\nu_2}_s\gamma^2\big\|_{\tilde L^k}\\ &\qquad+s^\theta\big\|\Phi^{\nu_1}_s\gamma^1\big\|_{\tilde L^{\frac{kp_0l}{k(p_0-l)-p_0l}}}\big\|\gamma^1_s-\gamma^2_s\big\|_{\tilde L^k}\Big]\Big\}^{\frac{q_0}{q_0-1}}\mathrm{d}s\bigg)^{\frac{q_0-1}{q_0}},\quad l\in\Big[1,\frac{kp_0}{k+p_0}\Big].\end{aligned}\tag{4.17}$$

Letting

$$\alpha_l:=\frac{q_0}{q_0-1}\Big(\frac{d(k-l)}{2kl}+\frac12\Big),\quad \beta_l:=\frac{kp_0l}{k(p_0-l)-p_0l},\tag{4.18}$$

by the definition of $\tilde d_{k,\lambda}$, this implies that for any $\lambda>0$ and $l\in[1,\frac{kp_0}{k+p_0}]$,

$$\begin{aligned}
&\tilde d_{k,\lambda}(\Phi^{\nu_1}\gamma^1,\Phi^{\nu_2}\gamma^2)\le c_1\|\nu_1-\nu_2\|_{\tilde L^k}\\
&+c_3\tilde d_{k,\lambda}(\Phi^{\nu_1}\gamma^1,\Phi^{\nu_2}\gamma^2)\sup_{t\in(0,T]}\bigg(\int_0^t(t-s)^{-\alpha_l}\mathrm{e}^{-\frac{\lambda q_0}{q_0-1}(t-s)}\mathrm{d}s\bigg)^{\frac{q_0-1}{q_0}}\\
&+c_3\tilde d_{k,\lambda}(\gamma^1,\gamma^2)\sup_{t\in(0,T]}\bigg(\int_0^t(t-s)^{-\alpha_l}\mathrm{e}^{-\frac{\lambda q_0}{q_0-1}(t-s)}\big(s^\theta\|\Phi_s^{\nu_1}\gamma^1\|_{\tilde L^{\beta_l}}\big)^{\frac{q_0}{q_0-1}}\mathrm{d}s\bigg)^{\frac{q_0-1}{q_0}}.
\end{aligned}\tag{4.19}$$

Below we complete the proof by considering two different situations respectively.

(a) Let $k<\infty$. By $(p_0,q_0)\in\mathscr{K}$ and $k>k_0:=\frac{d}{2\theta+1-dp_0^{-1}-2q_0^{-1}}$, α_l in (4.18) satisfies

$$\lim_{l\uparrow\frac{kp_0}{k+p_0}}\alpha_l+\frac{q_0}{q_0-1}\Big(\frac{d}{2k}-\theta\Big)^+=\frac{q_0}{q_0-1}\Big\{\frac{d}{2p_0}+\frac12+\Big(\frac{d}{2k}-\theta\Big)^+\Big\}<1.$$

So, we may take $l\in(1,\frac{kp_0}{k+p_0})$ such that

$$\alpha_l+\frac{q_0}{q_0-1}\Big(\frac{d}{2k}-\theta\Big)^+<1,\quad \beta_l\in(1,\infty).\tag{4.20}$$

By (4.7) for $k'=\infty$ and $p=\frac{\beta_l}{\beta_l-1}$, there exists a constant $c_4>0$ such that

$$\|\Phi_s^{\nu_1}\gamma^1\|_{L^{\beta_l}}\le c_4\|\nu_1\|_{\tilde L^k}s^{-\frac{d}{2k}}.$$

Combining this with (4.19) and (4.20), when λ is large enough increasing in $\|\nu_1\|_{\tilde L^k}(\le\|\nu_2\|_{\tilde L^k})$, we obtain

$$\tilde d_{k,\lambda}(\Phi^{\nu_1}\gamma^1,\Phi^{\nu_2}\gamma^2)\le c_1\|\nu_1-\nu_2\|_{\tilde L^k}+\frac14\tilde d_{k,\lambda}(\Phi^{\nu_1}\gamma^1,\Phi^{\nu_2}\gamma^2)+\frac14\tilde d_{k,\lambda}(\gamma^1,\gamma^2).$$

Taking $\nu_1=\nu_2=\nu$ we prove the contraction of Φ^ν on the complete metric space $(\tilde{\mathscr{P}}^k_{\nu,T},\tilde d_{k,\lambda})$, and hence Φ^ν has a unique fixed point. This implies the weak (also strong under (2.2)) well-posedness of (1.1). Moreover, for two solutions $(X^i)_{i=1,2}$ of this SDE with initial distribution densities $(\nu_i)_{i=1,2}$, by taking $\gamma^i=\mathscr{L}_{X^i_\cdot}$ we have $\gamma^i=\Phi^{\nu_i}\gamma^i$, so that this estimate implies (2.1) for some increasing function Λ.

(b) Let $k=\infty$. By taking $l=p_0$, we have $\beta_l=\infty$ and $\theta>\frac{2}{q_0}+\frac{d}{p_0}-1$ in (A_2) implies

$$\alpha_l+\frac{q_0}{q_0-1}\Big(\frac{d}{2k}-\theta\Big)^+=\frac{q_0}{q_0-1}\Big\{\frac{d}{2p_0}+\frac12+\theta^-\Big\}<1.$$

Combining (4.19) with (4.1) for $k=\infty$, we derive that for a large enough $\lambda>0$ increasing in $\|\nu_1\|_{\tilde L^\infty}$ $(\le\|\nu_2\|_{\tilde L^\infty})$,

$$\tilde d_{k,\lambda}(\Phi^{\nu_1}\gamma^1,\Phi^{\nu_2}\gamma^2)\le c_1\|\nu_1-\nu_2\|_{\tilde L^\infty}$$

$$
\begin{aligned}
&+ c_3\tilde{d}_{k,\lambda}(\Phi^{\nu_1}\gamma^1,\Phi^{\nu_2}\gamma^2)\sup_{t\in(0,T]}\bigg(\int_0^t (t-s)^{-\alpha_l}\mathrm{e}^{-\frac{\lambda q_0}{q_0-1}(t-s)}\mathrm{d}s\bigg)^{\frac{q_0-1}{q_0}}\\
&+ c_3\tilde{d}_{k,\lambda}(\gamma^1,\gamma^2)\sup_{t\in(0,T]}\bigg(\int_0^t (t-s)^{-\alpha_l}\mathrm{e}^{-\frac{\lambda q_0}{q_0-1}(t-s)}(s^\theta\|\nu_1\|_{\tilde{L}^\infty})^{\frac{q_0}{q_0-1}}\mathrm{d}s\bigg)^{\frac{q_0-1}{q_0}}\\
&\le c_1\|\nu_1-\nu_2\|_{\tilde{L}^\infty}+\frac{1}{4}\tilde{d}_{k,\lambda}(\Phi^{\nu_1}\gamma^1,\Phi^{\nu_2}\gamma^2)+\frac{1}{4}\tilde{d}_{k,\lambda}(\gamma^1,\gamma^2).
\end{aligned}
$$

Then we finish the proof as shown in step (a). □

Proof of Theorem 2.1(2). Let **(A)** hold for (A_2') replacing (A_2). By (3.8) and Hölder's inequality, we find constants $c_1, c_2 > 0$ such that for any $0 \le s < t \le T$ and $l \in [1, \frac{kp_0}{k+p_0}]$,

$$
\begin{aligned}
&\Big\|\hat{P}^\kappa_{s,t}\Big\{\big(C+f_0(s,\cdot)\big)\big(|\Phi^{\nu_1}_s\gamma^1-\Phi^{\nu_2}_s\gamma^2|+s^\theta(\Phi^{\nu_1}_s\gamma^1)|\gamma^1_s-\gamma^2_s|\big)\Big\}\Big\|_{L^k}\\
&\le c_1(t-s)^{-\frac{d(k-l)}{2kl}}\Big\{\|\Phi^{\nu_1}_s\gamma^1-\Phi^{\nu_2}_s\gamma^2\|_{L^l}+s^\theta\big\|(\Phi^{\nu_1}_s\gamma^1)|\gamma^1_s-\gamma^2_s|\big\|_{L^l}\\
&\qquad+\big\|f_0(s,\cdot)(\Phi^{\nu_1}_s\gamma^1-\Phi^{\nu_2}_s\gamma^2)\big\|_{L^l}+s^\theta\big\|f_0(s,\cdot)(\Phi^{\nu_1}_s\gamma^1)|\gamma^1_s-\gamma^2_s|\big\|_{L^l}\Big\}\\
&\le c_1(t-s)^{-\frac{d(k-l)}{2kl}}\Big\{\|\Phi^{\nu_1}_s\gamma^1-\Phi^{\nu_2}_s\gamma^2\|_{L^l}+s^\theta\big\|\Phi^{\nu_1}_s\gamma^1\big\|_{L^{\frac{kl}{k-l}}}\big\||\gamma^1_s-\gamma^2_s|\big\|_{L^k}\\
&\quad+\big\|f_0(s,\cdot)\big\|_{L^{p_0}}\Big(\big\|\Phi^{\nu_1}_s\gamma^1-\Phi^{\nu_2}_s\gamma^2\big\|_{L^{\frac{p_0l}{p_0-l}}}+s^\theta\big\|\Phi^{\nu_1}_s\gamma^1\big\|_{L^{\frac{p_0kl}{p_0k-kl-p_0l}}}\big\|\gamma^1_s-\gamma^2_s\big\|_{L^k}\Big)\Big\}.
\end{aligned}
$$

Noting that $l \in [1, \frac{kp_0}{k+p_0}]$ implies $l \vee \frac{p_0l}{p_0-l} \le k$ and $\frac{kl}{k-l} \le \frac{p_0kl}{p_0k-kl-p_0l}$, by combining this with (A_2'), (3.3), (4.10) and Hölder's inequality, we find constants $c_3, c_4 > 0$ such that

$$
\begin{aligned}
&\|\Phi^{\nu_1}_t\gamma^1-\Phi^{\nu_2}_t\gamma^2\|_{L^k}-c_1\|\nu_1-\nu_2\|_{L^k}\\
&\le c_3\int_0^t (t-s)^{-\frac{1}{2}}\Big\|\hat{P}^\kappa_{s,t}\Big\{\big(C+f_0(s,\cdot)\big)\big(|\Phi^{\nu_1}_s\gamma^1-\Phi^{\nu_2}_s\gamma^2|+s^\theta(\Phi^{\nu_1}_s\gamma^1)|\gamma^1_s-\gamma^2_s|\big)\Big\}\Big\|_{L^k}\mathrm{d}s\\
&\le c_4\big(1+\|f_0\|_{\tilde{L}^{p_0}_{q_0}}\big)\bigg(\int_0^t\Big\{\|(t-s)^{-\frac{d(k-l)}{2kl}-\frac{1}{2}}\Big[\|\Phi^{\nu_1}_s\gamma^1-\Phi^{\nu_2}_s\gamma^2\|_{L^k}\\
&\qquad\qquad+s^\theta\|\Phi^{\nu_1}_s\gamma^1\|_{L^{\frac{kp_0l}{k(p_0-l)-p_0l}}}|\gamma^1_s-\gamma^2_s\|_{L^k}\Big]\Big\}^{\frac{q_0}{q_0-1}}\mathrm{d}s\bigg)^{\frac{q_0-1}{q_0}},\quad l\in\Big[1,\frac{kp_0}{k+p_0}\Big].
\end{aligned}
$$

Then the remainder of the proof is similar to that of Theorem 2.1(1) from (4.17) with L replacing $\tilde{L}$. □

5 Proof of Theorem 2.2

Let $\nu \in \mathscr{P}^\infty$ with $\ell_\nu \in C_b^\beta$. By Theorem 2.1 and **(B)**, for any $\gamma \in \mathscr{P}^\infty_{\nu,T}$, the following density dependent SDE has a unique (weak and strong) solution with $\ell_{X^{\gamma,\nu}_\cdot} \in L^\infty_\infty$:

$$
\mathrm{d}X^{\gamma,\nu}_t = b_t(X^{\gamma,\nu}_t, \ell_{X^{\gamma,\nu}_t}(X^{\gamma,\nu}_t), \ell_{X^{\gamma,\nu}_t})\mathrm{d}t + \sigma^\gamma_t(X^{\gamma,\nu}_t),\quad \mathscr{L}_{X^{\gamma,\nu}_0} = \nu, t\in[0,T], \tag{5.1}
$$

and there exists a constant $c>0$ depending on C,α such that

$$\|\ell_{X_t^{\gamma,\nu}}\|_\infty \le c\|\ell_\nu\|_\infty,\quad \gamma\in L_\infty^\infty\cap\mathscr{D}_+^1. \tag{5.2}$$

We aim to show that the map

$$\gamma\mapsto \ell_{X_\cdot^{\gamma,\nu}}$$

has a unique fixed point in $L_\infty^\infty\cap\mathscr{D}_+^1$, such that the (weak and strong) well-posedness of (5.1) implies that of (1.1). As shown in the proof of Theorem 2.1, we will need heat kernel estimates presented in Section 2 for the operator $L_t^{a^\gamma,b^{\gamma,\nu}}$, where

$$a_t^\gamma := \frac{1}{2}\sigma_t^\gamma(\sigma_t^\gamma)^*,\quad b_t^{\gamma,\nu} := b_t\big(\cdot,\ell_{X_t^{\gamma,\nu}}(\cdot),\ell_{X_t^{\gamma,\nu}}\big),\quad t\in[0,T].$$

To this end, we first prove the Hölder continuity of $b_t^{\gamma,\nu}$. By **(B)**, this follows from the Hölder continuity of $\ell_{X_t^{\gamma,\nu}}$.

Lemma 5.1. *Assume* **(B)** *and let $\beta\in(0,1-\frac{d}{p_0}-\frac{2}{q_0})$. Then there exists a constant $c>0$ such that for any $\gamma\in L_\infty^\infty\cap\mathscr{D}_+^1$ and $\nu\in\mathscr{P}^\infty$ with $\ell_\nu\in C_b^\beta$,*

$$\|\ell_{X_t^{\gamma,\nu}}\|_{C_b^\beta} \le c\|\ell_\nu\|_{C_b^\beta},\quad t\in(0,T]. \tag{5.3}$$

Proof. Simply denote $\ell_t=\ell_{X_t^{\gamma,\nu}}$. Let $p_{s,t}^\gamma$ be the heat kernel for the operator

$$L_t^\gamma := \frac{1}{2}\mathrm{div}\big\{a_t^\gamma\nabla\big\} = L_t^{a^\gamma,\bar b^\gamma},$$

where

$$a_t^\gamma := \frac{1}{2}\sigma_t^\gamma(\sigma_t^\gamma)^*,\quad (\bar b_t^\gamma)_i := \frac{1}{2}\sum_{j=1}^d \partial_j(a_t^\gamma)_{ij}.$$

Then $p_{s,t}^\gamma(x,y)=p_{s,t}^\gamma(y,x)$, and by **(B)** and Theorem 3.1, there exist constants $c,\kappa>0$ depending on C,α,β such that for some diffeomorphisms $\{\psi_{s,t}\}_{0\le s\le t\le T}$ satisfying (3.2),

$$\begin{aligned}
&|\nabla^i p_{s,t}^\gamma(\cdot,y)(x)| \le c_1(t-s)^{-\frac{i}{2}} p_{t-s}^\kappa(\psi_{s,t}(x)-y),\quad i=0,1,2,\\
&|\nabla p_{s,t}^\gamma(x,\cdot)(y)| \le c_1(t-s)^{-\frac{1}{2}} p_{t-s}^\kappa(\psi_{s,t}(x)-y),\\
&|\nabla p_{s,t}^\gamma(\cdot,y)(x)-\nabla p_{s,t}^\gamma(\cdot,y')(x)|\\
&\le c_1|y-y'|^\beta(t-s)^{-\frac{1+\beta}{2}}\big\{p_{t-s}^\kappa(\psi_{s,t}(x)-y)+p_{t-s}^\kappa(\psi_{s,t}(x)-y')\big\},\\
&\qquad 0\le s<t\le T,\ x,y,y'\in\mathbb{R}^d.
\end{aligned} \tag{5.4}$$

By the argument leading to (4.10) for $\bar b^\gamma$ replacing $b^{(1)}$, we obtain

$$\ell_t(y) = \int_{\mathbb{R}^d} p_{0,t}^\gamma(x,y)\ell_\nu(x)\mathrm{d}x + \int_0^t \mathrm{d}s\int_{\mathbb{R}^d}\ell_s(x)\big\{\nabla_{b_s(x,\ell_s(x),\ell_s)-\bar b_s^\gamma(x)} p_{s,t}^\gamma(\cdot,y)\big\}(x)\mathrm{d}x. \tag{5.5}$$

By the symmetry of $p_{0,t}^\gamma(x,y)$ we have

$$\int_{\mathbb{R}^d} p_{0,t}^\gamma(x,y)\ell_\nu(x)\mathrm{d}x = \int_{\mathbb{R}^d} p_{0,t}^\gamma(y,x)\ell_\nu(x)\mathrm{d}x =: (P_{0,t}^\gamma\ell_\nu)(y). \tag{5.6}$$

Let X_t^x solve the SDE

$$\mathrm{d}X_t^x = \bar{b}_t^\gamma(X_t^x)\mathrm{d}t + \sigma_t^\gamma(X_t)\mathrm{d}W_t, \quad t \in [0,T], X_0 = x.$$

By [14, (4.8)], we find a constant $c_1 > 0$ depending on C, α in **(B)** such that

$$\mathbb{E}\Big[\sup_{t\in[0,T]} |X_t^x - X_t^y|\Big] \le c_1|x-y|, \quad x, y \in \mathbb{R}^d.$$

Then (5.6) implies

$$\begin{aligned}
&|(P_{0,t}^\gamma \ell_\nu)(y) - (P_{0,t}^\gamma \ell_\nu)(y')| = |\mathbb{E}[\ell_\nu(X_t^y) - \ell_\nu(X_t^{y'})]| \\
&\le \|\ell_\nu\|_{C_b^\beta}\mathbb{E}[|X_t^y - X_t^{y'}|^\beta] \le \|\ell_\nu\|_{C_b^\beta}(c_1|y-y'|)^\beta.
\end{aligned}$$

Since **(B)** implies $|b| + |\bar{b}^\gamma| \le cf_0$ for some constant $c > 0$, by combining this with (5.2), the last inequality in (5.4), and (5.5), we find a constant $c_2 > 0$ independent of γ, ν such that

$$\begin{aligned}
&|\ell_t(y) - \ell_t(y')| - c_2|y-y'|^\beta \\
&\le c_2\|\ell_\nu\|_\infty |y-y'|^\beta \int_0^t (t-s)^{-\frac{1+\beta}{2}} \big\{\tilde{P}_{s,t}^\kappa f_0(s,\cdot)(y) + \tilde{P}_{s,t}^\kappa f_0(s,\cdot)(y')\big\}\mathrm{d}s,
\end{aligned}$$

where $\tilde{P}_{s,t}$ is in (3.6). By (3.9) for $(p,q) = (p_0,q_0)$ and $p' = \infty$, we find a constant $c_3 > 0$ such that this implies

$$\begin{aligned}
&\frac{|\ell_t(y) - \ell_t(y')| - c_2|y-y'|^\beta}{c_2\|\ell_\nu\|_\infty|y-y'|^\beta} \\
&\le \int_0^t (t-s)^{-(\frac{1+\beta}{2}+\frac{d}{2p_0})}\Big(\tilde{P}_{s,t}^\kappa\big\{(t-s)^{\frac{d}{2p_0}} f_0(s,\cdot)\big\}(y) + \tilde{P}_{s,t}^\kappa\big\{(t-s)^{\frac{d}{2p_0}} f_0(s,\cdot)\big\}(y')\Big)\mathrm{d}s \\
&\le 2c_2\bigg(\int_0^t (t-s)^{-(\frac{1+\beta}{2}+\frac{d}{2p_0})\frac{q_0}{q_0-1}}\mathrm{d}s\bigg)^{\frac{q_0-1}{q_0}} \big\|\tilde{P}_{\cdot,t}^\kappa\{(t-\cdot)^{\frac{d}{2p_0}} f_0\}\big\|_{L_{q_0}^\infty(t)} \\
&\le c_3\|f\|_{\tilde{L}_{q_0}^{p_0}}, \quad y \ne y', t \in (0,T],
\end{aligned}$$

where we have used the fact that $\|\cdot\|_{\tilde{L}_{q_0}^\infty} = \|\cdot\|_{L_{q_0}^\infty}$ and $(\frac{1+\beta}{2} + \frac{d}{2p_0})\frac{q_0}{q_0-1} < 1$ due to $\beta \in (0, 1 - \frac{2}{q_0} - \frac{d}{p_0})$. Combining this with (5.2), we finish the proof. □

The next lemma contains two classical estimates on the operator $1-\Delta$ and the heat semigroup $P_t = \mathrm{e}^{t\Delta}$.

Lemma 5.2. *Let $P_t = \mathrm{e}^{t\Delta}$.*

(1) *For any $\beta > 0$, there exists a constant $c > 0$ such that*

$$\|(1-\Delta)^{\frac{\beta}{2}} f\|_\infty \le c\|f\|_{C_b^\beta}.$$

(2) *For any $\alpha, \beta, k \ge 0$, there exists a constant $c > 0$ such that*

$$\|(1-\Delta)^{-k}P_t f\|_{C_b^{\alpha+\beta}} \le ct^{-(\frac{\alpha}{2}-k)^+}\|f\|_{C_b^\beta}, \quad t > 0.$$

Proof of Theorem 2.2. For $\mathscr{L}_{X_0^i} = \nu_i$ with $\ell_{\nu_i} \in C_b^\beta(\mathbb{R}^d)$ and $\gamma^i \in L_\infty^\infty \cap \mathscr{D}_+^1$, simply denote

$$\ell_t^i = \ell_{X_t^{\gamma^i,\nu_i}}, \quad b_t^{\ell^i} := b_t(\cdot, \ell_t^i(\cdot), \ell_t^i), \quad t \in [0,T], \; i = 1,2.$$

Without loss of generality, let $\|\ell_{\nu_2}\|_{C_b^\beta} \le \|\ell_{\nu_1}\|_{C_b^\beta}$.

By (5.5) with $(\nu, \gamma) = (\nu_1, \gamma^1)$, we obtain

$$\ell_t^1(y) = P_{0,t}^{\gamma^1}\ell_{\nu_1}(y) + \int_0^t \mathrm{d}s \int_{\mathbb{R}^d} \ell_s^1(x)\big\{\nabla_{b_s^{\ell^1}(x) - \bar b_s^{\gamma^1}(x)} p_{s,t}^{\gamma^1}(\cdot, y)\big\}(x)\mathrm{d}x.$$

By the argument leading to (4.10) for $(p_{s,t}^{\gamma^1}, X_s^{\gamma^2,\nu_2})$ replacing $(p_{s,t}^{a^\gamma, b^{(1)}}, X_s^\gamma)$, we derive

$$\begin{aligned}\ell_t^2(y) = {} & P_{0,t}^{\gamma^1}\ell_{\nu_2}(y) + \int_0^t \mathrm{d}s \int_{\mathbb{R}^d} \ell_s^2(x)\big\{\nabla_{b_s^{\ell^2}(x) - \bar b_s^{\gamma^1}(x)} p_{s,t}^{\gamma^1}(\cdot, y)\big\}(x)\mathrm{d}x \\ & + \frac{1}{2}\sum_{i,j=1}^d \int_0^t \mathrm{d}s \int_{\mathbb{R}^d} \big\{\ell_s^2(a_s^{\gamma^2} - a_s^{\gamma^1})_{ij}\partial_i\partial_j p_{s,t}^{\gamma^1}(\cdot, y)\big\}(x)\mathrm{d}x.\end{aligned}$$

Thus,

$$\|\ell_t^1 - \ell_t^2\|_\infty \le I_1 + I_2 + \sum_{i,j=1}^d I_{ij}, \tag{5.7}$$

where

$$I_1 := \|P_{0,t}^{\gamma^1}\ell_{\nu_1} - P_{0,t}^{\gamma^1}\ell_{\nu_2}\|_\infty \le \|\ell_{\nu_1} - \ell_{\nu_2}\|_\infty, \tag{5.8}$$

and

$$\begin{aligned} I_2 &:= \int_0^t \mathrm{d}s \int_{\mathbb{R}^d} \Big|\Big\{\big[\ell_s^2(b_s^{\ell^2} - \bar b_s^{\gamma^1}) - \ell_s^1(b_s^{\ell^1} - \bar b_s^{\gamma^1})\big]\nabla p_{s,t}^{\gamma^1}(\cdot, y)\Big\}(x)\Big|\mathrm{d}x, \\ I_{ij} &:= \frac{1}{2}\sup_{y \in \mathbb{R}^d}\bigg|\int_0^t \mathrm{d}s \int_{\mathbb{R}^d} \big\{\ell_s^2(a_s^{\gamma^2} - a_s^{\gamma^1})_{ij}\partial_i\partial_j p_{s,t}^{\gamma^1}(\cdot, y)\big\}(x)\mathrm{d}x\bigg|.\end{aligned}$$

Below we estimate I_2 and I_{ij} respectively.

Firstly, by **(B)** and (5.2), we find a constant $c_1 > 0$ such that

$$\begin{aligned} &\big|\ell_s^2\{b_s^{\ell^2}(x) - \bar b_s^{\gamma^1}(x)\} - \ell_s^1(x)\{b_s^{\ell^1}(x) - \bar b_s^{\gamma^1}(x)\}\big| \\ &\le \|\ell_s^1 - \ell_s^2\|_\infty |b_s^{\ell^2}(x) - \bar b_s^{\gamma^1}(x)| + \|\ell_s^2\|_\infty |b_s^{\ell^2}(x) - b_s^{\ell^1}(x)| \\ &\le c_1\|\ell_{\nu_2}\|_\infty \|\ell_s^1 - \ell_s^2\|_\infty f_0(s,x), \quad s \in [0,T], x \in \mathbb{R}^d.\end{aligned}$$

Combining this with (5.4) for $i = 1$, (3.9) for $(p,q) = (p_0, q_0)$ and $p' = \infty$, and applying Hölder's inequality, we find constant $c_2, c_3 > 0$ such that

$$\begin{aligned} I_2 &\le c_2\|\ell_{\nu_2}\|_\infty \int_0^t (t-s)^{-\frac{1}{2}}\|\ell_s^1 - \ell_s^2\|_\infty \tilde P_{s,t}^\kappa f_0(s,\cdot)(y)\mathrm{d}s \\ &\le c_2\|\ell_{\nu_2}\|_\infty \bigg(\int_0^t \big\{(t-s)^{-(\frac{1}{2} + \frac{d}{2p_0})}\|\ell_s^1 - \ell_s^2\|_\infty\big\}^{\frac{q_0}{q_0-1}}\mathrm{d}s\bigg)^{\frac{q_0-1}{q_0}} \|(t-\cdot)^{\frac{d}{2p_0}} f_0\|_{\tilde L_{q_0}^\infty} \\ &\le c_3\|\ell_{\nu_2}\|_\infty \|f_0\|_{\tilde L_{q_0}^{p_0}} \bigg(\int_0^t (t-s)^{-\frac{q_0}{q_0-1}(\frac{1}{2} + \frac{d}{2p_0})}\|\ell_s^1 - \ell_s^2\|_\infty^{\frac{q_0}{q_0-1}}\mathrm{d}s\bigg)^{\frac{q_0-1}{q_0}}, \quad t \in [0,T].\end{aligned} \tag{5.9}$$

Next, by integration by parts formula, **(B)**, (5.3), (5.4) for $i=1$ and Lemma 5.2, for any $\delta := \alpha\wedge\beta$, we find constants $c_4, c_5 > 0$ such that

$$\begin{aligned}
&\bigg|\int_{\mathbb{R}^d}\big\{\ell_s^2(a_s^{\gamma^2}-a_s^{\gamma^1})_{ij}\partial_i\partial_j p_{s,t}^{\gamma^1}(\cdot,y)\big\}(x)\mathrm{d}x\bigg|\\
&=\bigg|\int_{\mathbb{R}^d}\Big[(1-\Delta)^{\frac{\delta}{2}}\big\{\ell_s^2(a_s^{\gamma^2}-a_s^{\gamma^1})_{ij}\big\}(x)\Big]\cdot\Big[\partial_i\partial_j(1-\Delta)^{-\frac{\delta}{2}}p_{s,t}^{\gamma^1}(\cdot,y)(x)\Big]\mathrm{d}x\bigg|\\
&\le \big\|(1-\Delta)^{\frac{\delta}{2}}\big\{\ell_s^2(a_s^{\gamma^2}-a_s^{\gamma^1})_{ij}\big\}\big\|_\infty\int_{\mathbb{R}^d}\big|\partial_i\partial_j(1-\Delta)^{-\frac{\delta}{2}}p_{s,t}^{\gamma^1}(\cdot,y)(x)\big|\mathrm{d}x\\
&\le c_4\big\|\ell_s^2(a_s^{\gamma^2}-a_s^{\gamma^1})_{ij}\big\|_{C_b^{\beta\wedge\alpha}}(t-s)^{\frac{\delta}{2}-1}\|\gamma_s^1-\gamma_s^2\|_\infty\\
&\le c_5\|\ell_{\nu_2}\|_{C_b^\beta}(t-s)^{\frac{\delta}{2}-1}\|\gamma_s^1-\gamma_s^2\|_\infty.
\end{aligned}$$

By combining this with (5.7), (5.8) and (5.9), we arrive at

$$\begin{aligned}
&\|\ell_t^1-\ell_t^2\|_\infty\le\|\ell_{\nu_1}-\ell_{\nu_2}\|_\infty\\
&+c_3\|\ell_{\nu_2}\|_\infty\bigg(\int_0^t(t-s)^{-\frac{q_0}{q_0-1}(\frac12+\frac{d}{2p_0})}\|\ell_s^1-\ell_s^2\|_\infty^{\frac{q_0}{q_0-1}}\mathrm{d}s\bigg)^{\frac{q_0-1}{q_0}}\\
&+\frac{d^2c_5}{2}\|\ell_{\nu_2}\|_{C_b^\beta}\int_0^t(t-s)^{\frac{\delta}{2}-1}\|\gamma_s^1-\gamma_s^2\|_\infty\mathrm{d}s,\quad t\in[0,T].
\end{aligned}$$

Consequently, for any $\lambda>0$,

$$\begin{aligned}
&d_{\infty,\lambda}(\ell_{X^{\gamma^1,\nu_1}},\ell_{X^{\gamma^2,\nu_2}}):=\sup_{t\in[0,T]}\mathrm{e}^{-\lambda t}\|\ell_t^1-\ell_t^2\|_\infty\\
&\le\|\ell_{\nu_1}-\ell_{\nu_2}\|_\infty+\varepsilon(\lambda)\big\{d_{\infty,\lambda}(\ell_{X^{\gamma^1,\nu_1}},\ell_{X^{\gamma^2,\nu_2}})+d_{\infty,\lambda}(\gamma^1,\gamma^2)\big\}
\end{aligned}$$

holds for

$$\begin{aligned}
\varepsilon(\lambda):=\sup_{t\in[0,T]}\bigg\{&c_3\|\ell_{\nu_2}\|_\infty\bigg(\int_0^t(t-s)^{-\frac{q_0}{q_0-1}(\frac12+\frac{d}{2p_0})}\mathrm{e}^{-\frac{q_0\lambda(t-s)}{q_0-1}}\mathrm{d}s\bigg)^{\frac{q_0-1}{q_0}}\\
&+\frac{d^2c_5}{2}\|\ell_{\nu_2}\|_{C_b^\beta}\int_0^t(t-s)^{\frac{\delta}{2}-1}\mathrm{e}^{-\lambda(t-s)}\mathrm{d}s\bigg\}.
\end{aligned}$$

Since $(p_0,q_0)\in\mathscr{K}$ implies $\frac{q_0}{q_0-1}(\frac12+\frac{d}{2p_0})<1$, and since $1-\frac{\delta}{2}<1$, by taking large enough $\lambda>0$ increasing in $\|\ell_{\nu_2}\|_{C_b^\beta}$, we obtain

$$d_{\infty,\lambda}(\ell_{X^{\gamma^1,\nu_1}},\ell_{X^{\gamma^2,\nu_2}})\le\|\ell_{\nu_1}-\ell_{\nu_2}\|_\infty+\frac14\big\{d_{\infty,\lambda}(\ell_{X^{\gamma^1,\nu_1}},\ell_{X^{\gamma^2,\nu_2}})+d_{\infty,\lambda}(\gamma^1,\gamma^2)\big\}. \tag{5.10}$$

Taking $\nu_1=\nu_2=\nu$, we see that the map $\gamma\mapsto\ell_{X^{\gamma,\nu}}$ is contractive on the complete metric space $(L_\infty^\infty\cap\mathscr{D}_+^1, d_{\infty,\lambda})$, so that it has a unique fixed point. Therefore, (1.1) is well-posed. Estimate (2.3) follows from Lemma 5.1 for $\gamma_t=\ell_{X_t}$ for the solution to (1.1), while (2.4) follows from (5.10) for $\gamma_t^i:=\ell_{X_t^i},\nu_i=\mathscr{L}_{X_0^i}, i=1,2$.

□

6 Density dependent reflecting SDEs

In this section, we extend Theorem 2.1 to density dependent reflecting SDEs on a domain D. There exists additional difficulty to extend Theorem 2.2, for instance, in the proof of Theorem 2.2 we used

$$(1-\Delta)^{-\frac{\delta}{2}}\partial_i\partial_j = \partial_i\partial_j(1-\Delta)^{-\frac{\delta}{2}}$$

which is no longer true for the Nuemann Laplacian in a domain.

Let $D \subset \mathbb{R}^d$ be a connected C^2-smooth open domain. Consider the following density dependent reflecting SDE on the closure $\bar{D}$ of D:

$$\mathrm{d}X_t = b_t(X_t, \ell_{X_t}(X_t), \ell_{X_t})\mathrm{d}t + \sigma_t(X_t)\mathrm{d}W_t + \mathbf{n}(X_t)\mathrm{d}l_t, \quad t \in [0,T], \tag{6.1}$$

where $\mathbf{n}$ is the unit inward normal vector field on the boundary ∂D, l_t is a continuous adapted increasing process with $\mathrm{d}l_t$ supported on $\{t : X_t \in \partial D\}$, and

$$b : [0,T] \times \bar{D} \times [0,\infty) \times \mathscr{D}_+^1 \to \mathbb{R}^d, \quad \sigma : [0,T] \times \bar{D} \to \mathbb{R}^d \otimes \mathbb{R}^m$$

are measurable. Here and in the following, $\mathscr{D}_+^1, \tilde{L}_q^p, L_q^p, L^p, \tilde{L}^p, \tilde{\mathscr{P}}^p, \mathscr{P}^p$ and $\mathscr{P}$ are defined as before for $\bar{D}$ replacing $\mathbb{R}^d$.

We will assume $\partial D \in C_b^{2,L}$ in the following sense: there exists a constant $r_0 > 0$ such that the polar coordinate map

$$\Psi : \partial D \times [-r_0, r_0] \ni (\theta, r) \mapsto \theta + r\mathbf{n}(\theta) \in \partial_{\pm r_0} D := \big\{x \in \mathbb{R}^d : \rho_\partial(x) := \mathrm{dist}(x, \partial D) \le r_0\big\}$$

is a C^2-diffeomorphism, such that $\Psi^{-1}(x)$ have bounded and continuous first and second order derivatives in $x \in \partial_{\pm r_0} D$, and $\nabla^2 \rho_\partial$ is Lipschitz continuous on $\partial_{\pm r_0} D$.

Note that $\partial D \in C_b^{2,L}$ does not imply the boundedness of D or ∂D, but any bounded $C^{2,L}$ domain satisfies $\partial D \in C_b^{2,L}$.

Definition 6.1. (1) A pair $(X_t, l_t)_{t\in[0,T]}$ is called a (strong) solution of (6.1), if $(X_t)_{t\in[0,T]}$ is a continuous adapted process on $\bar{D}$, $(l_t)_{t\in[0,T]}$ is a continuous adapted increasing process with $l_0 = 0$ and $\mathrm{d}l_t$ supported on $\{t \in [0,T] : X_t \in \partial D\}$, such that

$$\int_0^T \mathbb{E}\big[|b_s(X_s, \ell_{X_s}(X_s), \ell_{X_t})| + \|\sigma_s(X_s)\|^2\big]\mathrm{d}s < \infty$$

and $\mathbb{P}$-a.s.

$$X_t = X_0 + \int_0^t b_s(X_s, \ell_{X_s}(X_s), \ell_{X_s})\mathrm{d}s + \int_0^t \sigma_s(X_s)\mathrm{d}W_s + \int_0^t \mathbf{n}(X_s)\mathrm{d}l_s, \quad t \in [0,T].$$

(2) A triple $(X_t, l_t, W_t)_{t\in[0,T]}$ is called a weak solution of (6.1), if $(W_t)_{t\in[0,T]}$ an m-dimensional Brownian under a complete filtration probability space $(\Omega, \{\mathscr{F}_t\}_{t\in[0,T]}, \mathbb{P})$ such that $(X_t, l_t)_{t\in[0,T]}$ solves (6.1). We identify any two weak solutions (X_t, l_t, W_t) and $(\bar{X}_t, \bar{l}_t, \bar{W}_t)$ if $(X_t, l_t)_{t\in[0,T]}$ and $(\bar{X}_t, \bar{l}_t)_{t\in[0,T]}$ have the same distribution under the corresponding probability spaces.

To extend assumption **(A)** to the present setting, we introduce the the Neumann semigroup $\{P_{s,t}^{a,b^{(1)}}\}_{0\le s\le t\le T}$ generated by $L_t^{a,b^{(1)}}$ on $\bar D$ for $a_t := \sigma_t\sigma_t^*$, that is, for any $\phi\in C_b^2(\bar D)$, and any $t\in(0,T]$, $(P_{s,t}^{a,b^{(1)}}\phi)_{s\in[0,t]}$ is the unique solution of the PDE

$$\partial_s u_s = -L_s^{a,b^{(1)}} u_s,\quad \nabla_{\mathbf{n}} u_s|_{\partial D} = 0 \text{ for } s\in[0,t), u_t=\phi. \tag{6.2}$$

For any $t>0$, let $C_b^{1,2}([0,t]\times\bar D)$ be the set of functions $f\in C_b([0,t]\times\bar D)$ with bounded and continuous derivatives $\partial_t f, \nabla f$ and $\nabla^2 f$.

We now extend **(A)** to the domain setting as follows.

(C) Let $k\in[1,\infty]$, $\partial D\in C_b^{2,L}$, $\sigma_t(x,\rho)=\sigma_t(x)$ and $b_t(x,r,\rho)=b_t^{(1)}(x)+b_t^{(0)}(x,r,\rho)$ satisfy the following conditions.

(C_1) $a_t(x) := (\sigma_t\sigma_t^*)(x)$ is invertible for $(t,x)\in[0,T]\times\bar D$, $\|a\|_\infty+\|a^{-1}\|_\infty<\infty$, and

$$\lim_{\varepsilon\downarrow 0}\sup_{t\in[0,T]}\sup_{x,y\in\bar D,|x-y|\le\varepsilon}\|a_t(x)-a_t(y)\|=0.$$

Moreover, σ_t is weakly differentiable with $\|\nabla\sigma\|\le\sum_{i=1}^l f_i$ for some $l\in\mathbb{N}$, $0\le f_i\in\tilde L_{q_i}^{p_i}, (p_i,q_i)\in\mathscr{K}, 1\le i\le l$.

(C_2) (A_2) holds for $\bar D$ replacing $\mathbb{R}^d$.

(C_3) For any $\phi\in C_b^2(\bar D)$ and $t\in(0,T]$, the PDE (6.2) has a unique solution $P_{\cdot,t}^{\sigma,b^{(1)}}\phi\in C_b^{1,2}([0,t]\times\bar D)$, such that for some constants $c,\kappa>0$ and diffeomorphsims $\{\psi_{s,t}\}_{0\le s\le t\le T}$ on $\bar D$ satisfying (3.2), the hear kernel $p_{s,t}^{a,b^{(1)}}$ of $P_{s,t}^{a,b^{(1)}}$ satisfies

$$|\nabla^i p_{s,t}^{a,b^{(1)}}(\cdot,y)|(x)\le c(t-s)^{-\frac{i}{2}}p_{s,t}^\kappa(\psi_{s,t}(x)-y),\quad 0\le s<t\le T, x,y\in\bar D,\ i=1,2. \tag{6.3}$$

By [4, Theorem VI.3.1], (C_3) holds if D is bounded with $\partial D\in C^{2+\alpha}$ for some $\alpha\in(0,1)$, and there exists $c>0$ such that

$$\big\{|b_t^{(1)}(x)-b_s^{(1)}(y)|+\|a_t(x)-a_s(y)\|\big\}\le c(|t-s|^\alpha+|x-y|^\alpha),\quad s,t\in[0,T], x,y\in\bar D.$$

If moreover ∇a_t is Hölder continuous uniformly in $t\in[0,T]$, then for any $\beta\in(0,1)$ there exists a constant $c>0$ such that (3.5) holds for $p_{s,t}^{a,b^{(1)}}$:

$$\begin{aligned}&|\nabla p_{s,t}^{a,b^{(1)}}(\cdot,y)(x)-\nabla p_{s,t}^{a,b^{(1)}}(\cdot,y')(x)|\\ &\le c|y-y'|^\beta(t-s)^{-\frac{1+\beta}{2}}\big\{p_{t-s}^\kappa(\psi_{s,t}(x)-y)+p_{t-s}^\kappa(\psi_{s,t}(x)-y')\big\},\\ &\qquad 0\le s<t\le T,\ x,x',y\in\mathbb{R}^d.\end{aligned} \tag{6.4}$$

The following result extends Theorem 2.1 to the reflecting setting.

Theorem 6.1. *Assume* **(C)** *for some* $k\in[\frac{p_0}{p_0-1},\infty]\cap(k_0,\infty]$, *where* $k_0:=\frac{d}{2\theta+1-dp_0^{-1}-2q_0^{-1}}$.

(1) *For any* $\nu \in \tilde{\mathscr{P}}^k$, (6.1) *has a unique strong (respectively weak) solution with* $\mathscr{L}_{X_0} = \nu$ *satisfying* $\ell_{X.} \in \tilde{L}^k_\infty(\bar{D})$. *Moreover, there exists a constant* $c > 0$ *such that for any two solutions* X_t^1 *and* X_t^2 *of* (6.1) *with initial distributions* $\mathscr{L}_{X^1_\cdot}, \mathscr{L}_{X^2_\cdot} \in \tilde{L}^k_\infty$,

$$\sup_{t\in[0,T]} \|\ell_{X_t^1} - \ell_{X_t^2}\|_{\tilde{L}^k} \le c\|\mathscr{L}_{X_0^1} - \mathscr{L}_{X_0^2}\|_{\tilde{L}^k}.$$

(2) *Assertions in* (1) *hold for* $(\mathscr{P}^k, L^k_\infty, L^k)$ *replacing* $(\tilde{\mathscr{P}}^k, \tilde{L}^k_\infty, \tilde{L}^k)$, *provided in* (C_2) *the condition* (A_2) *is replaced by* (A_2') *for* $\bar{D}$ *replacing* $\mathbb{R}^d$.

Proof. As explained in the proof of Theorem 2.1(2), we only prove the first assertion.

According to [12, Theorem 2.2(ii)], for any $\gamma \in \tilde{L}^k_\infty \cap \mathscr{D}^1_+$, the reflecting SDE

$$\mathrm{d}X_t^\gamma = b_t^\gamma(X_t^\gamma)\mathrm{d}t + \sigma_t(X_t^\gamma)\mathrm{d}W_t + \mathbf{n}(X_t^\gamma)\mathrm{d}l_t^\gamma, \quad t\in[0,T] \tag{6.5}$$

is well-posed. Let $X_t^{\gamma,x}$ denote the solution with initial value $X_0^\gamma = x \in \bar{D}$, and simply denote X_t^γ for the solution with $X_0^\gamma = X_0$ for $\mathscr{L}_{X_0} = \nu$.

By Theorem 6.2.7(ii)-(iii) in [3], the distribution density function $\ell_{X_t^{\gamma,x}}$ exists for $t \in (0,T]$ and $x \in D$. Next, by [12, Theorem 4.1] for distribution independent drift, there exists $c > 0$ such that the following log-Harnack inequality holds for the associated semigroup:

$$P_t^\gamma \log f(x) \le \log P_t f(y) + \frac{c|x-y|^2}{t}, \quad t\in(0,T], x,y\in\mathbb{R}^d, f>0.$$

This implies that $\{\mathscr{L}_{X_t^{\gamma,x}}\}_{x\in\bar{D}}$ are mutually equivalent for $t \in (0,T]$. Thus, the existence of $\{\ell_{X_t^{\gamma,x}}\}_{t\in(0,T]}$ for $x \in D$ implies that for $x \in \bar{D}$. Consequently,

$$\Phi_t^\nu\gamma := \ell_{X_t^\gamma} = \int_{\bar{D}} \ell_{X_t^{\gamma,x}} \mathscr{L}_{X_0}(\mathrm{d}x)$$

exists for any $t \in (0,T]$. Since $\{\psi_{s,t}\}_{0\le s\le t\le T}$ are diffeomorphisms on $\bar{D}$ satisfying (3.2),

$$\hat{P}^\kappa_{s,t} f(y) := \int_{\bar{D}} p^\kappa_{t-s}(y-\psi_{s,t}(x)) f(x)\mathrm{d}x, \quad y\in\bar{D}$$

gives rise to a family of linear operators satisfying (3.8) and Lemma 3.2 for norms defined with $\bar{D}$ replacing $\mathbb{R}^d$. Then by repeating the proof of Lemma 4.1 using the present estimates, we conclude that Φ^ν maps $\tilde{\mathscr{P}}^k_{\nu,T}$ into $\tilde{\mathscr{P}}^k_{\nu,T}$ such that (4.1) and (4.10) hold for $\bar{D}$ replacing $\mathbb{R}^d$. With this result and using (C_3) replacing (3.3), we prove Theorem 6.1(1) by the means in used the proof of Theorem 2.1(1). □

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