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Exponential Ergodicity for Singular Reflecting McKean-Vlasov SDEs *

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Abstract

By refining a recent result of Xie and Zhang [27], we prove the exponential ergodicity under a weighted variation norm for singular SDEs with drift containing a local integrable term and a coercive term. This result is then extended to singular reflecting SDEs as well as singular McKean-Vlasov SDEs with or without reflection. The exponential ergodicity in the relative entropy and (weighted) Wasserstein distances are also studied for reflecting McKean-Vlasov SDEs. The main results are illustrated by non-symmetric singular granular media equations.

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1 Introduction

Let $D \subset \mathbb{R}^d$ be a connected open domain including the global situation $D = \mathbb{R}^d$, and let $\mathscr{P}$ denote the space of probability measures on $\bar{D}$, the closure of D. Consider the following distribution dependent (i.e. McKean-Vlasov) SDE on $\bar{D}$ with reflection if $D \neq \mathbb{R}^d$:

$$\mathrm{d}X_t = b(X_t, \mathscr{L}_{X_t})\mathrm{d}t + \sigma(X_t)\mathrm{d}W_t + \mathbf{n}(X_t)\mathrm{d}l_t, \quad t \geq 0, \tag{1.1}$$

where $(W_t)_{t\geq 0}$ is an m-dimensional Brownian motion on a complete filtration probability space $(\Omega, \{\mathscr{F}_t\}_{t\geq 0}, \mathbb{P})$, $\mathscr{L}_{X_t}$ is the distribution of X_t,

$$b : D \times \mathscr{P} \to \mathbb{R}^d, \quad \sigma : D \to \mathbb{R}^d \otimes \mathbb{R}^m$$

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are measurable, and when $D \neq \mathbb{R}^d$, $\mathbf{n}$ is the inward unit normal vector field of the boundary ∂D, and l_t is an adapted continuous increasing process which increases only when $X_t \in \partial D$.

In the case that $D = \mathbb{R}^d$, we have $l_t = 0$ so that (1.1) becomes the distribution dependent SDE (DDSDE)

$$\mathrm{d}X_t = b(X_t, \mathscr{L}_{X_t})\mathrm{d}t + \sigma(X_t)\mathrm{d}W_t, \quad t \geq 0. \tag{1.2}$$

If moreover $b(x, \mu) = b(x)$ does not depend on μ, it reduces to the classical Itô's SDE

$$\mathrm{d}X_t = b(X_t)\mathrm{d}t + \sigma(X_t)\mathrm{d}W_t, \quad t \geq 0. \tag{1.3}$$

In the recent work [25], the well-posedness and regularity estimates have been studied for solutions to (1.1) with b containing a locally integrable term and a Lipchitz continuous term. However, the ergodicity was only investigated under monotone or Lyapunov conditions excluding this singular situation. See also [5, 9, 10, 11, 13, 16, 19, 24] and references within for results on the ergodicity of McKean-Vlasov SDEs without reflection under monotone or Lyapunov conditions. On the other hand, by using Zvokin's transform, the exponential ergodicity was proved by Xie and Zhang [27] for the singular SDE (1.3). In this paper, we aim to refine the result of [27] and make extensions to singular SDEs with reflection and distribution dependent drift.

When the SDE (1.1) is well-posed, let $P_t^*\nu = \mathscr{L}_{X_t}$ for the solution with initial distribution $\nu \in \mathscr{P}$. We will study the exponential convergence of P_t^* under the weighted variation distance induced by a positive measurable function V:

$$\|\mu - \nu\|_V := |\mu - \nu|(V) = \sup_{|f| \leq V} |\mu(f) - \nu(f)|, \quad \mu, \nu \in \mathscr{P},$$

where $|\mu - \nu|$ is the total variation of $\mu - \nu$ and $\mu(f) := \int f \mathrm{d}\mu$ for a measure μ and $f \in L^1(\mu)$. When $V = 1$, $\|\cdot\|_V$ reduces to the the total variation norm $\|\cdot\|_{var}$.

We will consider $b(x, \mu) = b^{(0)}(x) + b^{(1)}(x, \mu)$, where $b^{(0)}$ is the singular term satisfying

$$\sup_{z \in \mathbb{R}^d} \int_{B(z,1) \cap D} |b^{(0)}(x)|^p (\mathrm{d}x) < \infty \tag{1.4}$$

for some $p > d \vee 2$, and $b^{(1)}(\cdot, \mu)$ is a coercive term such that

$$\limsup_{x \in \bar{D}, |x| \to \infty} \sup_{\mu \in \mathscr{P}} \langle b^{(1)}(x, \mu), \nabla V(x) \rangle = -\infty$$

holds for some compact function $V \in C^2(\mathbb{R}^d)$ (i.e. $\{V \leq r\}$ is compact for any $r > 0$). The later condition is trivial for bounded D by taking $V = 1$ and the convention that $\sup \emptyset = -\infty$.

To conclude this section, we present below an example for the L^1-exponential convergence of non-symmetric singular granular media equations, see [6, 10, 16] for the study of regular and symmetric models for $D = \mathbb{R}^d$.

Example 1.1. Let $D = \mathbb{R}^d$ or be a $C^{2,L}$-domain (see Definition 2.1 below). Consider the following nonlinear PDE for probability density functions on $\bar{D}$:

$$\partial_t \varrho_t = \Delta \varrho_t - \mathrm{div}\big\{\varrho_t b + \varrho_t (W * \varrho_t)\big\}, \quad \nabla_{\mathbf{n}} \varrho_t|_{\partial D} = 0 \text{ if } \partial D \neq \emptyset, \tag{1.5}$$

where

(i) W is a bounded measurable function on $\bar{D} \times \bar{D}$, and

$$(W * \varrho_t)(x) := \int_{\mathbb{R}^d} W(x,z)\varrho_t(z)\mathrm{d}z;$$

(ii) $b = b^{(0)} + b^{(1)}$ is a vector field such that (1.4) holds for some $p > d \vee 2$, and $b^{(1)}$ is locally bounded with $b^{(1)}(x) = -\phi(|x|^2)x$ for larger $|x|$ and some increasing function $\phi : [0,\infty) \to [1,\infty)$ with $\int_1^\infty \frac{\mathrm{d}s}{s\phi(s)} < \infty$.

In physics, ρ_t stands for the distribution density of particles, W describes the interaction among particles, and b refers to the potential of individual particles. When b and W are not of gradient type, the associated mean field particle systems are non-symmetric.

To characterize (1.5) using (1.1), let

$$b(x,\mu) = b(x) + (W * \mu)(x), \quad \sigma(x) = \sqrt{2}\mathbf{I}_d,$$

where $\mathbf{I}_d$ is the $d \times d$ identity matrix, and $(W * \mu)(x) := \int_{\bar{D}} W(x,z)\mu(\mathrm{d}z)$.

By (i) and (ii), **(A1)** holds for $V(x) := |x|^2$ when $D = \mathbb{R}^d$, while **(A2)** holds for $V = 1$ when D is a bounded $C_b^{2,L}$ domain. So, by Theorem 3.1, (1.1) is well-posed, and by Itô's formula, $\rho_t(x) := \frac{\mathrm{d}P_t^*\nu}{\mathrm{d}x}$ solves (1.5) for $\rho_0(x) := \frac{\mathrm{d}\nu}{\mathrm{d}x}$, see Subsection 1.2 in [25]. On the other hand, when $D = \mathbb{R}^d$ the superposition principle in [2] says that a solution of (1.5) is the distribution density of a weak solution to (1.1), such that (1.5) is well-posed as well. Moreover:

(a) By Theorem 3.1, when $\|W\|_\infty$ is small enough, P_t^* has a unique invariant probability measure μ satisfying (3.2), so that the solution $\rho_t := \frac{\mathrm{d}P_t^*\nu}{\mathrm{d}x}$ of (1.5) satisfies

$$\|\rho_t - \rho\|_{L^1} = \|P_t^*\nu - \mu\|_{var} \leq c\mathrm{e}^{-\lambda t}\|\rho_0 - \rho\|_{L^1}, \quad t \geq 0$$

for some constants $c, \lambda > 0$, where ρ is the density function of μ.

(b) Let $D = \mathbb{R}^d$ or D be convex. If there exists a constant $K > 0$ such that

$$\langle b(x) - b(y), x - y\rangle \leq -K|x-y|^2, \quad x, y \in D \tag{1.6}$$

holds, by Theorem 4.1, when $\|\nabla^2 W\|_\infty$ is small enough P_t^* is exponential ergodic in the relative entropy and the quadratic Wasserstein distance $\mathbb{W}_2$. If (1.6) only holds for large $|x-y|$, according to Theorem 4.3, P_t^* is exponential ergodic under a weighted Wasserstein distance provided $\|\nabla^2 W\|_\infty$ is small enough.

In the remainder of the paper, we study in Section 2 the exponential ergodicity for singular reflecting SDEs, then prove the uniform ergodicity for singular reflecting McKean-Vlasov SDEs in Section 3, and finally investigate in Section 4 the exponential ergodicity for reflecting McKean-Vlasov SDEs in relative entropy and (weighted) Wasserstein distances.

2 Exponential ergodicity for singular reflecting SDEs

To measure the singularity of the SDE, we introduce some functional spaces used in [26]. For any $p \geq 1$, let L^p be the class of measurable functions f on D such that

$$\|f\|_{L^p} := \bigg(\int_D |f(x)|^p \mathrm{d}x \bigg)^{\frac{1}{p}} < \infty.$$

For any $\epsilon > 0$ and $p \geq 1$, let $H^{\epsilon,p} := (1-\Delta)^{-\frac{\epsilon}{2}} L^p$ with

$$\|f\|_{H^{\epsilon,p}} := \|(1-\Delta)^{\frac{\epsilon}{2}} f\|_{L^p} < \infty, \quad f \in H^{\epsilon,p},$$

where Δ is the (Neumann if $\partial D \neq \emptyset$) Laplacian. For any $z \in \mathbb{R}^d$ and $r > 0$, let

$$B(z,r) := \{x \in \mathbb{R}^d : |x-z| \leq r\}$$

be the closed ball centered at z with radius r. We will simply denote $B_r = B(0,r)$ for $r > 0$. We write $f \in \tilde{L}^p$ if

$$\|f\|_{\tilde{L}^p} := \sup_{z \in \bar{D}} \|1_{B(z,1)} f\|_{L^p} < \infty.$$

Moreover, let $g \in C_0^\infty(\bar{D})$ with $g|_{B_1} = 1$ and the Neumann boundary condition $\nabla_{\mathbf{n}} g|_{\partial D} = 0$ if ∂D exists. We denote $f \in \tilde{H}^{\epsilon,p}$ if

$$\|f\|_{\tilde{H}^{\epsilon,p}} := \sup_{z \in \bar{D}} \|g(z+\cdot) f\|_{H^{\epsilon,p}} < \infty.$$

We note that the space $\tilde{H}^{\epsilon,p}$ does not depend on the choice of g. If a vector or matrix valued function has components in one of the above introduced spaces, then it is said in the same space with norm defined as the sum of components' norms.

In the following subsections, we first state the main results, then present some lemmas, and finally prove the main results.

2.1 Main results

We first consider the ergodicity of SDE (1.3) under the following assumption, where by the Sobolev embedding theorem σ (hence $\sigma\sigma^*$) is Hölder continuous by the boundedness of σ and $\|\nabla\sigma\| \in \tilde{L}^p$ for some $p > d$.

(A1) σ is weakly differentiable, $\sigma\sigma^*$ is invertible, and $b = b^{(0)} + b^{(1)}$ such that the following conditions hold.

(1) There exists $p > d \vee 2$ such that

$$\|\sigma\|_\infty + \|(\sigma\sigma^*)^{-1}\|_\infty + \|b^{(0)}\|_{\tilde{L}^p} + \|\nabla\sigma\|_{\tilde{L}^p} < \infty.$$

(2) $b^{(1)}$ is locally bounded, there exist constants $K > 0, \varepsilon \in (0,1)$, some compact function $V \in C^2(\mathbb{R}^d; [1,\infty))$, and a continuous increasing function $\Phi : [1,\infty) \to [1,\infty)$ with $\Phi(n) \to \infty$ as $n \to \infty$, such that

$$\begin{aligned} &\langle b^{(1)}, \nabla V\rangle(x) + \varepsilon |b^{(1)}(x)| \sup_{B(x,\varepsilon)} \{|\nabla V| + |\nabla^2 V|\} \le K - \varepsilon(\Phi \circ V)(x), \\ &\lim_{|x|\to\infty} \frac{\sup_{B(x,\varepsilon)} \{\|\nabla^2 V\| + |\nabla V|\}}{V(x) \wedge (\Phi \circ V)(x)} = 0. \end{aligned} \tag{2.1}$$

Theorem 2.1. *Assume* **(A1)**. *Then* (1.3) *is well-posed, the associated Markov semigroup P_t has a unique invariant probability measure μ such that $\mu(\Phi(\varepsilon_0 V)) < \infty$ for some $\varepsilon_0 \in (0,1)$, and*

$$\lim_{t\to\infty} \|P_t^*\nu - \mu\|_{var} = 0, \quad \nu \in \mathscr{P}. \tag{2.2}$$

Moreover:

(1) *If $\Phi(r) \ge \delta r$ for some constant $\delta > 0$ and all $r \ge 0$, then there exist constants $c > 1, \lambda > 0$ such that*

$$\|P_t^*\mu_1 - P_t^*\mu_2\|_V \le c\mathrm{e}^{-\lambda t}\|\mu_1 - \mu_2\|_V, \quad \mu_1, \mu_2 \in \mathscr{P}, t \ge 0. \tag{2.3}$$

In particular,

$$\|P_t^*\nu - \mu\|_V \le c\mathrm{e}^{-\lambda t}\|\nu - \mu\|_V, \quad \nu \in \mathscr{P}, t \ge 0.$$

(2) *Let $H(r) := \int_0^r \frac{\mathrm{d}s}{\Phi(s)} < \infty$ for $r \ge 0$. If Φ is convex, then there exist constants $k > 1, \lambda > 0$ such that*

$$\|P_t^*\delta_x - \mu\|_V \le k\big\{1 + H^{-1}(H(V(x)) - k^{-1}t)\big\}\mathrm{e}^{-\lambda t}, \quad x \in \mathbb{R}^d, t \ge 0, \tag{2.4}$$

where H^{-1} is the inverse of H with $H^{-1}(r) := 0$ for $r \le 0$. Consequently, if $H(\infty) < \infty$ then there exist constants $c, \lambda, t^* > 0$ such that

$$\|P_t^*\mu_1 - \mu_2\|_V \le c\mathrm{e}^{-\lambda t}\|\mu_1 - \mu_2\|_{var}, \quad t \ge t^*, \mu_1, \mu_2 \in \mathscr{P}. \tag{2.5}$$

To illustrate this result, we present below a consequence which covers the situation of [27, Theorem 2.10] where

$$\langle b^{(1)}(x), x\rangle \le c_1 - c_2|x|^{1+p}, \quad |b^{(1)}(x)| \le c_1(1+|x|)^p$$

holds for some constants $c_1, c_2 > 0$ and $p \ge 1$. Indeed, Corollary 2.2 implies the exponential ergodicity under the weaker condition

$$\langle b^{(1)}(x), x\rangle \le c_1 - c_2|x|^{1+p}, \quad |b^{(1)}(x)| \le c_1(1+|x|)^{p+1} \tag{2.6}$$

for some constants $p, c_1, c_2 > 0$ (p may be smaller than 1, $|b^{(1)}|$ may have higher order growth), since in this case, (2.7) and (2.8) hold for $\phi(r) := (1+r)^{\frac{1+p}{2}}$, and (2.9) holds for $\psi(r) := (1+r^2)^q$ for any $q > 0$ when $p \ge 1$.

Corollary 2.2. *Assume* **(A1)**(1) *and let* $b^{(1)}$ *satisfy*

$$\langle b^{(1)}(x), x\rangle \le c_1 - c_2\phi(|x|^2), \quad |b^{(1)}(x)| \le c_1\phi(|x|^2), \quad x \in \mathbb{R}^d \tag{2.7}$$

for some constants $c_1, c_2 > 0$ *and increasing function* $\phi : [0,\infty) \to [1,\infty)$ *with*

$$\alpha := \liminf_{r\to\infty} \frac{\log \phi(r)}{\log r} > \frac{1}{2}. \tag{2.8}$$

Then

(1) (1.3) *is well-posed,* P_t *has a unique invariant probability measure* μ *such that* $\mu(V) < \infty$ *and* (2.3) *hold for* $V := \mathrm{e}^{(1+|\cdot|^2)^\theta}$ *with* $\theta \in ((1-\alpha)^+, \frac{1}{2})$. *In general, for any increasing function* $1 \le \psi \in C^2([1,\infty))$ *satisfying*

$$\liminf_{r\to\infty} \frac{\psi'(r)\phi(r)}{\psi(r)} > 0, \quad \lim_{r\to\infty} \frac{\psi''(r)r}{\psi(r)} = 0, \tag{2.9}$$

$\mu(V) < \infty$ *and* (2.3) *hold for* $V := \psi(|\cdot|^2)$.

(2) *If* $\int_0^\infty \frac{\mathrm{d}s}{\phi(s)} < \infty$, *then* (2.5) *holds* $V := (1+|\cdot|^2)^q (q > 0)$ *and some constants* $c, \lambda, t^* > 0$.

Remark 2.1. We have the following assertions on the invariant probability measure μ and the ergodicity in Wasserstein distance and relative entropy.

(1) According to [4, Corollary 1.6.7 and Theorem 3.4.2], **(A1)** implies that μ has a strictly positive density function $\rho \in H^{1,p}_{loc}$, the space of functions f such that $fg \in H^{1,2}$ for all $g \in C_0^\infty(\mathbb{R}^d)$. Moreover, by [4, Theorem 3.1.2], when σ is Lipschitz continuous and $\mu(|b|^2) < \infty$, we have $\sqrt{\rho} \in H^{1,2}$. So, when (2.7) holds for $\phi(r) \sim r^p$ for some $p > \frac{1}{2}$ and large $r > 0$, Corollary 2.2(1) implies that μ has density with $\sqrt{\rho} \in H^{1,2}$. See also [22] and [23] for different type global regularity estimates on ρ under integrability conditions.

(2) Let $V := (1+|\cdot|^2)^{\frac{p}{2}}$ for some $p \ge 1$. By [18, Theorem 6.15], there exists a constant $c(p) > 0$ such that

$$\mathbb{W}_p(\mu,\nu)^p \le c(p)\|\mu - \nu\|_V,$$

where

$$\mathbb{W}_p(\mu_1,\mu_2) := \inf_{\pi\in\mathscr{C}(\mu_1,\mu_2)} \left(\int_{\mathbb{R}^d\times\mathbb{R}^d} |x-y|^p \pi(\mathrm{d}x,\mathrm{d}y)\right)^{\frac{1}{p}}$$

for $\mathscr{C}(\mu_1,\mu_2)$ being the set of couplings for μ_1 and μ_2. So, by Corollary 2.2, if **(A1)** holds with $\Phi(r) \ge \delta r$ for some $\delta > 0$, then there exist constants $c, \lambda > 0$ such that

$$\mathbb{W}_p(P_t^*\nu,\mu)^p \le c(1+\nu(|\cdot|^p))\mathrm{e}^{-\lambda t}, \quad t \ge 0, \nu \in \mathscr{P};$$

and if moreover Φ is convex with $\int_0^\infty \frac{\mathrm{d}s}{\Phi(s)} < \infty$, then there exist constants $c, \lambda, t^* > 0$ such that

$$\mathbb{W}_p(P_t^*\nu,\mu)^p \le c\mathrm{e}^{-\lambda t}\|\mu-\nu\|_{var}, \quad t \ge t^*, \nu \in \mathscr{P}.$$

(3) When $b^{(1)}$ is Lipschitz continuous, the log-Harnack inequality in [28, Theorem 4.1] implies

$$\mathrm{Ent}(P_t^*\nu|\mu) \le \frac{c'}{1\wedge t}\mathbb{W}_2(\nu,\mu)^2, \quad \nu\in\mathscr{P}, t>0$$

for some constant $c'>0$, where $\mathrm{Ent}(\nu|\mu)$ is the relative entropy. Thus, by Corollary 2.2, if **(A1)** holds for $V(x) := 1+|x|^2$ and $\Phi(r)\ge \delta r$ for some constant $\delta>0$, then there exist constants $c,\lambda>0$ such that

$$\mathrm{Ent}(P_t^*\nu|\mu) \le c(1+\nu(|\cdot|^2))\mathrm{e}^{-\lambda t}, \quad t\ge 1, \nu\in\mathscr{P};$$

and if moreover Φ is convex with $\int_0^\infty \frac{\mathrm{d}s}{\Phi(s)}<\infty$, then there exist $c,\lambda,t^*>0$ such that

$$\mathrm{Ent}(P_t^*\nu|\mu) \le c\mathrm{e}^{-\lambda t}\|\mu-\nu\|_{var}, \quad t\ge t^*, \nu\in\mathscr{P}.$$

Next, consider the following reflecting SDE on $D\ne\mathbb{R}^d$:

$$\mathrm{d}X_t = b(X_t)\mathrm{d}t + \sigma(X_t)\mathrm{d}W_t + \mathbf{n}(X_t)\mathrm{d}l_t, \quad t\ge 0, \tag{2.10}$$

where $\partial D\in C_b^{2,L}$ which is defined as follows.

Definition 2.1. Let ρ_∂ be the distance function to ∂D. For any $k\in\mathbb{N}$, we write $\partial D\in C_b^k$ if there exists a constant $r_0>0$ such that the polar coordinate around ∂D

$$\partial D\times[-r_0,r_0]\ni(\theta,r)\mapsto \theta+r\mathbf{n}(\theta)\in B_{r_0}(\partial D) := \{x\in\mathbb{R}^d : \rho_\partial(x)\le r_0\}$$

is a C^k-diffeomorphism. We write $\partial D\in C_b^{k,L}$, if it is C_b^k with $\nabla^k\rho_\partial$ being Lipschitz continuous on $B_{r_0}(\partial D)$.

We also need heat kernel estimates for the Neumann semigroup $\{P_t^\sigma\}_{t\ge0}$ generated by

$$L^\sigma := \frac{1}{2}\mathrm{tr}\big(\sigma_t\sigma_t^*\nabla^2\big).$$

For any $\varphi\in C_b^2(\bar D)$, let $P_t^\sigma\varphi$ be the solution of the PDE

$$\partial_t u_t = L^\sigma u_t, \quad \nabla_{\mathbf{n}} u_t|_{\partial D}=0 \text{ for } s>0, u_0=\varphi. \tag{2.11}$$

We will prove the exponential ergodicity of (2.10) under the following assumption.

(A2) $\partial D\in C_b^{2,L}$ and the following conditions hold.

(1) **(A1)** holds for $\bar D$ replacing $\mathbb{R}^d$, and there exists $r_0>0$ such that

$$\nabla_{\mathbf{n}(x)}V(y)\le 0, \quad x\in\partial D, |y-x|\le r_0. \tag{2.12}$$

(2) For any $\varphi\in C_b^2(\bar D)$, the PDE (2.11) has a unique solution $P_t^\sigma\varphi\in C_b^{1,2}(\bar D)$, such that for some constant $c>0$ we have

$$\|\nabla^i P_t^\sigma\varphi\|_\infty \le c(1\wedge t)^{-\frac{i}{2}}\|\nabla^{i-1}\varphi\|_\infty, \quad t>0,\ i=1,2, \varphi\in C_b^2(\bar D),$$

where $\nabla^0\varphi:=\varphi$.

As explained in [25, Remark 2.2(2)] that, **(A2)**(2) holds if D is bounded and σ is Hölder continuous. Moreover, (2.12) is trivial when ∂D is bounded, since in this case we may take $1 \le \tilde{V} \in C^2(\mathbb{R}^d)$ such that $\tilde{V} = 1$ on $\partial_{r_0}(\partial D)$ and $\tilde{V} = V$ outside a compact set, so that (2.1) remains true for $\tilde{V}$ replacing V. Similarly, (2.12) holds for $V(x_1, x_2) := V_1(x_1) + V_2(x_2)$ and $D = D_1 \times \mathbb{R}^l$ where $l \in \mathbb{N}$ is less than d, $\partial D_1 \subset \mathbb{R}^{d-l}$ is bounded, and $V_1 = 1$ in a neighborhood of ∂D_1.

Theorem 2.3. *Assume* **(A2)**. *Then all assertions in Theorem 2.1 hold for the reflecting SDE* (2.10).

2.2 Some lemmas

We first consider the following time dependent SDE with reflection when ∂D exists:

$$\mathrm{d}X_t = b_t(X_t)\mathrm{d}t + \sigma_t(X_t)\mathrm{d}W_t + \mathbf{n}(X_t)\mathrm{d}l_t, \quad t \ge 0. \tag{2.13}$$

For any $T > 0$ and $p, q > 1$, let $\tilde{L}_q^p(T)$ denote the class of measurable functions f on $[0,T] \times \bar{D}$ such that

$$\|f\|_{\tilde{L}_q^p(T)} := \sup_{z \in \bar{D}} \bigg(\int_0^T \|1_{B(z,1)} f_t\|_{L^p}^q \mathrm{d}t \bigg)^{\frac{1}{q}} < \infty.$$

For any $\epsilon > 0$, let $\tilde{H}_q^{\epsilon,p}(T)$ be the space of $f \in \tilde{L}_q^p$ with

$$\|f\|_{\tilde{H}_q^{\epsilon,p}(T)} := \sup_{z \in \bar{D}} \bigg(\int_0^T \|f_t\|_{\mathbb{H}^{\epsilon,p}}^q \mathrm{d}t \bigg)^{\frac{1}{q}} < \infty.$$

We will study the well-posedness, strong Feller property and irreducibility under the following assumptions for $D = \mathbb{R}^d$ and $D \ne \mathbb{R}^d$ respectively.

(A3) Let $T > 0, D = \mathbb{R}^d$, $a_t(x) := (\sigma_t \sigma_t^*)(x)$ and $b_t(x) = b_t^{(0)}(x) + b_t^{(1)}(x)$.

(1) a is invertible with $\|a\|_\infty + \|a^{-1}\|_\infty < \infty$ and

$$\lim_{\varepsilon \to 0} \sup_{|x-y| \le \varepsilon, t \in [0,T]} \|a_t(x) - a_t(y)\| = 0.$$

(2) There exist $l \ge 1$, $\{(p_i, q_i)\}_{0 \le i \le l} \in \mathscr{K} := \{(p,q) : p, q \in (2, \infty), \frac{d}{p} + \frac{2}{q} < 1\}$ and $1 \le f_i \in \tilde{L}_{q_i}^{p_i}$ such that

$$|b^{(0)}| \le f_0, \quad \|\nabla \sigma\| \le \sum_{i=1}^l f_i.$$

(3) $b^{(1)}$ is locally bounded, there exist constants $K, \varepsilon > 0$, increasing $\phi \in C^1([0,\infty); [1,\infty))$ with $\int_0^\infty \frac{\mathrm{d}s}{r + \phi(s)} = \infty$, and a compact function $V \in C^2(\mathbb{R}^d; [1,\infty))$ such that

$$\sup_{B(x,\varepsilon)} \big\{ |\nabla V| + \|\nabla^2 V\| \big\} \le K V(x),$$
$$\langle b_t^{(1)}(x), \nabla V(x) \rangle + \varepsilon |b_t^{(1)}(x)| \sup_{B(x,\varepsilon)} \|\nabla^2 V\| \le K \phi(V(x)), \quad (t,x) \in [0,T] \times \mathbb{R}^d.$$

When $D \neq \mathbb{R}^d$, we consider the following time dependent differential operator on $\bar{D}$:

$$L_t^\sigma := \frac{1}{2}\mathrm{tr}\big(\sigma_t\sigma_t^*\nabla^2\big), \quad t \in [0,T]. \tag{2.14}$$

Let $\{P_{s,t}^\sigma\}_{T\geq t_1\geq t\geq s\geq 0}$ be the Neumann semigroup on $\bar{D}$ generated by L_t^σ; that is, for any $\varphi \in C_b^2(\bar{D})$, and any $t \in (0,T]$, $(P_{s,t}^\sigma\varphi)_{s\in[0,t]}$ is the unique solution of the PDE

$$\partial_s u_s = -L_s^\sigma u_s, \quad \nabla_{\mathbf{n}} u_s|_{\partial D} = 0 \text{ for } s \in [0,t), u_t = \varphi. \tag{2.15}$$

For any $t > 0$, let $C_b^{1,2}([0,t]\times\bar{D})$ be the set of functions $f \in C_b([0,t]\times\bar{D})$ with bounded and continuous derivatives $\partial_t f, \nabla f$ and $\nabla^2 f$.

(A4) $D \in C_b^{2,L}$, **(A3)** holds with V satisfying (2.12) holds for some $r_0 > 0$. Moreover, for any $\varphi \in C_b^2(\bar{D})$ and $t \in (0,T]$, the PDE (2.15) has a unique solution $P_{\cdot,t}^\sigma\varphi \in C_b^{1,2}([0,t]\times\bar{D})$, such that for some constant $c > 0$ we have

$$\|\nabla^i P_{s,t}^\sigma\varphi\|_\infty \leq c(t-s)^{-\frac{1}{2}}\|\nabla^{i-1}\varphi\|_\infty, \quad 0 \leq s < t \leq T,\ i = 1,2, \varphi \in C_b^2(\bar{D}). \tag{2.16}$$

We have the following result, where the well-posedness for $D = \mathbb{R}^d$ has been addressed in [15].

Lemma 2.4. *Assume* **(A3)** *for $D = \mathbb{R}^d$ and* **(A4)** *for $D \neq \mathbb{R}^d$. Then* (2.13) *is well-posed up to time T. Moreover, for any $t \in (0,T]$,*

$$\lim_{\bar{D}\ni y\to x} \|P_t^*\delta_x - P_t^*\delta_y\|_{var} = 0, \quad t \in (0,T], x \in \bar{D}, \tag{2.17}$$

and P_t has probability density (i.e. heat kernel) $p_t(x,y)$ such that

$$\inf_{x,y\in\bar{D}\cap B_N,\ \rho_{\partial}(y)\geq N^{-1}} p_t(x,y) > 0, \quad N > 1, t \in (0,T], \tag{2.18}$$

where $\inf\emptyset := \infty$.

Proof. (a) The well-posedness. For any $n \geq 1$, let

$$b^n := 1_{B_n} b^{(1)} + b^{(0)}.$$

Since $b^{(1)}$ is locally bounded, by [26, Theorem 1.1] for $D = \mathbb{R}^d$ and [25, Theorem 2.2] for $D \neq \mathbb{R}^d$, for any $x \in \bar{D}$, the following SDE is well-posed:

$$\mathrm{d}X_t^{x,n} = b^n(X_t^{x,n})\mathrm{d}t + \sigma(X_t^{x,n})\mathrm{d}W_t + \mathbf{n}(X_t^{x,n})\mathrm{d}l_t^{x,n}, \quad X_0^{x,n} = x.$$

Let $\tau_n^x := \inf\{t \geq 0 : |X_t^{x,n}| \geq n\}$. Then $X_t^{x,n}$ solves (1.3) up to time τ_n^x, and by the uniqueness we have

$$X_t^{x,n} = X_t^{x,m}, \quad t \leq \tau_n^x \wedge \tau_m^x, n, m \geq 1.$$

So, it suffices to prove that $\tau_n^x \to \infty$ as $n \to \infty$.

Let $L_t^0 := L_t^\sigma + \nabla_{b_t^{(0)}}$. By [26, Theorem 3.1] for $D = \mathbb{R}^d$ and [25, Lemma 2.6] for $D \neq \mathbb{R}^d$, **(A3)** implies that for any $\lambda \geq 0$, the PDE

$$(\partial_t + L_t^0)u_t = \lambda u_t - b_t^{(0)}, \quad t \in [0,T], u_T = 0, \nabla_{\mathbf{n}} u_t|_{\partial D} = 0 \tag{2.19}$$

has a unique solution $u \in \tilde{H}_{q_0}^{p_0}(T)$, and there exist constants $\lambda_0, c, \theta > 0$ such that

$$\lambda^\theta(\|u\|_\infty + \|\nabla u\|_\infty) + \|\partial_t u\|_{\tilde{L}_{q_0}^{p_0}(T)} + \|\nabla^2 u\|_{\tilde{L}_{q_0}^{p_0}(T)} \leq c, \quad \lambda \geq \lambda_0. \tag{2.20}$$

So, we may take $\lambda \geq \lambda_0$ such that

$$\|u\|_\infty + \|\nabla u\|_\infty \leq \varepsilon, \tag{2.21}$$

where we take $\varepsilon \leq r_0$ when ∂D exists. Let $\Theta_t(x) = x + u_t(x)$. By (2.12) and (2.21) for $\varepsilon \leq r_0$ when ∂D exists, we have

$$\langle \nabla V(Y_t^{x,n}), \mathbf{n}(X_t^{x,n})\rangle \mathrm{d}l_t^{x,n} \leq 0.$$

So, by Itô's formula, $Y_t^{x,n} := \Theta_t(X_t^{x,n})$ satisfies

$$\mathrm{d}Y_t^{x,n} = \big\{1_{B_n} b_t^{(1)} + \lambda u_t + 1_{B_n}\nabla_{b_t^{(1)}} u_t\big\}(X_t^{x,n})\mathrm{d}t + \big\{(\nabla\Theta_t)\sigma_t\big\}(X_t^{x,n})\mathrm{d}W_t + \mathbf{n}(X_t^n)\mathrm{d}l_t^n, \tag{2.22}$$

where we have used the fact that $\nabla_{\mathbf{n}} u_t|_{\partial D} = 0$ implies that $\{\nabla\Theta_t\}\mathbf{n} = \mathbf{n}$ holds on ∂D. By (2.21) and **(A3)**(3) with (2.12) when $\partial D \neq \emptyset$, there exists a constant $c_0 > 0$ such that for some martingale M_t,

$$\begin{aligned}
&\mathrm{d}\{V(Y_t^{x,n}) + M_t\} \\
&\leq \Big[\big\langle\{b^{(1)} + \nabla_{b^{(1)}} u_t\}(X_t^{x,n}), \nabla V(Y_t^{x,n})\big\rangle + c_0(|\nabla V(Y_t^{x,n})| + \|\nabla^2 V(Y_t^{x,n})\|)\Big]\mathrm{d}t \\
&\leq \Big\{\langle b^{(1)}(X_t^{x,n}), \nabla V(X_t^{x,n})\rangle + \varepsilon|b^{(1)}(X_t^{x,n})| \sup_{B(X_t^{x,n},\varepsilon)} \|\nabla^2 V\| + c_0 K V(Y_t^{x,n})\Big\}\mathrm{d}t \\
&\leq \big\{K\phi(V(X_t^{x,n})) + c_0 K V(Y_t^{x,n})\big\}\mathrm{d}t \leq K\big\{\phi((1+\varepsilon K)V(Y_t^{x,n})) + c_0 V(Y_t^{x,n})\big\}\mathrm{d}t, \quad t \leq \tau_n^x.
\end{aligned}$$

Letting $H(r) := \int_0^r \frac{\mathrm{d}s}{r + \phi((1+\varepsilon K)s)}$, by Itô's formula and noting that $\phi' \geq 0$, we find a constant $c_1 > 0$ such that

$$\mathrm{d}H(V(Y_t^{x,n})) \leq c_1\mathrm{d}t + \mathrm{d}\tilde{M}_t, \quad t \in [0, \tau_n^x]$$

holds for some martingale $\tilde{M}_t$. Thus,

$$\mathbb{E}[(H \circ V)(Y_{t\wedge\tau_n^x}^{x,n})] \leq V(x + u(x)) + c_1 t, \quad t \geq 0, n \geq 1.$$

Since (2.21) and $|z| \geq n$ imply $|\Theta_t(z)| \geq |z| - |u(z)| \geq n - \varepsilon$, we derive

$$\mathbb{P}(\tau_n^x \leq t) \leq \frac{V(x + \Theta_0(x)) + c_1 t}{\inf_{|y| \geq n-\varepsilon} H(V(y))} =: \varepsilon_{t,n}(x), \quad t > 0. \tag{2.23}$$

Since $\lim_{|x|\to\infty} H(V)(x) = \int_0^\infty \frac{\mathrm{d}s}{s + \phi((1+\varepsilon K)s)} = \infty$, we obtain $\tau_n^x \to \infty (n \to \infty)$ as desired.

(b) Proof of (2.17). By [21, Proposition 1.3.8], the log-Harnack inequality

$$P_t \log f(y) \le \log P_t f(x) + c|x-y|^2, \quad x, y \in \bar{D}, 0 < f \in \mathscr{B}_b(\bar{D})$$

for some constant $c > 0$ implies the gradient estimate

$$|\nabla P_t f|^2 \le 2cP_t|f|^2, \quad f \in \mathscr{B}_b(\bar{D}),$$

and hence

$$\lim_{y \to x} \|P_t^* \delta_x - P_t^* \delta_y\|_{var} = 0, \quad x \in \bar{D}.$$

Let P_t^n be the Markov semigroup associated with X_t^n. Thus, by the log-Harnack inequality in [28, Theorem 4.1] for $D = \mathbb{R}^d$ and in [25, Theorem 4.1] for $D \ne \mathbb{R}^d$, we have

$$\lim_{y \to x} \|(P_t^n)^* \delta_x - (P_t^n)^* \delta_y\|_{var} = 0, \quad t \in (0, T]. \tag{2.24}$$

On the other hand, by (2.23) and $X_t = X_t^n$ for $t \le \tau_n$, we obtain

$$\begin{aligned} &\lim_{n \to \infty} \sup_{y \in \bar{D} \cap B(x,1)} \|P_t^* \delta_y - (P_t^n)^* \delta_y\|_{var} = \lim_{n \to \infty} \sup_{|f| \le 1, y \in \bar{D} \cap B(x,1)} |P_t f(y) - P_t^n f(y)| \\ &\le 2 \lim_{n \to \infty} \sup_{y \in \bar{D} \cap B(x,1)} \mathbb{P}(\tau_n^y \le t) = 0. \end{aligned}$$

Combining this with (2.24) and the triangle inequality, we prove (2.17).

(c) Finally, let $L_t := L_t^\sigma + \nabla_{b_t}$. By Itô's formula, for any $f \in C_0^2((0,T) \times D)$ we have

$$\mathrm{d} f_t(X_t) = (\partial_t + L_t) f_t(X_t) \mathrm{d}t + \mathrm{d}M_t$$

for some martingale M_t, so that $f_0 = f_T = 0$ yields

$$\int_{(0,T)} P_t \{(\partial_t + L) f_t\} \mathrm{d}t = 0, \quad f \in C_0^\infty((0,T) \times D).$$

This implies that the heat kernel $p_t(x, \cdot)$ of P_t solves the following PDE on $(0,T) \times D$ in the weak sense:

$$\partial_t u_t = L_t^* u_t = \mathrm{div} \mathscr{A}(t, \cdot, u_t, \nabla u_t) + \mathscr{B}(t, \cdot, \nabla u_t),$$

where $\mathscr{A} := (\mathscr{A}_1, \cdots, \mathscr{A}_d)$ and $\mathscr{B}$ are defined as

$$\begin{aligned} \mathscr{A}_i(t, \cdot, u, \nabla u) &:= \frac{1}{2} \sum_{j=1}^d (\sigma_t \sigma_t^*)_{ij} \partial_j u + \sum_{j=1}^d \Big\{ \frac{1}{2} \partial_j (\sigma_t \sigma_t^*)_{ij} - b_t^i \Big\} u, \\ \mathscr{B}(t, \cdot, \nabla u) &:= - \sum_{i,j=1}^d \big\{ \partial_j (\sigma_t \sigma_t^*)_{ij} \big\} \partial_i u. \end{aligned}$$

By the Harnack inequality as in [1, Theorem 3] (see also [17]), under the given conditions, for any $0 < s < t \le T$ and $N > 1$ with

$$\tilde{B}_N := \big\{ x \in \bar{D} \cap B_N : \ \rho_\partial(x) \ge N^{-1} \big\}$$

having positive volume, there exists a constant $c(s,t,N)>0$ such that satisfies

$$\sup_{\tilde B_N} p_s(x,\cdot) \le c(s,t,N) \inf_{\tilde B_N} p_t(x,\cdot), \quad x\in \bar D. \tag{2.25}$$

Since $\int_{\tilde B_N} p_s(x,y)\mathrm{d}y \to 1$ as $N\to\infty$, this implies $p_t(x,y)>0$ for any $(t,x,y)\in(0,T]\times\bar D\times D$. In particular, $P_t 1_{\tilde B_N}>0$. On the other hand, (2.17) implies that $P_t 1_{\tilde B_N}$ is continuous, so that

$$\inf_{x\in\bar D\cap B_N} P_t 1_{\tilde B_N}(x)>0, \quad t\in(0,T].$$

This together with (2.25) gives

$$\inf_{(\bar D\cap B_N)\times\tilde B_N} p_t \ge \frac{1}{c(s,t,N)} \inf_{x\in\bar D\cap\bar B_N} P_s 1_{\tilde B_N}(x)>0, \quad 0<s<t\le T.$$

Therefore, (2.18) holds.

□

To make Zvonkin's transform to kill the singular drift, we present the lemma which extends Theorem 2.10 in [27] for $D=\mathbb{R}^d$.

(A5) $D=\mathbb{R}^d$, σ and $b^{(0)}$ satisfy the following conditions.

(1) $a:=\sigma\sigma^*$ is invertible and uniformly continuous with $\|a\|_\infty+\|a^{-1}\|_\infty<\infty$.

(2) $|b^{(0)}|\in\tilde L^p$ for some $p>d$.

(A6) $\partial D\in C_b^{2,L}$, **(A5)** holds for $\bar D$ replacing $\mathbb{R}^d$, and **(A2)**(2) holds.

Lemma 2.5. *Assume* **(A5)** *for* $D=\mathbb{R}^d$ *and* **(A6)** *for* $D\ne\mathbb{R}^d$. *Let* $L^0=\frac{1}{2}\mathrm{tr}\{\sigma\sigma^*\nabla^2\}+\nabla_{b^{(0)}}$. *Then there exist constants* $\lambda_0>0$ *increasing in* $\|b^{(0)}\|_{\tilde L^p}$ *such that for any* $\lambda\ge\lambda_0$ *and any* $f\in\tilde L^k$ *for some* $k\in(1,\infty)$, *the elliptic equation*

$$(L^0-\lambda)u=f,\ \nabla_{\mathbf{n}}u|_{\partial D}=0 \text{ if } D\ne\mathbb{R}^d \tag{2.26}$$

has a unique solution $u\in\tilde H^{2,k}$. *Moreover, for any* $p'\in[k,\infty]$ *and* $\theta\in[0,2-\frac{d}{k}+\frac{d}{p'})$, *there exists a constant* $c>0$ *increasing in* $\|b^{(0)}\|_{\tilde L^p}$ *such that*

$$\lambda^{\frac{1}{2}(2-\theta+\frac{d}{p'}-\frac{d}{k})}\|u\|_{\tilde H^{\theta,p'}}+\|u\|_{\tilde H^{2,k}}\le c\|f\|_{\tilde L^k}, \quad f\in\tilde L^k. \tag{2.27}$$

Proof. (a) Let us verify the priori estimate (2.27) for a solution u to (2.26), which in particular implies the uniqueness, since the difference of two solutions solves the equation with $f=0$.

For $u\in\tilde H^{2,k}$ solving (2.26), let

$$\bar u_t=u(1-t), \quad t\in[0,1].$$

By (2.26) we have

$$(\partial_t+L^0-\lambda)\bar u_t=f(1-t)-u, \quad t\in[0,1], \bar u_1=0,\ \nabla_{\mathbf{n}}\bar u_t|_{\partial D}=0 \text{ if } D\ne\mathbb{R}^d.$$

By Theorem 2.1 with $q = q' = 2$ in [28] for $D = \mathbb{R}^d$, and Lemma 2.6 in [25] for $D \neq \mathbb{R}^d$, there exist constants $\lambda_1, c_1 > 1$ increasing in $\|b^{(0)}\|_{\tilde{L}^p}$ and sufficient large $q > 2$ such that

$$\lambda^{\frac{1}{2}(2-\theta+\frac{d}{p'}-\frac{d}{k})}\|\bar{u}\|_{\tilde{H}_q^{\theta,p'}} + \|\bar{u}\|_{\tilde{H}_q^{2,k}} \le c_1\|f(1-t)-u\|_{\tilde{L}_q^k} \le c_1\|f\|_{\tilde{L}^k} + c_1\|u\|_{\tilde{L}^k}. \tag{2.28}$$

Taking $\theta = 0, p = p'$ and $c_2 = \|1-\cdot\|_{L^q([0,1])}$, we obtain

$$\lambda^{\frac{1}{2}(2-\theta)}\|u\|_{\tilde{L}^p} \le \frac{c_1}{c_2}\big(\|f\|_{\tilde{L}^p} + \|u\|_{\tilde{L}^p}\big), \quad \lambda \ge \lambda_1.$$

Letting $\lambda_0 > \lambda_1$ such that

$$\lambda_0^{\frac{1}{2}(2-\theta)} \ge 2\frac{c_1}{c_2},$$

we obtain we obtain

$$\|u\|_{\tilde{L}^k} \le \|f\|_{\tilde{L}^k}, \quad \lambda \ge \lambda_0.$$

Combining this with (2.28) implies (2.27) for some constant $c > 0$.

(b) Existence of solution for $f \in \tilde{L}^k$. Let $\{f_n\}_{n\ge1} \subset C_b^\infty(\bar{D})$ with $\mathbf{n}f_n|_{\partial D} = 0$ is $\partial D \neq \emptyset$ such that $\|f_n - f\|_{\tilde{L}^k} \to 0$ as $n \to \infty$. Let P_t^0 be the Markov semigroup generated by L^0 and let

$$u_n = \int_0^\infty \mathrm{e}^{-\lambda t} P_t^0 f_n \mathrm{d}t.$$

By Kolmogorov equation we have

$$\partial_t P_t^0 f_n = L^0 P_t^0 f_n = P_t^0 L^0 f_n$$

so that

$$L^0 u_n = \int_0^\infty \mathrm{e}^{-\lambda t} L^0 P_t^0 f_n \mathrm{d}t = \int_0^\infty \mathrm{e}^{-\lambda t}\partial_t P_t^0 f_n \mathrm{d}t = \lambda u_n - f_n.$$

Then

$$(L^0 - \lambda)(u_n - u_m) = f_n - f_m, \quad n, m \ge 1.$$

By (2.27),

$$\lim_{n,m\to\infty}\big\{\|u_n - u_m\|_{\tilde{H}^{\theta,p'}} + \|\nabla^2(u_n - u_m)\|_{\tilde{L}^k}\big\} = 0,$$

so that $u := \lim_{n\to\infty} u_n$ exists in $\tilde{H}^{\theta,p'} \cap \tilde{H}^{2,k}$, which solves (2.26). □

2.3 Proofs of Theorems 2.1, 2.3 and Corollary 2.2

Proofs of Theorems 2.1, 2.3. It is easy to see that (2.1) implies **(A3)**(3) for any $T > 0$ and $\phi(r) = 1$, by Lemma 2.4, **(A1)** and **(A2)** imply the well-posedness, strong Feller property and irreducibility of (1.3) and (2.10) respectively. According to [7, Theorem 4.2.1], the strong Feller property and the irreducibility imply the uniqueness of invariant probability measure. So, it remains to prove the existence of the invariant probability measure μ and the claimed assertions on the ergodicity.

(a) Let u solve (2.26) for $b = -b^{(0)}$ and large enough $\lambda > 0$ such that (2.27) implies (2.21). Moreover, for $\Theta(x) := x + u(x)$, let $\hat{P}_t$ be the Markov semigroup associated with $Y_t := \Theta(X_t)$, so that

$$\hat{P}_t f(x) = \{P_t(f \circ \Theta)\}(\Theta^{-1}(x)), \quad t \geq 0, x \in \mathbb{R}^d, f \in \mathscr{B}_b(\mathbb{R}^d). \tag{2.29}$$

Since $\lim_{|x|\to\infty} \sup_{|y-x|\leq\varepsilon} \frac{|\nabla V(y)|}{V(x)} = 0$, by (2.21) and $V \geq 1$ we find a constant $\theta \in (0,1)$ such that

$$\|\nabla u(x)\| \vee |\Theta(x) - x| \leq \varepsilon, \quad \theta V(\Theta(x)) \leq V(x) \leq \theta^{-1} V(\Theta(x)), \quad x \in \bar{D}. \tag{2.30}$$

Thus, it suffices to prove the desired assertions for $\hat{P}_t$ replacing P_t, where the unique invariant probability measure $\hat{\mu}$ of $\hat{P}_t$ and that μ of P_t satisfies

$$\hat{\mu} = \mu \circ \Theta^{-1}. \tag{2.31}$$

(b) Let X_t^n, Y_t^n and τ_n be in the proof of Lemma 2.4 for the present time-homogenous setting. Since $Y_t^n = Y_t$ and $1_{B_n}(X_t^n) = 1$ for $t \leq \tau_n$, and since $\tau_n \to \infty$ as $n \to \infty$, (2.22) implies

$$\mathrm{d}Y_t = \big\{b^{(1)} + \lambda u + \nabla_{b^{(1)}} u\big\}(X_t)\mathrm{d}t + \{(\nabla\Theta)\sigma\}(X_t)\mathrm{d}W_t + \mathbf{n}(X_t)\mathrm{d}l_t,$$

so that for any $\varepsilon \in (0, 1 \wedge r_0)$, where $r_0 > 0$ is in (2.12) when $\partial D \neq \emptyset$, by Itô's formula and (2.12), we find a constant $c_\varepsilon > 0$ such that

$$\begin{aligned}
&\mathrm{d}\{V(Y_t) + M_t\} \leq \Big\{\big\langle\{b^{(1)} + \nabla_{b^{(1)}} u\}(X_t), \nabla V(Y_t)\big\rangle + c_\varepsilon(|\nabla V(Y_t)| + \|\nabla^2 V(Y_t)\|)\Big\}\mathrm{d}t \\
&\leq \Big\{\langle b^{(1)}(X_t), \nabla V(X_t)\rangle + \varepsilon|b^{(1)}(X_t)| \sup_{B(X_t,\varepsilon)} \{|\nabla V| + \|\nabla^2 V\|\} + c_\varepsilon \sup_{B(X_t,\varepsilon)} (|\nabla V| + \|\nabla^2 V\|)\Big\}\mathrm{d}t.
\end{aligned}$$

Combining this with (2.1) and (2.12) for $D \neq \mathbb{R}^d$, when $\varepsilon > 0$ is small enough we find constants $c_1, c_2 > 0$ such that

$$\mathrm{d}\{V(Y_t) + M_t\} \leq \{c_1 - c_2\Phi(V(X_t))\}\mathrm{d}t.$$

By (2.30), this implies that for some constant $c_4 > 0$,

$$\mathrm{d}V(Y_t) \leq \big\{c_4 - c_2\Phi(\theta V(Y_t))\big\}\mathrm{d}t - \mathrm{d}M_t. \tag{2.32}$$

Thus,

$$\int_0^t \mathbb{E}\Phi(\theta V(Y_s))\mathrm{d}s \leq \frac{c_4 + V(x)}{c_2} < \infty, \quad t > 0, Y_0 = x \in \Theta(\bar{D}).$$

Since $\Phi(\theta V)$ is a compact function, this implies the existence of invariant probability $\hat{\mu}$ according to the standard Bogoliubov-Krylov's tightness argument. Moreover, (2.32) implies $\hat{\mu}(\Phi(\theta V)) < \infty$, so that by (2.30) and (2.31), $\mu(\Phi(\varepsilon_0 V)) < \infty$ holds for $\varepsilon_0 = \theta^2$.

(c) By (2.18), (2.29) and (2.30), any compact set $\mathbf{K} \subset \Theta(\bar{D})$ is a petite set of $\hat{P}_t$, i.e. there exist $t > 0$ and a nontrivial measure ν such that

$$\inf_{x\in\mathbf{K}} \hat{P}_t^* \delta_x \geq \nu.$$

When $\Phi(r) \geq kr$ for some constant $k > 0$, (2.32) implies

$$\hat{P}_t V(x) \leq \frac{k_1}{k_2} + \mathrm{e}^{-k_2 t} V(x), \quad t \geq 0, x \in \Theta(\bar{D}) \tag{2.33}$$

for some constants $k_1, k_2 > 0$. Since $\lim_{|x|\to\infty} V(x) = \infty$ and as observed above that any compact set is a petite set for $\hat{P}_t$, by Theorem 5.2(c) in [8], we obtain

$$\|\hat{P}_t^* \delta_x - \hat{\mu}\|_V \leq c\mathrm{e}^{-\lambda t} V(x), \quad x \in \Theta(\bar{D}), t \geq 0$$

for some constants $c, \lambda > 0$. Thus,

$$\|\hat{P}_t^* \delta_x - \hat{P}_t^* \delta_y\|_V \leq c\mathrm{e}^{-\lambda t}(V(x) + V(y)), \quad t \geq 0, x, y \in \Theta(\bar{D}).$$

Therefore, for any probability measures μ_1, μ_2 on $\Theta(\bar{D})$,

$$\begin{aligned}
&\|\hat{P}_t^* \mu_1 - \hat{P}_t^* \mu_2\|_V = \|\hat{P}_t^*(\mu_1 - \mu_2)^+ - \hat{P}_t^*(\mu_1 - \mu_2)^-\|_V \\
&= \frac{1}{2}\|\mu_1 - \mu_2\|_{var} \Big\| \hat{P}_t^* \frac{2(\mu_1 - \mu_2)^+}{\|\mu_1 - \mu_2\|_{var}} - \hat{P}_t^* \frac{2(\mu_1 - \mu_2)^-}{\|\mu_1 - \mu_2\|_{var}} \Big\|_V \\
&\leq \frac{c}{2}\mathrm{e}^{-\lambda t}\|\mu_1 - \mu_2\|_{var} \Big(\frac{2(\mu_1 - \mu_2)^+}{\|\mu_1 - \mu_2\|_{var}} + \frac{2(\mu_1 - \mu_2)^-}{\|\mu_1 - \mu_2\|_{var}} \Big)(V) \\
&\leq c\mathrm{e}^{-\lambda t}\|\mu_1 - \mu_2\|_V.
\end{aligned}$$

This together with (2.29) and (2.30) implies (2.3) for some constants $c, \lambda > 0$.

(d) Let Φ be convex. By Jensen's inequality and (2.32), $\gamma_t := \theta \mathbb{E}[V(Y_t)]$ satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}\gamma_t \leq \theta c_4 - \theta c_2 \Phi(\gamma_t), \quad t \geq 0. \tag{2.34}$$

Let

$$H(r) := \int_0^r \frac{\mathrm{d}s}{\Phi(s)}, \quad r \geq 0.$$

We aim to prove that for some constant $k > 1$

$$\gamma_t \leq k + H^{-1}(H(\gamma_0) - tk^{-1}), \quad t \geq 0, \tag{2.35}$$

where $H^{-1}(r) := 0$ for $r \leq 0$. We prove this estimate by considering three situations.

(1) Let $\Phi(\gamma_0) \leq \frac{c_4}{c_2}$. Since (2.34) implies $\gamma_t' \leq 0$ for $\gamma_t \geq \Phi^{-1}(\frac{c_4}{c_2})$, so

$$\gamma_t \leq \Phi^{-1}(c_4/c_2), \quad t \geq 0. \tag{2.36}$$

(2) Let $\frac{c_4}{c_2} < \Phi(\gamma_0) \leq \frac{2c_4}{c_2}$. Then (2.34) implies $\gamma_t' \leq 0$ for all $t \geq 0$ so that

$$\gamma_t \leq \Phi^{-1}(2c_4/c_2), \quad t \geq 0. \tag{2.37}$$

(3) Let $\Phi(\gamma_0) > \frac{2c_4}{c_2}$. If

$$t \le t_0 := \inf\Big\{t \ge 0 : \Phi(\gamma_t) \le \frac{2c_4}{c_2}\Big\},$$

then (2.34) implies

$$\frac{\mathrm{d}H(\gamma_t)}{\mathrm{d}t} = \frac{\gamma_t'}{\Phi(\gamma_t)} \le -\frac{\theta c_2}{2},$$

so that

$$H(\gamma_t) \le H(\gamma_0) - \frac{\theta c_2}{2}t, \quad t \in [0, t_0], \tag{2.38}$$

which implies

$$\gamma_t \le H^{-1}(H(\gamma_0) - \theta c_2 t/2), \quad t \in [0, t_0].$$

Noting that when $t > t_0$, $(\gamma_t)_{t\ge t_0}$ satisfies (2.34) with γ_{t_0} satisfies $\frac{c_4}{c_2} < \Phi(\gamma_{t_0}) \le \frac{2c_4}{c_2}$, so that (2.36) holds, i.e.

$$\gamma_t \le \Phi^{-1}(2c_4/c_2).$$

In conclusion, we obtain

$$\gamma_t \le \Phi^{-1}(2c_4/c_2) + H^{-1}(H(\gamma_0) - \theta c_2 t/2), \quad t \ge 0.$$

Combining this with (1) and (2), we prove (2.35) for some constant $k > 1$.

(e) Since $1 \le \Phi(r) \to \infty$ as $r \to \infty$, when Φ is convex we find a constant $\delta > 0$ such that $\Phi(r) \ge \delta r, r \ge 0$. So, by step (b), (2.3) holds. Combining this with (2.35) and applying the semigroup property, we derive

$$\begin{aligned}\|\hat P_t^*\delta_x - \hat\mu\|_V &= \sup_{|f|\le V} |\hat P_{t/2}(\hat P_{t/2}f - \hat\mu(f))(x)| \\ &\le c\mathrm{e}^{-\lambda t/2}\hat P_{t/2}V(x) \le c\big\{k + H^{-1}(H(\theta V(x)) - (2k)^{-1}t)\big\}\mathrm{e}^{-\lambda t/2}.\end{aligned}$$

Combining this with (2.29), (2.30) and (2.31), we prove (2.4) for some constants $k, \lambda > 0$.

Finally, if $H(\infty) < \infty$, we take $t^* = kH(\infty)$ in (2.4) to derive

$$\sup_{x\in\bar D} \|P_t\delta_x - \mu\|_V \le c\mathrm{e}^{-\lambda t}, \quad t \ge t^*$$

for some constants $c, \lambda > 0$, which implies (2.5) by the argument leading to (2.3) in step (c). □

Proof of Corollary 2.2. By (2.8), for any $\theta \in ((1-\alpha)^+, \frac{1}{2})$ there exists a constant $c_3 > 0$ such that

$$\phi(r) \ge c_3(1+r)^{1-\theta}, \quad r \ge 0.$$

Then (2.1) holds for $V := \mathrm{e}^{(1+|\cdot|^2)^\theta}$ and $\Phi(r) = r$. So the first assertion in (1) follows from Theorem 2.1(1).

Next, (2.7) and (2.9) imply (2.1) for $V := \psi(|\cdot|^2)$ and $\Phi(r) = r$, so that the second assertion in (1) holds by Theorem 2.1(1).

Finally, if $\int_0^\infty \frac{\mathrm{d}s}{\phi(s)} < \infty$, then for any $q > 0$, (2.1) holds for $V := (1 + |\cdot|^2)^q$ and $\Phi(r) = (1+r)^{1-\frac{1}{q}}\phi(r^{\frac{1}{q}})$, so that $\int_0^\infty \frac{\mathrm{d}s}{\Phi(s)} < \infty$. Then the proof is finished by Theorem 2.1(2). □

3 Uniform ergodicity for singular reflecting McKean-Vlasov SDEs

We now consider the SDE (1.1) for $D = \mathbb{R}^d$ or D being a $C_b^{2,L}$ domain. To prove the uniform ergodicity, compare (1.1) with the following classical SDE with fixed distribution parameter $\gamma \in \mathscr{P}$:

$$\mathrm{d}X_t^\gamma = b(X_t^\gamma, \gamma) + \sigma(X_t^\gamma)\mathrm{d}W_t + \mathbf{n}(X_t^\gamma)\mathrm{d}l_t^\gamma, \tag{3.1}$$

where we set $l_t^\gamma = 0$ if $D = \mathbb{R}^d$. If for any $\nu \in \mathscr{P}$, **(A1)** for $D = \mathbb{R}^d$ or **(A2)** for $D \neq \mathbb{R}^d$ holds for $b(\cdot, \nu)$ replacing b, then the well-posedness follows from that of (3.1) and [25, Theorem 3.2] for $k = 0$. Let $(P_t^\gamma)^*\nu = \mathscr{L}_{X_t^\gamma}$ for $\mathscr{L}_{X_0^\gamma} = \nu$.

Let $\zeta(\gamma_1, \gamma_2) := \sigma^*(\sigma\sigma^*)^{-1}\big[b(\cdot, \gamma_2) - b(\cdot, \gamma_1)\big]$, $\gamma_1, \gamma_2 \in \mathscr{P}$. We make the following assumption on the dependence of distribution.

(H1) **(A1)** for $D = \mathbb{R}^d$ or **(A2)** for $D \neq \mathbb{R}^d$ holds with $b(\cdot, \nu)$ replacing b uniformly in $\nu \in \mathscr{P}$.

(H2) There exist constants $q \geq 2$ and $k > 0$ such that for any $\gamma \in \mathscr{P}$ and $\nu \in C([0,1]; \mathscr{P})$,

$$\int_0^t \mathrm{d}s \int_{\bar{D}} |\zeta(\gamma, \nu_s)|^2 \mathrm{d}(P_s^\gamma)^*\nu_0 \leq k^2 \bigg(\int_0^t \|\nu_s - \gamma\|_{var}^q \mathrm{d}s \bigg)^{\frac{2}{q}},$$
$$\int_{\bar{D}} \mathrm{e}^{\frac{1}{2}\int_0^t |\zeta(\gamma,\nu_s)|^2} \mathrm{d}(P_s^\gamma)^*\nu_0 < \infty, \quad t \geq 0.$$

Remark 3.1. (1) Obviously, **(H2)** holds for $q = 2$ if there exists a constant $\kappa > 0$ such that

$$|b(x, \gamma_1) - b(x, \gamma_2)| \leq \kappa \|\gamma_1 - \gamma_2\|_{var}, \quad x \in \bar{D}, \gamma_1, \gamma_2 \in \mathscr{P}.$$

By a standard fixed point argument, this together with (H_1) implies the well-posedness of (1.1) for any initial value, see [25].

(2) In general, **(H2)** follows from Krylov's estimate for $p, q > 2$ with $\frac{d}{p} + \frac{2}{q} < 1$ and the condition

$$\|\zeta(\gamma_1, \gamma_2)\|_{\tilde{L}^p} \leq k\|\gamma_1 - \gamma_2\|_{var}, \quad \gamma_1, \gamma_2 \in \mathscr{P}.$$

Note that under **(H1)**, the Krylov's estimate holds when b contains an $\tilde{L}^p$ term for $p > d$ and a Lipschitz continuous term, see [25] for details.

Theorem 3.1. *Assume* **(H1)** *and* **(H2)** *and let* (1.1) *be well-posed. If k is small enough and Φ is convex with $\int_0^\infty \frac{\mathrm{d}s}{\Phi(s)} < \infty$, then P_t^* has a unique invariant probability measure μ, $\mu(\Phi(\varepsilon_0 V)) < \infty$ holds for some constant $\varepsilon_0 > 0$, and there exist constants $c, \lambda > 0$ such that*

$$\|P_t^*\nu - \mu\|_{var} \leq c\mathrm{e}^{-\lambda t}\|\mu - \nu\|_{var}, \quad t \geq 0, \nu \in \mathscr{P}. \tag{3.2}$$

To prove this result, we first present a general result deducing the uniform ergodicity of McKean-Vlasov SDEs from that of classical ones.

The following result says that if (3.1) is uniformly ergodic uniformly in γ, and if the dependence of $b(x, \mu)$ on μ is weak enough, then (1.1) is uniformly ergodic.

Lemma 3.2. *Assume* (H_1) *and that for any* $\gamma \in \mathscr{P}$, $(P_t^\gamma)^*$ *has a unique invariant probability measure* μ_γ *such that*

$$\|(P_t^\gamma)^*\mu - \mu_\gamma\|_{var} \le c\mathrm{e}^{-\lambda t}\|\mu - \mu_\gamma\|_{var}, \quad t \ge 0, \gamma, \mu \in \mathscr{P} \tag{3.3}$$

holds for some constants $c, \lambda > 0$. *Then* (1.1) *is well-posed and the following assertions hold.*

(1) *If there exists a constant* $\kappa \in (0, \kappa_1)$ *for*

$$\kappa_1 := \sup_{t > (\log c)/\lambda} \frac{1 - c\mathrm{e}^{-\lambda t}}{\sqrt{t}},$$

such that

$$\int_{\bar{D}} |\zeta(\gamma_1, \gamma_2)|^2 \mathrm{d}\mu_{\gamma_2} \le \kappa^2 \|\gamma_1 - \gamma_2\|_{var}^2, \quad \gamma_1, \gamma_2 \in \mathscr{P}, \tag{3.4}$$

then P_t^* *associated with* (1.1) *has a unique invariant probability measure* μ.

(2) *Let* μ *be* P_t^**-invariant. If there exist constants* $q \ge 2$ *and* $k \in (0, k_q)$, *where*

$$k_q := \sup\left\{k > 0 : \ \frac{4^{q-1}(ck)^q \mathrm{e}^{2^{q-1}k^q t}}{q\lambda + 2^{q-1}k^q} \le \frac{1}{2}\right\},$$

such that for $\hat{t} := \frac{\log(2c)}{\lambda}$,

$$\mathbb{E}\int_0^t |\zeta(\mu, P_s^*\nu)(X_s^\mu)|^2 \mathrm{d}s \le k^2\bigg(\int_0^t \|\mu - P_s^*\nu\|_{var}^q \mathrm{d}s\bigg)^{\frac{1}{q}}, \quad t \in (0, \hat{t}], \nu \in \mathscr{P}, \tag{3.5}$$

then there exists a constant $c' > 0$ *such that*

$$\|P_t^*\nu - \mu\|_{var}^q \le c'\mathrm{e}^{-\lambda' t}\|\nu - \mu\|_{var}^q, \quad t \ge 0, \nu \in \mathscr{P} \tag{3.6}$$

holds for

$$\lambda' := -\frac{\lambda}{\log(2c)} \log\Big(\frac{1}{2} + \frac{4^{q-1}(ck)^q \mathrm{e}^{2^{q-1}k^q t}}{q\lambda + 2^{q-1}k^q}\Big) > 0.$$

Proof. (a) Existence and uniqueness of μ. For any $\gamma \in \mathscr{P}$, (3.3) implies that P_t^γ has a unique invariant probability measure μ_γ. It suffices to prove that the map $\gamma \mapsto \mu_\gamma$ has a unique fixed point μ, which is the unique invariant probability measure of P_t^*.

For $\gamma_1, \gamma_2 \in \mathscr{P}$, (3.1) implies

$$\|(P_t^{\gamma_1})^*\mu_{\gamma_2} - \mu_{\gamma_1}\|_{var} \le c\mathrm{e}^{-\lambda t}\|\mu_{\gamma_2} - \mu_{\gamma_1}\|_{var}, \quad t \ge 0. \tag{3.7}$$

On the other hand, let (X_t^1, X_t^2) solve the SDEs

$$\mathrm{d}X_t^i = b(X_t^i, \gamma_i) + \sigma(X_t^i)\mathrm{d}W_t + \mathbf{n}(X_t^i)\mathrm{d}l_t^i, \quad i = 1, 2$$

with $X_0^1 = X_0^2$ having distribution μ_{γ_2}. Since μ_{γ_2} is $(P_t^{\gamma_2})^*$-invariant, we have

$$\mathscr{L}_{X_t^2} = (P_t^{\gamma_2})^*\mu_{\gamma_2} = \mu_{\gamma_2}, \quad \mathscr{L}_{X_t^1} = (P_t^{\gamma_1})^*\mu_{\gamma_2}, \quad t \ge 0. \tag{3.8}$$

By (H_2),

$$R_t = \mathrm{e}^{\int_0^t \langle \zeta(\gamma_1,\gamma_2)(X_s^1), \mathrm{d}W_s\rangle - \frac{1}{2}\int_0^t |\zeta(\gamma_1,\gamma_2)(X_s^1)|^2 \mathrm{d}s}, \quad t \geq 0$$

is a martingale, and by Girsanov's theorem, for any $t > 0$,

$$\tilde{W}_r := W_r - \int_0^r \zeta(\gamma_1, \gamma_2)(X_s^1)\mathrm{d}s, \quad r \in [0, t]$$

is a Brownian motion under $\mathbb{Q}_t := R_t\mathbb{P}$. Reformulating the SDE for X_r^1 as

$$\mathrm{d}X_r^1 = b(X_r^1, \gamma_2)\mathrm{d}r + \sigma(X_r^1)\mathrm{d}\tilde{W}_r + \mathbf{n}(X_r^1)\mathrm{d}l_r^1, \quad r \in [0, t],$$

by $X_0^1 = X_0^2$ and the weak uniqueness, the law of X_t^1 under $\mathbb{Q}_t$ satisfies

$$\mathscr{L}_{X_t^1|\mathbb{Q}_t} = \mathscr{L}_{X_t^2} = (P_t^{\gamma_2})^*\mu_{\gamma_2} = \mu_{\gamma_2}.$$

Combining this with (3.8) and Pinsker's inequality, we obtain

$$\begin{aligned} &\|(P_t^{\gamma_1})^*\mu_{\gamma_2} - \mu_{\gamma_2}\|_{var}^2 = \|(P_t^{\gamma_1})^*\mu_{\gamma_2} - (P_t^{\gamma_2})^*\mu_{\gamma_2}\|_{var}^2 \\ &= \sup_{|f|\leq 1} \big|\mathbb{E}[f(X_t^1)] - \mathbb{E}[f(X_t^1)R_t]\big|^2 \leq \big(\mathbb{E}|R_t - 1|\big)^2 \leq 2\mathbb{E}[R_t \log R_t] \\ &= 2\mathbb{E}_{\mathbb{Q}_t}[\log R_t] = \mathbb{E}_{\mathbb{Q}_t}\int_0^t \big|\zeta(\gamma_1,\gamma_2)|^2(X_s^1)\big|^2\mathrm{d}s = t\int_{\bar{D}} |\zeta(\gamma_1,\gamma_2)|^2\mathrm{d}\mu_{\gamma_2}. \end{aligned} \tag{3.9}$$

Then (3.4) implies

$$\|(P_t^{\gamma_1})^*\mu_{\gamma_2} - \mu_{\gamma_2}\|_{var}^2 \leq \kappa^2 t\|\gamma_1 - \gamma_2\|_{var}^2.$$

Combining this with (3.7) and taking $t = \frac{\log(2c)}{\lambda}$, we derive

$$\begin{aligned} \|\mu_{\gamma_1} - \mu_{\gamma_2}\|_{var} &\leq \|(P_t^{\gamma_1})^*\mu_{\gamma_2} - \mu_{\gamma_1}\|_{var} + \|(P_t^{\gamma_1})^*\mu_{\gamma_2} - \mu_{\gamma_2}\|_{var} \\ &\leq \ c\mathrm{e}^{-\lambda t}\|\mu_{\gamma_1} - \mu_{\gamma_2}\|_{var} + \kappa\sqrt{t}\|\gamma_1 - \gamma_2\|_{var}, \quad t > 0. \end{aligned}$$

Thus,

$$\|\mu_{\gamma_1} - \mu_{\gamma_2}\|_{var} \leq \inf_{t>(\log c)/\lambda} \frac{\kappa\sqrt{t}}{1 - c\mathrm{e}^{-\lambda t}}\|\gamma_1 - \gamma_2\|_{var} = \frac{\kappa}{\kappa_1}\|\gamma_1 - \gamma_2\|_{var}.$$

Since $\kappa < \kappa_1$, μ_γ is contractive in γ, hence has a unique fixed point.

(b) Uniform ergodicity. Let μ be the unique invariant probability measure of P_t^*, and for any $\nu \in \mathscr{P}$ let $(\bar{X}_0, X_0)$ be $\mathscr{F}_0$-measurable such that

$$\mathbb{P}(\bar{X}_0 \neq X_0) = \frac{1}{2}\|\mu - \nu\|_{var}, \quad \mathscr{L}_{\bar{X}_0} = \mu, \quad \mathscr{L}_{X_0} = \nu.$$

Let $\bar{X}_t$ and X_t solve the following SDEs with initial values $\bar{X}_0$ and X_0 respectively:

$$\begin{aligned} \mathrm{d}\bar{X}_t &= b(\bar{X}_t, \mu)\mathrm{d}t + \sigma(\bar{X}_t)\mathrm{d}W_t + \mathbf{n}(\bar{X}_t)\mathrm{d}\bar{l}_t, \\ \mathrm{d}X_t &= b(X_t, P_t^*\nu)\mathrm{d}t + \sigma(X_t)\mathrm{d}W_t + \mathbf{n}(X_t)\mathrm{d}l_t. \end{aligned}$$

Since μ is P_t^*-invariant, we have

$$\mathscr{L}_{\bar{X}_t} = (P_t^\mu)^*\mu = P_t^*\mu = \mu. \tag{3.10}$$

Moreover, $\mathscr{L}_{X_t} = P_t^*\nu$ by the definition of P_t^*. Let

$$\bar{R}_t := \mathrm{e}^{\int_0^t \langle \zeta(P_s^*\nu,\mu)(X_s), \mathrm{d}W_s\rangle - \frac{1}{2}\int_0^t |\zeta(P_s^*\nu,\mu)(X_s)|^2 \mathrm{d}s}.$$

Similarly to (3.9), by Girsanov's theorem we have $\mathscr{L}_{X_t|\bar{R}_t\mathbb{P}} = (P_t^\mu)^*\nu$, so that Pinsker's inequality and (3.5) yield

$$\begin{aligned}\|(P_t^\mu)^*\nu - P_t^*\nu\|_{var}^2 &= \sup_{|f|\le 1} \big|\mathbb{E}[f(X_t)\bar{R}_t] - \mathbb{E}[f(X_t)]\big|^2 \\ &\le k^2\bigg(\int_0^t \|\mu - P_s^*\nu\|_{var}^q\bigg)^{\frac{2}{q}}\mathrm{d}s, \quad t\in[0,\hat{t}].\end{aligned}$$

This together with (3.7) for $\gamma_1 = \mu$ and (3.10) gives

$$\begin{aligned}\|P_t^*\nu - \mu\|_{var}^q &\le 2^{q-1}\big(\|P_t^*\nu - (P_t^\mu)^*\nu\|_{var}^q + \|(P_t^\mu)^*\nu - \mu\|_{var}^q\big) \\ &\le 2^{q-1}k^q\int_0^t \|\mu - P_s^*\nu\|_{var}^q\mathrm{d}s + 2^{q-1}c^q\mathrm{e}^{-q\lambda t}\|\nu-\mu\|_{var}^q, \quad t\in[0,\hat{t}].\end{aligned}$$

By Gronwall's inequality we obtain

$$\begin{aligned}\|P_t^*\nu - \mu\|_{var}^q &\le \|\mu-\nu\|_{var}^q\bigg(2^{q-1}c^q\mathrm{e}^{-q\lambda t} + 4^{q-1}k^qc^q\int_0^t \mathrm{e}^{-q\lambda s + 2^{q-1}k^q(t-s)}\mathrm{d}s\bigg) \\ &\le \bigg\{2^{q-1}c^q\mathrm{e}^{-q\lambda t} + \frac{4^{q-1}(ck)^q\mathrm{e}^{2^{q-1}k^qt}}{q\lambda + 2^{q-1}k^q}\bigg\}\|\mu-\nu\|_{var}^q, \quad t\in[0,\hat{t}].\end{aligned}$$

Taking $t = \hat{t} := \frac{\log(2c)}{\lambda}$, we arrive at

$$\|P_{\hat{t}}^*\nu - \mu\|_{var}^2 \le \delta_k\|\mu-\nu\|_{var}^2, \quad \nu\in\mathscr{P}$$

for

$$\delta_k := \Big(\frac{1}{2} + \frac{4^{q-1}(ck)^q\mathrm{e}^{2^{q-1}k^qt}}{q\lambda + 2^{q-1}k^q}\Big) < 1, \quad k\in(0,k_p).$$

So, (3.6) holds for some constant $c'>0$ due to the semigroup property $P_{t+s}^* = P_t^*P_s^*$. □

To verify condition (3.3), we present below a Harris type theorem on the uniform ergodicity for a family of Markov processes.

Lemma 3.3. *Let (E,ρ) be a metric space and let $\{(P_t^i)_{t\ge0} : i\in I\}$ be a family of Markov semigroups on $\mathscr{B}_b(E)$. If there exist $t_0, t_1 > 0$ and measurable set $B\subset E$ such that*

$$\alpha := \inf_{i\in I, x\in E} P_{t_0}^i 1_B(x) > 0, \tag{3.11}$$

$$\beta := \sup_{i\in I, x,y\in B} \|(P_{t_1}^i)^*\delta_x - (P_{t_1}^i)^*\delta_y\|_{var} < 2, \tag{3.12}$$

then there exists $c > 0$ such that

$$\sup_{i\in I, x,y\in E} \|(P_t^i)^*\delta_x - (P_t^i)^*\delta_y\|_{var} \le c\mathrm{e}^{-\lambda t}, \quad t \ge 0 \tag{3.13}$$

holds for $\lambda := \frac{1}{t_0+t_1}\log\frac{2}{2-\alpha^2(2-\beta)} > 0$.

Proof. The proof is more or less standard. By the semigroup property, we have

$$\begin{aligned}
&\|(P_{t_0+t_1}^i)^*\delta_x - (P_{t_0+t_1}^i)^*\delta_y\|_{var} \\
&= \sup_{|f|\le 1}\left|\int_{E\times E}\big(P_{t_1}^i f(x') - P_{t_1}^i f(y')\big)\{(P_{t_0}^i)^*\delta_x\}(\mathrm{d}x')\{(P_{t_0}^i)^*\delta_y\}(\mathrm{d}y')\right| \\
&\le \int_{B\times B}\|(P_{t_1}^i)^*\delta_{x'} - (P_{t_1}^i)^*\delta_{y'}\|_{var}\{(P_{t_0}^i)^*\delta_x\}(\mathrm{d}x')\{(P_{t_0}^i)^*\delta_y\}(\mathrm{d}y') \\
&\quad + 2\int_{(B\times B)^c}\{(P_{t_0}^i)^*\delta_x\}(\mathrm{d}x')\{(P_{t_0}^i)^*\delta_y\}(\mathrm{d}y') \\
&\le \beta\{P_{t_0}^i 1_B(x)\}P_{t_0}^i 1_B(y) + 2\big[1 - \{P_{t_0}^i 1_B(x)\}P_{t_0}^i 1_B(y)\big] \le 2 - \alpha^2(2-\beta).
\end{aligned}$$

Thus, for $\delta := \frac{2-\alpha^2(2-\beta)}{2} < 1$, we have

$$\|(P_{t_0+t_1}^i)^*\delta_x - (P_{t_0+t_1}^i)^*\delta_y\|_{var} \le \delta\|\delta_x - \delta_y\|_{var}, \quad x, y \in E.$$

Combining this with the semigroup property, we find constants $c > 0$ such that (3.13) holds for the claimed $\lambda > 0$. □

Proof Theorem 3.1. According to Theorems 2.1-2.3 and Lemma 3.2, it suffices to prove (3.3) for small $k > 0$. By Lemma 3.3, we only need to prove (3.11) and (3.12) for the family $\{P_t^\gamma : \gamma \in \mathscr{P}\}$, where $P_t^\gamma f(x) := \int_{\bar D} f\mathrm{d}(P_t^\gamma)^*\delta_x$.

(a) Proof of (3.12). Let us fix $\gamma \in \mathscr{P}$, and let $X_t^{x,\gamma}$ solve (3.1) with $X_0^\gamma = x$. For any $\nu \in \mathscr{P}$, by Girsanov's theorem we have

$$P_t^\nu f(x) = \mathbb{E}[f(X_t^{x,\gamma})R_t^{x,\gamma,\nu}], \quad t \ge 0,$$

where

$$R_t^{x,\gamma,\nu} := \mathrm{e}^{\int_0^t\langle\zeta(\gamma,\nu)(X_s^{x,\gamma}),\mathrm{d}W_s\rangle - \frac{1}{2}|\zeta(\gamma,\nu)(X_s^{x,\gamma})|^2\mathrm{d}s}.$$

By **(H2)**, Girsanov's theorem and Pinsker's inequality, we obtain

$$\|(P_t^\gamma)^*\delta_z - (P_t^\nu)^*\delta_z\|_{var}^2 \le (\mathbb{E}|R_t^{\gamma,\nu} - 1|)^2 \le k^2 t^{\frac{2}{q}}\|\gamma - \nu\|_{var}^2 \le 4k^2 t^{\frac{2}{q}}, \quad t \ge 0, z \in \bar D, \nu \in \mathscr{P}.$$

Taking $t = t_1 = (4k)^{-q}$, we obtain

$$\sup_{\nu\in\mathscr{P}}\|(P_{t_1}^\gamma)^*\delta_z - (P_{t_1}^\nu)^*\delta_z\|_{var} \le \frac{1}{2}, \quad z \in \bar D, \nu \in \mathscr{P}. \tag{3.14}$$

On the other hand, by (2.17), there exists $x_0 \in D$ and a constant $\varepsilon > 0$ such that $B(x_0, \varepsilon) \subset D$ and

$$\|(P_{t_1}^\gamma)^*\delta_x - (P_{t_1}^\gamma)^*\delta_y\|_{var} \le \frac{1}{4}, \quad x, y \in B(x_0, \varepsilon).$$

Combining this with (3.14) we derive

$$\sup_{\nu\in\mathscr{P}} \|(P_{t_1}^\nu)^*\delta_x - (P_{t_1}^\nu)^*\delta_y\|_{var} \le \frac{3}{2} < 2, \quad x, y \in B(x_0, \varepsilon).$$

So, (3.12) holds for $B = B(x_0, \varepsilon)$.

(b) Let u solve (2.26) for $f = -b^{(0)}$ and large $\lambda > 0$ such that (2.21) holds, and let $\Theta(x) = x + u(x)$. By **(H1)**, we see that (2.32) holds with $Y_t^{x,\nu} := \Theta(X_t^{x,\nu})$ replacing Y_t for all $\nu \in \mathscr{P}$. So, by $H(\infty) < \infty$ and the argument leading to (2.35), we obtain

$$\sup_{\nu\in\mathscr{P}, x\in\bar{D}} \mathbb{E}[V(Y_t^{x,\nu})] \le \theta^{-1}k, \quad t \ge kH(\infty) =: t_2.$$

This together with (2.30) implies

$$\sup_{\nu\in\mathscr{P}, x\in\bar{D}} \mathbb{E}[V(X_t^{x,\nu})] \le \theta^{-2}k, \quad t \ge t_2.$$

Letting $\mathbf{K} := \{V \le 2\theta^{-2}k\}$, we derive

$$\inf_{\nu\in\mathscr{P}, x\in\bar{D}} P_{t_2}^\nu 1_{\mathbf{K}}(x) \ge \frac{1}{2}. \tag{3.15}$$

On the other hand, by Girsanov's theorem and Schwartz's inequality, we find a constant $c_0 > 0$ such that

$$P_1^\nu 1_{B(x_0,\varepsilon)}(x) = \mathbb{E}\big[1_{B(x_0,\varepsilon)}(X_1^{x,\gamma})R_1^{x,\gamma,\nu}\big] \ge \frac{\{\mathbb{E}1_{B(x_0,\varepsilon)}(X_1^{x,\gamma})\}^2}{\mathbb{E}R_1^{x,\gamma,\nu}} \ge c_0(P_1^\gamma 1_{B(x_0,\varepsilon)}(x))^2.$$

Since $\mathbf{K}$ is bounded, combining this with Lemma 2.4 for P_t^γ, we find a constant $c_1 > 0$ such that

$$\inf_{\nu\in\mathscr{P}, x\in\mathbf{K}} P_1^\nu 1_{B(x_0,\varepsilon)}(x) \ge c_1.$$

This together with (3.15) and the semigroup property yields

$$P_{t_2+1}^\nu 1_{B(x_0,\varepsilon)}(x) \ge P_{t_2}^\nu\big\{1_{\mathbf{K}}P_1^\nu 1_{B(x_0,\varepsilon)}\big\}(x) \ge c_1 P_{t_2}^\nu 1_{\mathbf{K}}(x) \ge \frac{c_1}{2} > 0, \quad x \in \bar{D}, \nu \in \mathscr{P}.$$

Therefore, (3.11) holds for $t_0 = t_2 + 1$. □

4 Exponential ergodicity in entropy and Wasserstein distance

In this section, we consider the following reflecting McKean-Vlasov SDE where the noise may also be distribution dependent:

$$\mathrm{d}X_t = b_t(X_t, \mathscr{L}_{X_t})\mathrm{d}t + \sigma_t(X_t, \mathscr{L}_{X_t})\mathrm{d}W_t + \mathbf{n}(X_t)\mathrm{d}l_t, \quad t \ge 0, \tag{4.1}$$

where

$$b : [0,\infty) \times \bar{D} \times \mathscr{P} \to \mathbb{R}^d, \quad \sigma : [0,\infty) \times \bar{D} \times \mathscr{P} \to \mathbb{R}^d \otimes \mathbb{R}^n$$

are measurable. We study the exponential ergodicity under entropy and weighted Wasserstein distance for dissipative and partially dissipative cases respectively, such that the corresponding results in [16] and [24] are extended to the reflecting setting. For simplicity, we only consider convex D, for which the local time on boundary does not make trouble in the study.

(A7) *Let $k > 1$, $\mathscr{P}_k := \{\mu \in \mathscr{P} : \mu(|\cdot|^k) < \infty\}$. D is convex, b and σ are bounded on bounded subsets of $[0,\infty) \times \bar{D} \times \mathscr{P}_k(\bar{D})$, and the following two conditions hold.*

(1) *For any $T > 0$ there exists a constant $K > 0$ such that*

$$\begin{aligned}&\|\sigma_t(x,\mu) - \sigma_t(y,\nu)\|_{HS}^2 + 2\langle x-y, b_t(x,\mu) - b_t(y,\nu)\rangle^+ \\ &\le K\big\{|x-y|^2 + |x-y|\mathbb{W}_k(\mu,\nu) + 1_{\{k\ge 2\}}\mathbb{W}_k(\mu,\nu)^2\big\}, \quad t \in [0,T], x, y \in \bar{D}, \mu, \nu \in \mathscr{P}_k(\bar{D}).\end{aligned}$$

(2) *There exists a subset $\tilde{\partial} D \subset \partial D$ such that*

$$\langle y - x, \mathbf{n}(x)\rangle \ge 0, \quad x \in \partial D \setminus \tilde{\partial} D,\ y \in \bar{D}, \tag{4.2}$$

and when $\tilde{\partial} D \ne \emptyset$, there exists $\tilde{\rho} \in C_b^2(\bar{D})$ such that $\tilde{\rho}|_{\partial D} = 0$, $\langle \nabla \tilde{\rho}, \mathbf{n}\rangle|_{\partial D} \ge 1_{\tilde{\partial} D}$ and

$$\sup_{(t,x)\in[0,T]\times\bar{D}} \big\{\|(\sigma_t^\mu)^*\nabla\tilde{\rho}\|^2(x) + \langle b_t^\mu, \nabla\tilde{\rho}\rangle^-(x)\big\} < \infty, \quad \mu \in C([0,T]; \mathscr{P}_k(\bar{D})). \tag{4.3}$$

According to [25], this assumption implies the well-posedness of (4.1) for distributions in $\mathscr{P}_k$. Let $P_t^*\mu = \mathscr{L}_{X_t}$ for the solution with $\mathscr{L}_{X_0} = \mu \in \mathscr{P}_k$.

4.1 Dissipative case: exponential convergence in entropy and $\mathbb{W}_2$

In this part, we study the exponential ergodicity of P_t^* in entropy and $\mathbb{W}_2$. For probability measures μ_1, μ_2 on $\bar{D}$, let

$$\mathrm{Ent}(\mu_1|\mu_2) := \begin{cases} \mu_2(f\log f), & \text{if } \mathrm{d}\mu_1 = f\mathrm{d}\mu_2, \\ \infty, & \text{if } \mu_1 \text{ is not absolutely continuous w.r.t. } \mu_2 \end{cases}$$

be the relative entropy of μ_1 w.r.t. μ_2, and let

$$\mathbb{W}_2(\mu_1,\mu_2) := \inf_{\pi\in\mathscr{C}(\mu_1,\mu_2)} \left(\inf \int_{\bar{D}\times\bar{D}} |x-y|^2 \pi(\mathrm{d}x, \mathrm{d}y)\right)^{\frac{1}{2}}$$

be the quadratic Wasserstein distance, where $\mathscr{C}(\mu_1,\mu_2)$ is the set of all couplings for μ_1 and μ_2. The following result extends the corresponding one derived in [16] for McKean-Vlasov SDEs without reflection.

Theorem 4.1. *Let D be convex and (σ, b) satisfy* **(A7)** *with $k = 2$. Let $K_1, K_2 \in L^1_{loc}([0,\infty);\mathbb{R})$ such that*

$$\begin{aligned} &2\langle b_t(x,\mu) - b_t(y,\nu), x-y\rangle + \|\sigma_t(x,\mu) - \sigma_t(y,\nu)\|_{HS}^2 \\ &\le K_1(t)|x-y|^2 + K_2(t)\mathbb{W}_2(\mu,\nu)^2, \quad t \ge 0. \end{aligned} \tag{4.4}$$

Then and P_t^* satisfies

$$\mathbb{W}_2(P_t^*\mu, P_t^*\nu)^2 \le \mathrm{e}^{\int_0^t (K_1+K_2)(r)\mathrm{d}r}\mathbb{W}_2(\mu,\nu)^2, \quad \mu,\nu \in \mathscr{P}_2(\bar{D}),\ t \ge 0. \tag{4.5}$$

Consequently, if (b_t, σ_t) does not depend on t and $\lambda := -(K_1 + K_2) > 0$, then P_t^* has a unique invariant probability measure $\bar{\mu}$ such that

$$\mathbb{W}_2(P_t^*\mu, \bar{\mu})^2 \le \mathrm{e}^{-\lambda t}\mathbb{W}_2(\mu,\bar{\mu})^2, \quad \mu \in \mathscr{P}_2(\bar{D}),\ t \ge 0, \tag{4.6}$$

and the following assertions hold:

(1) *When $\sigma_t(x,\mu) = \sigma_t(x)$ does not depend on μ and $\sigma\sigma^*$ is invertible with $\|\sigma\|_\infty + \|(\sigma\sigma^*)^{-1}\|_\infty < \infty$, there exists a constant $c > 0$ such that*

$$\mathrm{Ent}(P_t^*\mu|\bar{\mu}) \le c\mathrm{e}^{-\lambda t}\mathbb{W}_2(\mu,\bar{\mu})^2, \quad t \ge 1, \mu \in \mathscr{P}_2(\bar{D}). \tag{4.7}$$

(2) *When $\sigma(x,\mu) = \sigma(\mu)$ does not depend on x, there exists a constant $c > 0$ such that $\bar{\mu}$ satisfies the following log-Sobolev inequality and Talagrand inequality:*

$$\bar{\mu}(f^2 \log f^2) \le c\bar{\mu}(|\nabla f|^2), \quad f \in C_b^1(\mathbb{R}^d), \bar{\mu}(f^2) = 1, \tag{4.8}$$

$$\mathbb{W}_2(\mu,\bar{\mu})^2 \le c\mathrm{Ent}(\mu|\bar{\mu}), \quad \mu \in \mathscr{P}_2. \tag{4.9}$$

(3) *When $\sigma(x,\mu) = \sigma$ is constant with $\sigma\sigma^*$ invertible, there exists a constant $c > 0$ such that*

$$\begin{aligned} &\mathbb{W}_2(P_t^*\mu,\bar{\mu})^2 + \mathrm{Ent}(P_t^*\mu|\bar{\mu}) \\ &\le c\mathrm{e}^{-\lambda t}\min\big\{\mathbb{W}_2(\mu,\bar{\mu})^2, \mathrm{Ent}(\mu|\bar{\mu})\big\}, \quad t \ge 1, \mu \in \mathscr{P}_2(\bar{D}). \end{aligned} \tag{4.10}$$

Proof. The well-posedness is ensured by [24, Theorem 3.3]. Since D is convex, by Remark 2.1 in [24],

$$\langle y - x, \mathbf{n}(x)\rangle \ge 0, \quad y \in \bar{D}, x \in \partial D, \mathbf{n}(x) \in \mathscr{N}_x. \tag{4.11}$$

For any $\mu,\nu \in \mathscr{P}_2(\bar{D})$, let X_0^μ and X_0^ν be $\mathscr{F}_0$-measurable such that

$$\mathscr{L}_{X_0^\mu} = \mu, \quad \mathscr{L}_{X_0^\nu} = \nu, \quad \mathbb{E}|X_0^\mu - X_0^\nu|^2 = \mathbb{W}_2(\mu,\nu)^2. \tag{4.12}$$

By (4.4), (4.11), and applying Itô's formula to $|X_t^\mu - X_t^\nu|^2$, where $(X_t^\mu)_{t\ge0}$ and $(X_t^\nu)_{t\ge0}$ solve (4.1), we obtain

$$\mathrm{d}|X_t^\mu - X_t^\nu|^2 \le \big\{K_1(t)|X_t^\mu - X_t^\nu|^2 + K_2(t)\mathbb{W}_2(\mathbb{P}_t^*\mu, P_t^*\nu)^2\big\}\mathrm{d}t + \mathrm{d}M_t$$

for some martingale M_t. Combining this with (4.12), $\mathbb{W}_2(P_t^*\mu, P_t^*\nu)^2 \le \mathbb{E}|X_t^\mu - X_t^\nu|^2$, and Gronwall's lemma, we prove (4.5).

Let (b_t, σ_t) do not depend on t and $\lambda := -(K_1 + K_2) > 0$. Then (4.5) implies the uniqueness of P_t^*-invariant probability measure $\bar\mu \in \mathscr{P}_2(\bar D)$ and (4.6).

The existence of $\bar\mu$ follows from a standard argument by showing that for $x_0 \in D$, $\{P_t^*\delta_{x_0}\}_{t\ge 0}$ is a $\mathbb{W}_2$-Cauchy family as $t \to \infty$. Since the term of local time does not make trouble due to (4.11), the proof is completely similar to that of [23, Theorem 3.1] for the case $D = \mathbb{R}^d$, so we skip the details to save space. Below we prove statements (1)-(3) respectively.

(1) When $\sigma_t(x,\mu) = \sigma_t(x)$ and $\sigma\sigma^*$ is invertible with $\|\sigma\|_\infty + \|(\sigma\sigma^*)^{-1}\|_\infty < \infty$, by Theorem 4.2 in [24], **(A7)** with $k = 2$ implies the log-Harnack inequality

$$\mathrm{Ent}(P_1^*\mu|\bar\mu) \le c\mathbb{W}_2(\mu,\bar\mu)^2, \quad \mu \in \mathscr{P}_2(\bar D)$$

for some constant $c > 0$. So, (4.7) follows from (4.6) and $P_t^* = P_1^* P_{t-1}^*$ for $t \ge 1$.

(2) Let $\sigma(x,\mu) = \sigma(\mu)$ be independent of x. Consider the SDE

$$\mathrm{d}\bar X_t^x = b(\bar X_t^x, \bar\mu)\mathrm{d}t + \sigma(\bar\mu)\mathrm{d}W_t + \mathbf{n}(\bar X_t^x)\mathrm{d}l_t, \quad t \ge s, \bar X_0^x = x \in \bar D. \tag{4.13}$$

The associated Markov semigroup $\{\bar P_t\}_{t\ge 0}$ is given by

$$\bar P_t f(x) := \mathbb{E} f(\bar X_t^x), \quad t \ge 0, f \in \mathscr{B}_b(\bar D), x \in \bar D.$$

Let $\bar P_t^*$ be given by

$$(\bar P_t^*\mu)(f) := \mu(\bar P_t f), \quad \mu \in \mathscr{P}, t \ge 0, f \in \mathscr{B}_b(\bar D).$$

Since (4.4) with $x = y$ implies $K_2 \ge 0$, we have

$$K_1 \le -\lambda < 0. \tag{4.14}$$

As explained in the above proofs of (4.5) and (4.6), this implies that $\bar P_t^*$ has a unique invariant probability measure $\tilde\mu$ such that

$$\lim_{t\to\infty} \bar P_t f(x) = \tilde\mu(f), \quad f \in C_b(\bar D), x \in \bar D. \tag{4.15}$$

Since $\bar\mu$ is the unique invariant probability measure of P_t^*, and when the initial distribution is $\bar\mu$, the SDE (4.13) coincides with (4.1), we conclude that $\tilde\mu = \bar\mu$. Hence, (4.15) yields

$$\bar\mu(f) = \lim_{t\to\infty} P_t f(x_0), \quad f \in C_b(\bar D), x_0 \in D. \tag{4.16}$$

Now, by Itô's formula, (4.11) and (4.4) with (b_t, σ_t) independent of t, we obtain

$$|\bar X_t^x - \bar X_t^y|^2 \le \mathrm{e}^{K_1 t}|x-y|^2, \quad x, y \in \bar D, t \ge 0.$$

This and (4.14) imply

$$\begin{aligned} |\nabla \bar P_t f(x)| &:= \limsup_{y\to x} \frac{|\bar P_t f(x) - \bar P_t f(y)|}{|x-y|} \le \limsup_{y\to x} \frac{\mathbb{E}|f(\bar X_t^x) - f(\bar X_t^y)|}{|x-y|} \\ &\le \mathrm{e}^{-\frac{\lambda t}{2}} \limsup_{y\to x} \mathbb{E}\frac{|f(\bar X_t^x) - f(\bar X_t^y)|}{|\bar X_t^x - \bar X_t^y|} = \mathrm{e}^{-\lambda t/2}\bar P_t|\nabla f|(x), \quad t \ge 0, f \in C_b^1(\bar D). \end{aligned} \tag{4.17}$$

On the other hand, we have

$$\partial_t \bar{P}_t f = \bar{L}\bar{P}_t f, \quad \langle \mathbf{n}, \nabla \bar{P}_t f\rangle|_{\partial D} = 0, \quad t \geq 0, f \in C_N^2(\bar{D}),$$

where $C_N^2(\bar{D})$ is the set of $f \in C_b^2(\bar{D})$ satisfying with $\langle \mathbf{n}, \nabla f\rangle|_{\partial D} = 0$, and

$$\bar{L} := \frac{1}{2}\mathrm{tr}\{(\bar{\sigma}\bar{\sigma}^*)\nabla^2\} + \nabla_{b(\cdot,\bar{\mu})}, \quad \bar{\sigma} := \sigma(\bar{\mu}), \quad s \geq 0.$$

So, by Itô's formula, for any $\varepsilon > 0$ and $f \in C_N^2(\bar{D})$,

$$\mathrm{d}\big\{(\bar{P}_{t-s}(\varepsilon + f^2))\log \bar{P}_{t-s}(\varepsilon + f^2)\big\}(\bar{X}_s) = \Big\{\frac{|\bar{\sigma}^*\nabla \bar{P}_{t-s} f^2|^2}{\varepsilon + \bar{P}_{t-s} f^2}\Big\}\mathrm{d}t + \mathrm{d}M_s^\varepsilon, \quad s \in [0,t]$$

holds for some martingale $(M_s^\varepsilon)_{s\in[0,t]}$. Combining this with (4.17), we find a constant $c > 0$ such that for any $f \in C_N^2(\mathbb{R}^d)$,

$$\begin{aligned}
&\bar{P}_t\big\{(\varepsilon + f^2)\log(\varepsilon + f^2)\big\} - (\varepsilon + \bar{P}_t f^2)\log(\varepsilon + \bar{P}_t f^2) \\
&= \int_0^t \bar{P}_s \frac{|\bar{\sigma}^*\nabla \bar{P}_{t-s} f^2|^2}{\varepsilon + \bar{P}_{t-s} f^2}\mathrm{d}s \leq 4(c_1\|\bar{\sigma}\|_\infty)^2 \int_0^t \mathrm{e}^{-\lambda(t-s)}\bar{P}_s\bar{P}_{t-s}|\nabla f|^2\mathrm{d}s \\
&= 4(c_1\|\bar{\sigma}\|_\infty)^2(\bar{P}_t|\nabla f|^2)\int_0^t \mathrm{e}^{-\lambda(t-s)}\mathrm{d}s \leq c\bar{P}_t|\nabla f|^2, \quad t \geq 0, \varepsilon > 0.
\end{aligned}$$

By letting first $\varepsilon \downarrow 0$ then $t \to \infty$, we deduce from this and (4.16) that

$$\bar{\mu}(f^2\log f^2) \leq c_2\bar{\mu}(|\nabla f|^2), \quad f \in C_N^2(\bar{D}), \bar{\mu}(f^2) = 1$$

holds for some constant $c_2 > 0$. This implies (4.8) by an approximation argument, indeed the inequality holds for $f \in H^{1,2}(\bar{\mu})$ with $\bar{\mu}(f^2) = 1$. According to Lemma 4.2 below, (4.9) holds.

(3) Let σ be constant with $\sigma\sigma^*$ invertible. Then (4.10) follows from (4.6), (4.7) and (4.9). □

The following result on the Talagrand inequality is known by [3] when $\bar{\mu}(\mathrm{d}x) = \mathrm{e}^{V(x)}\mathrm{d}x$ for some $V \in C(\mathbb{R}^d)$, which is first proved in [14] on Riemannian manifolds under a curvature condition, see also [20] for more general results. We extend it to general probability measures for the above application to $\bar{\mu}$ which is supported on $\bar{D}$ rather than $\mathbb{R}^d$.

Lemma 4.2. *Let $c > 0$ be a constant and $\bar{\mu} \in \mathscr{P}_2(\mathbb{R}^d)$. Then the log-Sobolev inequality* (4.8) *implies* (4.9).

Proof. By an approximation argument, we only need to prove for $\mu = \varrho\bar{\mu}$ for some density $\varrho \in C_b(\mathbb{R}^d)$ Let $P_t^{(0)}$ be the Ornstein-Uhlenbeck semigroup generated by $\Delta - x\cdot\nabla$ on $\mathbb{R}^d$. We have

$$|\nabla P_t^{(0)} f| \leq P_t^{(0)}|\nabla f|, \quad P_t^{(0)}(f^2\log f^2) \leq tP_t^{(0)}|\nabla f|^2 + (P_t^{(0)} f^2)\log P_t^{(0)} f^2, \quad f \in C_b^1(\mathbb{R}^d).$$

Combining this with (4.8), we see that $\bar{\mu}_t := (P_t^{(0)})^*\bar{\mu}$ satisfies

$$\begin{aligned}&\bar{\mu}_t(f^2\log f^2) = \bar{\mu}(P_t^{(0)}(f^2\log f^2)) \le t\bar{\mu}_t(|\nabla f|^2) + \bar{\mu}((P_t^{(0)}f^2)\log P_t^{(0)}f^2)\\ &\le t\bar{\mu}_t(|\nabla f|^2) + c\bar{\mu}\Big(\Big|\nabla\sqrt{P_t^{(0)}f^2}\Big|^2\Big) + \bar{\mu}_t(f^2)\log\bar{\mu}_t(f^2)\\ &\le (t+c)\bar{\mu}_t(|\nabla f|^2) + \bar{\mu}_t(f^2)\log\bar{\mu}_t(f^2),\quad f\in C_b^1(\mathbb{R}^d),\quad t>0,\end{aligned}$$

where the last step follows from the gradient estimate $|\nabla P_t^{(0)}f| \le P_t^{(0)}|\nabla f|$, which and the Schwarz inequality imply

$$\Big|\nabla\sqrt{P_t^{(0)}f^2}\Big|^2 = \frac{|\nabla P_t^{(0)}f^2|^2}{4P_t^{(0)}f^2} \le \frac{\{P_t^{(0)}(|f\nabla f|)\}^2}{P_t^{(0)}f^2} \le P_t^{(0)}|\nabla f|^2.$$

Therefore, $\bar{\mu}_t$ satisfies the log-Sobolev inequality with constant $t+c$ and has smooth strictly positive density. According to [3], we have

$$\mathbb{W}_2(\mu,\bar{\mu}_t)^2 \le (t+c)\mathrm{Ent}(\mu|\bar{\mu}_t),\quad \mu\in\mathscr{P}_2(\mathbb{R}^d).$$

Since $\mathbb{W}_2(\bar{\mu}_t,\bar{\mu})\to 0$ as $t\to 0$, and $\mu=\varrho\bar{\mu}$ with $\varrho\in C_b(\mathbb{R}^d)$, this implies

$$\begin{aligned}\mathbb{W}_2(\mu,\bar{\mu})^2 &= \lim_{t\downarrow 0}\mathbb{W}_2(\mu,\bar{\mu}_t)^2 \le \lim_{t\downarrow 0}(t+c)\mathrm{Ent}(\mu|\bar{\mu}_t)\\ &= \lim_{t\downarrow 0}(t+c)\bar{\mu}((P_t^{(0)}\varrho)\log P_t^{(0)}\varrho) = c\bar{\mu}(\varrho\log\varrho).\end{aligned}$$

Therefore, (4.9) holds. □

4.2 Partially dissipative case: exponential convergence in $\mathbb{W}_\psi$

In this part, we consider the partially dissipative case such that [24, Theorem 3.1] is extended to the reflecting setting. For any $\kappa>0$, let

$$\begin{aligned}\Psi_\kappa := \big\{&\psi\in C^2((0,\infty))\cap C^1([0,\infty)) : \psi(0)=0,\ \psi'|_{(0,\infty)}>0, \|\psi'\|_\infty<\infty\\ &r\psi'(r)+r^2\{\psi''\}^+(r)\le\kappa\psi(r) \text{ for } r>0\big\}.\end{aligned}\tag{4.18}$$

For $\psi\in\Psi_\kappa$, we introduce the associated Wasserstein "distance" (also called transportation cost)

$$\mathbb{W}_\psi(\mu,\nu) := \inf_{\pi\in\mathscr{C}(\mu,\nu)}\int_{\bar{D}\times\bar{D}}\psi(|x-y|)\pi(\mathrm{d}x,\mathrm{d}y),\quad \mu,\nu\in\mathscr{P}_\psi.\tag{4.19}$$

Then $\mathbb{W}_\psi$ is a complete quasi-metric on the space

$$\mathscr{P}_\psi := \big\{\mu\in\mathscr{P} : \mu(\psi(|\cdot|))<\infty\big\}.$$

(A8) *$\sigma_t(x,\mu)=\sigma_t(x)$ does not depend on μ so that 4.1 reduces to* (1.1).

(1) (*Ellipticity*) *There exist* $\alpha \in C([0,\infty);(0,\infty))$ *and* $\hat{\sigma} \in \mathscr{B}([0,\infty)\times\bar{D};R^d\otimes\mathbb{R}^d)$ *such that*

$$\sigma_t(x)\sigma_t(x)^* = \alpha_t\mathbf{I}_d + \hat{\sigma}_t(x)\hat{\sigma}_t(x)^*, \quad t\geq 0, x\in\bar{D}.$$

(2) (*Partial dissipativity*) *Let* $\psi \in \Psi_\kappa$ *in* (4.18) *for some* $\kappa > 0$, $\gamma \in C([0,\infty))$ *with* $\gamma(r) \leq Kr$ *for some constant* $K > 0$ *and all* $r \geq 0$, *such that*

$$2\alpha_t\psi''(r) + (\gamma\psi')(r) \leq -\zeta_t\psi(r), \quad r\geq 0, t\geq 0 \tag{4.20}$$

holds for some for some $\zeta \in C([0,\infty);\mathbb{R})$. *Moreover,* $b \in C([0,\infty)\times\bar{D}\times\mathscr{P}_\psi)$, *and there exists* $\theta \in C([0,\infty);[0,\infty))$ *such that*

$$\begin{aligned}&\langle b_t(x,\mu)-b_t(y,\nu), x-y\rangle + \frac{1}{2}\|\hat{\sigma}_t(x)-\hat{\sigma}_t(y)\|_{HS}^2\\ &\quad\leq |x-y|\big\{\theta_t\mathbb{W}_\psi(\mu,\nu)+\gamma(|x-y|)\big\}, \quad t\geq 0, x,y\in\bar{D}, \mu,\nu\in\mathscr{P}_\psi.\end{aligned} \tag{4.21}$$

Theorem 4.3. *Let* D *be convex and assume* **(A8)**, *where* $\psi''\leq 0$ *if* $\hat{\sigma}$ *is non-constant. Then* (1.1) *is well-posed for distributions in* $\mathscr{P}_\psi$, *and* P_t^* *satisfies*

$$\mathbb{W}_\psi(P_t^*\mu, P_t^*\nu) \leq \mathrm{e}^{-\int_0^t\{\zeta_s-\theta_s\|\psi'\|_\infty\}\mathrm{d}s}\mathbb{W}_\psi(\mu,\nu), \quad t\geq 0, \mu,\nu\in\mathscr{P}_\psi. \tag{4.22}$$

Consequently, if $(b_t,\sigma_t,\zeta_t,\theta_t)$ *do not depend on* t *and* $\zeta > \theta\|\psi'\|_\infty$, *then* P_t^* *has a unique invariant probability measure* $\bar{\mu}\in\mathscr{P}_\psi$ *such that*

$$\mathbb{W}_\psi(P_t^*\mu,\bar{\mu}) \leq \mathrm{e}^{-(\zeta-\theta\|\psi'\|_\infty)t}\mathbb{W}_\psi(\mu,\bar{\mu}), \quad t\geq 0, \mu\in\mathscr{P}_\psi. \tag{4.23}$$

Proof. Since D is convex, the proof is similar to that of [24, Theorem 3.1]. We outline it below for complement.

By Theorem 3.1, the well-posedness follows from **(A8)**(1) and **(A8)**(2). Next, according to the proof of Theorem 4.1(2) with $\mathbb{W}_\psi$ replacing $\mathbb{W}_2$, the second assertion follows from the first. So, in the following we only prove (4.22).

For any $s\geq 0$, let (X_s,Y_s) be $\mathscr{F}_s$-measurable such that

$$\mathscr{L}_{X_s} = P_s^*\mu, \quad \mathscr{L}_{Y_s} = P_s^*\nu, \quad \mathbb{W}_\psi(P_s^*\mu,P_s^*\nu) = \mathbb{E}\psi(|X_s-Y_s|). \tag{4.24}$$

Let $W_t^{(1)}$ and $W_t^{(2)}$ be two independent d-dimensional Brownian motions and consider the following SDE:

$$\mathrm{d}X_t = b_t(X_t,P_t^*\mu)\mathrm{d}t + \sqrt{\alpha_t}\mathrm{d}W_t^{(1)} + \hat{\sigma}_t(X_t)\mathrm{d}W_t^{(2)} + \mathbf{n}(X_t)\mathrm{d}l_t^X, \quad t\geq s, \tag{4.25}$$

where l_t^X is the local time of X_t on ∂D. By Theorem 4.1, **(A8)**(1) and **(A8)**(2) imply that this SDE. By $\sigma_t\sigma_t^* = \alpha_t\mathbf{I}_d + \hat{\sigma}_t\hat{\sigma}_t^*$, we have

$$\begin{aligned}&\sigma_t^*(\sigma_t\sigma_t^*)^{-1}\big\{\alpha_t+\hat{\sigma}_t\hat{\sigma}_t^*\big\}(\sigma_t\sigma_t^*)^{-1}\sigma_t + \big\{\mathbf{I}_m - \sigma_t^*(\sigma_t\sigma_t^*)^{-1}\sigma_t\big\}^2\\ &= \sigma_t^*(\sigma_t\sigma_t^*)^{-1}\sigma_t + \mathbf{I}_m - \sigma_t^*(\sigma_t\sigma_t^*)^{-1}\sigma_t = \mathbf{I}_m.\end{aligned}$$

So, for an m-dimensional Brownian motion $W^{(3)}$ independent of $(W^{(1)}, W^{(2)}$,

$$W_t := \int_0^t \big\{\sigma_s^*(\sigma_s\sigma_s^*)^{-1}\big\}(X_s)\big\{\sqrt{\alpha_s}\mathrm{d}W_s^{(1)} + \hat{\sigma}_s(X_s)\mathrm{d}W_s^{(2)}\big\} + \int_0^t big\{I_m - \sigma_s^*(\sigma_s\sigma_s^*)^{-1}\sigma_s\big\}(X_s)\mathrm{d}W_s^3$$

is an m-dimensional Brownian motion such that

$$\sigma_t(X_t)\mathrm{d}W_t = \sqrt{\alpha_t}\mathrm{d}W_t^{(1)} + \hat{\sigma}_t(X_t)\mathrm{d}W_t^{(2)}.$$

Thus, by the weak uniqueness of (1.1), we have $\mathscr{L}_{X_t} = P_{s,t}^* P_s^*\mu = P_t^*\mu, t \geq s$, where for $\gamma \in \mathscr{P}_\psi$ we denote $P_{s,t}^*\gamma = \mathscr{L}_{X_t}$ for X_t solving (4.25) with $\mathscr{L}_{X_s} = \gamma$.

To construct the coupling with reflection, let

$$u(x,y) = \frac{x-y}{|x-y|}, \quad x \neq y \in \mathbb{R}^d.$$

We consider the SDE for $t \geq s$:

$$\mathrm{d}Y_t = b_t(Y_t, P_t^*\nu)\mathrm{d}t + \sqrt{\alpha_t}\big\{\mathbf{I}_d - 2u(X_t,Y_t)\otimes u(X_t,Y_t)1_{\{t<\tau\}}\big\}\mathrm{d}W_t^{(1)} + \hat{\sigma}_t(Y_t)\mathrm{d}W_t^{(2)} + \mathrm{d}l_t^Y, \tag{4.26}$$

where

$$\tau := \inf\{t \geq s : Y_t = X_t\}$$

is the coupling time. Since the coefficients in noises are Lipschitz continuous outside a neighborhood of the diagonal, by [12, Theorem 1.1], (4.26) has a unique solution up to the coupling time τ. When $t \geq \tau$, the equation of Y_t becomes

$$\mathrm{d}Y_t = b_t(Y_t, P_t^*\nu)\mathrm{d}t + \sqrt{\alpha_t}\mathrm{d}W_t^{(1)} + \hat{\sigma}_t(Y_t)\mathrm{d}W_t^{(2)} + \mathrm{d}l_t^Y, \tag{4.27}$$

which is well-posed under **(A8)**(1) and **(A8)**(2) according to Theorem 3.1. So, (4.26) is well-posed and $\mathscr{L}_{Y_t} = P_t^*\nu$ by the same reason leading to $\mathscr{L}_{X_t} = P_t^*\mu$. Since D is convex, (4.11) holds. So, by **(A8)**(1) and **(A8)**(2) for $\psi \in \Psi$ with $\psi'' \leq 0$ when $\hat{\sigma}_t$ is non-constant, and applying Itô's formula, we obtain

$$\begin{aligned}\mathrm{d}\psi(|X_t - Y_t|) \leq& \big\{\theta_t\psi'(|X_t-Y_t|)\mathbb{W}_\psi(P_t^*\mu, P_t^*\nu) - \zeta_t\psi(|X_t-Y_t|)\big\}\mathrm{d}t \\ &+ \psi'(|X_t-Y_t|)\Big[2\sqrt{\alpha_t}\Big\langle u(X_t,Y_t), \mathrm{d}W_t^{(1)}\Big\rangle \\ &+ \Big\langle u(X_t,Y_t), (\hat{\sigma}_t(X_t) - \hat{\sigma}_t(Y_t))\mathrm{d}W_t^{(2)}\Big\rangle\Big], \quad s \leq t < \tau.\end{aligned} \tag{4.28}$$

By a standard argument and noting that $\psi(|X_{t\wedge\tau}, Y_{t\wedge\tau}|)1_{\{\tau\leq t\}} = 0$, this implies

$$\begin{aligned}&\mathrm{e}^{\int_s^t \zeta_p\mathrm{d}p}\mathbb{E}\big[\psi(|X_{t\wedge\tau} - Y_{t\wedge\tau}|)\big] = \mathbb{E}\big[\mathrm{e}^{\int_s^{t\wedge\tau}\zeta_p\mathrm{d}p}\psi(|X_{t\wedge\tau} - Y_{t\wedge\tau}|)\big] \\ &\leq \mathbb{E}\psi(|X_s - Y_s|) + \|\psi'\|_\infty \int_s^{t\wedge\tau} \theta_r \mathrm{e}^{\int_s^r \zeta_p\mathrm{d}p}\mathbb{W}_\psi(P_r^*\mu, P_r^*\nu)\mathrm{d}r, \quad t \geq s.\end{aligned}$$

Consequently,

$$\begin{aligned}&\mathbb{E}\psi(|X_{t\wedge\tau} - Y_{t\wedge\tau}|) \\ &\leq \mathrm{e}^{-\int_s^t \zeta_r\mathrm{d}r}\mathbb{E}\psi(|X_s - Y_s|) + \|\psi'\|_\infty \int_s^{t\wedge\tau} \theta_r \mathrm{e}^{-\int_r^t \zeta_p\mathrm{d}p}\mathbb{W}_\psi(P_r^*\mu, P_r^*\nu)\mathrm{d}r, \quad t \geq s.\end{aligned} \tag{4.29}$$

On the other hand, when $t \geq \tau$, by **(A8)**(2) and applying Itô's formula for (4.25) and (4.27), we find a constant $C > 0$ such that

$$\begin{aligned}\mathrm{d}\psi(|X_t - Y_t|) \leq &\big\{C\psi(|X_t - Y_t|)\mathrm{d}t + \theta_t\|\psi'\|_\infty \mathbb{W}_\psi(P_t^*\mu, P_t^*\nu)\big\}\mathrm{d}t \\ &+ \psi'(|X_t - Y_t|)\langle\{\hat\sigma_t(X_t) - \hat\sigma_t(Y_t)\}^* u(X_t, Y_t), \mathrm{d}W_t^{(2)}\rangle.\end{aligned}$$

Noting that $\psi(|X_\tau - Y_\tau|) = 0$, we obtain

$$\mathbb{E}\big[1_{\{t>\tau\}}\psi(|X_t - Y_t|)\big] \leq \|\psi'\|_\infty \mathrm{e}^{C(t-s)}\mathbb{E}\int_{t\wedge\tau}^t \theta_r \mathbb{W}_\psi(P_r^*\mu, P_r^*\nu)\mathrm{d}r, \quad t \geq s.$$

Combining this with (4.29) and (4.24), we derive

$$\begin{aligned}\mathbb{W}_\psi(P_t^*\mu, P_t^*\nu) &\leq \mathbb{E}\psi(|X_t - Y_t|) = \mathbb{E}\psi(|X_{t\wedge\tau} - Y_{t\wedge\tau}|) + \mathbb{E}\big[1_{\{t>\tau\}}\psi(|X_t - Y_t|)\big] \\ &\leq \mathrm{e}^{-\int_s^t \zeta_r \mathrm{d}r}\mathbb{E}\psi(|X_s - Y_s|) + \|\psi'\|_\infty \mathrm{e}^{C(t-s)}\int_s^t \theta_r \mathbb{W}_\psi(P_r^*\mu, P_r^*\nu)\mathrm{d}r \\ &= \mathrm{e}^{-\int_s^t \zeta_r \mathrm{d}r}\mathbb{W}_\psi(P_s^*\mu, P_s^*\nu) + \|\psi'\|_\infty \mathrm{e}^{C(t-s)}\int_s^t \theta_r \mathbb{W}_\psi(P_r^*\mu, P_r^*\nu)\mathrm{d}r, \quad t \geq s.\end{aligned}$$

Therefore,

$$\begin{aligned}\frac{\mathrm{d}^+}{\mathrm{d}s}\mathbb{W}_\psi(P_s^*\mu, P_s^*\nu) &:= \limsup_{t\downarrow s} \frac{\mathbb{W}_\psi(P_t^*\mu, P_t^*\nu) - \mathbb{W}_\psi(P_s^*\mu, P_s^*\nu)}{t - s} \\ &\leq -(\zeta_s - \theta_s\|\psi'\|_\infty)\mathbb{W}_\psi(P_s^*\mu, P_s^*\nu), \quad s \geq 0.\end{aligned}$$

This implies (4.22). □

As a consequence of Theorem 4.3, we consider the non-dissipative case where $\nabla b_t(\cdot, \mu)(x)$ is positive definite in a possibly unbounded set but with bounded "one-dimensional puncture mass" in the sense of (4.32) below.

Let $\mathbb{W}_1 = \mathbb{W}_\psi$ and $\mathscr{P}_1(\bar D) = \mathscr{P}_\psi$ for $\psi(r) = r$, and define

$$S_b(x) := \sup\big\{\langle\nabla_v b_t(\cdot, \mu)(x), v\rangle : \ t \geq 0, |v| \leq 1, \mu \in \mathscr{P}_1(\bar D)\big\}, \quad x \in \bar D.$$

(A8) (3) There exist constants $\theta_0, \theta_1, \theta_2, \beta \geq 0$ such that

$$\frac{1}{2}\|\sigma_t(x) - \sigma_t(y)\|_{HS}^2 \leq \theta_0|x - y|^2, \quad t \geq 0, x, y \in \bar D; \tag{4.30}$$

$$S_b(x) \leq \theta_1, \quad |b_t(x, \mu) - b_t(x, \nu)| \leq \beta\mathbb{W}_1(\mu, \nu), \quad t \geq 0, x \in \bar D, \mu, \nu \in \mathscr{P}_1(\bar D); \tag{4.31}$$

$$\zeta := \sup_{x, v\in\bar D, |v|=1}\int_{\mathbb{R}} 1_{\{S_b(x+sv) > -\theta_2\}}\mathrm{d}s < \infty. \tag{4.32}$$

According to the proof of [24, Corollary 3.2], the following result follows from Theorem 4.3.

Corollary 4.4. *Let D be convex. Assume* **(A8)**(1) *and* **(A8)**(3). *Let*

$$\begin{aligned} &\gamma(r) := (\theta_1+\theta_2)\big\{(\zeta r^{-1})\wedge r\big\} - (\theta_2-\theta_0)r, \quad r\geq 0,\\ &k := \frac{2\alpha}{\int_0^\infty t\,\mathrm{e}^{\frac{1}{2\beta}\int_0^t \gamma(u)\mathrm{d}u}\mathrm{d}t} - \frac{\beta(\theta_2-\theta_0)}{2\alpha}\int_0^\infty t\mathrm{e}^{\frac{1}{2\alpha}\int_0^t\gamma(u)\mathrm{d}u}\mathrm{d}t. \end{aligned} \tag{4.33}$$

Then there exists a constant $c>0$ such that

$$\mathbb{W}_1(P_t^*\mu, P_t^*\nu) \leq c\mathrm{e}^{-kt}\,\mathbb{W}_1(\mu,\nu), \quad t\geq 0, \mu,\nu\in\mathscr{P}_1(\bar{D}).$$

If (b_t,σ_t) does not depend on t and $\theta_2>\theta_0$ with

$$\beta < \frac{4\alpha^2}{(\theta_2-\theta_0)(\int_0^\infty t\,\mathrm{e}^{\frac{1}{2\alpha}\int_0^t\gamma(u)\mathrm{d}u}\mathrm{d}t)^2},$$

then $k>0$ and P_t^* has a unique invariant probability measure $\bar{\mu}\in\mathscr{P}_1(\bar{D})$ satisfying

$$\mathbb{W}_1(P_t^*\mu,\bar{\mu}) \leq c\mathrm{e}^{-kt}\,\mathbb{W}_1(\mu,\bar{\mu}), \quad t\geq 0, \mu\in\mathscr{P}_1(\bar{D}).$$

Remark 4.1 We note that [24, Theorem 2.1] presents an ergodicity result for the non-dissipative case, which also holds for present setting with convex D. We drop the detailed statement to save space.

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