

LOWER BOUNDS ON LATTICE COVERING DENSITIES OF SIMPLICES*

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Abstract. This paper presents new lower bounds for the lattice covering densities of simplices by studying the Degree-Diameter Problem for abelian Cayley digraphs. In particular, it proves that the density of any lattice covering of a tetrahedron is at least $25/18$ and the density of any lattice covering of a four-dimensional simplex is at least $343/264$.

Key words. lattice covering, simplex, the degree-diameter problem

MSC codes. 52C17, 52B10, 52C07, 05C12

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1. Introduction. More than 2,300 years ago, Aristotle claimed that *congruent regular tetrahedra can tile the whole space*. In other words, he believed that congruent regular tetrahedra can fill the whole space with neither gap nor overlap. Unfortunately, his claim was wrong: *Congruent regular tetrahedra can not tile the whole space*, which was discovered by Regiomontanus in the fifteenth century (see [16]). Then, two natural questions arose immediately: *What is the density of the densest tetrahedron packing and what is the density of the thinnest tetrahedron covering?* In fact, the packing case was emphasized by Hilbert [12] as a part of his 18th problem.

Tetrahedron packings have been studied by many scholars, including mathematicians, physicists, and chemical engineers. Its history is dramatic and eventful. Let S_3 be a regular tetrahedron (a three-dimensional regular simplex). Let $\delta^c(S_3)$ and $\delta^l(S_3)$ denote the densities of the densest congruent packing and the densest lattice packing of S_3 , respectively. Let's recall several events here, just as comparisons to the covering case. In 1904, Minkowski [18] claimed that $\delta^l(S_3) = \frac{9}{38}$. In fact, he also made a mistake. The correct answer is $\delta^l(S_3) = \frac{18}{49}$, which was proved by Hoylman [13] in 1970. Although Aristotle's claim was disproved by Regiomontanus in the fifteenth century which implied that $\delta^c(S_3) < 1$, up to now we only know that $\delta^c(S_3) < 1 - 2.6 \times 10^{-25}$ which was discovered by Gravel, Elser, and Kallus [11] in 2011. For comprehensive surveys on packings and tetrahedron packings, we refer to [6] and [16], respectively.

Covering is often regarded as a counterpart of packing. At least, their concepts, problems, and results are often pairwise indeed. Nevertheless, covering came to mathematics much later than packing and our knowledge about it is still very limited. Let K denote a convex body in $\mathbb{E}^n$, and let C denote a centrally symmetric one. Let $\theta^c(K)$, $\theta^t(K)$, and $\theta^l(K)$ denote the densities of the thinnest congruent covering, the

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thinnest translative covering, and the thinnest lattice covering of $\mathbb{E}^n$ with K , respectively. The definitions of lattice and $\theta^l(K)$ can be found at the beginning of section 3. Clearly, for every convex body K we have

$$1 \leq \theta^c(K) \leq \theta^t(K) \leq \theta^l(K).$$

Moreover, both $\theta^t(K)$ and $\theta^l(K)$ are invariant under nonsingular linear transformations on K . Therefore, in this paper we will work on the simplex with vertices $(0, 0, \dots, 0)$, $(1, 0, \dots, 0)$, $(0, 1, \dots, 0)$, $\dots$, $(0, 0, \dots, 1)$, instead of the regular one.

As usual, let B_n denote the n -dimensional unit ball centered at the origin, and let S_n denote an n -dimensional regular simplex with unit edges and centered at the origin. In the plane, our covering knowledge is comparatively complete. In 1939, Kershner [15] proved that

$$\theta^t(B_2) = \theta^l(B_2) = \frac{2\pi}{\sqrt{27}}.$$

In 1946 and 1950, Fejes Tóth [7, 8] showed that

$$\theta^t(C) = \theta^l(C) \leq \frac{2\pi}{\sqrt{27}}$$

holds for all centrally symmetric convex domains C and the equality in the second inequality holds if and only if C is an ellipse. In 1950, Fáry [5] proved that

$$\theta^l(K) \leq \frac{3}{2}$$

holds for all convex domains K , where the equality holds if and only if K is a triangle. It is rather surprising that

$$\theta^t(S_2) = \frac{3}{2}$$

was proved only in 2010 by Januszek [14]. It is even more surprising that, up to now some basic covering problems in the plane are still open (see [2, 24]). For example, we do not know yet if $\theta^t(K) = \theta^l(K)$ holds for all convex domains.

In $\mathbb{E}^3$, besides the space-fillers, we know one exact covering result

$$\theta^l(B_3) = \frac{5\sqrt{5}\pi}{24},$$

which was discovered in 1954 by Bambah [1]. For tetrahedron coverings, several bounds have been achieved. In the lattice case, we have

$$\frac{2^{16} + 1}{2^{16}} \leq \theta^l(S_3) \leq \frac{125}{63},$$

where the upper bound was discovered by Fiduccia, Forcade and Zito [9] and Dougherty and Faber [4] in the 1990s by constructing a particular lattice covering and the lower bound was achieved by Xue and Zong [22] in 2018. In fact, it was proved by Forcade and Lamoreaux [10] and Dougherty and Faber [4] that the upper bound is a local minimum. They even conjectured it to be the exact value of $\theta^l(S_3)$. In the congruent case, in 2006 Conway and Torquato [3] obtained

$$\theta^c(S_3) \leq \frac{9}{8}$$

by constructing a particular tetrahedron covering. Nothing nontrivial is known about $\theta^t(S_3)$.

In $\mathbb{E}^n$, covering has been studied by Bambah, Coxeter, Davenport, Erdős, Few, Watson, and, in particular, by Rogers (see [20]). They proved that

$$\begin{aligned}\theta^l(K) &\leq n^{\log_2 \log_e n + c}, \\ \theta^t(K) &\leq n \log n + n \log \log n + 5n,\end{aligned}$$

and

$$\frac{n}{e\sqrt{e}} \ll \theta^t(B_n) \leq \theta^l(B_n) \leq c \cdot n(\log_e n)^{\frac{1}{2} \log_2 2\pi e}.$$

In 2018, Xue and Zong [22] discovered that

$$(1.1) \quad \theta^l(S_n) \geq 1 + \frac{1}{2^{3n+7}}.$$

In 2021, Ordentlich, Regev, and Weiss [19] improved Rogers' upper bound to

$$\theta^l(K) \leq cn^2,$$

where c is a suitable positive constant.

In this paper, we prove the following results.

THEOREM 1.1. *If $S_3 + \Lambda$ is a lattice covering of $\mathbb{E}^3$, then its density is at least $25/18$. In other words, we have*

$$\theta^l(S_3) \geq \frac{25}{18}.$$

THEOREM 1.2. *If $S_4 + \Lambda$ is a lattice covering of $\mathbb{E}^4$, then its density is at least $343/264$. In other words, we have*

$$\theta^l(S_4) \geq \frac{343}{264}.$$

Our method is based on the *Degree-Diameter Problem* for *abelian Cayley digraphs*. Let G be a finite *abelian group*, and let $E = \{\mathbf{g}_1, \mathbf{g}_2, \dots, \mathbf{g}_n\}$ be a set of generators for G . Then the abelian Cayley digraph of G and E has the elements of G as its vertices and directed edges from each vertex $\mathbf{u}$ to all vertices $\mathbf{v} = \mathbf{u} + \mathbf{g}_i$, $1 \leq i \leq n$. Note that every vertex in the digraph has out-degree n , and the order of the digraph is $|G|$. Then, the Degree-Diameter Problem for abelian Cayley digraphs can be stated as follows: *Given positive integers n and d , find the largest order $f(n, d)$ among all abelian Cayley digraphs of G and E , where $|E| = n$ and the diameters of these digraphs are at most d .*

Both the Degree-Diameter Problem and Cayley graphs are well-known in the graph theory community. For a survey and recent progress, we refer to Miller and Sirán [17] and Zhang and Ge [23], respectively. In 1974, Wong and Coppersmith [21] showed that

$$f(2, d) = \left\lfloor \frac{(d+2)^2}{3} \right\rfloor.$$

In 1998, Fiduccia, Forcade, and Zito [9] proved that

$$(1.2) \quad f(3, d) \leq \frac{3(d+3)^3}{25}$$

and asked for a similar bound for $f(4, d)$. In 2004, Dougherty and Faber [4] obtained that

$$(1.3) \quad \frac{cd^n}{n!n(\ln n)^{1+\log_2 e}} + O(d^{n-1}) \leq f(n, d) \leq \binom{d+n}{n}.$$

In this paper, we also prove the following result.

THEOREM 1.3. *If an abelian Cayley digraph has degree at most 4 and diameter at most d , then it has at most $\frac{11(d+4)^4}{343}$ elements. In other words,*

$$f(4, d) \leq \frac{11(d+4)^4}{343}.$$

Remark 1.1. Theorem 1.3 will be useful in the proof of Theorem 1.2. As one can check, Theorem 1.3 is better than (1.3) when d is large. In fact, by similar method the upper bound in (1.3) can be improved to

$$f(n, d) \leq \frac{(d+n)^n}{n \cdot n!} \left(n-1 + \left(\frac{n-1}{2n-1} \right)^{n-1} \right).$$

Consequently, the lower bound in (1.1) can be improved to

$$\theta^l(S_n) \geq \frac{n}{n-1 + \left(\frac{n-1}{2n-1} \right)^{n-1}} > 1 + \frac{1}{n} - \frac{1}{(n-1)2^{n-1}}.$$

However, since the improvements are not essential, we will not include their proofs here.

2. The degree-diameter problem. The Degree-Diameter Problem for abelian Cayley digraphs is closely related to lattice coverings (see [4] and [9]). Let $\mathbb{Z}^n$ be the integer lattice generated by the standard basis $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$. From the algebraic point of view, $\mathbb{Z}^n$ is a free abelian group with n generators. Thus for any finite abelian group G generated by $\{\mathbf{g}_1, \mathbf{g}_2, \dots, \mathbf{g}_n\}$ (in additive), there is a homomorphism $\phi: \mathbb{Z}^n \rightarrow G$ defined by

$$\phi \left(\sum_{i=1}^n z_i \mathbf{e}_i \right) = \sum_{i=1}^n z_i \mathbf{g}_i, \quad z_i \in \mathbb{Z}.$$

Clearly, ϕ is surjective. Let L_n be the kernel of ϕ . It is well-known and easy to check that L_n is an n -dimensional sublattice of $\mathbb{Z}^n$. Then there is an isomorphism

$$\bar{\phi}: \mathbb{Z}^n / L_n \rightarrow G$$

and

$$|G| = [\mathbb{Z}^n : L_n] = d(L_n),$$

where $d(L_n)$ denote the determinant of L_n .

Let $\mathbf{o}$ be the origin of $\mathbb{E}^n$, and let $\mathcal{O}_n$ denote the positive orthant in $\mathbb{E}^n$. We start from $\mathbf{o}$ and perform a breadth-first search in $\mathbb{Z}^n \cap \mathcal{O}_n$ until we find all representative elements of the group G . Let $\mathbf{x} = (x_1, x_2, \dots, x_n)$ and $\mathbf{y} = (y_1, y_2, \dots, y_n)$ be two

distinct points in $\mathbb{Z}^n \cap \mathcal{O}_n$, and let $|\mathbf{o}, \mathbf{x}|_M$ denote the *Manhattan distance* between $\mathbf{o}$ and $\mathbf{x}$. In other words,

$$|\mathbf{o}, \mathbf{x}|_M = \sum_{i=1}^n x_i.$$

Then, we write $\mathbf{x} \prec \mathbf{y}$ if either $|\mathbf{o}, \mathbf{x}|_M < |\mathbf{o}, \mathbf{y}|_M$ or $|\mathbf{o}, \mathbf{x}|_M = |\mathbf{o}, \mathbf{y}|_M$ but $\mathbf{x}$ precedes $\mathbf{y}$ lexicographically. For example, in $\mathbb{Z}^2 \cap \mathcal{O}_2$ we have

$$(0, 0) \prec (0, 1) \prec (1, 0) \prec (0, 2) \prec (1, 1) \prec (2, 0) \prec (0, 3) \prec (1, 2) \prec (2, 1) \prec (3, 0) \prec \cdots$$

and in $\mathbb{Z}^3 \cap \mathcal{O}_3$ we have

$$(0, 0, 0) \prec (0, 0, 1) \prec (0, 1, 0) \prec (1, 0, 0) \prec (0, 0, 2) \prec (0, 1, 1) \prec (0, 2, 0) \prec (1, 0, 1) \prec \cdots$$

Clearly, the linear order $\prec$ is compatible with addition. In other words, if $\mathbf{x} \prec \mathbf{y}$, then

$$\mathbf{x} + \mathbf{z} \prec \mathbf{y} + \mathbf{z}$$

holds for all $\mathbf{z} \in \mathbb{Z}^n \cap \mathcal{O}_n$.

For $\mathbf{g} \in G$, let $\phi^*(\mathbf{g})$ be the first lattice point $\mathbf{x}$ in the ordered sequence satisfying $\phi(\mathbf{x}) = \mathbf{g}$. Then

$$(2.1) \quad T_n = \{\phi^*(\mathbf{g}) : \mathbf{g} \in G\}$$

is a complete set of coset representatives for the kernel L_n . From the geometric point of view, $T_n + L_n$ is a *tiling* of $\mathbb{Z}^n$, i.e., $T_n + L_n = \mathbb{Z}^n$ and the translates are pairwise disjoint. Of course, we have

$$(2.2) \quad |G| = [\mathbb{Z}^n : L_n] = |T_n|.$$

Note that the diameter of the abelian Cayley digraph of G and $E = \{\mathbf{g}_1, \mathbf{g}_2, \dots, \mathbf{g}_n\}$ is equal to the greatest Manhattan distance of any point in T_n from $\mathbf{o}$. We will accordingly call it the *M-diameter* of T_n . Thus we have the following result.

PROPOSITION 2.1. *The abelian Cayley digraph of G and E has a diameter of at most d if and only if $T_n + L_n$ is a tiling of $\mathbb{Z}^n$, where T_n has a M-diameter of at most d . Moreover, the order of the graph is equal to $|T_n|$, i.e., $[\mathbb{Z}^n : L_n]$.*

If $T_n + L_n$ is a tiling of $\mathbb{Z}^n$, then $\overline{T_n} + L_n$ is a tiling of $\mathbb{E}^n$, where

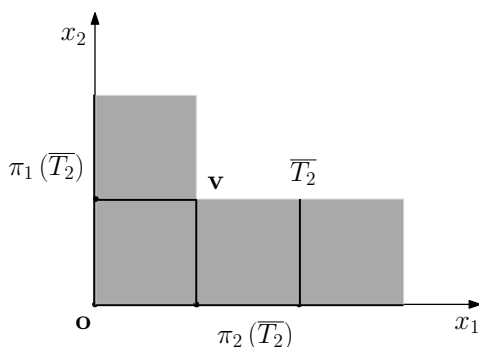
$$\overline{T_n} = T_n + [0, 1)^n$$

is called a *Cayley tile* of $\mathbb{E}^n$. Clearly, we have

$$(2.3) \quad \text{vol}(\overline{T_n}) = |T_n| = d(L_n),$$

where $\text{vol}(\overline{T_n})$ denotes the volume of $\overline{T_n}$. If d is the M-diameter of T_n , then the greatest Manhattan distance between a point of $\overline{T_n}$ and $\mathbf{o}$ is $d + n$. We say that the *M-diameter* of $\overline{T_n}$ is $d + n$. Based on Proposition 2.1 and (2.3), the order of the graph is equal to $\text{vol}(\overline{T_n})$, and the Degree-Diameter Problem for abelian Cayley digraphs is equivalent to the following problem:

Given positive integers n and d , find the largest $\text{vol}(\overline{T_n})$ among all Cayley tiles $\overline{T_n}$ of $\mathbb{E}^n$ with M-diameter at most $d + n$.


 FIG. 1. A Cayley tile in $\mathbb{E}^2$ and its silhouette.

Let's recall some useful definitions and notions introduced by Fiduccia, Forcade, and Zito [9]. Assume that both $\mathbf{x} = (x_1, x_2, \dots, x_n)$ and $\mathbf{y} = (y_1, y_2, \dots, y_n)$ are points in $\mathcal{O}_n$. We say that $\mathbf{x} \preceq \mathbf{y}$ if $x_i \leq y_i$ holds for all i . A *notch* on the boundary of $\overline{T_n}$ is a point $\mathbf{x}$ such that

$$\mathbf{x} - \epsilon \mathbf{e}_i \in \text{int}(\overline{T_n})$$

hold for any small $\epsilon > 0$ and all $i = 1, 2, \dots, n$, where $\text{int}(\overline{T_n})$ denotes the interior of $\overline{T_n}$. For example, the point $\mathbf{v}$ in Figure 1 is a notch. The *silhouette* of $\overline{T_n}$ is the set of points $\mathbf{p}$ with at least one zero coordinate such that $\mathbf{p} \preceq \mathbf{q}$ holds for some point $\mathbf{q} \in \overline{T_n}$. In fact, as one can see from the next lemma, it is the union of the projections of $\overline{T_n}$ onto the coordinate hyperplanes. Let π_i denote the projection from $\mathbb{E}^n$ to the hyperplane $H_i = \{\mathbf{x} : x_i = 0\}$. Then, the silhouette of $\overline{T_n}$ is $\bigcup_{i=1}^n \pi_i(\overline{T_n})$.

LEMMA 2.1 (Fiduccia, Forcade, and Zito [9]). *The Cayley tile $\overline{T_n}$ has the following properties:*

- (1) *If $\mathbf{x} \in \overline{T_n}$ and $\mathbf{o} \preceq \mathbf{y} \preceq \mathbf{x}$, then $\mathbf{y} \in \overline{T_n}$.*
- (2) *It has at most one notch.*
- (3) *It is uniquely determined by its silhouette and, if it has a notch, the coordinates of the notch.*

Remark 2.1. Let $\mathbf{v}$ be a notch of $\overline{T_n}$. The last property of Lemma 2.1 implies that if all projections $\pi_i(\mathbf{p})$ of a point $\mathbf{p}$ are in the silhouette $\overline{T_n}$ and $\mathbf{v} \not\preceq \mathbf{p}$, then $\mathbf{p} \in \overline{T_n}$. Let $[\mathbf{p}, \pi_i(\mathbf{p})]$ denote the segment connecting $\mathbf{p}$ and $\pi_i(\mathbf{p})$. It follows from the first property of Lemma 2.1 that $\pi_i(\mathbf{p})$ belongs to the silhouette of $\overline{T_n}$ if and only if

$$[\mathbf{p}, \pi_i(\mathbf{p})] \cap \overline{T_n} \neq \emptyset.$$

Recall that $\overline{T_4}$ is a Cayley tile of $\mathbb{E}^4$ with M -diameter of d^* , where $d^* \leq d + 4$. We proceed to prove Theorem 1.3 by dealing with two cases.

Case 1. The Cayley tile $\overline{T_4}$ has no notch

By Lemma 2.1, $\overline{T_4}$ is completely determined by its silhouette. For each point $\mathbf{p} \notin \overline{T_4}$ in $\mathcal{O}_4$, according to Remark 2.1, there must be a π_i satisfying

$$[\mathbf{p}, \pi_i(\mathbf{p})] \cap \overline{T_4} = \emptyset.$$

For convenience, then we say $\mathbf{p}$ is of π_i -type. Note that there are at most four types and a point can be of more than one type simultaneously.

LEMMA 2.2. Assume that $\mathbf{p}$ and $\mathbf{q}$ are two points in $\mathcal{O}_4$. If $\mathbf{p} \preceq \mathbf{q}$ and $\mathbf{p}$ is of π_i -type, then $\mathbf{q}$ is also of π_i -type.

Proof. Suppose that $\mathbf{p}$ is π_i -type. Then, we have

$$[\mathbf{p}, \pi_i(\mathbf{p})] \cap \overline{T_4} = \emptyset.$$

Therefore, we have

$$\pi_i(\mathbf{p}) \notin \overline{T_4}.$$

Since $\mathbf{p} \preceq \mathbf{q}$, for every point $\mathbf{w} \in [\mathbf{q}, \pi_i(\mathbf{q})]$ one can deduce that

$$\pi_i(\mathbf{p}) \preceq \mathbf{w}.$$

Thus $\mathbf{w}$ cannot be a point of $\overline{T_4}$, which implies that

$$[\mathbf{q}, \pi_i(\mathbf{q})] \cap \overline{T_4} = \emptyset.$$

Therefore, $\mathbf{q}$ is also π_i -type. The lemma is proved. $\square$

For convenience, we define

$$S_{4,d^*} = \{(x_1, x_2, x_3, x_4) \in \mathbb{E}^4: x_i \geq 0, x_1 + x_2 + x_3 + x_4 \leq d^*\}.$$

Note that the greatest Manhattan distance between any point of S_{4,d^*} and $\mathbf{o}$ is d^* , which implies that

$$(2.4) \quad \overline{T_4} \subseteq S_{4,d^*}.$$

Clearly,

$$F = \{(x_1, x_2, x_3, x_4) \in \mathbb{E}^4: x_i \geq 0, x_1 + x_2 + x_3 + x_4 = d^*\}$$

is a facet of S_{4,d^*} , which, in fact, is a regular tetrahedron.

Let F_1 denote the subset of F including all points of π_1 -type and successively define F_i to be the subset of F including all points of π_i -type which are not in $F_1 \cup \cdots \cup F_{i-1}$. Since $\overline{T_4}$ has no notch, it is easy to see that F_1, F_2, F_3 , and F_4 are not empty, they are pairwise disjoint, and

$$F = F_1 \cup F_2 \cup F_3 \cup F_4.$$

For $1 \leq i \leq 4$, we define

$$D_i = \bigcup_{\mathbf{p} \in F_i} [\mathbf{p}, \pi_i(\mathbf{p})].$$

Since $[\mathbf{p}, \pi_i(\mathbf{p})] \cap \overline{T_4} = \emptyset$ holds for all $\mathbf{p} \in F_i$, we have

$$D_i \cap \overline{T_4} = \emptyset,$$

which implies that D_i is in the complement of $\overline{T_4}$.

LEMMA 2.3. The sets D_i , where $1 \leq i \leq 4$, are pairwise disjoint.

Proof. If, on the contrary, D_i and D_j have a common point $\mathbf{p}$, where $i < j$. Then there is a point $\mathbf{p}_i \in F_i$ and a point $\mathbf{p}_j \in F_j$ such that

$$\mathbf{p} \in [\mathbf{p}_i, \pi_i(\mathbf{p}_i)] \cap [\mathbf{p}_j, \pi_j(\mathbf{p}_j)],$$

where $[\mathbf{p}_i, \pi_i(\mathbf{p}_i)] \cap \overline{T_4} = \emptyset$ and $[\mathbf{p}_j, \pi_j(\mathbf{p}_j)] \cap \overline{T_4} = \emptyset$. Consequently, $\mathbf{p} \preceq \mathbf{p}_i$, $\mathbf{p} \preceq \mathbf{p}_j$, and $\mathbf{p}$ is both π_i -type and π_j -type. According to Lemma 2.2, $\mathbf{p}_j$ is also of π_i -type, i.e., $\mathbf{p}_j \in F_i$, which contradicts the fact

$$F_i \cap F_j = \emptyset.$$

The lemma is proved. □

By (2.4) and Lemma 2.3 we have

$$(2.5) \quad \text{vol}(\overline{T_4}) \leq \text{vol}(S_{4,d^*}) - \sum_{i=1}^4 \text{vol}(D_i) = \frac{d^{*4}}{4!} - \sum_{i=1}^4 \text{vol}(D_i).$$

Notice that

$$(2.6) \quad \sum_{i=1}^4 \text{vol}(D_i) = \iiint_F \lambda f(\mathbf{x}) d\mathbf{x} = \sum_{i=1}^4 \iiint_{\pi_i(F_i)} (d^* - |\mathbf{o}, \mathbf{x}|_M) d\mathbf{x},$$

where λ is a constant because we are integrating over F instead of over the coordinate hyperplanes, $f(\mathbf{x})$ is the distance from $\mathbf{x}$ to an appropriate coordinate hyperplane. In fact, if $\mathbf{x} = (x_1, x_2, x_3, x_4) \in F_i$, then

$$f(\mathbf{x}) = x_i.$$

To maximize the right-hand side of (2.5), we need to minimize the value of (2.6). For any $\mathbf{x} = (x_1, x_2, x_3, x_4)$ in F , we define

$$g(\mathbf{x}) = \min \{x_1, x_2, x_3, x_4\}.$$

Furthermore, we define

$$F'_i = \{\mathbf{x} \in F: g(\mathbf{x}) = x_i\}.$$

Of course, $\pi_i(F'_i)$ is the projection of F'_i onto the nearest coordinate hyperplane. Thus, we have

$$\begin{aligned} \text{vol}(\overline{T_4}) &\leq \frac{d^{*4}}{4!} - \iiint_F \lambda g(\mathbf{x}) d\mathbf{x} \\ &= \frac{d^{*4}}{24} - \sum_{i=1}^4 \iiint_{\pi_i(F'_i)} (d^* - |\mathbf{o}, \mathbf{x}|_M) d\mathbf{x} \\ &= \frac{d^{*4}}{24} - 4 \iiint_{\pi_4(F'_4)} (d^* - x_1 - x_2 - x_3) d\mathbf{x}. \end{aligned}$$

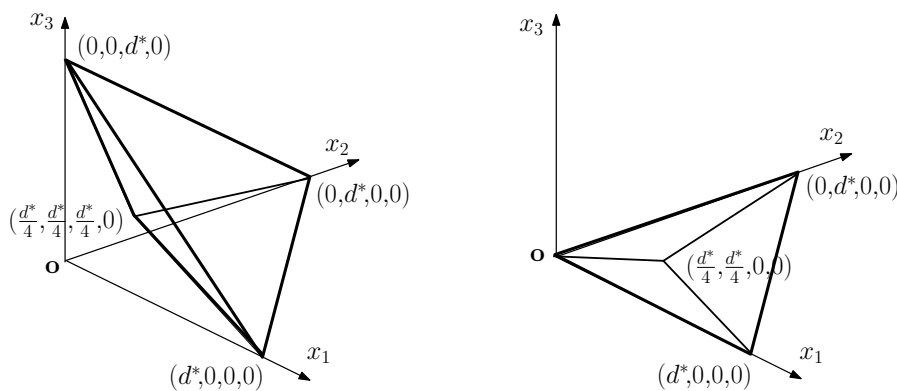


FIG. 2. The integral region $\pi_4(F'_4)$ and its projection on the x_1x_2 -plane.

The integral region $\pi_4(F'_4)$ and its projection on the x_1x_2 -coordinate plane are shown in Figure 2. It follows that

$$\begin{aligned} \iiint_{\pi_4(F'_4)} (d^* - x_1 - x_2 - x_3) \, d\mathbf{x} &= \int_0^{\frac{d^*}{4}} dx_1 \int_{d^*-3x_1}^{d^*-x_1} dx_2 \int_{\frac{1}{2}(d^*-x_1-x_2)}^{d^*-x_1-x_2} (d^* - x_1 - x_2 - x_3) \, dx_3 \\ &\quad + \int_{\frac{d^*}{4}}^{d^*} dx_1 \int_{\frac{1}{3}(d^*-x_1)}^{d^*-x_1} dx_2 \int_{\frac{1}{2}(d^*-x_1-x_2)}^{d^*-x_1-x_2} (d^* - x_1 - x_2 - x_3) \, dx_3 \\ &\quad + \int_0^{\frac{d^*}{4}} dx_1 \int_{x_1}^{d^*-3x_1} dx_2 \int_{d^*-2x_1-x_2}^{d^*-x_1-x_2} (d^* - x_1 - x_2 - x_3) \, dx_3 \\ &\quad + \int_0^{\frac{d^*}{4}} dx_2 \int_{x_2}^{d^*-3x_2} dx_1 \int_{d^*-x_1-2x_2}^{d^*-x_1-x_2} (d^* - x_1 - x_2 - x_3) \, dx_3 \\ &= \frac{d^{*4}}{384}. \end{aligned}$$

Thus, with no notch in $\overline{T}_4$, we have

$$\text{vol}(\overline{T}_4) \leq \frac{d^{*4}}{32}.$$

Case 2. The Cayley tile $\overline{T}_4$ has a notch

From Remark 2.1, if $\overline{T}_4$ has a notch $\mathbf{v}$, then not all points $\mathbf{x} \in F$ are of π_1 -, π_2 -, π_3 -, and π_4 -type, and the exceptional points $\mathbf{x}$ must satisfy $\mathbf{v} \preceq \mathbf{x}$. Then, we define

$$Q = \{\mathbf{x} \in F: \mathbf{v} \preceq \mathbf{x}\}.$$

Following the previous argument, let F_1 denote the subset of $F \setminus Q$ including all points of π_1 -type and successively define F_i to be the subset of $F \setminus Q$ including all points of π_i -type which are not in $F_1 \cup \cdots \cup F_{i-1}$. Then, we write

$$D_i = \bigcup_{\mathbf{x} \in F_i} [\mathbf{x}, \pi_i(\mathbf{x})].$$

It is easy to see that

$$F = F_1 \cup F_2 \cup F_3 \cup F_4 \cup Q$$

is a disjoint union, and the sets D_i are in the complement of $\overline{T_4}$. Furthermore, similar to Lemma 2.3, one can prove that D_i are pairwise disjoint. Then, we define

$$P = \{\mathbf{p} \in S_{4,d^*} : \mathbf{v} \preceq \mathbf{p}\}.$$

In other words, P is the union of all segments from Q to the notch point $\mathbf{v}$. Thus, we have

$$(2.7) \quad \text{vol}(\overline{T_4}) \leq \frac{d^{*4}}{4!} - \sum_{i=1}^4 \text{vol}(D_i) - \text{vol}(P).$$

Similar to (2.6), we have

$$(2.8) \quad \sum_{i=1}^4 \text{vol}(D_i) = \iiint_{F \setminus Q} \lambda f(\mathbf{x}) d\mathbf{x} = \sum_{i=1}^4 \iiint_{\pi_i(F_i)} (d^* - |\mathbf{o}, \mathbf{x}|_M) d\mathbf{x},$$

where λ is a constant because we are integrating over F instead of over the coordinate hyperplanes, $f(\mathbf{x})$ is the distance from $\mathbf{x}$ to an appropriate coordinate hyperplane. In fact, if $\mathbf{x} = (x_1, x_2, x_3, x_4) \in F_i$, then

$$f(\mathbf{x}) = x_i.$$

First, let's fix the volumes of Q and P and to minimize the value of (2.8). For a point $\mathbf{x} = (x_1, x_2, x_3, x_4)$ in $F \setminus Q$, we define

$$g(\mathbf{x}) = \min \{x_1, x_2, x_3, x_4\},$$

which is the distance from $\mathbf{x}$ to the nearest coordinate hyperplane, and

$$F'_i = \{\mathbf{x} \in F \setminus Q : g(\mathbf{x}) = x_i\}.$$

Of course, then $\pi_i(F'_i)$ is the projection of F'_i onto the nearest coordinate hyperplane. Therefore, we have

$$\sum_{i=1}^4 \text{vol}(D_i) \geq \iiint_{F \setminus Q} \lambda g(\mathbf{x}) d\mathbf{x}.$$

Let $g_1(\mathbf{x})$ denote the distance from $\mathbf{x}$ to the nearest face of F (see Figure 3 (1)), then

$$\iiint_{F \setminus Q} \lambda g(\mathbf{x}) d\mathbf{x} = \iiint_{F \setminus Q} \lambda_1 g_1(\mathbf{x}) d\mathbf{x}$$

holds with a suitable constant λ_1 . Obviously, the four sets F'_1 , F'_2 , F'_3 , and F'_4 are subsets of the four pyramids with the center $\mathbf{c}$ of F as their common vertices and with the four facets of F as their bases, respectively. Now, we proceed to minimize the integral

$$I = \iiint_{F \setminus Q} \lambda_1 g_1(\mathbf{x}) d\mathbf{x}.$$

For convenience, we write

$$h = g_1(\mathbf{c})$$

and define that

$$\chi(\mathbf{x}, t) = \begin{cases} 1, & g_1(\mathbf{x}) \geq t, \\ 0, & g_1(\mathbf{x}) < t. \end{cases}$$

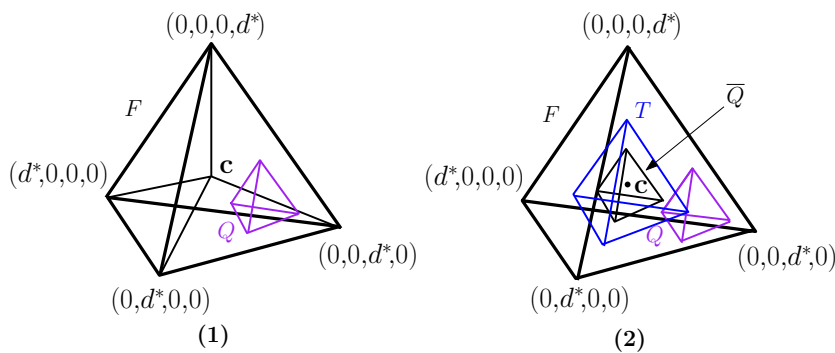


FIG. 3. The F_i' regions and the placement of Q . (Figure in color online.)

Then, we have

$$I = \lambda_1 \iiint_{F \setminus Q} \int_0^h \chi(\mathbf{x}, t) dt d\mathbf{x} = \lambda_1 \int_0^h \iiint_{F \setminus Q} \chi(\mathbf{x}, t) d\mathbf{x} dt.$$

It is easy to see, for any $t \in [0, h]$, the set

$$T = \{\mathbf{x}: g_1(\mathbf{x}) \geq t\}$$

is a tetrahedron centered at $\mathbf{c}$, i.e., the blue tetrahedron in Figure 3 (2). Thus, we have

$$\iiint_{F \setminus Q} \chi(\mathbf{x}, t) d\mathbf{x} = \text{vol}((F \setminus Q) \cap T).$$

Since the volume of Q is fixed, the integral is minimized when Q is centered at $\mathbf{c}$. In other words,

$$I = \lambda_1 \int_0^h \iiint_{F \setminus Q} \chi(\mathbf{x}, t) d\mathbf{x} dt \geq \lambda_1 \int_0^h \iiint_{F \setminus \overline{Q}} \chi(\mathbf{x}, t) d\mathbf{x} dt,$$

where $\overline{Q}$ is the translate of Q centered at $\mathbf{c}$.

Now the only question remaining is, when Q is centered at $\mathbf{c}$, how large should Q be in order to maximize

$$\frac{d^{*4}}{24} - I - \text{vol}(P).$$

Suppose that the notch $\mathbf{v} = (v, v, v, v)$, where $0 \leq v \leq \frac{d^*}{4}$, then we have

$$\text{vol}(P) = \frac{(d^* - 4v)^4}{4!}.$$

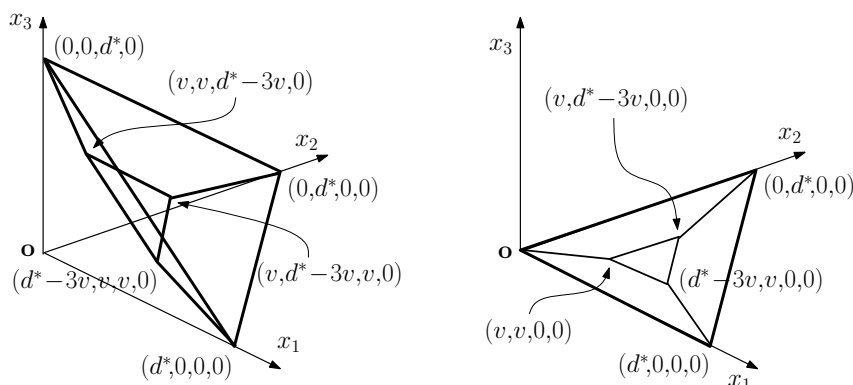


FIG. 4. The integral region $\pi_4(F'_4)$ and its projection on the x_1x_2 -coordinate plane when there is a notch.

The integral region $\pi_4(F'_4)$ and its projection on the x_1x_2 -plane are shown in Figure 4. Then, one can deduce that

$$\begin{aligned}
 \iiint_{\pi_4(F'_4)} (d^* - x_1 - x_2 - x_3) d\mathbf{x} &= \int_0^v dx_1 \int_{d^*-3x_1}^{d^*-x_1} dx_2 \int_{\frac{1}{2}(d^*-x_1-x_2)}^{d^*-x_1-x_2} (d^* - x_1 - x_2 - x_3) dx_3 \\
 &\quad + \int_v^{d^*-3v} dx_1 \int_{d^*-x_1-2v}^{d^*-x_1} dx_2 \int_{\frac{1}{2}(d^*-x_1-x_2)}^{d^*-x_1-x_2} (d^* - x_1 - x_2 - x_3) dx_3 \\
 &\quad + \int_{d^*-3v}^{d^*} dx_1 \int_{\frac{1}{3}(d^*-x_1)}^{d^*-x_1} dx_2 \int_{\frac{1}{2}(d^*-x_1-x_2)}^{d^*-x_1-x_2} (d^* - x_1 - x_2 - x_3) dx_3 \\
 &\quad + \int_v^{d^*-3v} dx_1 \int_v^{d^*-x_1-2v} dx_2 \int_{d^*-x_1-x_2-v}^{d^*-x_1-x_2} (d^* - x_1 - x_2 - x_3) dx_3 \\
 &\quad + \int_0^v dx_1 \int_{x_1}^{d^*-3x_1} dx_2 \int_{d^*-2x_1-x_2}^{d^*-x_1-x_2} (d^* - x_1 - x_2 - x_3) dx_3 \\
 &\quad + \int_0^v dx_2 \int_{x_2}^{d^*-3x_2} dx_1 \int_{d^*-x_1-2x_2}^{d^*-x_1-x_2} (d^* - x_1 - x_2 - x_3) dx_3 \\
 &= 2v^4 - \frac{4d^*}{3}v^3 + \frac{d^{*2}}{4}v^2.
 \end{aligned}$$

Therefore, in the case that $\overline{T}_4$ has a notch, we have shown that

$$(2.9) \quad \text{vol}(\overline{T}_4) \leq -\frac{56}{3}v^4 + 16d^*v^3 - 5d^{*2}v^2 + \frac{2d^{*3}}{3}v.$$

To find the maximum of the right-hand side of (2.9), taking its derivative and setting it equal to zero, we get

$$\left(v - \frac{d^*}{4}\right)^2 \left(v - \frac{d^*}{7}\right) = 0.$$

It is easy to check that the right-hand side of (2.9) attains a local minimum $\frac{d^{*4}}{32}$ at $v = \frac{d^*}{4}$ and attains its maximum $\frac{11d^{*4}}{343}$ at $v = \frac{d^*}{7}$. In other words, we have

$$\text{vol}(\overline{T}_4) \leq \frac{11d^{*4}}{343} \leq \frac{11(d+4)^4}{343}.$$

As a conclusion of these two cases, by (2.2) and (2.3) we have shown that

$$f(4, d) \leq \frac{11(d + 4)^4}{343}.$$

Theorem 1.3 is proved.

3. Lattice coverings of simplices. If $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n$ are n linearly independent vectors in $\mathbb{E}^n$, then the discrete set

$$\Lambda = \left\{ \sum z_i \mathbf{a}_i : z_i \in \mathbb{Z} \right\}$$

is called an n -dimensional lattice. The set $\{\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n\}$ is called a basis of Λ . Assume that A is the $n \times n$ matrix whose i th row is the coordinates of $\mathbf{a}_i$, then $|\det A|$ is called the *determinant* of Λ . It is written as $d(\Lambda)$. If $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n$ are n linearly independent vectors in $\mathbb{Z}^n$, then Λ is an n -dimensional sublattice of $\mathbb{Z}^n$. In addition, we have

$$d(\Lambda) = [\mathbb{Z}^n : \Lambda].$$

For a convex body K in $\mathbb{E}^n$, we call $K + \Lambda$ a *lattice covering* of $\mathbb{E}^n$ if

$$\mathbb{E}^n = \bigcup_{\mathbf{v} \in \Lambda} (K + \mathbf{v}).$$

The value

$$\theta(K, \Lambda) = \frac{\text{vol}(K)}{d(\Lambda)}$$

is called the density of the covering. Let $\mathcal{L}$ denote the family of all lattices Λ such that $K + \Lambda$ is a lattice covering of $\mathbb{E}^n$. Then we call

$$\theta^l(K) = \min_{\Lambda \in \mathcal{L}} \theta(K, \Lambda) = \min_{\Lambda \in \mathcal{L}} \frac{\text{vol}(K)}{d(\Lambda)}$$

the *lattice covering density* of K . Clearly, it is the density of the thinnest lattice covering of K .

Given positive integers n and d , we define

$$S_{n,d}^\circ = \{(z_1, \dots, z_n) \in \mathbb{Z}^n : z_i \geq 0, z_1 + \dots + z_n \leq d\}.$$

Note that the greatest Manhattan distance between a point of $S_{n,d}^\circ$ and $\mathbf{o}$ is d , and

$$(3.1) \qquad |S_{n,d}^\circ| = \binom{d+n}{n} = \frac{d^n}{n!} + O(d^{n-1}).$$

We call $S_{n,d}^\circ + L_n$ a *lattice covering* of $\mathbb{Z}^n$ if L_n is a lattice and

$$\mathbb{Z}^n = \bigcup_{\mathbf{v} \in L_n} (S_{n,d}^\circ + \mathbf{v}).$$

For such a lattice covering we define

$$\theta^*(S_{n,d}^\circ, L_n) = \frac{|S_{n,d}^\circ|}{d(L_n)}.$$

Let $\mathcal{L}_n^*$ denote the family of all L_n such that $S_{n,d}^\circ + L_n$ is a lattice covering of $\mathbb{Z}^n$. Then we call

$$\theta^*(S_{n,d}^\circ) = \min_{L_n \in \mathcal{L}_n^*} \theta^*(S_{n,d}^\circ, L_n) = \min_{L_n \in \mathcal{L}_n^*} \frac{|S_{n,d}^\circ|}{d(L_n)}$$

the *lattice covering density* of $S_{n,d}^\circ$.

Clearly, $S_{n,d}$ is the convex hull of $S_{n,d}^\circ$. To show a useful connection between $\theta^*(S_{n,d}^\circ)$ and $\theta^l(S_n)$, we need the following lemma.

LEMMA 3.1 (Dougherty and Faber [4]). *Let L_n be a sublattice of $\mathbb{Z}^n$, and let Λ be a lattice in $\mathbb{E}^n$ which is generated by $\{\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n\}$. Then the following holds:*

- (1) *If $S_{n,d}^\circ + L_n = \mathbb{Z}^n$, then $S_{n,d+n} + L_n = \mathbb{E}^n$.*
- (2) *If $S_{n,d} + \Lambda = \mathbb{E}^n$, then there is a constant c such that for all sufficiently large real numbers k , if $\mathbf{b}_i$ is obtained from $k\mathbf{a}_i$ by rounding all coordinates to the nearest integer, and L_n is the lattice generated by $\{\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_n\}$, then*

$$S_{n,kd+c} + L_n = \mathbb{E}^n.$$

LEMMA 3.2. *For fixed n and large d , we have*

$$\theta^*(S_{n,d}^\circ) = \theta^l(S_n) + O(d^{-1}).$$

Proof. Suppose that $S_{n,d}^\circ + L_n$ is a lattice covering of $\mathbb{Z}^n$ with density $\theta^*(S_{n,d}^\circ)$. By Lemma 3.1 (1), we have

$$S_{n,d+n} + L_n = \mathbb{E}^n.$$

Then, by (3.1) we get

$$\theta^l(S_n) = \theta^l(S_{n,d+n}) \leq \theta(S_{n,d+n}, L_n) = \frac{\text{vol}(S_{n,d+n})\theta^*(S_{n,d}^\circ)}{|S_{n,d}^\circ|} = \frac{\theta^*(S_{n,d}^\circ)}{1 + O(d^{-1})},$$

which implies

$$(3.2) \quad \theta^*(S_{n,d}^\circ) \geq \theta^l(S_n) + O(d^{-1}).$$

On the other hand, suppose that $S_{n,1} + \Lambda$ is a lattice covering of $\mathbb{E}^n$ with density $\theta^l(S_n)$. Let $\{\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n\}$ and c as chosen in Lemma 3.1. Assuming that d is a large integer, we take $k = d - c$ and define L_n to be the lattice approximating $k\Lambda$ as in Lemma 3.1 (2), generated by $\{\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_n\}$. Now L_n is a sublattice of $\mathbb{Z}^n$ satisfying

$$S_{n,k+c} + L_n = \mathbb{E}^n,$$

so we have

$$S_{n,d}^\circ + L_n = \mathbb{Z}^n.$$

Let A and B be the $n \times n$ matrices with rows $\mathbf{a}_i$ and $\mathbf{b}_i$, respectively. Then, we have

$$\theta^*(S_{n,d}^\circ) \leq \theta^*(S_{n,d}^\circ, L_n) = \frac{|S_{n,d}^\circ|}{|\det B|},$$

which together with (3.1) implies

$$(3.3) \quad \theta^*(S_{n,d}^\circ) \leq \left(\frac{1}{n!} + O(d^{-1}) \right) \frac{1}{|\det(d^{-1}B)|}.$$

By the definitions of $A = (a_{i,j})$ and $B = (b_{i,j})$, we have

$$b_{i,j} = ka_{i,j} + O(1) = da_{i,j} + O(1), \quad i, j = 1, 2, \dots, n,$$

which implies that

$$d^{-1}b_{i,j} = a_{i,j} + O(d^{-1}), \quad i, j = 1, 2, \dots, n.$$

Combined with the assumption that

$$\theta^l(S_n) = \frac{\text{vol}(S_{n,1})}{|\det A|} = \frac{1}{n!|\det A|},$$

we get

$$(3.4) \quad |\det(d^{-1}B)| = |\det A| + O(d^{-1}) = \frac{1}{n!\theta^l(S_n)} + O(d^{-1}).$$

Therefore, by (3.3) and (3.4) we get

$$(3.5) \quad \theta^*(S_{n,d}^\circ) \leq \theta^l(S_n) + O(d^{-1}).$$

Clearly, (3.2) and (3.5) together imply

$$\theta^*(S_{n,d}^\circ) = \theta^l(S_n) + O(d^{-1}).$$

The lemma is proved. □

COROLLARY 3.1.

$$\theta^l(S_n) = \theta^l(S_{n,1}) = \lim_{d \rightarrow \infty} \theta^*(S_{n,d}^\circ).$$

Recall that T_n was defined by (2.1) with M-diameter d . To show Theorems 1.1 and 1.2, we need another result.

LEMMA 3.3. $S_{n,d}^\circ + L_n$ is a covering of $\mathbb{Z}^n$ if and only if $T_n + L_n$ is a tiling of $\mathbb{Z}^n$, where

$$T_n = S_{n,d}^\circ \setminus \bigcup_{\mathbf{v} \in L_n, \mathbf{o} \prec \mathbf{v}} (S_{n,d}^\circ + \mathbf{v}).$$

Proof. Since $T_n \subseteq S_{n,d}^\circ$, the sufficiency is obvious. Now we proceed to show the necessary part. In other words, if $S_{n,d}^\circ + L_n$ is a covering of $\mathbb{Z}^n$, for every point $\mathbf{x} \in \mathbb{Z}^n$ there is exact one point $\mathbf{u} \in L_n$ satisfying $\mathbf{x} \in T_n + \mathbf{u}$.

Without loss of generality, we only deal with the points $\mathbf{x} = (x_1, x_2, \dots, x_n) \in \mathbb{Z}^n$ with $x_i \geq d$ for all $i = 1, 2, \dots, n$. Let $\mathbf{u}$ be the $\prec$ -last point of L_n such that $\mathbf{x} \in S_{n,d}^\circ + \mathbf{u}$ and write

$$\mathbf{y} = \mathbf{x} - \mathbf{u} \in S_{n,d}^\circ.$$

Then, for any point $\mathbf{v} \in L_n$ satisfying $\mathbf{o} \prec \mathbf{v}$ we have

$$\mathbf{y} \notin S_{n,d}^\circ + \mathbf{v}.$$

Otherwise, we obtain that

$$\mathbf{x} = \mathbf{y} + \mathbf{u} \in S_{n,d}^\circ + \mathbf{v} + \mathbf{u},$$

where $\mathbf{u} \prec \mathbf{v} + \mathbf{u}$, which contradicts the $\prec$ assumption on $\mathbf{u}$. Thus, we must have $\mathbf{y} \in T_n$ and

$$\mathbf{x} \in T_n + \mathbf{u}.$$

Suppose that

$$\mathbf{x} = \mathbf{y} + \mathbf{u} = \mathbf{y}' + \mathbf{u}',$$

where $\mathbf{u}$ and $\mathbf{u}'$ are distinct points of L_n and both $\mathbf{y}$ and $\mathbf{y}'$ are in T_n (also in $S_{n,d}^\circ$). Then, one can deduce that

$$\mathbf{v} = \mathbf{y} - \mathbf{y}' = \mathbf{u}' - \mathbf{u}$$

is a nonzero point of L_n . Therefore, we have either $\mathbf{o} \prec \mathbf{v}$ or $\mathbf{v} \prec \mathbf{o}$. In the former case, we have

$$\mathbf{y} = \mathbf{y}' + \mathbf{v} \in S_{n,d}^\circ + \mathbf{v},$$

which contradicts to $\mathbf{y} \in T_n$. In the latter case, we get

$$\mathbf{y}' = \mathbf{y} - \mathbf{v} \in S_{n,d}^\circ + (-\mathbf{v}),$$

which contradicts to $\mathbf{y}' \in T_n$. The lemma is proved. $\square$

Proof of Theorem 1.1. Based on Proposition 2.1 and Lemma 3.3, we know that the abelian Cayley digraph of G and $\{\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3\}$ has a diameter of at most d if and only if $S_{3,d}^\circ + L_3 = \mathbb{Z}^3$. By (1.2) and (2.2), we have

$$[\mathbb{Z}^3 : L_3] \leq \frac{3(d+3)^3}{25},$$

which implies

$$\theta^*(S_{3,d}^\circ) = \min_{L_3 \in \mathcal{L}_3^*} \frac{|S_{3,d}^\circ|}{[\mathbb{Z}^3 : L_3]} \geq \frac{25 \binom{d+3}{3}}{3(d+3)^3},$$

where $\mathcal{L}_3^*$ is the family of all L_3 such that $S_{3,d}^\circ + L_3$ is a lattice covering of $\mathbb{Z}^3$. By Corollary 3.1, we have

$$\theta^l(S_3) = \lim_{d \rightarrow \infty} \theta^*(S_{3,d}^\circ) \geq \lim_{d \rightarrow \infty} \frac{25 \binom{d+3}{3}}{3(d+3)^3} = \frac{25}{18}.$$

Theorem 1.1 is proved. $\square$

Proof of Theorem 1.2. Similar to Theorem 1.1, by (2.2), and Theorem 1.3, we have

$$[\mathbb{Z}^4 : L_4] \leq \frac{11(d+4)^4}{343},$$

which implies

$$\theta^*(S_{4,d}^\circ) = \min_{L_4 \in \mathcal{L}_4^*} \frac{|S_{4,d}^\circ|}{[\mathbb{Z}^4 : L_4]} \geq \frac{343 \binom{d+4}{4}}{11(d+4)^4},$$

where $\mathcal{L}_4^*$ is the family of all L_4 such that $S_{4,d}^\circ + L_4$ is a lattice covering of $\mathbb{Z}^4$. By Corollary 3.1, we have

$$\theta^l(S_4) = \lim_{d \rightarrow \infty} \theta^*(S_{4,d}^\circ) \geq \lim_{d \rightarrow \infty} \frac{343 \binom{d+4}{4}}{11(d+4)^4} = \frac{343}{264}.$$

Theorem 1.2 is proved. $\square$

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