



Asymptotic behavior of the basic reproduction number in an age-structured SIS epidemic patch model

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Abstract

In this paper we consider an age-structured SIS epidemic patch model. We define the principal eigenvalue λ_0^d , basic reproduction number R_0^d and analyze the effects of the diffusion rate d on λ_0^d and R_0^d . More precisely, we investigate their asymptotic behavior for small and large diffusion rates. Finally we study the global behavior of this age-structured SIS epidemic patch model.

Keywords Age-structure · Patch environment · Principal eigenvalue · Basic reproduction number · Spectral radius · Global behavior

Mathematics Subject Classification 35Q92 · 46N60 · 92D30

1 Introduction

Recently studies on patch epidemic models or patch population models, which can describe the movement of multigroups efficiently, attract great attention of many researchers, see Allen et al. (2007), Gao (2019), Gao and Dong (2020), Chen et al. (2020, 2022), Li and Peng (2019), Liang and Zhang (2021), Liu et al. (2022), Zhang and Zhao (2022) and the references cited therein. Among these papers, the asymptotic behavior of the principal eigenvalue and/or the basic reproduction number for small and large diffusion rates have been studied frequently, for instance, Allen et al. (2007) established the monotonicity of the basic reproduction number with respect to the diffusion rate of infectious population and obtained the asymptotic behavior for small and large diffusion rates, under the assumption that the connectivity matrix is symmetric and irreducible. Later, Gao (2019), Gao and Dong (2020) and Chen et al. (2020) showed the similar asymptotic behavior without the symmetry condition. Recently,

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Zhang and Zhao (2022) further generalized the result even without the irreducible assumption in both autonomous and time-periodic environment.

To our best knowledge, there is few study on the patch or multigroup age-structured epidemic models (see Kuniya et al. 2018; Feng et al. 2005), in particular, on the asymptotic behavior of the principal eigenvalue and the basic reproduction number for small and large diffusion rates. Based on an observation of similarity between age-structured models and time-periodic models under some special assumptions, in this paper we consider an age-structured SIS epidemic patch model. First we follow Kuniya et al. (2018) to provide a derivation of the model. Assume that the population in each patch N obeys the following Lotka–McKendrick system:

$$\begin{cases} \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial a} \right) N_j(t, a) = d \sum_{m=1}^n L_{jm}(a) N_m(t, a) - \mu_j(a) N_j(t, a), & t > 0, a \in (0, a^+), \\ N_j(t, 0) = \int_0^{a^+} \beta_j(a) N_j(t, a) da, & t > 0, \\ N_j(0, a) = N_{j0}(a), & a \in (0, a^+). \end{cases} \quad (1.1)$$

Here $a^+ < \infty$ denotes the maximum age, $j = 1, \dots, n$ and $n \geq 2$ is the number of patches, $\mu_j(a)$ and $\beta_j(a)$ are the age-dependent death rate and birth rate in patch j , respectively. Further, $L = (L_{jm}(a))_{n \times n}$, called connectivity matrix, is a nonnegative matrix depending on age a that denotes the degree of movement from patch m to patch j for $j \neq m$ and $-L_{jj}(\cdot) = \sum_{m=1, m \neq j}^n L_{mj}(\cdot)$ in $[0, a^+]$ is the degree of movement from patch j to all other patches.

In this paper we assume that the population is in the demographic steady states. For this purpose, we make the following assumptions.

Assumption 1.1 (i) $\mu_j \in L^1_{\text{loc},+}(0, a^+)$ and $\int_0^{a^+} \mu_j(a) da = \infty$ for all $j = 1, \dots, n$.

In addition, there exists $\tilde{\mu} > 0$ such that $\mu_j(a) \geq \tilde{\mu}$ a.e. $a \in (0, a^+)$ for $j = 1, \dots, n$. Denote $\mu(a) = \text{diag}\{\mu_1(a), \dots, \mu_n(a)\}$;

- (ii) $\beta_j \in L^1_{+,+}(0, a^+)$ for all $j = 1, \dots, n$. Denote $\beta(a) = \text{diag}\{\beta_1(a), \dots, \beta_n(a)\}$;
- (iii) The net reproduction matrix $\int_0^{a^+} \beta(a) \mathcal{U}_d(a, 0) da$ is irreducible and its Frobenius root is 1, where $\{\mathcal{U}_d(a, \tau)\}_{0 \leq \tau \leq a \leq a^+}$ is the evolution family associated to the following equation

$$\begin{cases} \frac{du(a)}{da} = dL(a)u(a) - \mu(a)u(a), & a \in (0, a^+), \\ u(\tau) = \psi \in \mathbb{R}^n. \end{cases} \quad (1.2)$$

- (iv) $L_{jm} \in L^\infty_+(0, a^+)$ for all $j, m = 1, \dots, n$ with $j \neq m$ and with zero column sums.

Under Assumption 1.1, it has been shown in Inaba (1988) that (1.1) has a strongly positive (see the definition in the following) demographic steady state $N^*(a) = (N_1^*(a), \dots, N_n^*(a))$, which satisfies

$$N^*(0) = \int_0^{a^+} \beta(a) \mathcal{U}_d(a, 0) da N^*(0).$$

It also has been shown in Inaba (1988, 2017) that N^* is globally asymptotically stable with intrinsic growth rate 0. Therefore, in the following we assume that the demographic steady state has already been reached at $t = 0$, so

$$N(t, a) \equiv N^*(a) = \mathcal{U}_d(a, 0)N^*(0), \text{ for all } t > 0. \quad (1.3)$$

Moreover, it is easy to see that $N^*(0)$ is actually the strongly positive eigenvector of the net reproduction matrix corresponding to 1. In addition, we can normalize N_d^* such that $N_d^*(a) = N^*(a)$ is uniformly bounded with respect to $d > 0$.

In this paper we focus on the disease transmission described by SIS epidemic models, that is, recovered individuals do not have immunity and remain susceptible, for instance, some sexually transmitted diseases, see Lajmanovich and Yorke (1976). Thus we consider the following equation,

$$\begin{cases} \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial a} \right) S_j(t, a) = d_S \sum_{m=1}^n L_{jm}(a) S_m(t, a) \\ \quad - \Gamma_j(t, a) S_j(t, a) + \gamma_j(a) I_j(t, a) - \mu_j(a) S_j(t, a), & t > 0, a \in (0, a^+), \\ \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial a} \right) I_j(t, a) = d_I \sum_{m=1}^n L_{jm}(a) I_m(t, a) \\ \quad + \Gamma_j(t, a) S_j(t, a) - (\gamma_j(a) + \mu_j(a)) I_j(t, a), & t > 0, a \in (0, a^+), \\ S_j(t, 0) = \int_0^{a^+} \beta_j(a) N_j^*(a) da, \quad I_j(t, 0) = 0, & t > 0, \\ S_j(0, a) = S_{j0}(a), \quad I_j(0, a) = I_{j0}(a), & a \in (0, a^+), \end{cases} \quad (1.4)$$

The variables $S_j(t, a)$ and $I_j(t, a)$ represent the susceptible and infective populations, respectively, in patch j at time t and age a , with maximum age $a^+ \in (0, \infty)$. The parameters Γ_j are the forces of infection to susceptible individuals of age a at time t and γ_j are the age-dependent recovery rates. Moreover, d_S and d_I are positive diffusion rates for the susceptible and infective population respectively. Here we assume them to be equal ($d_S = d_I = d$) and this assumption will allow us to reduce the system into a single equation. S_{j0} and I_{j0} denote the initial data of susceptible and infective populations in patch j , respectively, which are assumed in $L_+^1(0, a^+)$ in general. We assume that $\Gamma_j(t, a)$ are given in the following integral forms:

$$\Gamma_j(t, a) = \sum_{m=1}^n \int_0^{a^+} K_{jm}(a, s) I_m(t, s) ds,$$

where $K_{jm}(a, s)$ denotes the age-dependent diseases transmission rates, which describes that an infective individuals of age s in group m comes into a disease transmitting contact with a susceptible individual of age a in group j .

The following assumptions recovery rates γ_j and transmission rates K_{jm} are made:

- Assumption 1.2** (i) $\gamma_j \in L_+^\infty(0, a^+)$ for all $j = 1, \dots, n$;
(ii) $K_{jm} \in L_+^\infty((0, a^+) \times (0, a^+))$ for all $j, m = 1, \dots, n$.

We mention that such models (1.4) and (1.1) are basically from Kuniya et al. (2018), where they considered the transmission matrix $K := \{K_{jm}\}_{n \times n}$ as a diagonal matrix. Moreover, we put a diffusion rate in front of the connectivity matrix L and we focus on the effects of diffusion rate d on the principal eigenvalue and basic reproduction number via studying their asymptotic behavior for small and large diffusion rates.

Similar multigroup (without movements between groups) SIS epidemic age-structured models were also studied by Feng et al. (2005), where they considered a more general infection force motivated by Busenberg et al. (1991), that is

$$\Gamma_j(t, a) := K_j(a)I_j(t, a) + \sum_{m=1}^n \int_0^{a^+} K_{jm}(a, s)I_m(t, s)ds, \quad (1.5)$$

where $K_j(a)$ is the infection rate for pure intracohort interaction in group j and $K_{jm}(a, s)$ are the transmission rates between groups as same as above.

Let $u = (u_1, \dots, u_n)^T$ and $v = (v_1, \dots, v_n)$ be two vectors. We write $u \geq v$ if $u_i \geq v_i$ for all $i = 1, \dots, n$. We write $u > v$ if $u_i \geq v_i$ for all $i = 1, \dots, n$ and there exists i_0 such that $u_{i_0} > v_{i_0}$. We write $u \gg v$ if $u_i > v_i$ for all $i = 1, \dots, n$. We say that u is *strongly positive* if $u \gg 0_{\mathbb{R}^n}$.

The paper is organized as follows. In Sect. 2 we establish the well-posedness of (1.4), define the principal eigenvalue and basic reproduction number for (1.4). Next in Sect. 3 we study their asymptotic behavior for small and large diffusion rates, which is based on the methodology of Zhang and Zhao (2022). In Sect. 4, we apply our established results to investigate the global behavior of (1.4). Finally, we end up with a discussion in Sect. 5. Note that Assumptions 1.1 and 1.2 are assumed to be held throughout the whole paper, thus we will not repeat them in the following results.

2 Preliminaries

In this section we consider the well-posedness of (1.4). To this aim, we first reduce system (1.4) to a single equation by setting

$$S_j(t, a) = N_j^*(a) - I_j(t, a) \text{ and } \Lambda_j(a) = \gamma_j(a) + \mu_j(a), \quad \forall 1 \leq j \leq n.$$

Now (1.4) is rewritten as follows,

$$\begin{cases} \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial a} \right) I_j(t, a) = d \sum_{m=1}^n L_{jm}(a) I_m(t, a) \\ \quad + \Gamma_j(t, a) (N_j^*(a) - I_j(t, a)) - \Lambda_j(a) I_j(t, a), & t > 0, a \in (0, a^+), \\ I_j(t, 0) = 0, & t > 0, \\ I_j(0, a) = I_{j0}(a), & a \in (0, a^+). \end{cases} \quad (2.1)$$

We emphasize that here we do not normalize Eq. (2.1) as shown in Kuniya et al. (2018), otherwise, the new connectivity matrix L will involve N_j^* and thus is dependent on

a even when the original L is independent on a , and then lost some properties of the constant connectivity matrix, for instance Theorem 3.3.

2.1 Well-posedness

Kuniya et al. (2018) showed well-posedness of (2.1) with normalization and that the semiflow generated by (2.1) is monotone. Here we can use a similar argument as in Kuniya et al. (2018), but define a different Cauchy problem, which will be useful to our study in Sect. 3, to show the wellposedness of (2.1) itself. Let $E := L^1(0, a^+; \mathbb{R}^n)$, $E_+ := L^1_+(0, a^+; \mathbb{R}^n)$ and define the state space

$$M := \{\psi = (\psi_1, \dots, \psi_n)^T \in E_+ : 0 \leq \psi_j(a) \leq N_j^*(a), \text{ a.e. for all } j = 1, \dots, n\}.$$

Denote $L(a) = (L_{ij}(a))_{n \times n}$ and $\Lambda(a) = \text{diag}\{\Lambda_1(a), \dots, \Lambda_n(a)\}$ for $a \in (0, a^+)$ and consider the following problem,

$$\begin{cases} \frac{du(a)}{da} = dL(a)u(a) - \Lambda(a)u(a), & a \in (0, a^+), \\ u(s) = \psi \in \mathbb{R}^n. \end{cases} \quad (2.2)$$

By the constant of variation formula, (2.2) generates an evolution family $\{U_d(a, s)\}_{0 \leq s \leq a \leq a^+} \subset \mathcal{L}(\mathbb{R}^n)$. Denote the growth bound of the evolution family $\{U_d(a, s)\}_{0 \leq s \leq a \leq a^+}$ by $\omega(U_d)$, then we have $\omega(U_d) < 0$ for all $d \geq 0$. In fact, by Lemma 2.1 in Zhang and Zhao (2022), the connectivity matrix $L(\cdot)$ can be split into a block lower triangular matrix via a permutation as follows,

$$L(\cdot) = \begin{pmatrix} L_{11}(\cdot) & \cdots & L_{1\alpha}(\cdot) \\ \vdots & \ddots & \vdots \\ L_{\alpha 1}(\cdot) & \cdots & L_{\alpha\alpha}(\cdot) \end{pmatrix} \quad (2.3)$$

such that $L_{ll}(\cdot)$ is an $n_l \times n_l$ irreducible matrix for $1 \leq l \leq \alpha$ with $\sum_{l=1}^{\alpha} n_l = n$ and satisfies $\omega(L_{ll}) \leq 0$ for any $1 \leq l \leq \alpha$. Now we can write down Λ as the form of L in (2.3) and then consider the evolution family $\{U_d^l(a, s)\}_{0 \leq s \leq a \leq a^+}$ generated by each block $dL_{ll} + \Lambda_l$ for $1 \leq l \leq \alpha$. It follows that $\omega(U_d^l) \leq -\tilde{\mu}$ for each $1 \leq l \leq \alpha$ and thus $\omega(U_d) \leq -\tilde{\mu} < 0$.

We also define a family of bounded linear operators $\{W_\lambda\}_{\lambda > -\tilde{\mu}} \subset \mathcal{L}(E)$ for $g \in E$ by

$$[W_\lambda g](a) := \int_0^a e^{-\lambda(a-s)} U_d(a, s) g(s) ds. \quad (2.4)$$

We will show that this provides a family of positive pseudoresolvents. To this aim, one can make some computations to obtain

$$W_\nu W_\lambda g = \int_0^a e^{-\nu(a-s)} U_d(a, s) \int_0^s e^{-\lambda(s-\tau)} U_d(s, \tau) g(\tau) d\tau ds$$

$$= \int_0^a \int_0^s e^{\lambda\tau - \nu a} e^{-(\lambda - \nu)s} U_d(a, \tau) g(\tau) d\tau ds.$$

Hence for $\nu \neq \lambda$, we have

$$\begin{aligned} W_\nu W_\lambda g &= \frac{1}{\nu - \lambda} \left(e^{-(\lambda - \nu)a} - e^{-(\lambda - \nu)\tau} \right) \int_0^a e^{\lambda\tau - \nu a} U_d(a, \tau) g(\tau) d\tau \\ &= \frac{1}{\nu - \lambda} (W_\lambda - W_\nu) g. \end{aligned}$$

Moreover, one see (for example Magal and Ruan 2018 Lemma 3.8.3) that for all $\lambda > -\tilde{\mu}$,

$$W_\lambda g = 0 \text{ only occurs if } g = 0$$

and

$$\lim_{\lambda \rightarrow \infty} \lambda W_\lambda g = g, \quad \forall g \in E.$$

Moreover, one has

$$\|W_\lambda\|_{\mathcal{L}(E)} \leq \frac{1}{\lambda + \tilde{\mu}}.$$

Thus, by Section 1.9 in Pazy (1983) there exists a unique closed Hille–Yosida operator $B : D(B) \subset E \rightarrow E$ with $\overline{D(B)} = E$ such that

$$(\lambda I - B)^{-1} = W_\lambda \text{ for all } \lambda > -\tilde{\mu}. \quad (2.5)$$

On the other hand define an operator $C : E \rightarrow E$ by,

$$\begin{aligned} C(\psi) &:= N^* \Gamma(\psi) - \Gamma(\psi) \cdot \psi, \text{ with } \Gamma(\psi) := \int_0^{a^+} K(\cdot, s) \psi(s) ds, \quad \psi \in E, \\ \text{and } K(a, s) &= \{K_{jm}(a, s)\}_{n \times n}, \quad N^*(a) = \text{diag}\{N_1^*(a), \dots, N_n^*(a)\}. \end{aligned} \quad (2.6)$$

We mention that N^* denotes the diagonal matrix or a column vector according to the text and

$$\Gamma(\psi) \cdot \psi := (\Gamma_1(\psi) \psi_1, \dots, \Gamma_n(\psi) \psi_n)^T. \quad (2.7)$$

The system (2.1) can then be rewritten as follows,

$$\frac{dI(t)}{dt} = BI(t) + C(I(t)), \quad t > 0, \quad I(0) = I_0 \in E, \quad (2.8)$$

where $I(t) = (I_1(t), \dots, I_n(t))^T$ and $I_0 = (I_{10}, \dots, I_{n0})^T$. Then we have the following result.

Proposition 2.1 *The problem (2.8) has a global classical solution $I(t) \in M$ for any $I_0 \in M \cap D(B)$. Furthermore, (2.8) defines a positive semiflow $\{T(t)\}_{t \geq 0}$ satisfying*

$$T(t)(M) \subset M, \text{ and } T(t)\psi \leq T(t)\phi \text{ for all } \psi, \phi \in M \text{ such that } \psi \leq \phi, \\ \xi T(t)\psi \leq T(t)\xi\psi \text{ for all } \psi \in M \text{ and } \xi \in (0, 1). \quad (2.9)$$

Proof The proof is very similar to Proposition 3.1 in Kuniya et al. (2018), but for the completeness we include it.

Since L is cooperative, we have that $\{U_d(a, s)\}_{0 \leq s \leq a^+}$ satisfies comparison principle, that is,

$$U_d(a, s)\psi(s) \leq U_d(a, s)\phi(s) \leq U_d(a, s)N^*(s) \\ \leq N^*(a), \quad \forall \psi \leq \phi \leq N^*, \text{ with } \psi, \phi \in M. \quad (2.10)$$

Observe that the operator B defined by (2.5) generates the following C_0 -semigroup $\{e^{Bt}\}_{t \geq 0} : E \rightarrow E$

$$[e^{tB}\psi](a) := \begin{cases} U_d(a, a-t)\psi(a-t), & a-t \geq 0, \\ 0, & t-a \geq 0, \end{cases}, \quad \psi \in E. \quad (2.11)$$

Then by (2.10) it is easy to see that $e^{tB}(M) \subset M$ and $e^{tB}\psi \leq e^{tB}\phi$ for all $\psi, \phi \in M$ such that $\psi \leq \phi$. For any positive constant $\frac{1}{\alpha} > \omega(U_d)$, by (2.4) the resolvent $(I - \alpha B)^{-1} : E \rightarrow E$ is given as follows,

$$(I - \alpha B)^{-1}\psi(a) = \frac{1}{\alpha} \int_0^a e^{-\frac{1}{\alpha}(a-s)} U_d(a, s)\psi(s) ds, \quad \psi \in E.$$

Hence for all $\psi, \phi \in M$ such that $\psi \leq \phi$, it holds by (2.10) that

$$(I - \alpha B)^{-1}\psi \leq (I - \alpha B)^{-1}\phi.$$

Let $\frac{1}{\alpha} > \omega(U_d)$ satisfy

$$\alpha < \frac{1}{\max_{1 \leq j \leq n} \left\{ \sum_{k=1}^n K_{jk}^+ \|N_k^*\|_{L^1(0, a^+)} \right\}},$$

where $K_{jk}^+ = \|K_{jk}\|_{L^\infty((0, a^+)^2)}$. Note that such α exists due to $\omega(U_d) < 0$. For any $\psi, \phi \in M$ such that $\psi \leq \phi$ we have

$$(I + \alpha C)\psi = \psi + \alpha N^*\Gamma(\psi) - \alpha \Gamma(\psi) \cdot \psi \\ \leq \psi + \alpha N^*\Gamma(\phi) - \alpha \Gamma(\phi) \cdot \psi \\ = \alpha N^*\Gamma(\phi) + (e - \alpha \Gamma(\phi)) \cdot \psi \\ \leq \alpha N^*\Gamma(\phi) + (e - \alpha \Gamma(\phi)) \cdot \phi,$$

$$= (I + \alpha C)\phi, \text{ with } e = (1, \dots, 1)^T.$$

It implies that $I + \alpha C$ is a monotone operator. For any $\psi \in M$ and $\xi \in (0, 1)$, we have

$$\begin{aligned}\xi(I + \alpha C)\psi &= \xi\psi + \xi\alpha N^*\Gamma(\psi) - \xi\alpha\Gamma(\psi) \cdot \psi \\ &\leq \xi\psi + \alpha N^*\Gamma(\xi\psi) - \alpha\Gamma(\xi\psi) \cdot \xi\psi \\ &= (I + \alpha C)\xi\psi.\end{aligned}$$

It follows that

$$\xi(I + \alpha C)\psi \leq (I + \alpha C)\xi\psi, \quad \forall \psi \in M, \quad \xi \in (0, 1).$$

In addition, for any $\psi \in M$, one has

$$\begin{aligned}0_E &\leq (I - \alpha B)^{-1}0_E \leq (I - \alpha B)^{-1}\psi \leq (I - \alpha B)^{-1}N^* \leq (1 - e^{-\frac{1}{\alpha}a})N^* \leq N^*. \\ 0_E &\leq (I + \alpha C)0_E \leq (I + \alpha C)\psi \leq (I + \alpha C)N^* = N^*.\end{aligned}$$

These inequalities imply that $(I - \alpha B)^{-1}(M) \subset M$ and $(I + \alpha C)(M) \subset M$. Consequently, by Theorem 3.2 in Busenberg et al. (1991) or Lemma 2.1 in Kuniya et al. (2018), we see that the abstract Cauchy problem (2.8) has a global classical solution $I(t) \in M$ for any $I_0 \in M \cap D(B)$, and it defines a semiflow $\{T(t)\}_{t \geq 0}$ satisfying (2.9). This completes the proof. $\square$

Before ending up this section, we mention that here we split the problem (2.1) in a different way, compared with Kuniya et al. (2018), since our setting is more natural to study the asymptotic behavior of principal eigenvalue and basic reproduction number with respect to diffusion rate, which is shown in Sect. 3.2. On the other hand, in our setting, the evolution family has strong monotone property provided L is irreducible, which is crucial to obtain the existence, uniqueness and global stability of the endemic equilibrium via monotone arguments, see Sect. 4.

2.2 Existence and uniqueness of the endemic equilibrium

In this section we investigate the existence and uniqueness of the endemic equilibrium of (2.1), which satisfies the following equation,

$$\begin{cases} \frac{dI_j(a)}{da} = d \sum_{m=1}^n L_{jm}(a)I_m(a) + \Gamma_j(a)(N_j^*(a) - I_j(a)) - \Lambda_j(a)I_j(a), & a \in (0, a^+), \\ I_j(0) = 0, \\ \Gamma_j(a) = \int_0^{a^+} \sum_{m=1}^n K_{jm}(a, s)I_m(s)ds. \end{cases} \quad (2.12)$$

It is easy to see that zero is always an equilibrium of (2.12). Thus in the following, we are interested in the existence, uniqueness and stability of a nontrivial positive

equilibrium. To this aim, by a linearization at the disease-free equilibrium, we can obtain the following linear equation

$$\begin{cases} \frac{dI_j(a)}{da} = d \sum_{m=1}^n L_{jm}(a)I_m(a) + \Gamma_j(a)N_j^*(a) - \Lambda_j(a)I_j(a), & a \in (0, a^+), \\ I_j(0) = 0, \\ \Gamma_j(a) = \int_0^{a^+} \sum_{m=1}^n K_{jm}(a, s)I_m(s)ds. \end{cases} \quad (2.13)$$

Now we study the eigenvalue problem associated with (2.13), that is,

$$\begin{cases} \frac{dI_j(a)}{da} = d \sum_{m=1}^n L_{jm}I_m(a) + \Gamma_j(a)N_j^*(a) - \Lambda_j(a)I_j(a) - \lambda I_j(a), & a \in (0, a^+), \\ I_j(0) = 0, \\ \Gamma_j(a) = \int_0^{a^+} \sum_{m=1}^n K_{jm}(a, s)I_m(s)ds. \end{cases} \quad (2.14)$$

Further, (2.14) can be rewritten as a vector form as follows,

$$\begin{cases} \frac{dI(a)}{da} = dL(a)I(a) + N^*(a)\Gamma(a) - \Lambda(a)I(a) - \lambda I(a), & a \in (0, a^+), \\ I(0) = 0_{\mathbb{R}^n}, \\ \Gamma(a) = \int_0^{a^+} K(a, s)I(s)ds, \end{cases} \quad (2.15)$$

Recalling (2.5) and (2.6), the eigenvalue problem (2.15) is rewritten as

$$(B + C'[0_E])\psi = \lambda\psi, \quad \psi \in D(B).$$

which is equivalent to

$$C'[0_E](\lambda I - B)^{-1}\phi = \phi, \quad \phi \in E, \quad \lambda > s(B).$$

It follows by (2.5) and (2.6) again that

$$\phi(a) = N^*(a) \int_0^{a^+} K(a, s) \int_0^s e^{-\lambda(s-\tau)} U_d(s, \tau) \phi(\tau) d\tau ds. \quad (2.16)$$

By setting $\phi(a) = N^*(a)\varphi(a)$, the above equation is equivalent to

$$\varphi(a) = \int_0^{a^+} K(a, s) \int_0^s e^{-\lambda(s-\tau)} U_d(s, \tau) N^*(\tau) \varphi(\tau) d\tau ds. \quad (2.17)$$

Since E is an abstract L space, one has that $s(B) = \omega(U_d) < 0$. Then by a standard procedure as usual, we can define the basic reproduction number as follows (see Thieme 2009 or Inaba 2012),

$$R_0 := r\left(C'[0_E](-B)^{-1}\right) = r(\mathcal{F}_0), \text{ with } \mathcal{F}_\lambda \in \mathcal{L}(E), \quad (2.18)$$

$$[\mathcal{F}_\lambda \psi](a) := \int_0^{a^+} K(a, s) \int_0^s e^{-\lambda(s-\tau)} U_d(s, \tau) N^*(\tau) \psi(\tau) d\tau ds, \quad \psi \in E, \lambda \in \mathbb{C}. \quad (2.19)$$

Here $r(\mathcal{F}_0)$ denotes the spectral radius of $\mathcal{F}_0$ and $\mathcal{L}(E)$ denotes the space of linear bounded operators from E to E .

In order to study the existence and uniqueness of the endemic equilibrium, we give the following assumptions.

Assumption 2.2 (i) There holds

$$\lim_{h \rightarrow 0} \int_0^{a^+} |K(a+h, s) - K(a, s)| ds = 0,$$

uniformly in $s \in [0, a^+]$;

(ii) Assume that there exists a diagonal operator $\underline{K} \in \mathcal{L}(L_+^\infty(0, a^+; \mathbb{R}^n))$ such that $K_{jj}(a, s) \geq \underline{K}_{jj}(s) > 0$ for all $(a, s) \in (0, a^+) \times (0, a^+)$ and $j = 1, \dots, n$;

Note that Assumption 2.2-(i) often appears in the literature of age-structured models, which is employed to show the compactness of operators. In addition, Assumption 2.2-(ii) is also required in Kuniya et al. (2018) to show the uniqueness of the endemic equilibrium. Here it will be used to first show some strong positive property of operators and then imply the uniqueness.

Next we provide some definitions which will be used later. A positive operator $L \in \mathcal{L}(E)$ is said to be *nonsupporting* if for every pair $\psi \in E_+ \setminus \{0_E\}$ and $f \in E_+^* \setminus \{0_{E^*}\}$, there exists a positive integer $p = p(\psi, f)$ such that

$$\langle f, L^q \psi \rangle := \sum_{i=1}^n \int_0^{a^+} f_i(s) [L^q \psi(s)]_i ds > 0, \text{ for all } q \geq p,$$

where E_+^* denotes the dual space of E_+ . Nonsupporting property is a kind of generalization of strong positivity or irreducibility, which is needed in the strong version of Krein–Rutman theorem. Further, $f \in E_+^*$ is said to be *strictly positive* if $\langle f, \psi \rangle = \sum_{i=1}^n \int_0^{a^+} f_i(s) \psi_i(s) ds > 0$ for all $\psi \in E_+ \setminus \{0_E\}$. The interested readers can refer to Sawashima (1964) and Marek (1970) for more details.

Lemma 2.3 *Let Assumption 2.2 hold, then the operator $\mathcal{F}_\lambda$ is compact for all $\lambda \in \mathbb{C}$. In addition, if $L(a)$ is irreducible a.e. $a \in (0, a^+)$, then $\mathcal{F}_\lambda$ is nonsupporting for all $\lambda \in \mathbb{R}$.*

Proof Compactness follows from Assumption 2.2-(i) due to Kolmogorov–Fréchet–Riesz theorem. Next we show the nonsupporting property. To this aim, define a positive

functional $\mathbb{F}_\lambda$ for $\lambda \in \mathbb{R}$ by

$$\langle \mathbb{F}_\lambda, \psi \rangle := \min_{1 \leq i \leq n} \left(\int_0^{a^+} \underline{K}(s) \int_0^s e^{-\lambda(s-\tau)} U_d(s, \tau) N^*(\tau) \psi(\tau) d\tau ds \right)_i, \quad \forall \psi \in E_+.$$

Since $L(a)$ is irreducible, the evolution family $\{U_d(a, \tau)\}_{0 \leq \tau \leq a \leq a^+}$ is strong monotone (see Chapter 4, Theorem 1.1 in Smith 2008), it follows that

$$\int_0^s U_d(s, \tau) N^*(\tau) \psi(\tau) d\tau \gg 0_{\mathbb{R}^n}, \quad \forall \psi \in E_+ \setminus \{0_E\}, \quad s > 0.$$

Thus combining Assumption 2.2-(ii), one has $\langle \mathbb{F}_\lambda, \psi \rangle > 0$ for all $\psi \in E_+ \setminus \{0_E\}$. It implies that $\mathbb{F}_\lambda$ is a strictly positive functional. Further, we have

$$\mathcal{F}_\lambda \psi \geq \langle \mathbb{F}_\lambda, \psi \rangle e \text{ and } \lim_{\lambda \rightarrow -\infty} \langle \mathbb{F}_\lambda, e \rangle = \infty, \quad (2.20)$$

where $e = (1, \dots, 1)^T$ is a quasi-interior point in $L^1(0, a^+; \mathbb{R}^n)$. In addition, we have

$$\begin{aligned} \mathcal{F}_\lambda^{q+1} \psi &\geq \langle \mathbb{F}_\lambda, \mathcal{F}_\lambda^q \psi \rangle e \geq \langle \mathbb{F}_\lambda, \langle \mathbb{F}_\lambda, \mathcal{F}_\lambda^{q-1} \psi \rangle e \rangle \\ &= \langle \mathbb{F}_\lambda, e \rangle \langle \mathbb{F}_\lambda, \mathcal{F}_\lambda^{q-1} \psi \rangle e \\ &\geq \langle \mathbb{F}_\lambda, e \rangle^q \langle \mathbb{F}_\lambda, \psi \rangle e, \quad q = 1, 2, \dots \end{aligned}$$

It follows that for any $\mathcal{F}_\lambda^* \in E_+^* \setminus \{0_{E^*}\}$ and $\psi \in E_+ \setminus \{0_E\}$, we have

$$\langle \mathcal{F}_\lambda^*, \mathcal{F}_\lambda^q \psi \rangle \geq \langle \mathbb{F}_\lambda, e \rangle^{(q-1)} \langle \mathbb{F}_\lambda, \psi \rangle \langle \mathcal{F}_\lambda^*, e \rangle > 0, \quad q = 1, 2, \dots$$

Hence $\mathcal{F}_\lambda$ is nonsupporting, and the proof is complete. $\square$

Thus it follows from the strong version of Krein–Rutman theorem (see Marek 1970) that $r(\mathcal{F}_\lambda)$ is the principal eigenvalue of $\mathcal{F}_\lambda$ associated with a strongly positive eigenvector in E , denoted by Γ_λ . Due to (2.20), one has that $r(\mathcal{F}_\lambda) \rightarrow \infty$ as $\lambda \rightarrow -\infty$. On the other hand, it is easy to see that $r(\mathcal{F}_\lambda) \rightarrow 0$ as $\lambda \rightarrow \infty$. Therefore, by the continuity and strict monotonicity of $\lambda \rightarrow r(\mathcal{F}_\lambda)$ (see Kang et al. 2022), there exists a unique $\lambda_0 \in \mathbb{R}$ such that $r(\mathcal{F}_{\lambda_0}) = 1$.

In the following, we will use directly a well-known result related to the spectrum properties between $\mathcal{F}_\lambda$ and $B + C'[0_E]$ in age-structured models without providing proofs. The interested readers can refer to Webb (1984), Kuniya et al. (2018), Thieme (2009) or Kang and Ruan (2021) for more details. There holds

$$\{\lambda \in \mathbb{C} : \lambda \in \sigma_P(B + C'[0_E])\} = \{\lambda \in \mathbb{C} : 1 \in \sigma_P(\mathcal{F}_\lambda)\}, \quad (2.21)$$

where $\sigma_P(A)$ denotes the point spectrum of A . Moreover, the multiplicity of eigenvalues are preserved. It follows that λ_0 is the principal eigenvalue of (2.15) corresponding

to a strongly positive eigenvector given as follows,

$$I(a) = \int_0^a e^{-\lambda_0(a-\tau)} U_d(a, \tau) N^*(\tau) \Gamma_0(\tau) d\tau,$$

where Γ_0 is the principal eigenvector of $\mathcal{F}_{\lambda_0}$ corresponding to eigenvalue 1, that is $\mathcal{F}_{\lambda_0}\Gamma_0 = \Gamma_0$. Now we have the following result, which one can refer to Kuniya et al. (2018) or Kang and Ruan (2021) for more details. We leave the proof in Sect. 4.

Proposition 2.4 *Assume $R_0 > 1$, then system (2.1) has a unique strongly positive endemic equilibrium, denoted by I^* , i.e. $I^* \gg 0_E$.*

3 Asymptotic behavior

3.1 An equivalent transformation

In this subsection, let us state an assumption on K to make sure that we can concentrate on the eigenvalue problem (2.15) itself instead of the operator form defined in (2.17), and this is very useful for us to study the asymptotic behavior of principal eigenvalues with respect to d . Moreover, under this assumption, the principal eigenfunction I_d can attain its positive maximum and minimum on $[0, a_2]$ for some $a \in (0, a_2)$.

Assumption 3.1 There exists $a_2 \in (0, a^+)$ such that $K(a, s) = 0$ for $(a, s) \in [a_2, a^+) \times [a_2, a^+)$.

Now let us rewrite the function space as follows,

$$E = L^1(0, a_2; \mathbb{R}^n) \times L^1(a_2, a^+; \mathbb{R}^n)$$

with a function $\psi \in L^1(0, a^+; \mathbb{R}^n)$ identified to $(\psi|_{(0, a_2)}, \psi|_{(a_2, a^+)}) \in L^1(0, a_2; \mathbb{R}^n) \times L^1(a_2, a^+; \mathbb{R}^n)$. Define an operator $\widehat{B}$ in $L^1(0, a_2; \mathbb{R}^n)$ by

$$\widehat{B}\psi = -\frac{d\psi}{da} + dL\psi - \Lambda\psi, \text{ for } \psi \in W^{1,1}(0, a_2; \mathbb{R}^n) \text{ with } \psi(0) = 0_{\mathbb{R}^n}.$$

Note that $\widehat{B}$ is a closed operator under Assumption 3.1, since now $\Lambda_j \in L_+^\infty(0, a_2)$ holds for all $j = 1, \dots, n$ due to Assumption 1.1-(i). Moreover, define a bounded operator $\widehat{C}$ as follows,

$$\widehat{C}\psi = N^*|_{(0, a_2)} \int_0^{a_2} K(\cdot, s)\psi(s)ds - \psi \cdot \int_0^{a_2} K(\cdot, s)\psi(s)ds, \text{ for } \psi \in L^1(0, a_2; \mathbb{R}^n),$$

so that $\widehat{C}$ maps $L^1(0, a_2; \mathbb{R}^n)$ into itself.

Next let us show that $\sigma(B + C'[0_E]) = \sigma(\widehat{B} + \widehat{C}'[0_E])$. To do so, it suffices to show that $\rho(B + C'[0_E]) = \rho(\widehat{B} + \widehat{C}'[0_E])$. Recalling (2.21), it says that

$$\{\lambda \in \mathbb{C} : \lambda \in \sigma_P(B + C'[0_E])\} = \{\lambda \in \mathbb{C} : 1 \in \sigma_P(\mathcal{F}_\lambda)\}.$$

Similarly, applying (2.21) to $a^+ = a_2$ yields that

$$\{\lambda \in \mathbb{C} : \lambda \in \sigma_P(\widehat{B} + \widehat{C}'[0_E])\} = \{\lambda \in \mathbb{C} : 1 \in \sigma_P(\widehat{\mathcal{F}}_\lambda)\},$$

where $\widehat{\mathcal{F}}_\lambda \in \mathcal{L}(L^1(0, a_2; \mathbb{R}^n))$ is defined for $\lambda \in \mathbb{C}$ by

$$\widehat{\mathcal{F}}_\lambda \psi = \int_0^{a_2} K(\cdot, s) \int_0^s e^{-\lambda(s-\tau)} U_d(s, \tau) N^*(\tau) \varphi(\tau) d\tau ds, \quad \forall \psi \in L^1(0, a_2; \mathbb{R}^n).$$

But under Assumption 3.1, we have $\widehat{\mathcal{F}}_\lambda = \mathcal{F}_\lambda$ for all $\lambda \in \mathbb{C}$. It follows that $\sigma(B + C'[0_E]) = \sigma(\widehat{B} + \widehat{C}'[0_E])$ holds, thus we can study the principal spectral theory of $\widehat{B} + \widehat{C}'[0_E]$ instead of $B + C'[0_E]$ in the following, provided Assumption 3.1 is true. Further, in order to not introduce too many notations, we still denote B and C under Assumption 3.1.

Remark 3.2 We mention that Assumption 3.1 is more or less equivalent to (A6) in Kuniya et al. (2018), which makes sure $\Lambda_j \in L_+^\infty(0, a_2)$ for all $j = 1, \dots, n$ due to Assumption 1.1-(i). Finally, we will let Assumption 3.1 hold throughout the following sections, which allows us to study the eigenvalue problem (2.15) defined in $[0, a_2]$.

3.2 The connectivity matrix is independent on a

In this subsection, under the assumption $L(a) \equiv L$, we first establish the asymptotic behavior of λ_0 and R_0 with respect to diffusion rate. In order to include more results, we emphasize that here we do not need L to be irreducible, but this does not affect the existence of λ_0 provided that there exists $\lambda_1 \in \mathbb{R}$ such that $r(\mathcal{F}_{\lambda_1}) > 1$. The argument finding λ_0 with $r(\mathcal{F}_{\lambda_0}) = 1$ remains valid by monotonicity as above, just noting the fact that $\lim_{\lambda \rightarrow -\infty} r(\mathcal{F}_\lambda) = \infty$ induced by nonsupporting property was replaced by the assumption that there exists $\lambda_1 \in \mathbb{R}$ such that $r(\mathcal{F}_{\lambda_1}) > 1$. The only difference is that the corresponding eigenvector associated to λ_0 may not be strongly positive, and this will only have effects on the existence, uniqueness and global stability of the endemic equilibrium, see Sect. 4.

From now on, let λ_0^d be the principal eigenvalue of (2.15) and $I_d = (I_{d,1}, \dots, I_{d,n})$ be the associated positive eigenvector corresponding to λ_0^d for any given $d \geq 0$. Before investigating the asymptotic behavior, we recall some results from Zhang and Zhao (2022).

Theorem 3.3 (Zhang and Zhao 2022, Theorem A) *Let Assumption 1.1-(iv) hold. Let α_0 be the algebraic multiplicity of the zero eigenvalue of L . Then the following statements are valid:*

- (i) *There exists nonnegative matrices $P = (p_{hj})_{\alpha_0 \times n}$ and $Q = (q_{il})_{n \times \alpha_0}$ such that PL and LQ are zero matrices and PQ is an $\alpha_0 \times \alpha_0$ identity matrix.*
- (ii) *If M is an $n \times n$ cooperative matrix, then PMQ is an $\alpha_0 \times \alpha_0$ cooperative matrix.*
- (iii) *Let $\hat{P} = (\hat{p}_{hj})_{\alpha_0 \times n}$ and $\hat{Q} = (\hat{q}_{il})_{n \times \alpha_0}$ be two nonnegative matrices such that $\hat{P}L$ and $L\hat{Q}$ are zero matrices and $\hat{P}\hat{Q}$ is an $\alpha_0 \times \alpha_0$ identity matrix. Then PMQ is similar to $\hat{P}M\hat{Q}$.*

Before stating the main theorem in this paper, we first provide a uniform estimate of λ_0^d independent on d . For convenience, we normalize I_d by $\|I_d\|_{L^1(0,a_2;\mathbb{R}^n)} = 1$.

Lemma 3.4 *Let Assumptions 2.2 and 3.1 hold. Then there exists a constant $C_1 > 0$ independent of d such that $|\lambda_0^d| \leq C_1$.*

Proof Let

$$\begin{aligned}\overline{K} &= (K_{\max})_{n \times n} \text{ with } K_{\max} = \max_{1 \leq i, j \leq n} K_{ij}, \\ \underline{K} &= \text{diag}\{K_{\min}, \dots, K_{\min}\} \text{ with } K_{\min} = \min_{1 \leq j \leq n} K_{jj}, \\ \overline{\Lambda} &= \text{diag}\{\Lambda_{\max}, \dots, \Lambda_{\max}\} \text{ with } \Lambda_{\max} = \max_{1 \leq j \leq n} \Lambda_j, \\ \underline{\Lambda} &= \text{diag}\{\Lambda_{\min}, \dots, \Lambda_{\min}\} \text{ with } \Lambda_{\min} = \min_{1 \leq j \leq n} \Lambda_j, \\ \overline{N}^* &= \text{diag}\{N_{\max}^*, \dots, N_{\max}^*\} \text{ with } N_{\max}^* := \max_{1 \leq j \leq n} N_j^*, \\ \underline{N}^* &= \text{diag}\{N_{\min}^*, \dots, N_{\min}^*\} \text{ with } N_{\min}^* = \min_{1 \leq j \leq n} N_j^*.\end{aligned}$$

Now we consider the following eigenvalue problems of scalar age-structured operators,

$$\begin{cases} \frac{du(a)}{da} = N_{\min}^*(a)v(a) - \Lambda_{\max}\underline{u}(a) - \underline{\lambda}\underline{u}(a), & a \in (0, a_2), \\ \underline{u}(0) = 0, \\ v(a) = \int_0^{a_2} K_{\min}(a, s)\underline{u}(s)ds, \end{cases} \quad (3.1)$$

and

$$\begin{cases} \frac{d\overline{u}(a)}{da} = N_{\max}^*(a)w(a) - \Lambda_{\min}(a)\overline{u}(a) - \overline{\lambda}\overline{u}(a), & a \in (0, a_2), \\ \overline{u}(0) = 0, \\ w(a) = \int_0^{a_2} nK_{\max}(a, s)\overline{u}(s)ds, \end{cases} \quad (3.2)$$

Following the same argument as in Sect. 2.2, we obtain that (3.1) and (3.2) admit two principal eigenvalues $\underline{\lambda}_0$ and $\overline{\lambda}_0$ respectively satisfying

$$\begin{aligned}r(\mathcal{F}_{\underline{\lambda}_0}) &:= r\left(\int_0^{a_2} K_{\min}(\cdot, s) \int_0^s e^{-\underline{\lambda}_0(s-\tau)} e^{-\int_\tau^s \Lambda_{\max}(x)dx} N_{\min}^*(\tau) d\tau ds\right) = 1, \quad \mathcal{F}_{\underline{\lambda}_0} v = v, \\ r(\mathcal{F}_{\overline{\lambda}_0}) &:= r\left(\int_0^{a_2} nK_{\max}(\cdot, s) \int_0^s e^{-\overline{\lambda}_0(s-\tau)} e^{-\int_\tau^s \Lambda_{\min}(x)dx} N_{\max}^*(\tau) d\tau ds\right) = 1, \quad \mathcal{F}_{\overline{\lambda}_0} w = w.\end{aligned}$$

Moreover, v and w are positive in $L^1(0, a_2; \mathbb{R})$. Denote $V = (v, \dots, v)$ and $W = (w, \dots, w)$. Next consider the following auxiliary eigenvalue problems respectively

$$\begin{cases} \frac{dI(a)}{da} = I(a)(dL - \bar{\Lambda}(a) - \lambda) + V(a)\underline{N}^*(a), & a \in (0, a_2), \\ I(0) = 0_{\mathbb{R}^n}^T, \\ V(a) = \int_0^{a_2} I(s)\underline{K}(a, s)ds. \end{cases} \quad (3.3)$$

and

$$\begin{cases} \frac{dI(a)}{da} = I(a)(dL - \underline{\Lambda}(a) - \lambda) + W(a)\bar{N}^*(a), & a \in (0, a_2), \\ I(0) = 0_{\mathbb{R}^n}^T, \\ W(a) = \int_0^{a_2} I(s)\bar{K}(a, s)ds, \end{cases} \quad (3.4)$$

Denote $e = (1, \dots, 1)^T$. Due to (3.1) and (3.2), we see that (3.3) and (3.4) admit the principal eigenvalues $\underline{\lambda}_0$ and $\bar{\lambda}_0$ associated to strongly positive eigenvectors $\underline{u}(a)e^T$ and $\bar{u}(a)e^T$ respectively, which are given as follows

$$\begin{aligned} \underline{u}(a) &= \int_0^a e^{-\underline{\lambda}_0(a-s)} e^{-\int_s^a \Lambda_{\max}(x)dx} N_{\min}^*(s)v(s)ds, \\ \bar{u}(a) &= \int_0^a e^{-\bar{\lambda}_0(a-s)} e^{-\int_s^a \Lambda_{\min}(x)dx} N_{\max}^*(s)w(s)ds. \end{aligned}$$

Finally, by comparing (3.3) and (3.4) with (2.15) and recalling the monotonicity of $\lambda \rightarrow r(\mathcal{F}_\lambda)$, one has $\underline{\lambda}_0 \leq \lambda_0^d \leq \bar{\lambda}_0$. Thus the proof is complete. $\square$

Next we provide the limit of the total population $N_d^*(a)$ as $d \rightarrow \infty$.

Lemma 3.5 *Let Assumptions 1.1 and 3.1 hold, then the limit $\lim_{d \rightarrow \infty} N_d^*(a)$ exists and is denoted by $N_\infty^*(a)$. Moreover, it is satisfied*

$$PN_\infty^*(0) = \int_0^{a_2} P\beta(a)Qe^{-\int_0^a P\mu(s)Qds} da PN_\infty^*(0), \quad (3.5)$$

where P and Q are defined as in Theorem 3.3.

Note that N_∞^* is independent on the choice of P and Q , see the following argument of (3.7). We omit the proof, since it is almost the same with the one of the following Theorem 3.6, when computing the limit of I_d . One can also refer to Lemma 3.9 in Zhang and Zhao (2022).

Now let $\tilde{\Lambda}(a) = P\Lambda(a)Q$ and $\tilde{K}(a, s) = PN_\infty^*(a)K(a, s)Q$ for a.e. $a, s \in (0, a_2)$, where P and Q are defined as in Theorem 3.3. In addition, denote

$$P := (p_1, \dots, p_{\alpha_0})^T \text{ and } Q := (q_1, \dots, q_{\alpha_0}).$$

Next let us provide the main theorem of this paper.

Theorem 3.6 *Let Assumptions 2.2 and 3.1 hold and $L(a) \equiv L$ in $(0, a_2)$. In addition, assume that the operator $\psi \rightarrow \int_0^{a_2} \tilde{K}(\cdot, s)\psi(s)ds$ is nonsupporting, then one has*

$$\lim_{d \rightarrow 0^+} \lambda_0^d = \lambda_0^0, \quad \lim_{d \rightarrow \infty} \lambda_0^d = \tilde{\lambda},$$

where λ_0^0 and $\tilde{\lambda}$ satisfy respectively,

$$\begin{aligned} r \left(\int_0^{a_2} K(\cdot, s) \int_0^s e^{-\lambda_0^0(s-\tau)} e^{-\int_\tau^s \Lambda(x)dx} N^*(\tau) d\tau ds \right) &= 1, \\ r \left(\int_0^{a_2} \tilde{K}(\cdot, s) \int_0^s e^{-\tilde{\lambda}(s-\tau)} e^{-\int_\tau^s \tilde{\Lambda}(x)dx} d\tau ds \right) &= 1. \end{aligned} \quad (3.6)$$

Proof The idea of the proof below was traced back to Zhang and Zhao (2021, 2022). For the completeness and reader's convenience, we include the necessary modifications and provide a detailed proof.

First let us show the limit for $d \rightarrow 0$. Since the solutions of (2.15) depend continuously on parameters, it follow that

$$\begin{aligned} \mathcal{F}_\lambda^d &:= \int_0^{a_2} K(\cdot, s) \int_0^s e^{-\lambda(s-\tau)} U_d(s, \tau) N^*(\tau) d\tau ds \text{ converges to} \\ \mathcal{F}_\lambda^0 &:= \int_0^{a_2} K(\cdot, s) \int_0^s e^{-\lambda(s-\tau)} e^{-\int_\tau^s \Lambda(x)dx} N^*(\tau) d\tau ds, \text{ in the operator norm as } d \rightarrow 0^+. \end{aligned}$$

Moreover, both the operators $\mathcal{F}_\lambda^d$ and $\mathcal{F}_\lambda^0$ are compact and positive, thus their spectral radius are principal eigenvalues. Therefore, the desired limit for $d = 0$ follows from the continuity of spectral radius of compact operators, see Theorem 2.1 in Degla (2008).

Before proceeding, we show that the value $\tilde{\lambda}$ defined in (3.6) is independent of the choice of P and Q . Assume that there are matrices $\hat{P}$ and $\hat{Q}$ satisfying Theorem 3.3-(iii), then by Lemma 2.4 in Zhang and Zhao (2022) there exists an $\alpha_0 \times \alpha_0$ invertible matrix H such that $HP = \hat{P}$ and $QH^{-1} = \hat{Q}$. Thus the operator defined in (3.6) associated with $\hat{P}$ and $\hat{Q}$ becomes

$$\begin{aligned} &\int_0^{a_2} H \tilde{K}(\cdot, s) H^{-1} \int_0^s e^{-\tilde{\lambda}(s-\tau)} e^{-\int_\tau^s H \tilde{\Lambda}(x) H^{-1} dx} d\tau ds \\ &= \int_0^{a_2} H \tilde{K}(\cdot, s) H^{-1} \int_0^s e^{-\tilde{\lambda}(s-\tau)} H e^{-\int_\tau^s \tilde{\Lambda}(x) dx} H^{-1} d\tau ds \\ &= H \int_0^{a_2} \tilde{K}(\cdot, s) \int_0^s e^{-\tilde{\lambda}(s-\tau)} e^{-\int_\tau^s \tilde{\Lambda}(x) dx} H^{-1} d\tau ds. \end{aligned} \quad (3.7)$$

It follows that the spectral radius of the operator defined in (3.6) is invariant upon the choices of P and Q .

Next let us evaluate the limit for $d \rightarrow \infty$. To this aim, define

$$X_1 := \text{span}\{q_l\}_{1 \leq l \leq \alpha_0} \text{ and } X_2 := \{q \in \mathbb{R}^n : p_l^T q = 0, 1 \leq l \leq \alpha_0\}.$$

It then follows that

$$\mathbb{R}^n = X_1 \oplus X_2.$$

Denote the semigroup generated by dL by $S_d(a)$, i.e. $S_d(a) = e^{dLa}$. It is easy to see that $S_d(a)X_1 \subset X_1$ and $S_d(a)X_2 \subset X_2$. According to Theorem 7.3 in Daners and Medina (1992), we have

$$\|S_d(a)\phi\|_{\mathbb{R}^n} \leq C e^{-\omega_0 da} \|\phi\|_{\mathbb{R}^n}, \quad \forall \phi \in X_2,$$

for some $\omega_0 > 0$ and $C > 0$ independent on a and d . For any $d > 0$, we define

$$\tilde{I}_d^l(a) := p_l^T I_d(a), \quad 1 \leq l \leq \alpha_0, \quad \tilde{I}_d(a) = \sum_{l=1}^{\alpha_0} \tilde{I}_d^l(a) q_l \text{ and } \widehat{I}_d(a) := I_d(a) - \tilde{I}_d(a).$$

Multiplying the first equation of (2.15) by p_l^T from left, we obtain

$$\frac{d\tilde{I}_d^l(a)}{da} = p_l^T N_d^*(a) \Gamma_d(a) - p_l^T \Lambda(a) I_d(a) - \lambda_0^d \tilde{I}_d^l(a), \quad 1 \leq l \leq \alpha_0, \quad (3.8)$$

where

$$\Gamma_d(a) = \int_0^{a_2} K(a, s) I_d(s) ds.$$

Then multiplying (3.8) by q_l gives

$$\frac{d\tilde{I}_d^l(a) q_l}{da} = [p_l^T N_d^*(a) \Gamma_d(a)] q_l - [p_l^T \Lambda(a) I_d(a)] q_l - \lambda_0^d \tilde{I}_d^l(a) q_l, \quad 1 \leq l \leq \alpha_0.$$

Adding them together yields

$$\frac{d\tilde{I}_d(a)}{da} = \sum_{l=1}^{\alpha_0} [p_l^T N_d^*(a) \Gamma_d(a)] q_l - \sum_{l=1}^{\alpha_0} [p_l^T \Lambda(a) I_d(a)] q_l - \lambda_0^d \tilde{I}_d(a). \quad (3.9)$$

Subtracting (3.9) from (2.15), we have

$$\frac{d\widehat{I}_d(a)}{da} = dL\widehat{I}_d(a) + N_d^*(a) \Gamma_d(a) - \sum_{l=1}^{\alpha_0} [p_l^T N_d^*(a) \Gamma_d(a)] q_l$$

$$-\Lambda(a)I_d(a) + \sum_{l=1}^{\alpha_0} \left[p_l^T \Lambda(a)I_d(a) \right] q_l - \lambda_0^d \widehat{I}_d(a).$$

Note that for any $1 \leq l \leq \alpha_0$ and a.e. $a \in (0, a_2)$, we have

$$p_l^T L\widehat{I}_d(a) = 0, \quad p_l^T \widehat{I}_d(a) = p_l^T (I_d(a) - \widetilde{I}_d(a)) = \bar{I}_d(a) - \bar{I}_d(a)(p_l^T q_l) = 0,$$

and

$$\begin{aligned} & p_l^T \left(N_d^*(a)\Gamma_d(a) - \sum_{l=1}^{\alpha_0} \left[p_l^T N_d^*(a)\Gamma_d(a) \right] q_l \right) \\ &= p_l^T N_d^*(a)\Gamma_d(a) - p_l^T N_d^*(a)\Gamma_d(a)(p_l^T q_l) = 0, \\ & p_l^T \left(\Lambda(a)I_d(a) - \sum_{l=1}^{\alpha_0} \left[p_l^T \Lambda(a)I_d(a) \right] q_l \right) \\ &= p_l^T \Lambda(a)I_d(a) - p_l^T \Lambda(a)I_d(a)(p_l^T q_l) = 0, \end{aligned}$$

i.e. for a.e. $a \in (0, a_2)$, $L\widehat{I}_d(a) \in X_2$, $\widehat{I}_d(a) \in X_2$, $N_d^*(a)\Gamma_d(a) - \sum_{l=1}^{\alpha_0} [p_l^T N_d^*(a)\Gamma_d(a)]q_l \in X_2$ and $\Lambda(a)I_d - \sum_{l=1}^{\alpha_0} [p_l^T \Lambda(a)I_d]q_l \in X_2$. By Lemma 3.4 and the normalization of I_d , there exist $C_1 > 0$ and $C > 0$ independent of d and a such that $|\lambda_0^d| \leq C_1$ and

$$\begin{aligned} & \left\| \Lambda I_d - \sum_{l=1}^{\alpha_0} \left[p_l^T \Lambda I_d \right] q_l \right\|_{L^1(0, a_2; \mathbb{R}^n)} \leq C \\ \text{and} \quad & \left\| N_d^* \Gamma_d - \sum_{l=1}^{\alpha_0} \left[p_l^T N_d^* \Gamma_d \right] q_l \right\|_{L^1(0, a_2; \mathbb{R}^n)} \leq C. \end{aligned}$$

Via the constant variation formula, we obtain

$$\begin{aligned} \widehat{I}_d(a) = & \int_0^a S_d(a-s) \left\{ N_d^*(s)\Gamma_d(s) - \sum_{l=1}^{\alpha_0} \left[p_l^T N_d^*(s)\Gamma_d(s) \right] q_l \right. \\ & \left. - \Lambda(s)I_d(s) + \sum_{l=1}^{\alpha_0} p_l^T \Lambda(s)I_d(s)q_l - \lambda_0^d \widehat{I}_d(s) \right\} ds. \end{aligned}$$

By the direct computation and Lemma 3.4 ($|\lambda_0^d| \leq C_1$), we obtain

$$\|\widehat{I}_d\|_{L^1(0, a_2; \mathbb{R}^n)} \leq C(\omega_0 d)^{-1} + C_1(\omega_0 d)^{-1} \|\widehat{I}_d\|_{L^1(0, a_2; \mathbb{R}^n)}.$$

Now letting $d \rightarrow \infty$ yields $\|\widehat{I}_d\|_{L^1(0,a_2;\mathbb{R}^n)} \rightarrow 0$. On the other hand, due to $I_d \in W^{1,1}(0, a_2; \mathbb{R}^n) \subset C([0, a_2]; \mathbb{R}^n)$, we have

$$\max_{a \in [0, a_2]} \widehat{I}_d(a) \rightarrow 0, \text{ as } d \rightarrow \infty. \quad (3.10)$$

Finally, we show the desired limits. To this aim, revisit (3.8) given as follows,

$$\frac{d\bar{I}_d^l(a)}{da} = p_l^T N_d^*(a) \Gamma_d(a) - p_l^T \Lambda(a) I_d(a) - \lambda_0^d \bar{I}_d^l(a), \quad 1 \leq l \leq \alpha_0. \quad (3.11)$$

Observing (3.11), one has $\|\frac{d\bar{I}_d^l}{da}\|_{L^1(0,a_2)} \leq C$ for $1 \leq l \leq \alpha_0$, where $C > 0$ denotes some constant that may vary from line to line but is independent of d . It follows that $\bar{I}_d^l$ for $1 \leq l \leq \alpha_0$ is bounded in $W^{1,1}(0, a_2)$ which is continuously embedded into $L^\infty(0, a_2)$ so that $\|\bar{I}_d^l\|_{L^\infty(0,a_2)} \leq C$. Again by (3.11), there holds

$$\left\| \frac{d\bar{I}_d^l}{da} \right\|_{L^\infty(0,a_2)} \leq C, \quad 1 \leq l \leq \alpha_0.$$

By the Sobolev compact embedding ($W^{1,\infty}(0, a_2) \hookrightarrow C([0, a_2])$), it follows that there exists a sequence $d_m \rightarrow \infty$ such that $\lambda_0^{d_m} \rightarrow \lambda_0^\infty$ and $\bar{I}_{d_m}^l \rightarrow \bar{I}_\infty^l$ uniformly in $a \in [0, a_2]$, $1 \leq l \leq \alpha_0$ as $m \rightarrow +\infty$, for some λ_∞ and $\bar{I}_\infty^l \in C([0, a_2])$, $1 \leq l \leq \alpha_0$. We integrate (3.11) from 0 to a to obtain

$$\bar{I}_d^l(a) = \int_0^a \left[p_l^T N_d^*(s) \Gamma_d(s) - p_l^T \Lambda(s) \tilde{I}_d(s) - p_l^T \Lambda(s) \widehat{I}_d(s) - \lambda_0^d \bar{I}_d^l(s) \right] ds.$$

By (3.10), letting $d_m \rightarrow \infty$, for any $1 \leq l \leq \alpha_0$, we have

$$\bar{I}_\infty^l(a) = \int_0^a \left[p_l^T N_\infty^*(s) \Gamma_\infty(s) - p_l^T \Lambda(s) \left(\sum_{h=1}^{\alpha_0} \bar{I}_\infty^h(s) q_h \right) - \lambda_0^\infty \bar{I}_\infty^l(s) \right] ds,$$

where

$$\Gamma_\infty(a) = \sum_{h=1}^{\alpha_0} \int_0^{a_2} K(a, s) \bar{I}_\infty^h(s) q_h ds.$$

Hence one has

$$\frac{d\bar{I}_\infty^l(a)}{da} = p_l^T N_\infty^*(a) \Gamma_\infty(a) - \sum_{h=1}^{\alpha_0} \left(p_l^T \Lambda(a) q_h \right) \bar{I}_\infty^h(a) - \lambda_0^\infty \bar{I}_\infty^l(a), \quad 1 \leq l \leq \alpha_0.$$

Applying the constant variation formula again to above equation, we get

$$\bar{I}_\infty(a) = \int_0^a e^{-\int_s^a (\lambda_0^\infty + \tilde{\Lambda}(x))dx} P N_\infty^*(s) \Gamma_\infty(s) ds. \quad (3.12)$$

Next multiplying the third equation of (2.15) by $p_l^T N^*(a)$, one has

$$\begin{aligned} p_l^T N_d^*(a) \Gamma_d(a) &= \int_0^{a_2} p_l^T N_d^*(a) K(a, s) I_d(s) ds \\ &= \int_0^{a_2} p_l^T N_d^*(a) K(a, s) (\widehat{I}_d(s) + \tilde{I}_d(s)) ds. \end{aligned}$$

Letting $d_m \rightarrow \infty$ for all $1 \leq l \leq \alpha_0$ in the above equality, we get

$$\begin{aligned} p_l^T N_\infty^*(a) \Gamma_\infty(a) &= \int_0^{a_2} p_l^T N_\infty^*(a) K(a, s) \left(\sum_{h=1}^{\alpha_0} \bar{I}_\infty^h(s) q_h \right) ds \\ &= \int_0^{a_2} \sum_{h=1}^{\alpha_0} \left(p_l^T N_\infty^*(a) K(a, s) q_h \right) \bar{I}_\infty^h(s) ds. \end{aligned} \quad (3.13)$$

Now plugging (3.12) into (3.13), one obtains

$$P N_\infty^*(a) \Gamma_\infty(a) = \int_0^{a_2} \tilde{K}(a, s) \int_0^s e^{-\int_\tau^s (\lambda_0^\infty + \tilde{\Lambda}(x))dx} P N_\infty^*(\tau) \Gamma_\infty(\tau) d\tau ds. \quad (3.14)$$

Finally, we consider an operator given as follows,

$$[\tilde{\mathcal{F}}_{\lambda_0^\infty} \psi](a) := \int_0^{a_2} \tilde{K}(a, s) \int_0^s e^{-\int_\tau^s (\lambda_0^\infty + \tilde{\Lambda}(x))dx} \psi(\tau) d\tau ds, \quad \psi \in L^1(0, a_2; \mathbb{R}^{\alpha_0}).$$

Observe that (3.14) can be rewritten as follows by setting $\psi_\infty(a) = P N_\infty^*(a) \Gamma_\infty(a)$,

$$\psi_\infty(a) = \int_0^{a_2} \tilde{K}(a, s) \int_0^s e^{-\int_\tau^s (\lambda_0^\infty + \tilde{\Lambda}(x))dx} \psi_\infty(\tau) d\tau ds = [\tilde{\mathcal{F}}_{\lambda_0^\infty} \psi_\infty](a).$$

It follows from the nonsupporting property, compactness of the operator $\tilde{\mathcal{F}}_{\lambda_0^\infty} \in \mathcal{L}(L^1(0, a_2; \mathbb{R}^{\alpha_0}))$ and Krein-Rutman theorem that $\tilde{\lambda} = \lambda_0^\infty$. This leads to the desired result. $\square$

Remark 3.7 We mention that the assumption appeared in Theorem 3.6 that the operator $\psi \rightarrow \int_0^{a_2} \tilde{K}(\cdot, s) \psi(s) ds$ is nonsupporting can be achieved by assuming $K(a, s)$ being a positive matrix a.e. $(a, s) \in (0, a_2)^2$ (i.e. $K_{jm}(a, s) > 0$ for all $j, m = 1, \dots, n$). In fact, now the matrix $\tilde{K}(a, s) = P N_\infty^*(a) K(a, s) Q$ is positive a.e. $(a, s) \in (0, a_2)^2$.

Finally, we provide two corollaries when L is irreducible with and without symmetry respectively.

Corollary 3.8 *Let Assumptions 2.2 and 3.1 hold. In addition, assume that $L(a) \equiv L$ in $(0, a_2)$ is irreducible, then one has*

$$\lim_{d \rightarrow \infty} \lambda_0^d = \tilde{\lambda}^*,$$

where $\tilde{\lambda}^*$ satisfies,

$$r \left(\int_0^{a_2} \sum_{j,m=1}^n q_m N_{\infty,j}^*(\cdot) K_{jm}(\cdot, s) \int_0^s e^{-\tilde{\lambda}^*(s-\tau)} e^{-\int_\tau^s \sum_{m=1}^n q_m \Lambda_m(x) dx} d\tau ds \right) = 1.$$

Here $q = (q_i)_{n \times 1}$ is a right and positive eigenvector of L corresponding to the eigenvalue 0 such that $\sum_{i=1}^n q_i = 1$.

Proof When L is irreducible, there always exist a left eigenvector $e^T = (1, \dots, 1)$ and a right eigenvector of L , denoted by $q = (q_i)_{n \times 1}$, respectively, corresponding to eigenvalue 0 such that $\sum_{i=1}^n q_i = 1$ with all $q_i > 0$.

Thus now $\alpha_0 = 1$ and the matrices $P = e^T$ and $Q = (q_i)_{n \times 1}$. Moreover, one has

$$\tilde{K}(a, s) = \sum_{j,m=1}^n q_m N_{\infty,j}^*(a) K_{jm}(a, s) \text{ and } \tilde{\Lambda}(a) = \sum_{m=1}^n q_m \Lambda_m(a).$$

Then let us replace P by e^T and Q by $(q_i)_{n \times 1}$ appeared in the proof of Theorem 3.6, and all the computations in the proof of Theorem 3.6 remain valid. Hence, we can obtain the desired result. $\square$

3.3 The connectivity matrix is dependent on a

In this subsection we provide the second corollary where we relax L to be dependent on a , but we need to require $L(a)$ to be irreducible and symmetric for a.e. $a \in (0, a_2)$.

Corollary 3.9 *Let Assumptions 2.2 and 3.1 hold. In addition, assume that L is dependent on a , irreducible and symmetric, then one has*

$$\lim_{d \rightarrow \infty} \lambda_0^d = \tilde{\lambda}^{**},$$

where $\tilde{\lambda}^{**}$ satisfies,

$$r \left(\int_0^{a_2} \sum_{j,m=1}^n N_{\infty,j}^*(\cdot) K_{jm}(\cdot, s) \int_0^s e^{-\tilde{\lambda}^{**}(s-\tau)} e^{-\int_\tau^s \sum_{m=1}^n \Lambda_m(x) dx} d\tau ds \right) = 1.$$

Note that when $L = L(a)$ is irreducible and symmetric, e^T and e are always the left and right eigenvectors of L respectively, corresponding to eigenvalue 0. Thus replacing $P = e^T$ and $Q = e$ will yield the desired conclusion, we omit the details.

Remark 3.10 We would like to mention that for $L = L(a)$ being only irreducible, we can not obtain the asymptotic behavior as shown in Corollary 3.8, since now the right eigenvector q is also dependent on a , then uniform estimate in Lemma 3.4 and the computations (containing ∂_a) in Theorem 3.6 are not valid any more. But if in addition, $L = L(a)$ is also symmetric, then e can be chosen as the right eigenvector, which is independent on a .

Finally, as a direct corollary of Theorem 3.6 and Corollaries 3.8 and 3.9, we have the asymptotic behavior of basic reproduction number R_0^d with respect to d as follows.

Theorem 3.11 *Let Assumptions 2.2, 3.1 hold and $L(a) \equiv L$ in $(0, a_2)$. In addition, assume that the operator $\psi \rightarrow \int_0^{a_2} \tilde{K}(\cdot, s) \psi(s) ds$ is nonsupporting, then we have*

$$\begin{aligned} \lim_{d \rightarrow 0^+} R_0^d &= r \left(\int_0^{a_2} K(\cdot, s) \int_0^s e^{-\int_\tau^s \Lambda(x) dx} N^*(\tau) d\tau ds \right), \\ \lim_{d \rightarrow \infty} R_0^d &= r \left(\int_0^{a_2} \tilde{K}(\cdot, s) \int_0^s e^{-\int_\tau^s \tilde{\Lambda}(x) dx} d\tau ds \right). \end{aligned}$$

In addition, if $L(a) \equiv L$ in $(0, a_2)$ is irreducible, then

$$\lim_{d \rightarrow \infty} R_0^d = r \left(\int_0^{a_2} \sum_{j,m=1}^n q_m N_{\infty,j}^*(\cdot) K_{jm}(\cdot, s) \int_0^s e^{-\int_\tau^s \sum_{m=1}^n q_m \Lambda_m(x) dx} d\tau ds \right),$$

where $q = (q_i)_{n \times 1}$ is a right and positive eigenvector of L corresponding to the eigenvalue 0 such that $\sum_{i=1}^n q_i = 1$. Finally, if $L = L(a)$, $a \in (0, a_2)$ is irreducible and symmetric, then

$$\lim_{d \rightarrow \infty} R_0^d = r \left(\int_0^{a_2} \sum_{j,m=1}^n N_{\infty,j}^*(\cdot) K_{jm}(\cdot, s) \int_0^s e^{-\int_\tau^s \sum_{m=1}^n \Lambda_m(x) dx} d\tau ds \right).$$

4 Global behavior

In this section we study the global behavior of (2.1) via the sign of $R_0 - 1$. We mention that the global behavior of (2.1) was already studied by Kuniya et al. (2018) by the sign of their basic reproduction number $\tilde{R}_0 - 1$. Here if we can prove that our $R_0 - 1$ has the same sign with theirs, then the global behavior will be clear via our R_0 .

4.1 Equivalent basic reproduction number

Let us recall that in Kuniya et al. (2018), they split the Eq. (2.1) as follows,

$$A\psi := -\frac{d\psi}{da}, \text{ with } D(A) := \{\psi \in W^{1,1}(0, a^+; \mathbb{R}^n), \psi(0) = 0_{\mathbb{R}^n}\},$$

$$F(\psi) := N^*\Gamma(\psi) - \Gamma(\psi) \cdot \psi - \Lambda\psi + dL\psi, \quad \psi \in E. \quad (4.1)$$

Here we mention that in Kuniya et al. (2018), they define the connectivity matrix L involving the death rate μ , and then assumed that L after normalization satisfies $L_{ij} \in L_+^\infty(0, a^+)$ for all $i \neq j$. Hence the operator F above is well defined. While for our case, since μ only satisfies Assumption 1.1-(i), we can not define an operator directly involving μ as follows,

$$B_1\psi := -\frac{d\psi}{da} + dL\psi - (\mu + \gamma)\psi, \\ \text{with } D(B_1) := \{\psi \in W^{1,1}(0, a^+; \mathbb{R}^n), \psi(0) = 0_{\mathbb{R}^n}\}. \quad (4.2)$$

Note that here B_1 is not well defined, since $\mu\psi$ is not well defined. This is why we start from pseudoresolvent operators to define an implicit operator B shown in (2.5). Such methodology is often used in age-structured models to deal with the non-densely operators, for instance, see Thieme (2009), Magal and Ruan (2018). However, under Assumption 3.1, (4.1) and (4.2) are equivalent.

Next let us recall that the basic reproduction number $\tilde{R}_0$ in Kuniya et al. (2018) is given by,

$$\tilde{R}_0 := r\left(F'[0_E](-A)^{-1}\right).$$

While in our case, recalling (2.18), the basic reproduction number R_0 is given as follows,

$$R_0 = r\left(C'[0_E](-B)^{-1}\right).$$

On the one hand, it is shown in Kuniya et al. (2018) Lemma 2.3 that,

$$\text{sign}(\tilde{R}_0 - 1) = \text{sign}(\omega(A + F'[0_E])), \quad (4.3)$$

where $\omega(A + F'[0_E])$ denotes the growth bound of $A + F'[0_E]$. Here applying Lemma 2.3 in Kuniya et al. (2018) to our situation (note that $s(B) < 0$), the following sign relation holds,

$$\text{sign}(R_0 - 1) = \text{sign}(\omega(B + C'[0_E])). \quad (4.4)$$

On the other hand, observing (2.15) we have $A + F'[0_E] = B + C'[0_E]$. Thus combining (4.3) and (4.4), we have

$$\text{sign}(R_0 - 1) = \text{sign}(\tilde{R}_0 - 1).$$

It follows that our index R_0 can be used to obtain the following global behavior of (2.1) via combining the results in Theorem 3.1 of Kuniya et al. (2018) with some modifications in next section.

Theorem 4.1 *Let Assumptions R_0 be defined by (2.18) and assume that $L(a)$ is irreducible in $(0, a^+)$.*

- (i) *If $R_0 < 1$, then the disease-free equilibrium 0_E of system (2.1) is globally attractive;*
- (ii) *If $R_0 > 1$, then system (2.1) has a unique endemic equilibrium $I^* \gg 0_E$ and it is globally attractive.*

Remark 4.2 We would like to mention that here we established an equivalent relation with the existing one in the literature Kuniya et al. (2018). However, one can follow the monotone dynamical system approach as same as in Kuniya et al. (2018), for instance, to define a new map

$$\Phi := (I - \alpha B)^{-1}(I + \alpha C), \text{ with } \frac{1}{\alpha} > s(B), \quad (4.5)$$

and, under the assumption that $L(a)$ is irreducible, then employ its monotonicity (Proposition 2.1) to investigate the global behavior of (2.1) (including existence, uniqueness i.e. Proposition 2.4 and global stability of endemic equilibrium), as did in Kuniya et al. (2018) or Kang and Ruan (2021). We will explain the assumption that $L(a)$ is irreducible, which is more required compared with Kuniya et al. (2018), in next section.

4.2 A comment to Kuniya et al. (2018)

In this subsection, we comment that there is a mistake in Kuniya et al. (2018), where they proved the uniqueness and global stability of the endemic equilibrium. In inequality (3.17) of Kuniya et al. (2018), the authors said that for all $\varphi \in E_+ \setminus \{\mathbf{0}\}$:

$$\lambda_j[a|\varphi] \geq \int_0^{a+} \kappa_j^-(\sigma) P_j^*(\sigma) \varphi_j(\sigma) d\sigma =: \lambda_j^-(\varphi) > 0, \quad a \in (0, a_+), \quad j \in \mathcal{N}. \quad (4.6)$$

Actually, this inequality is not true for any $\varphi \in E_+ \setminus \{\mathbf{0}\}$, for example, taking

$$\varphi = (0, \dots, 1, \dots, 0)^T, \text{ where } 1 \text{ only appears in the } j - \text{th component,}$$

then only $\lambda_j[a|\varphi] > 0$ and the remaining ones $\lambda_i[a|\varphi] = 0, i \neq j$. Thus the quantity $\underline{\lambda}(v) := \min_{j \in \mathcal{N}} \lambda_j^-(\varphi)$ could be zero and so does $c(v)$.

Fortunately, we can modify the inequality (4.6) by requiring $\varphi \in E_+$ with φ being strongly positive. Then it follows that $\lambda_j[a|\varphi] > 0$ for all $j \in \mathcal{N}$, and the proofs of Lemma 3.1 in Kuniya et al. (2018) remain valid. But in order to allow $\lambda_j^-(j^*) > 0, \forall j = 1, \dots, n$ hold in Proposition 3.3 of Kuniya et al. (2018), the endemic equilibrium j^* is required to be strongly positive. Thus to our best knowledge, it seems that the irreducibility of L is needed to obtain strong positivity and thus the uniqueness of endemic equilibrium.

For the global stability of endemic equilibrium, a kind of persistence result was shown in Lemma 3.2 of Kuniya et al. (2018) where (4.6) was used again. It seems not

correct because now $\lambda_j^-(i)$ may not be positive for all $j = 1, \dots, n$. In order to fix it, to our best knowledge, the irreducibility of L is needed again, to guarantee the strong monotonicity of the semigroup $\{T(t)\}_{t \geq 0}$. Moreover, it is reasonable since the kind of persistence result stated in Lemma 3.2 of Kuniya et al. (2018) could be understood as strong positivity, that is, $U(t_0)I_0 \gg 0_{\mathbb{R}^n}$ for some $t_0 > 0$.

4.3 Proof of Theorem 4.1

We now provide a new proof based on the comparison principle to prove Theorem 4.1 via assuming that $L(a)$ is irreducible a.e. in $(0, a^+)$ and that Assumption 3.1 holds. We mention that the following proofs essentially use the monotonicity of the system (2.1), which is the same idea with Kuniya et al. (2018). The only difference between them is that our arguments are more based on the equation itself, while Kuniya et al. (2018) used functional analysis language via the mapping Φ defined in (4.5).

Recall that if $L(a)$ is irreducible, then by Lemma 2.3 the principal eigenvector φ_0 associated to λ_0 is strongly positive. We revisit (2.12) as follows,

$$\begin{cases} \frac{dI(a)}{da} = dL(a)I(a) + \int_0^{a_2} K(a, s)I(s)ds \cdot (N^*(a) - I(a)) - \Lambda(a)I(a), & a \in (0, a_2), \\ I(0) = 0. \end{cases} \quad (4.7)$$

For (1), $R_0 < 1$ is equivalent to $\omega(B + C'[0_E]) < 0$ by (4.4), which implies that 0_E is globally attractive. For (2), $R_0 > 1$ implies $\lambda_0 > 0$. Observe that comparison principle holds in (4.7). Let $\underline{I}(a) = \epsilon \varphi_0(a)$ with $\epsilon > 0$ small enough and plug it into (4.7), then we get

$$\begin{aligned} \frac{d\underline{I}(a)}{da} - dL(a)\underline{I}(a) - \int_0^{a_2} K(a, s)\underline{I}(s)ds \cdot (N^*(a) - \underline{I}(a)) + \Lambda(a)\underline{I}(a) \\ = \left(\epsilon \int_0^{a_2} K(a, s)\varphi_0(s)ds - \lambda_0 \right) \epsilon \varphi_0 \ll 0_{\mathbb{R}^n}. \end{aligned} \quad (4.8)$$

It follows that $\underline{I}(a)$ is a subsolution of (4.7). On the other hand, $N^*(a)$ is always a supersolution of (4.7). Then by the monotone iteration, there exists at least one equilibrium $I^* \gg 0_E$ satisfying (4.7).

Next we show the uniqueness by a sliding argument. Assume that there exist two strongly positive equilibria u and v . By the similar estimates as in Proposition 3.3 of Kuniya et al. (2018), we can find some $\alpha \in (0, 1)$ such that $\alpha u \leq v$ in $(0, a_2)$. Let $\alpha_* = \sup\{\alpha > 0 : \alpha u \leq v \text{ in } (0, a_2)\}$. By the assumption on u and v , the number α_* is well defined and positive. Claim that $u \leq v$ holds. If $\alpha_* \geq 1$, then we are done. So we assume $\alpha_* < 1$.

Set $w = v - \alpha_* u$, then $w \geq 0_{\mathbb{R}^n}$ and by the definition of α_* , we can find

$$a_0 := \min\{a \in [0, a_2] : \exists i \in \{1, \dots, n\} \text{ such that } w_i(a_0) = 0\} \text{ and } a_0 \neq 0.$$

It means that a_0 is the leftmost nonzero point in $[0, a_2]$ such that there exists $i \in \{1, \dots, n\}$ satisfying $w_i(a_0) = 0$. We mention that if there is an interval $[0, a_1]$ with

$a_1 < a_2$ such that for some $i \in \{1, \dots, n\}$ satisfying $w_i(s) \equiv 0$ in $[0, a_1]$. Then such a_0 can be chosen to be any number in $(0, a_1)$.

Due to the fact that $N^* \in L_+^\infty(0, a^+)$ and $K_{ij} \in L_+^\infty(0, a^+)$ for all $i, j = 1, \dots, n$, one can define the Lipschitz constant of the function $\mathcal{G}(u) := \int_0^{a_2} K(\cdot, s)u(s)ds(N^* - u)$ by V , that is

$$\left| \int_0^{a_2} K(a, s)u(s)ds(N^*(a) - u(a)) - \int_0^{a_2} K(a, s)v(s)ds(N^*(a) - v(a)) \right| \leq V|u(a) - v(a)|, \text{ a.e. in } (0, a_2), \forall u, v \in M. \quad (4.9)$$

Next consider w which satisfies

$$\begin{aligned} \frac{dw(a)}{da} &= dL(a)w(a) - \Lambda(a)w(a) - \tilde{M}w(a) + [\mathcal{G}(v)](a) \\ &\quad + \tilde{M}v(a) - [\alpha_* \mathcal{G}(u)](a) - \alpha_* \tilde{M}u(a) \\ &\gg dL(a)w(a) - \Lambda(a)w(a) - \tilde{M}w(a) + [\mathcal{G}(v)](a) \\ &\quad + \tilde{M}v(a) - [\mathcal{G}(\alpha_* u)](a) - \tilde{M}\alpha_* u(a). \end{aligned}$$

where $\tilde{M} > V$ and we used $\alpha_* < 1$ in the second inequality. Applying constant variation formula to the above equation, we obtain

$$w(a) \gg \int_0^a U_d(a, s)e^{-\tilde{M}(a-s)} \left[[\mathcal{G}(v)](s) + \tilde{M}v(s) - [\mathcal{G}(\alpha_* u)](s) - \tilde{M}\alpha_* u(s) \right] ds.$$

Due to (4.9), we see the the right hand side of the above inequality is nonnegative for $a \in (0, a_0]$. But considering the above inequality at a_0 , we immediately get a contradiction because $w_i(a_0) = 0$. Thus we have $u \leq v$. Now switch u and v in the above argument, we also have $v \leq u$, which shows the uniqueness of the endemic equilibrium.

Finally, we show the global stability. By integrating the first equation of (2.1) along the characteristic line $t - a = c$, we have

$$I(t, a) = \begin{cases} U_d(a, a-t)I_0(a-t) + \int_0^t U_d(a, a-t+s)C(I(s, \cdot))(a-t+s)ds, & a \geq t, \\ \int_{t-a}^t U_d(a, a-t+s)C(I(s, \cdot))(a-t+s)ds, & a < t. \end{cases} \quad (4.10)$$

Observe that $I(t, a) \gg 0_{\mathbb{R}^n}$ when $t \leq a$, since $I_0 \geq 0_E$, $I_0 \neq 0_E$. We now focus on $t > a$, by contradiction, assume that there exists $0 < a_0 < t_0$ such that $I(t_0, a_0)$ has some components being zero. By the strong monotonicity of $\{U_d(a, s)\}_{0 \leq s \leq a \leq a_2}$, there must hold $C(I(s, \cdot)) = 0_{\mathbb{R}^n}$ for $s \in [t_0 - a_0, t_0]$. It implies that

$$C(I(t_0 - a_0, \cdot)) = \Gamma(I(t_0 - a_0, \cdot))(N^* - I(t_0 - a_0, \cdot)) = 0_{\mathbb{R}^n}.$$

Note that $I \neq N^*$, if so, by (2.1) N^* will satisfy

$$\frac{\partial N^*}{\partial a} = \frac{\partial I}{\partial a} = dLI - \Lambda I = dLN^* - \Lambda N^*,$$

which contradicts (1.1). Hence there must hold

$$\Gamma(I(t_0 - a_0, \cdot)) = \int_0^{a_2} K(\cdot, \tau) I(t_0 - a_0, \tau) d\tau = 0_{\mathbb{R}^n}.$$

By Assumption 2.2-(ii), we have $I(t_0 - a_0, \tau) \equiv 0_{\mathbb{R}^n}$ all for $\tau \in [0, a_2]$.

Now replace (t_0, a_0) by $(t_0 - a_0, a_0)$ in (4.10). If $t_0 - a_0 \leq a_0$, we have $I(t_0 - a_0, a_0) \gg 0_{\mathbb{R}^n}$ which is a contradiction. Thus we have $t_0 - a_0 > a_0$. Then by the same argument as above, we have $I(t_0 - 2a_0, a_0) = 0_{\mathbb{R}^n}$. Next repeating the above process by induction, there exists an integer $O > 0$ such that $t_0 - Oa_0 \leq a_0$, which implies $I(t_0 - Oa_0, a_0) = 0_{\mathbb{R}^n}$. But this is a contradiction with the case $a \geq t$ in (4.10).

Hence $I(t, a) \gg 0_{\mathbb{R}^n}$ for any $t > 0$ if $I_0 \in E_+ \setminus \{0_E\}$. Then the persistence result shown in Lemma 3.2 of Kuniya et al. (2018) remains true. Alternatively, one can also use the monotone method to prove directly the global stability of the endemic equilibrium, see Theorem 4.2 of Feng et al. (2005).

5 Discussion

In this paper we studied the effects of diffusion rate on the principal eigenvalue and basic reproduction number of an age-structured SIS epidemic patch model. We obtained the asymptotic behavior of principal eigenvalue and basic reproduction number for small and large diffusion rates.

We mention that here in order to study the global behavior of this age-structured SIS epidemic patch model, we assumed the diffusion rates to be equal for susceptible and infective population. Actually, even without age structure, the global behavior of the patch SIS epidemic ODE models, in particular, the global stability of endemic equilibrium is quite complicated and subtle, see Li and Peng (2019), Chen et al. (2020). We expect to study the case $d_S \neq d_I$ for our models with age structure involved in the future.

In addition, motivated by the studies of time-periodic patch models or reaction diffusion equations by Zhang and Zhao (2022) and Liu and Lou (2022), we do not expect the monotonicity of basic reproduction number with respect to diffusion rate in our patch age-structured models. In fact, when we look at the eigenvalue problem associated to (1.1), it is equivalent to study the following problem,

$$\begin{cases} \frac{dN_j(a)}{da} = d \sum_{m=1}^n L_{jm} N_m(a) - \mu_j(a) N_j(a) - \lambda N_j(a), & a \in (0, a^+), \\ N_j(0) = \int_0^{a^+} \beta_j(a) N_j(a) da. \end{cases} \quad (5.1)$$

Note that when the birth rate and death rate matrix in (5.1) are chosen as follows, respectively,

$$\beta_j(a) = \delta_{a^+}(a), \text{ and } \mu_j(a) = \mu_j(a + a^+), \quad \forall 1 \leq j \leq n$$

where δ_{a^+} is the delta function which is equal to 1 only at a^+ and zero otherwise, then (5.1) becomes a time-periodic patch eigenvalue problem with the period being a^+ given as follow,

$$\begin{cases} \frac{dN_j(a)}{da} = d \sum_{m=1}^n L_{jm} N_m(a) - \mu_j(a) N_j(a) - \lambda N_j(a), & a \in (0, a^+), \\ N_j(0) = N_j(a^+). \end{cases} \quad (5.2)$$

However, it is discovered in Zhang and Zhao (2022) or in Liu and Lou (2022) where L is replaced by the Laplace diffusion operator that the basic reproduction number is not monotone with respect to diffusion rate.

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