INSTABILITY OF THE STANDING WAVES FOR THE NONLINEAR KLEIN-GORDON EQUATIONS IN ONE DIMENSION

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Abstract. In this paper, we consider the nonlinear Klein-Gordon equation

$$\partial_{tt}u - \Delta u + u = |u|^{p-1}u, \qquad t \in \mathbb{R},\ x \in \mathbb{R}^d,$$

with $1 < p < 1 + \frac{4}{d}$. The equation has the standing wave solutions $u_\omega = e^{i\omega t}\phi_\omega$ with the frequency $\omega \in (-1, 1)$, where ϕ_ω is the solution of

$$-\Delta\phi + (1 - \omega^2)\phi - \phi^p = 0.$$

It was proved by Shatah (1983), and Shatah-Strauss (1985) that there exists a critical frequency $\omega_c \in (0, 1)$ such that the standing waves solution u_ω is orbitally stable when $\omega_c < |\omega| < 1$, and orbitally unstable when $|\omega| < \omega_c$. Furthermore, the strong instability for the critical frequency $|\omega| = \omega_c$ in the high dimensions $d \geq 2$ was proved by Ohta-Todorova (2007). In this paper, we settle the only remaining problem when $|\omega| = \omega_c$, $p > 1$, and $d = 1$, in which case we prove that the standing waves solution u_ω is orbitally unstable.

1. Introduction

In this paper, we consider the stability theory of the following nonlinear Klein-Gordon equation

$$\partial_{tt}u - \Delta u + u = |u|^{p-1}u, \qquad t \in \mathbb{R},\ x \in \mathbb{R}^d, \tag{1.1}$$

with the initial data

$$u(0, x) = u_0(x), \quad u_t(0, x) = u_1(x). \tag{1.2}$$

Here $d \geq 1$ and $1 < p < 1 + \frac{4}{d-2}$ ($1 < p < \infty$ when $d = 1, 2$). The $H^1 \times L^2$-solution (u, u_t) of (1.1)–(1.2) obeys the following charge, momentum and energy conservation laws,

$$Q(u, u_t) = \operatorname{Im} \int u\bar{u}_t\, dx = Q(u_0, u_1); \tag{1.3}$$

$$P(u, u_t) = \operatorname{Re} \int \nabla u\bar{u}_t\, dx = P(u_0, u_1); \tag{1.4}$$

$$E(u, u_t) = \frac{1}{2}\|u_t\|_{L^2}^2 + \frac{1}{2}\|\nabla u\|_{L^2}^2 + \frac{1}{2}\|u\|_{L^2}^2 - \frac{1}{p+1}\|u\|_{L^{p+1}}^{p+1} = E(u_0, u_1). \tag{1.5}$$

The well-posedness for the Cauchy problem (1.1)–(1.2) is well-understood in the energy space $H^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$. More precisely, for any $(u_0, u_1) \in H^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$, there exists a unique solution $(u, u_t) \in C([0, T); H^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)) \cap X$ of (1.1)–(1.2), with the maximal lifetime $T = T(\|(u_0, u_1)\|_{H^1 \times L^2})$. Here X is some suitable

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auxiliary space. We say that the Cauchy problem (1.1)–(1.2) is globally well-posed when $T=\infty$, and that it blows up in finite time when $T<\infty$. See for examples Ginibre-Velo's results [6, 7] for the local and global well-posedness, and Payne-Sattinger [22] for the blow-up. We also refer the readers to [11, 12, 13] and the references therein for the scattering.

The equation (1.1) has the standing waves solution $e^{i\omega t}\phi_\omega$, where ϕ_ω is the ground state solution of the following elliptic equation,

$$-\Delta\phi+(1-\omega^2)\phi-\phi^p=0. \tag{1.6}$$

When the parameter $|\omega|<1$, there exists an H^1-solution to (1.6), see [26] for example. Furthermore, there uniquely (up to some symmetries) exists a positive radial solution ϕ_ω, say the ground state solution, which decays exponentially at infinity. Particularly, in one-dimensional case, the solution to (1.6) is unique up to the symmetries of the rotation and the spatial transformation. See also [1, 4, 5] for some examples on the existence of the multi-solitary waves of the nonlinear Klein-Gordon equations.

The stability theory of the standing waves solution $e^{i\omega t}\phi_\omega$ of Klein-Gordon equations has been widely studied. In particular, Berestycki and Cazenave [2] proved that it is strongly unstable by blow-up when $\omega=0$ and $1<p<1+\frac{4}{d-2}$, see also Shatah [24]. In the case of $\omega\neq 0$, Shatah [23] proved that it is orbitally stable when $1<p<1+\frac{4}{d}$ and $\omega_c<|\omega|<1$. Here

$$\omega_c=\sqrt{\frac{p-1}{4-(d-1)(p-1)}}.$$

Later, Shatah and Strauss [25] showed that when $1<p<1+\frac{4}{d}, |\omega|<\omega_c$ or $1+\frac{4}{d}<p<1+\frac{4}{d-2}, |\omega|<1$, the standing waves solution $e^{i\omega t}\phi_\omega$ is orbitally unstable. See also Stuart [27] for the stability of the solitary waves. Hence, the above results show that ω_c is the threshold of dichotomy between stability and instability.

The critical cases, $|\omega|=\omega_c$ when $1<p<1+\frac{4}{d}$, are degenerate based on the theory of Grillakis, Shatah and Strauss [8, 9]. These degenerate cases were further investigated by Comech-Pelinovsky [3], Maeda [15], and Ohta [19]. In particular, as an application of the theorems established in [3, 15], the standing waves solution $e^{i\omega t}\phi_\omega$ is orbitally unstable in the critical cases $|\omega|=\omega_c$ when $2\le p<1+\frac{4}{d}$. However, the range $1<p<2$ in the critical case is not covered in the previously mentioned works, since the nonlinear term is not regular enough. Furthermore, Ohta and Todorova [21] proved the strong instability when $1<p<1+\frac{4}{d}, |\omega|\le\omega_c$ or $1+\frac{4}{d}\le p<1+\frac{4}{d-2}, |\omega|<1$ in high dimensions $d\ge 2$, which cover the entire instability region in the case of $d\ge 2$, see also [10, 14, 20] for some companion results.

As the summary of the results above, the region of index that decides the stability and instability has been completely proved, except for the final open case: $1<p<2$, $|\omega|=\omega_c$ and $d=1$. The present paper aims to settle this remaining problem .

Before stating our theorem, we recall some definitions. Let $v=u_t$, $\vec{u}=(u,v)^T$, $\vec{u}_0=(u_0,u_1)^T$, and $\overrightarrow{\Phi_\omega}=(\phi_\omega,i\omega\phi_\omega)^T$. For $\varepsilon>0$, we denote the set $U_\varepsilon\big(\overrightarrow{\Phi_\omega}\big)$ as

$$U_\varepsilon\big(\overrightarrow{\Phi_\omega}\big)=\{\vec{u}\in H^1(\mathbb{R})\times L^2(\mathbb{R}):\inf_{(\theta,y)\in\mathbb{R}^2}\|\vec{u}-e^{i\theta}\overrightarrow{\Phi_\omega}(\cdot-y)\|_{H^1\times L^2}<\varepsilon\}.$$

Definition 1.1. *We say that the solitary wave solution $u_\omega = e^{i\omega t}\phi_\omega$ of (1.1) is stable if for any $\varepsilon > 0$ there exists $\delta > 0$ such that if $\|\vec{u}_0 - \overrightarrow{\Phi_\omega}\|_{H^1\times L^2} < \delta$, then the solution $\vec{u}(t)$ of (1.1) with $\vec{u}(0) = \vec{u}_0$ exists for all $t \in \mathbb{R}$, and $\vec{u}(t) \in U_\varepsilon\big(\overrightarrow{\Phi_\omega}\big)$ for all $t \in \mathbb{R}$. Otherwise, u_ω is said to be unstable.*

Then the main result in the present paper is

Theorem 1.2. *Let $d = 1$, $1 < p < 5$, $\omega \in (-1, 1)$ and ϕ_ω be a solution of (1.6). If $|\omega| = \sqrt{\frac{p-1}{4}}$, then the standing waves solution $e^{i\omega t}\phi_\omega$ is orbitally unstable.*

Then together with the results in [23, 25, 3, 15, 21] and Theorem 1.2, we have the following classification of stability theory for the standing waves solution $e^{i\omega t}\phi_\omega(x)$.

Corollary 1.3. *Let $d \geq 1$, $1 < p < 1 + \frac{4}{d}$, $|\omega| < 1$. Then*

(1) *If $|\omega| > |\omega_c|$, then the standing waves solution $e^{i\omega t}\phi_\omega(x)$ is orbitally stable;*
(2) *If $|\omega| \leq |\omega_c|$, then the standing waves solution $e^{i\omega t}\phi_\omega(x)$ is orbitally unstable.*

The main new ingredient in this paper is that we establish a new monotonic inequality based on the virial identity and the modulation theory. Our argument is completely different from [3, 15, 21]. First, our method does not require the regularity of nonlinear term, while the previous ones in [3, 15] are based on the high-order derivatives of the Lyapunov functional. Second, we overcome the difficulty that the virial identity is too weak to prove the blow-up directly in 1D case due to the lack of symmetry. Ohta and Todorova's argument in [21] relies heavily on the radial Sobolev inequality, so it can only handle the higher dimensional case when $d \geq 2$.

Now, we briefly describe the framework of proof. We argue by contradiction, assuming that the solution is close to the standing wave solution for any time. Then, the modulation argument gives the smallness of perturbation function up to the rotation, spatial translation and scaling. Finally, we use the localized virial identity to preclude that both the perturbation function and the scaling parameter keep the initial sizes for any time, which leads to a contradiction.

There are two observations that play an important role in our argument. The first one is the flatness of functional $E\big(\overrightarrow{\Phi_\omega}\big) + \lambda\omega Q\big(\overrightarrow{\Phi_\omega}\big)$ respect to the scaling parameter λ in the degenerate case. This gives a nice bound of the perturbation function ε in the following way,

$$\|\vec{\varepsilon}\|^2_{H^1\times L^2} \lesssim |\lambda - 1|\|\vec{\varepsilon}_0\|_{H^1\times L^2} + o((\lambda - 1)^2).$$

Since we assume that the solution is stable, then the square of the perturbation equation is roughly controlled by its first power at the initial moment. This is inspired by the work of [16].

The second observation is a specific form of the localized virial identity $I(t)$. By suitable definition, we can prove that

$$I'(t) = P(\vec{u}(t))$$

for some quantity $P(\vec{u})$. Using the modulation theory, we decompose $P(\vec{u})$ as

$$\begin{aligned} A(\vec{u}_0) + \langle B(\vec{\Phi}_\omega), \varepsilon\rangle + C(\vec{\Phi}_\omega)(\lambda - 1) &+ D(\vec{\Phi}_\omega)(\lambda - 1)^2 \\ &+ O(\|\vec{\varepsilon}\|^2_{H^1\times L^2}) + o((\lambda - 1)^2). \end{aligned}$$

Then, we shall prove that the quantity $P(\vec{u})$ satisfies the following structure: there exist $c_1 > 0$ and $c_2 > 0$ such that

$$A(\vec{u}_0) \geq c_1 \|\varepsilon(0)\|_X, \quad D(\vec{\Phi}_\omega) \geq c_2, \text{ and } \langle B(\vec{\Phi}_\omega), \varepsilon \rangle = C(\vec{\Phi}_\omega) = 0.$$

In fact, we are able to obtain the first inequality by combining suitable initial datum and the conservation laws, the second one by using the structure of $P(\vec{u})$, and the last identity by choosing suitable orthogonal conditions and applying the conservation laws. Combining with this structure and the smallness estimate of the perturbation ε given in the previous observation, we can prove that $I(t)$ goes to infinity as $t \to +\infty$, which contradicts its uniform boundedness.

The following is the organization of the paper. In Section 2, we give some preliminaries. It includes some basic definitions and properties, the coercivity property of the Hessian, and the modulation statement. In Section 3, we give the virial identities, control the remainder function and the scaling parameter, and finally prove Theorem 1.2.

2. Preliminary

2.1. **Notations.** For $f, g \in L^2(\mathbb{R}) = L^2(\mathbb{R}, \mathbb{C})$, we define

$$\langle f, g \rangle = \operatorname{Re} \int_{\mathbb{R}} f(x) \overline{g(x)} \, dx$$

and regard $L^2(\mathbb{R})$ as a real Hilbert space. Similarly, for $\vec{f}, \vec{g} \in \big(L^2(\mathbb{R})\big)^2 = \big(L^2(\mathbb{R}, \mathbb{C})\big)^2$, we define

$$\langle \vec{f}, \vec{g} \rangle = \operatorname{Re} \int_{\mathbb{R}} \vec{f}(x)^T \cdot \overline{\vec{g}(x)} \, dx.$$

For a function $f(x)$, its L^q-norm $\|f\|_{L^q} = \Big(\int_{\mathbb{R}} |f(x)|^q dx \Big)^{\frac{1}{q}}$ and its H^1-norm $\|f\|_{H^1} = (\|f\|_{L^2}^2 + \|\partial_x f\|_{L^2}^2)^{\frac{1}{2}}$.

Further, we write $X \lesssim Y$ or $Y \gtrsim X$ to indicate $X \leq CY$ for some constant $C > 0$. We use the notation $X \sim Y$ whenever $X \lesssim Y \lesssim X$. Also, we use $O(Y)$ to denote any quantity X such that $|X| \lesssim Y$; and use $o(Y)$ to denote any quantity X such that $X/Y \to 0$, if $Y \to 0$.

2.2. **Some basic definitions and properties.** In the following, we only consider one dimension problem and the case of $1 < p < 5$, in which $\omega_c = \sqrt{\frac{p-1}{4}}$. Let $\vec{u} = (u, v)^T, \overrightarrow{\Phi_\omega} = (\phi_\omega, i\omega\phi_\omega)^T$. Recall that the conserved qualities,

$$Q(\vec{u}) = \operatorname{Im} \int u \bar{v} \, dx,$$
$$E(\vec{u}) = \frac{1}{2}\|v\|_{L^2}^2 + \frac{1}{2}\|u_x\|_{L^2}^2 + \frac{1}{2}\|u\|_{L^2}^2 - \frac{1}{p+1}\|u\|_{L^{p+1}}^{p+1}.$$

First, we give some basic properties on the charge and energy.

Lemma 2.1. *The following equalities hold,*

(1) $\frac{d}{d\omega} Q\left(\overrightarrow{\Phi_\omega}\right)\Big|_{\omega=\pm\omega_c} = 0$;

(2) *If* $|\omega| = \omega_c$, *then* $(p+3)E\left(\overrightarrow{\Phi_\omega}\right) + 8\omega Q\left(\overrightarrow{\Phi_\omega}\right) = 0$.

Proof. Note that

$$Q\left(\overrightarrow{\Phi_\omega}\right) = -\omega\|\phi_\omega\|_{L^2}^2.$$

Moreover, by rescaling, we find,

$$\phi_\omega(x) = (1-\omega^2)^{\frac{1}{p-1}}\phi_0\big(\sqrt{1-\omega^2}x\big).$$

This implies that

$$Q\left(\overrightarrow{\Phi_\omega}\right) = -\omega(1-\omega^2)^{\frac{2}{p-1}-\frac{1}{2}}\|\phi_0\|_{L^2}^2.$$

Hence by a direct computation, we have

$$\frac{d}{d\omega}Q\left(\overrightarrow{\Phi_\omega}\right) = -(1-\omega^2)^{\frac{2}{p-1}-\frac{3}{2}}\Big[1-\frac{4}{p-1}\omega^2\Big]\|\phi_0\|_{L^2}^2.$$

This gives (1). For (2), we have

$$E\left(\overrightarrow{\Phi_\omega}\right) = \frac{1}{2}\|\partial_x\phi_\omega\|_{L^2}^2 + \frac{1}{2}(1+\omega^2)\|\phi_\omega\|_{L^2}^2 - \frac{1}{p+1}\|\phi_\omega\|_{L^{p+1}}^{p+1}.$$

From the equation (1.6), we obtain that

$$\|\partial_x\phi_\omega\|_{L^2}^2 + (1-\omega^2)\|\phi_\omega\|_{L^2}^2 - \|\phi_\omega\|_{L^{p+1}}^{p+1} = 0;$$
$$\|\partial_x\phi_\omega\|_{L^2}^2 - (1-\omega^2)\|\phi_\omega\|_{L^2}^2 + \frac{2}{p+1}\|\phi_\omega\|_{L^{p+1}}^{p+1} = 0.$$

These give that

$$E\left(\overrightarrow{\Phi_\omega}\right) = \frac{1}{p+3}\big(p-1+4\omega^2\big)\|\phi_\omega\|_{L^2}^2.$$

Combining the value of $Q\left(\overrightarrow{\Phi_\omega}\right)$ above, we obtain (2). □

Now we define the functional S_ω as

$$S_\omega(\vec{u}) = E(\vec{u}) + \omega Q(\vec{u}).$$

Then we have

$$S_\omega'(\vec{u}) = \begin{pmatrix} -u_{xx}+u-|u|^{p-1}u \\ v \end{pmatrix} + i\omega\begin{pmatrix} v \\ -u \end{pmatrix}.$$

Note that $S_\omega'(\overrightarrow{\Phi_\omega}) = 0$. Moreover, for the vector $\vec{f} = (f,g)^T$, a direct computation shows that

$$S_\omega''(\overrightarrow{\Phi_\omega})\vec{f} = \begin{pmatrix} -f_{xx}+f-p\phi_\omega^{p-1}\mathrm{Re}f - i\phi_\omega^{p-1}\mathrm{Im}f \\ g \end{pmatrix} + i\omega\begin{pmatrix} g \\ -f \end{pmatrix}. \tag{2.1}$$

From the invariance of $S_\omega'\big(\overrightarrow{\Phi_\omega}\big)$ in the rotation and spatial transformations, we have

$$S_\omega''\big(\overrightarrow{\Phi_\omega}\big)i\overrightarrow{\Phi_\omega} = 0, \qquad S_\omega''\big(\overrightarrow{\Phi_\omega}\big)\partial_x\overrightarrow{\Phi_\omega} = 0. \tag{2.2}$$

Indeed, from

$$S_\omega'\big(e^{i\theta}\overrightarrow{\Phi_\omega}(\cdot-y)\big) = 0, \quad \text{for any } \theta\in\mathbb{R}, y\in\mathbb{R},$$

we find that

$$S_\omega''\big(\overrightarrow{\Phi_\omega}\big)i\overrightarrow{\Phi_\omega} = \partial_\theta S_\omega'\big(e^{i\theta}\overrightarrow{\Phi_\omega}\big)\Big|_{\theta=0} = 0,$$

and

$$S''_\omega(\overrightarrow{\Phi_\omega})\partial_x\overrightarrow{\Phi_\omega} = -\partial_y S'_\omega\big(\overrightarrow{\Phi_\omega}(\cdot - y)\big)\Big|_{y=0} = 0.$$

This gives (2.2).

Moreover, taking the derivative of $S'_\omega(\overrightarrow{\Phi_\omega}) = 0$ gives that

$$S''_\omega(\overrightarrow{\Phi_\omega})\partial_\omega\overrightarrow{\Phi_\omega} = -Q'(\overrightarrow{\Phi_\omega}). \tag{2.3}$$

Then a consequence of Lemma 2.1 (1) is

Corollary 2.2. *Let $\lambda \in \mathbb{R}^+$, $\omega = \pm\omega_c$, then*

$$S_{\lambda\omega}(\overrightarrow{\Phi_{\lambda\omega}}) - S_{\lambda\omega}(\overrightarrow{\Phi_\omega}) = o\big((\lambda-1)^2\big).$$

Proof. From the definition and the Taylor's type expansion,

$$\begin{aligned}
&S_{\lambda\omega}(\overrightarrow{\Phi_{\lambda\omega}}) - S_{\lambda\omega}(\overrightarrow{\Phi_\omega})\\
&\quad= S_\omega(\overrightarrow{\Phi_{\lambda\omega}}) - S_\omega(\overrightarrow{\Phi_\omega}) + (\lambda-1)\omega\Big(Q(\overrightarrow{\Phi_{\lambda\omega}}) - Q(\overrightarrow{\Phi_\omega})\Big)\\
&\quad= \frac{1}{2}\Big\langle S''_\omega(\overrightarrow{\Phi_\omega})\Big(\overrightarrow{\Phi_{\lambda\omega}} - \overrightarrow{\Phi_\omega}\Big), \Big(\overrightarrow{\Phi_{\lambda\omega}} - \overrightarrow{\Phi_\omega}\Big)\Big\rangle\\
&\qquad + (\lambda-1)\omega\Big(Q(\overrightarrow{\Phi_{\lambda\omega}}) - Q(\overrightarrow{\Phi_\omega})\Big) + o\big((\lambda-1)^2\big).
\end{aligned}$$

Note that

$$\overrightarrow{\Phi_{\lambda\omega}} - \overrightarrow{\Phi_\omega} = (\lambda-1)\omega\partial_\omega\overrightarrow{\Phi_\omega} + o(\lambda-1),$$

we find that

$$\begin{aligned}
&\Big\langle S''_\omega(\overrightarrow{\Phi_\omega})\Big(\overrightarrow{\Phi_{\lambda\omega}} - \overrightarrow{\Phi_\omega}\Big), \Big(\overrightarrow{\Phi_{\lambda\omega}} - \overrightarrow{\Phi_\omega}\Big)\Big\rangle\\
&\quad= (\lambda-1)^2\omega^2\Big\langle S''_\omega(\overrightarrow{\Phi_\omega})\partial_\omega\overrightarrow{\Phi_\omega}, \partial_\omega\overrightarrow{\Phi_\omega}\Big\rangle + o\big((\lambda-1)^2\big)\\
&\quad= -(\lambda-1)^2\omega^2\Big\langle Q'(\overrightarrow{\Phi_\omega}), \partial_\omega\overrightarrow{\Phi_\omega}\Big\rangle + o\big((\lambda-1)^2\big)\\
&\quad= -(\lambda-1)^2\omega^2\frac{d}{d\lambda}Q(\overrightarrow{\Phi_{\lambda\omega}})\Big|_{\lambda=1} + o\big((\lambda-1)^2\big),
\end{aligned}$$

where we use (2.3) for the second identity. Using Lemma 2.1 (1), we have

$$\frac{d}{d\lambda}Q(\overrightarrow{\Phi_{\lambda\omega}})\Big|_{\lambda=1} = 0.$$

Hence,

$$\Big\langle S''_\omega(\overrightarrow{\Phi_\omega})\Big(\overrightarrow{\Phi_{\lambda\omega}} - \overrightarrow{\Phi_\omega}\Big), \Big(\overrightarrow{\Phi_{\lambda\omega}} - \overrightarrow{\Phi_\omega}\Big)\Big\rangle = o\big((\lambda-1)^2\big),$$

and

$$Q(\overrightarrow{\Phi_{\lambda\omega}}) - Q(\overrightarrow{\Phi_\omega}) = o\big(\lambda-1\big).$$

Thus we obtain the desired estimate. □

2.3. **Coercivity.** First, we need the following lemma.

Lemma 2.3. *Let* $\vec{\psi}_\omega = (\partial_\omega \phi_\omega, i\omega\partial_\omega\phi_\omega)^T$, $\overrightarrow{\Psi_\omega} = (2\omega\phi_\omega, 0)^T$, *then*

$$S''_\omega(\overrightarrow{\Phi_\omega})\vec{\psi}_\omega = \overrightarrow{\Psi_\omega}. \tag{2.4}$$

Moreover, if $|\omega| = \omega_c$, *then*

$$\left\langle S''_\omega(\overrightarrow{\Phi_\omega})\vec{\psi}_\omega, \vec{\psi}_\omega \right\rangle < 0. \tag{2.5}$$

Proof. Note that from the equation (1.6), we have

$$\big(- \partial_{xx} + (1-\omega^2) - p\phi_\omega^{p-1}\mathrm{Re} - i\phi_\omega^{p-1}\mathrm{Im}\big)\partial_\omega\phi_\omega = 2\omega\phi_\omega.$$

Then (2.4) follows from a straightforward computation.

For (2.5), we have

$$\begin{aligned}\left\langle S''_\omega(\overrightarrow{\Phi_\omega})\vec{\psi}_\omega, \vec{\psi}_\omega \right\rangle &= \left\langle \overrightarrow{\Psi_\omega}, \vec{\psi}_\omega \right\rangle \\ &= 2\omega \int \phi_\omega\, \partial_\omega\phi_\omega\, dx = \omega \frac{d}{d\omega}\|\phi_\omega\|_{L^2}^2 \\ &= -\frac{d}{d\omega} Q\left(\overrightarrow{\Phi_\omega}\right) - \|\phi_\omega\|_{L^2}^2.\end{aligned}$$

Using Lemma 2.1 (1), when $|\omega| = \omega_c$,

$$\left\langle S''_\omega(\overrightarrow{\Phi_\omega})\vec{\psi}_\omega, \vec{\psi}_\omega \right\rangle = -\|\phi_\omega\|_{L^2}^2 < 0.$$

This proves the lemma. □

Now we have the following coercivity property.

Lemma 2.4. *Let* $\omega = \pm\omega_c$. *Suppose that* $\vec{\xi} \in H^1(\mathbb{R}) \times L^2(\mathbb{R})$ *satisfies*

$$\left\langle \vec{\xi}, i\overrightarrow{\Phi_\omega} \right\rangle = \left\langle \vec{\xi}, \partial_x\overrightarrow{\Phi_\omega} \right\rangle = \left\langle \vec{\xi}, \overrightarrow{\Psi_\omega} \right\rangle = 0.$$

Then

$$\left\langle S''_\omega(\overrightarrow{\Phi_\omega})\vec{\xi}, \vec{\xi} \right\rangle \gtrsim \|\vec{\xi}\|_{H^1\times L^2}^2.$$

Proof. First, we show that

$$\mathrm{Ker}\Big(S''_\omega(\overrightarrow{\Phi_\omega})\Big) = \mathrm{Span}\big\{i\overrightarrow{\Phi_\omega}, \partial_x\overrightarrow{\Phi_\omega}\big\}. \tag{2.6}$$

Indeed, from (2.2), we have

$$\big\{i\overrightarrow{\Phi_\omega}, \partial_x\overrightarrow{\Phi_\omega}\big\} \subset \mathrm{Ker}\Big(S''_\omega(\overrightarrow{\Phi_\omega})\Big).$$

Hence, to prove (2.6), we now turn to show that if

$$S''_\omega(\overrightarrow{\Phi_\omega})\vec{f} = 0, \tag{2.7}$$

then

$$\vec{f} = c_1 i\overrightarrow{\Phi_\omega} + c_2\partial_x\overrightarrow{\Phi_\omega}. \tag{2.8}$$

Let $\vec{f} = (f, g)$, then from (2.1), the equality (2.7) is equivalent to

$$\begin{cases} -f_{xx} + f - p\phi_\omega^{p-1}\mathrm{Re}f - i\phi_\omega^{p-1}\mathrm{Im}f + i\omega g = 0, \\ g - i\omega f = 0 \end{cases}$$

This implies that f obeys the equation

$$-f_{xx} + (1-\omega^2)f - p\phi_\omega^{p-1}\mathrm{Re}f - i\phi_\omega^{p-1}\mathrm{Im}f = 0.$$

Then from Proposition 2.8 in Weinstein [28], we obtain that there exist $c_1 \in \mathbb{R}, c_2 \in \mathbb{R}$,

$$f = c_1\partial_x\phi_\omega + c_2 i\phi_\omega.$$

This yields that

$$g = i\omega f = c_1 i\omega\partial_x\phi_\omega + c_2 i\omega \cdot i\phi_\omega.$$

Hence we have (2.8) and thus we prove (2.6).

Second, we claim that

$$S''_\omega(\overrightarrow{\Phi_\omega}) \text{ has exactly one negative eigenvalue.} \tag{2.9}$$

To prove (2.9), we need some well-known facts. It is known that the operator

$$-\partial_{xx} + (1-\omega^2) - \phi_\omega^{p-1} \quad \text{is non-negative,} \tag{2.10}$$

and the operator

$$-\partial_{xx} + (1-\omega^2) - p\phi_\omega^{p-1}$$

has exactly one negative eigenvalue (see Page 489 in Weinstein [28]). That is, there uniquely exists a pair $(\lambda_{-1}, f_{-1}) \in \mathbb{R}^- \times H^1(\mathbb{R})$ such that

$$-\partial_{xx}f_{-1} + (1-\omega^2)f_{-1} - p\phi_\omega^{p-1}f_{-1} = \lambda_{-1}f_{-1}.$$

Moreover, the formula (2.5) implies that $S''_\omega(\overrightarrow{\Phi_\omega})$ at least exists one negative eigenvalue. That is, there is at least one negative eigenvalue and its associated eigenvector, say $(\tilde\lambda_{-1}, \vec\xi_{-1}) \in \mathbb{R}^- \times H^1(\mathbb{R})$, such that

$$S''_\omega(\overrightarrow{\Phi_\omega})\vec\xi_{-1} = \tilde\lambda_{-1}\vec\xi_{-1}. \tag{2.11}$$

Using (2.1), this yields that

$$\begin{cases} -\partial_{xx}\xi_{-1} + \xi_{-1} - p\phi_\omega^{p-1}\mathrm{Re}\xi_{-1} - i\phi_\omega^{p-1}\mathrm{Im}\xi_{-1} + i\omega\eta_{-1} = \tilde\lambda_{-1}\xi_{-1}, \\ \eta_{-1} - i\omega\xi_{-1} = \tilde\lambda_{-1}\eta_{-1}, \end{cases}$$

where $\vec\xi_{-1} = (\xi_{-1}, \eta_{-1})$. This further implies that

$$\begin{cases} -\partial_{xx}\xi_{-1} + (1-\omega^2)\xi_{-1} - p\phi_\omega^{p-1}\mathrm{Re}\xi_{-1} - i\phi_\omega^{p-1}\mathrm{Im}\xi_{-1} = \tilde\lambda_{-1}\Big(\frac{\omega^2}{1-\tilde\lambda_{-1}} + 1\Big)\xi_{-1}, \\ \eta_{-1} = \frac{i\omega}{1-\tilde\lambda_{-1}}\xi_{-1}. \end{cases}$$

Now we use facts (2.10) and (2.11), to obtain that

$$\tilde\lambda_{-1}\Big(\frac{\omega^2}{1-\tilde\lambda_{-1}} + 1\Big) = \lambda_{-1}, \quad \text{and} \quad \xi_{-1} = f_{-1}. \tag{2.12}$$

Then we find that given $\lambda_{-1} < 0$, there exactly exists one negative solution $\tilde\lambda_{-1} < 0$, satisfying the first equation in (2.12). This implies $S''_\omega(\overrightarrow{\Phi_\omega})$ has exactly one negative eigenvalue. That is, there uniquely exists $(\tilde\lambda_{-1}, \vec\xi_{-1})$ satisfying (2.11). This proves (2.9).

Now we are ready to prove the lemma. Since ϕ_ω is exponentially localized, $S''_\omega(\overrightarrow{\Phi_\omega})$ can be considered as compact perturbation of

$$2\begin{pmatrix} -\partial_{xx}+1 & i\omega \\ -i\omega & 1 \end{pmatrix}.$$

Therefore its essential spectrum is $[2(1-\omega^2),\infty)$ and by Weyl's Theorem its spectrum in $(-\infty,2(1-\omega^2))$ consists of isolated eigenvalues. Without loss of generality, we may assume that $\vec{\xi}_{-1}$ is the $L^2\times L^2$-normalized eigenvector associated to the negative eigenvalue $\tilde{\lambda}_{-1}$, that is

$$S''_\omega(\overrightarrow{\Phi_\omega})\vec{\xi}_{-1}=\tilde{\lambda}_{-1}\vec{\xi}_{-1},\quad \text{and}\quad \|\vec{\xi}_{-1}\|_{L^2\times L^2}=1. \tag{2.13}$$

According to these, we may write the decomposition of $\vec{\xi}$ along the spectrum of $S''_\omega(\overrightarrow{\Phi_\omega})$ as

$$\vec{\xi}=a_{-1}\vec{\xi}_{-1}+a_{0,1}i\overrightarrow{\Phi_\omega}+a_{0,2}\partial_x\overrightarrow{\Phi_\omega}+\vec{\eta},$$

with $a_{-1},a_{0,1},a_{0,2}\in\mathbb{R}$, and $\vec{\eta}$ verifying $\langle\vec{\eta},\vec{\xi}_{-1}\rangle=\langle\vec{\eta},i\overrightarrow{\Phi_\omega}\rangle=\langle\vec{\eta},\partial_x\overrightarrow{\Phi_\omega}\rangle=0$ and

$$\left\langle S''_\omega(\overrightarrow{\Phi_\omega})\vec{\eta},\vec{\eta}\right\rangle\gtrsim\|\vec{\eta}\|^2_{H^1\times L^2}. \tag{2.14}$$

Since $\langle\vec{\xi},i\overrightarrow{\Phi_\omega}\rangle=\langle\vec{\xi},\partial_x\overrightarrow{\Phi_\omega}\rangle=0$, we have $a_{0,1}=a_{0,2}=0$, and thus

$$\vec{\xi}=a_{-1}\vec{\xi}_{-1}+\vec{\eta}. \tag{2.15}$$

Similarly, noting that $\langle\vec{\psi}_\omega,i\overrightarrow{\Phi_\omega}\rangle=\langle\vec{\psi}_\omega,\partial_x\overrightarrow{\Phi_\omega}\rangle=0$, we write

$$\vec{\psi}_\omega=b_{-1}\vec{\xi}_{-1}+\vec{g}, \tag{2.16}$$

with $b_{-1}\in\mathbb{R}$ and $\vec{g}$ verifying

$$\langle\vec{g},\vec{\xi}_{-1}\rangle=\langle\vec{g},i\overrightarrow{\Phi_\omega}\rangle=\langle\vec{g},\partial_x\overrightarrow{\Phi_\omega}\rangle=0,\ \text{and}\ \left\langle S''_\omega(\overrightarrow{\Phi_\omega})\vec{g},\vec{g}\right\rangle\gtrsim\|\vec{g}\|^2_{H^1\times L^2}.$$

From (2.15), we find

$$\left\langle S''_\omega(\overrightarrow{\Phi_\omega})\vec{\xi},\vec{\xi}\right\rangle=\tilde{\lambda}_{-1}a_{-1}^2+\left\langle S''_\omega(\overrightarrow{\Phi_\omega})\vec{\eta},\vec{\eta}\right\rangle. \tag{2.17}$$

Hence by (2.14), we only need to estimate $\tilde{\lambda}_{-1}a_{-1}^2$. To this end, we shall use the third orthogonality condition.

For simplicity, we denote

$$\delta_0=-\left\langle S''_\omega(\overrightarrow{\Phi_\omega})\vec{\psi}_\omega,\vec{\psi}_\omega\right\rangle,$$

then from (2.5), we have $\delta_0>0$. Moreover, using (2.16) we obtain the relationship

$$\tilde{\lambda}_{-1}b_{-1}^2=-\delta_0-\langle S''_\omega(\overrightarrow{\Phi_\omega})\vec{g},\vec{g}\rangle. \tag{2.18}$$

Furthermore, the formulas (2.4) and (2.16) imply

$$\overrightarrow{\Psi_\omega}=\tilde{\lambda}_{-1}b_{-1}\vec{\xi}_{-1}+S''_\omega(\overrightarrow{\Phi_\omega})\vec{g}.$$

Hence, with combination of (2.15) and the orthogonality condition $\langle\vec{\xi},\overrightarrow{\Psi_\omega}\rangle=0$, we have

$$\tilde{\lambda}_{-1}a_{-1}b_{-1}+\langle S''_\omega(\overrightarrow{\Phi_\omega})\vec{g},\vec{\eta}\rangle=0. \tag{2.19}$$

Together with (2.18) and (2.19), and using the Cauchy-Schwartz inequality, we obtain that

$$\begin{aligned}-\tilde{\lambda}_{-1}a_{-1}^2 = \frac{\tilde{\lambda}_{-1}^2 a_{-1}^2 b_{-1}^2}{-\tilde{\lambda}_{-1}b_{-1}^2} &= \frac{\langle S''_\omega(\overrightarrow{\Phi_\omega})\vec{g},\vec{\eta}\rangle^2}{\delta_0+\langle S''_\omega(\overrightarrow{\Phi_\omega})\vec{g},\vec{g}\rangle}\\ &\le \frac{\langle S''_\omega(\overrightarrow{\Phi_\omega})\vec{\eta},\vec{\eta}\rangle\langle S''_\omega(\overrightarrow{\Phi_\omega})\vec{g},\vec{g}\rangle}{\delta_0+\langle S''_\omega(\overrightarrow{\Phi_\omega})\vec{g},\vec{g}\rangle}. \end{aligned}\tag{2.20}$$

Hence this combining with (2.17) and (2.14), gives

$$\begin{aligned}\Big\langle S''_\omega(\overrightarrow{\Phi_\omega})\vec{\xi},\vec{\xi}\Big\rangle &\ge -\frac{\langle S''_\omega(\overrightarrow{\Phi_\omega})\vec{\eta},\vec{\eta}\rangle\langle S''_\omega(\overrightarrow{\Phi_\omega})\vec{g},\vec{g}\rangle}{\delta_0+\langle S''_\omega(\overrightarrow{\Phi_\omega})\vec{g},\vec{g}\rangle} + \Big\langle S''_\omega(\overrightarrow{\Phi_\omega})\vec{\eta},\vec{\eta}\Big\rangle\\ &= \delta_0\frac{\langle S''_\omega(\overrightarrow{\Phi_\omega})\vec{\eta},\vec{\eta}\rangle}{\delta_0+\langle S''_\omega(\overrightarrow{\Phi_\omega})\vec{g},\vec{g}\rangle} \gtrsim \big\|\vec{\eta}\big\|_{H^1\times L^2}^2. \end{aligned}\tag{2.21}$$

Using (2.20) again, and by Hölder's inequality, we have

$$a_{-1}^2 \lesssim \big\|\vec{\eta}\big\|_{H^1\times L^2}^2.$$

Hence, from (2.15),

$$\big\|\vec{\xi}\big\|_{L^2\times L^2}^2 \lesssim a_{-1}^2 + \big\|\vec{\eta}\big\|_{H^1\times L^2}^2 \lesssim \big\|\vec{\eta}\big\|_{H^1\times L^2}^2.$$

This together with (2.21), yields

$$\Big\langle S''_\omega(\overrightarrow{\Phi_\omega})\vec{\xi},\vec{\xi}\Big\rangle \gtrsim \big\|\vec{\xi}\big\|_{L^2\times L^2}^2. \tag{2.22}$$

Lastly, from the definition of $S''_\omega(\overrightarrow{\Phi_\omega})$ in (2.1), we have

$$\big\|\vec{\xi}\big\|_{H^1\times L^2}^2 \lesssim \langle S''_\omega(\overrightarrow{\Phi_\omega})\vec{\xi},\vec{\xi}\rangle + \|\vec{\xi}\|_{L^2\times L^2}^2.$$

Therefore, followed from (2.22), we obtain that

$$\Big\langle S''_\omega(\overrightarrow{\Phi_\omega})\vec{\xi},\vec{\xi}\Big\rangle \gtrsim \big\|\vec{\xi}\big\|_{H^1\times L^2}^2. \tag{2.23}$$

This finishes the proof of the lemma. □

2.4. **Modulation.** The modulation method was first introduced by Weinstein [28], and strengthened by the mathematicians such as Martel, Merle, Raphaël [16, 17, 18]. We use the modulation argument inspired by these works. Particularly, in the Klein-Gordon setting, we use a similar form established by Bellazzini1, Ghimenti, and Le Coz in [1], who considered the total linearized action. The following modulation lemma says that if the standing wave solution is stable, then after suitably choosing the parameters, the orthogonality conditions in Lemma 2.4 can be verified.

Lemma 2.5. *Let $\omega = \pm\omega_c$. There exists $\varepsilon_0 > 0$, such that for any $\varepsilon \in (0,\varepsilon_0)$, if $\vec{u}(t) \in U_\varepsilon(\vec{\Phi}_\omega)$ for any $t \in \mathbb{R}$, then the following properties is verified. There exist C^1-functions*

$$(\theta, y) : \mathbb{R}^2 \to \mathbb{R}, \quad \lambda : \mathbb{R} \to \mathbb{R}^+,$$

such that if we define $\vec{\xi}$ by

$$\vec{\xi}(t) = e^{-i\theta(t)}\vec{u}\big(t, \cdot - y(t)\big) - \overrightarrow{\Phi_{\lambda(t)\omega}}, \tag{2.24}$$

then $\vec{\xi}$ satisfies the following orthogonality conditions for any $t \in \mathbb{R}$,

$$\left\langle \vec{\xi}, i\overrightarrow{\Phi_{\lambda(t)\omega}} \right\rangle = \left\langle \vec{\xi}, \partial_x \overrightarrow{\Phi_{\lambda(t)\omega}} \right\rangle = \left\langle \vec{\xi}, \overrightarrow{\Psi_{\lambda(t)\omega}} \right\rangle = 0. \tag{2.25}$$

Moreover, the following estimates verify

$$\|\vec{\xi}\|_{H^1\times L^2} + |\lambda - 1| \lesssim \varepsilon,$$

and for any $t \in \mathbb{R}$,

$$|\dot{\theta} - \lambda\omega| + |\dot{y}| + |\dot{\lambda}| = O\big(\|\vec{\xi}\|_{H^1\times L^2}\big).$$

Proof. Since the argument is standard, see c.f. Proposition 1 in [16] and Proposition 9 in [1], we give the proof much briefly. The existence of the parameters follows from classical arguments involving the implicit function theorem. More precisely, fixing $t \in \mathbb{R}$ and writting $\vec{u} = \vec{u}(t)$ for short, we denote $F_j, j = 1, 2, 3 : U_1(\vec{\Phi}_\omega) \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^+$ by

$$F_1(\vec{u}, \theta, y, \lambda) = \left\langle \vec{\xi}, i\overrightarrow{\Phi_{\lambda\omega}} \right\rangle; \; F_2(\vec{u}, \theta, y, \lambda) = \left\langle \vec{\xi}, \partial_x \overrightarrow{\Phi_{\lambda\omega}} \right\rangle; \; F_3(\vec{u}, \theta, y, \lambda) = \left\langle \vec{\xi}, \overrightarrow{\Psi_{\lambda\omega}} \right\rangle.$$

Then

$$F_j\big(\overrightarrow{\Phi_\omega}, 0, 0, 1\big) = 0, \quad \text{for } j = 1, 2, 3.$$

Moreover, a direct computation gives that

$$\begin{vmatrix} \partial_\theta F_1 & \partial_y F_1 & \partial_\lambda F_1 \\ \partial_\theta F_2 & \partial_y F_2 & \partial_\lambda F_2 \\ \partial_\theta F_3 & \partial_y F_3 & \partial_\lambda F_3 \end{vmatrix}_{(\vec{u},\theta,y,\lambda)=\left(\overrightarrow{\Phi_{\lambda\omega}},0,0,1\right)}$$

$$= \begin{vmatrix} -\big\|\overrightarrow{\Phi_\omega}\big\|_{L^2\times L^2} & 0 & 0 \\ 0 & -\big\|\partial_x\overrightarrow{\Phi_\omega}\big\|_{L^2\times L^2} & 0 \\ 0 & 0 & \|\phi_\omega\|_{L^2}^2 \end{vmatrix} \neq 0.$$

Therefore, the implicit function theorem implies that there exists $\varepsilon_0 > 0$, such that for any $\varepsilon \in (0, \varepsilon_0)$, for any $\vec{u} \in U_\varepsilon(\vec{\Phi}_\omega)$, there exist continuity functions

$$(\theta, y) : U_\varepsilon(\vec{\Phi}_\omega) \to \mathbb{R}^2, \quad \lambda : U_\varepsilon(\vec{\Phi}_\omega) \to \mathbb{R}^+,$$

such that $F_j(\vec{u}, \theta, y, \lambda) = 0$ for $j = 1, 2, 3$.

The parameters $(\theta, y, \lambda) \in C^1$ in time can be followed from the regularization arguments, see c.f. Lemma 4 in [16]. Now we consider the dynamic of the parameters. From (2.24), we have

$$\vec{u}(t) = e^{i\theta(t)}\big(\vec{\xi} + \overrightarrow{\Phi_{\lambda(t)\omega}}\big)\big(t, \cdot + y(t)\big).$$

Then using this equality, the equations

$$u_t = v, \quad v_t = \Delta u - u + |u|^{p-1}u,$$

and (1.6), we obtain that

$$\partial_t\vec{\xi} + i(\dot{\theta} - \lambda\omega)\big(\vec{\xi} + \overrightarrow{\Phi_{\lambda(t)\omega}}\big) + \dot{y}\partial_x\big(\vec{\xi} + \overrightarrow{\Phi_{\lambda(t)\omega}}\big) + \dot{\lambda}\omega\partial_\lambda\overrightarrow{\Phi_{\lambda(t)\omega}} = \mathcal{N}(\vec{\xi}). \tag{2.26}$$

Here we have used the notations $\dot{f} = \partial_t f$ for the time dependent function f, and $\mathcal{N}(\vec{\xi})$ verifying

$$\langle \mathcal{N}(\vec{\xi}), \vec{f} \rangle = O\big(\|\vec{\xi}\|_{H^1\times L^2}\big)\|\vec{f}\|_{H^1\times L^2}, \quad \text{for any } f \in H^1 \times L^2.$$

Now multiplying (2.26) by $i\overrightarrow{\Phi_{\lambda(t)\omega}}$, $\partial_x\overrightarrow{\Phi_{\lambda(t)\omega}}$ and $\overrightarrow{\Psi_{\lambda(t)\omega}}$, respectively, integrating by parts and then using the orthogonal conditions (2.25), we obtain that

$$(\dot\theta-\lambda\omega)\Big(\|\overrightarrow{\Phi_{\lambda\omega}}\|_{L^2\times L^2}^2+\langle\vec\xi,\overrightarrow{\Phi_{\lambda\omega}}\rangle\Big)+\dot y\langle\partial_x\vec\xi,i\overrightarrow{\Phi_{\lambda\omega}}\rangle-\dot\lambda\omega\langle\vec\xi,i\partial_\lambda\overrightarrow{\Phi_{\lambda\omega}}\rangle=O\big(\|\vec\xi\|_{H^1\times L^2}\big);$$

$$\begin{aligned}(\dot\theta-\lambda\omega)\langle i\vec\xi,\partial_x\overrightarrow{\Phi_{\lambda\omega}}\rangle&+\dot y\Big(\|\partial_x\overrightarrow{\Phi_{\lambda\omega}}\|_{L^2\times L^2}^2+\langle\partial_x\vec\xi,\partial_x\overrightarrow{\Phi_{\lambda\omega}}\rangle\Big)\\&-\dot\lambda\omega\langle\vec\xi,\partial_x\partial_\lambda\overrightarrow{\Phi_{\lambda\omega}}\rangle=O\big(\|\vec\xi\|_{H^1\times L^2}\big);\end{aligned}$$

and

$$\begin{aligned}(\dot\theta-\lambda\omega)\langle i\vec\xi,\overrightarrow{\Psi_{\lambda\omega}}\rangle&+\dot y\langle\partial_x\vec\xi,\overrightarrow{\Psi_{\lambda\omega}}\rangle\\&-\dot\lambda\omega\Big(2\|\phi_{\lambda\omega}\|_{L^2}^2+\langle\vec\xi,\partial_\lambda\overrightarrow{\Psi_{\lambda\omega}}\rangle\Big)=O\big(\|\vec\xi\|_{H^1\times L^2}\big);\end{aligned}$$

With combination of these three estimates, we obtain that

$$|\dot\theta-\lambda\omega|+|\dot y|+|\dot\lambda|=O\big(\|\vec\xi\|_{H^1\times L^2}\big).$$

This finishes the proof of the lemma. □

3. Proof of the main theorem

3.1. **Localized virial identities.** To prove main Theorem 1.2, one of the key ingredient is the localized virial identities.

Lemma 3.1. *Let* $\varphi\in C^1(\mathbb{R})$, *then*

$$\frac{d}{dt}Re\int u\bar u_t\,dx=\int\big[|u_t|^2-|u_x|^2-|u|^2+|u|^{p+1}\big]\,dx;$$

$$Re\int\varphi\frac{d}{dt}\Big(u_x\bar u_t\Big)\,dx=-\frac12\int\varphi'\big[|u_t|^2+|u_x|^2-|u|^2+\frac{2}{p+1}|u|^{p+1}\big]\,dx.$$

Proof. It follows from a direct calculation. See [21] for the details. □

Now we define the smooth cutoff function $\varphi_R\in C^\infty(\mathbb{R})$ as

$$\varphi_R(x)=x,\quad\text{when}\quad|x|\le R;\qquad\varphi_R(x)=0,\quad\text{when}\quad|x|\ge 2R,$$

and $|\varphi_R'|\le1$ for any $x\in\mathbb{R}$. Moreover, we denote

$$I(t)=\frac{4}{p-1}\mathrm{Re}\int u\bar u_t\,dx+2\mathrm{Re}\int\varphi_R\big(x-y(t)\big)u_x\bar u_t\,dx.$$

Then from Lemma 3.1 we have the following lemma.

Lemma 3.2. *Let* $R>0$, *if* $|\dot y|\lesssim1$, *then*

$$\begin{aligned}I'(t)=-\frac{p+3}{p-1}\cdot2E(u_0,u_1)-\frac{16\omega}{p-1}Q(u_0,u_1)-2\dot yP(u_0,u_1)+\frac{8}{p-1}\|u_t-i\omega u\|_{L^2}^2\\+O\Big(\int_{|x-y(t)|\ge R}|u_t|^2+|u_x|^2+|u|^2+|u|^{p+1}\,dx\Big).\end{aligned}$$

Proof. First, we have

$$\frac{d}{dt}\mathrm{Re}\int \varphi_R\big(x-y(t)\big)u_x\bar{u}_t\,dx = -\dot{y}\mathrm{Re}\int \varphi_R'\big(x-y(t)\big)u_x\bar{u}_t\,dx + \mathrm{Re}\int \varphi_R\big(x-y(t)\big)\frac{d}{dt}\big(u_x\bar{u}_t\big)\,dx.$$

Then from Lemma 3.1 and the momentum conservation law, we obtain

$$\begin{aligned}\frac{d}{dt}\mathrm{Re}\int \varphi_R\big(x-y(t)\big)u_x\bar{u}_t\,dx &= -\dot{y}\mathrm{Re}\int \varphi_R'\big(x-y(t)\big)u_x\bar{u}_t\,dx\\ &\quad -\frac{1}{2}\int \varphi_R'\big(x-y(t)\big)\Big[|u_t|^2+|u_x|^2-|u|^2+\frac{2}{p+1}|u|^{p+1}\Big]\,dx\\ &= -\dot{y}P(u_0,u_1)-\dot{y}\mathrm{Re}\int \Big[\varphi_R'\big(x-y(t)\big)-1\Big]u_x\bar{u}_t\,dx\\ &\quad -\frac{1}{2}\int \Big(|u_t|^2+|u_x|^2-|u|^2+\frac{2}{p+1}|u|^{p+1}\Big)\,dx\\ &\quad -\frac{1}{2}\int \Big[\varphi_R'\big(x-y(t)\big)-1\Big]\Big(|u_t|^2+|u_x|^2-|u|^2+\frac{2}{p+1}|u|^{p+1}\Big)\,dx.\end{aligned}$$

Since $\mathrm{supp}\big[\varphi_R'\big(x-y(t)\big)-1\big]\subset\{x:|x-y(t)|\geq R\}$, $|\varphi_R'|\leq 1$ and $|\dot{y}|\lesssim 1$, we get

$$\begin{aligned}&\frac{d}{dt}\mathrm{Re}\int \varphi_R\big(x-y(t)\big)u_x\bar{u}_t\,dx\\ &\quad= -\dot{y}P(u_0,u_1)-\frac{1}{2}\int \Big(|u_t|^2+|u_x|^2-|u|^2+\frac{2}{p+1}|u|^{p+1}\Big)\,dx\\ &\qquad+O\Big(\int_{|x-y(t)|\geq R}|u_t|^2+|u_x|^2+|u|^2+|u|^{p+1}\,dx\Big).\end{aligned}$$

Moreover, from Lemma 3.1,

$$\frac{d}{dt}\mathrm{Re}\int u\bar{u}_t\,dx = \int \Big[|u_t|^2-|u_x|^2-|u|^2+|u|^{p+1}\Big]\,dx.$$

Combining the two estimates above, we obtain that

$$\begin{aligned}I'(t) =& -2\dot{y}P(u_0,u_1)+\Big(\frac{4}{p-1}-1\Big)\|u_t\|_{L^2}^2-\frac{p+3}{p-1}\|u_x\|_{L^2}^2\\ &+\frac{p-5}{p-1}\|u\|_{L^2}^2+2\frac{p+3}{p^2-1}\|u\|_{L^{p+1}}^{p+1}\\ &+O\Big(\int_{|x-y(t)|\geq R}|u_t|^2+|u_x|^2+|u|^2+|u|^{p+1}\,dx\Big).\end{aligned}\tag{3.1}$$

Note that when $|\omega|=\omega_c$,

$$\begin{aligned}&\Big(\frac{4}{p-1}-1\Big)\|u_t\|_{L^2}^2-\frac{p+3}{p-1}\|u_x\|_{L^2}^2+\frac{p-5}{p-1}\|u\|_{L^2}^2+2\frac{p+3}{p^2-1}\|u\|_{L^{p+1}}^{p+1}\\ &\quad=\frac{8}{p-1}\|u_t-i\omega u\|_{L^2}^2-\frac{p+3}{p-1}\cdot 2E(u_0,u_1)-\frac{16\omega}{p-1}Q(u_0,u_1).\end{aligned}$$

Inserting this equality into (3.1), we prove the lemma. □

3.2. **The choice of initial data.** In this subsection, we choose the initial data such that it is close to the standing waves solution but leads the instability. We set

$$\vec{u}_0 = (1+a)\overrightarrow{\Phi_\omega}. \tag{3.2}$$

Here, $a \in (0, a_0)$ is an arbitrary small constant, and a_0 will be decided later. Then we have

Lemma 3.3. *Let $\vec{u}_0$ be defined in* (3.2)*, then*

$$P(\vec{u}_0) = 0,$$

and

$$Q(\vec{u}_0) - Q\left(\overrightarrow{\Phi_\omega}\right) = -2a\omega\|\phi_\omega\|_{L^2}^2 + O(a^2).$$

Proof. It follows from the definition that $P(\vec{u}_0) = 0$. Now consider $Q(\vec{u}_0)$. We write

$$\begin{aligned} Q(\vec{u}_0) - Q\left(\overrightarrow{\Phi_\omega}\right) =& \left\langle Q'\left(\overrightarrow{\Phi_\omega}\right), \vec{u}_0 - \overrightarrow{\Phi_\omega}\right\rangle + O\big(\|\vec{u}_0 - \overrightarrow{\Phi_\omega}\|_{H^1\times L^2}^2\big) \\ =& -\omega\left\langle \phi_\omega, u_0 - \phi_\omega\right\rangle - \left\langle i\phi_\omega, u_1 - i\omega\phi_\omega\right\rangle + O\big(a^2\big) \\ =& -2a\omega\|\phi_\omega\|_{L^2}^2 + O(a^2). \end{aligned}$$

This finishes the proof of the lemma. □

Using the above lemma, we can scale the main part in $I'(t)$.

Lemma 3.4. *Let $\vec{u}_0$ be defined in* (3.2)*, then*

$$-\frac{p+3}{p-1}\cdot 2E(\vec{u}_0) - \frac{16\omega}{p-1}Q(\vec{u}_0) = \frac{5-p}{p-1}\cdot 4a\omega^2\|\phi_\omega\|_{L^2}^2 + O(a^2).$$

Proof. Making use of Lemma 2.1 (2), we have

$$\begin{aligned} &-\frac{p+3}{p-1}\cdot 2E(\vec{u}_0) - \frac{16\omega}{p-1}Q(\vec{u}_0) \\ &\quad= -\frac{p+3}{p-1}\cdot 2\Big[E(\vec{u}_0) - E\left(\overrightarrow{\Phi_\omega}\right)\Big] - \frac{16\omega}{p-1}\Big[Q(\vec{u}_0) - Q\left(\overrightarrow{\Phi_\omega}\right)\Big] \\ &\qquad -\frac{p+3}{p-1}\cdot 2E\left(\overrightarrow{\Phi_\omega}\right) - \frac{16\omega}{p-1}Q\left(\overrightarrow{\Phi_\omega}\right) \\ &\quad= -\frac{p+3}{p-1}\cdot 2\Big[E(\vec{u}_0) - E\left(\overrightarrow{\Phi_\omega}\right)\Big] - \frac{16\omega}{p-1}\Big[Q(\vec{u}_0) - Q\left(\overrightarrow{\Phi_\omega}\right)\Big]. \end{aligned}$$

Since

$$E(\vec{u}_0) - E\left(\overrightarrow{\Phi_\omega}\right) = \Big[S_\omega(\vec{u}_0) - S_\omega\left(\overrightarrow{\Phi_\omega}\right)\Big] - \omega\Big[Q(\vec{u}_0) - Q\left(\overrightarrow{\Phi_\omega}\right)\Big],$$

we further write

$$\begin{aligned} &-\frac{p+3}{p-1}\cdot 2E(\vec{u}_0) - \frac{16\omega}{p-1}Q(\vec{u}_0) \\ &\quad= -\frac{p+3}{p-1}\cdot 2\Big[S_\omega(\vec{u}_0) - S_\omega\left(\overrightarrow{\Phi_\omega}\right)\Big] - \frac{5-p}{p-1}\cdot 2\omega\Big[Q(\vec{u}_0) - Q\left(\overrightarrow{\Phi_\omega}\right)\Big]. \end{aligned}$$

By Taylor's type extension, we have

$$S_\omega(\vec{u}_0) - S_\omega\left(\overrightarrow{\Phi_\omega}\right) = O\big(\|\vec{u}_0 - \overrightarrow{\Phi_\omega}\|_{H^1\times L^2}^2\big) = O(a^2).$$

Now using Lemma 3.3, we prove the lemma. □

Similar computation also gives

Lemma 3.5. *Let* $\lambda \in \mathbb{R}^+$ *with* $\lambda \lesssim 1$, $\vec{u}_0$ *be defined in* (3.2)*, then*

$$S_{\lambda\omega}(\vec{u}_0) - S_{\lambda\omega}\left(\overrightarrow{\Phi_\omega}\right) = -2(\lambda-1)a\omega^2\|\phi_\omega\|_{L^2}^2 + O(a^2).$$

Proof. By the definition of S_ω, we have

$$\begin{aligned} &S_{\lambda\omega}(\vec{u}_0) - S_{\lambda\omega}\left(\overrightarrow{\Phi_\omega}\right) \\ &\quad= S_\omega(\vec{u}_0) - S_\omega\left(\overrightarrow{\Phi_\omega}\right) + (\lambda-1)\omega\Big[Q_\omega(\vec{u}_0) - Q_\omega\left(\overrightarrow{\Phi_\omega}\right)\Big]. \end{aligned}$$

Since

$$S(\vec{u}_0) - S\left(\overrightarrow{\Phi_\omega}\right) = O(a^2),$$

then by Lemma 3.3, we prove the lemma. □

Now we control the rest terms of the virial identity in Lemma 3.2. We argue by contradiction, and suppose that the standing wave solution u_ω is stable. That is, for any $\varepsilon > 0$, there exists a constant $a_0 > 0$, such that for any $a \in (0, a_0)$, if $\vec{u}_0 \in U_a(\overrightarrow{\Phi_\omega})$, then $\vec{u}(t) \in U_\varepsilon(\overrightarrow{\Phi_\omega})$ for any $t \in \mathbb{R}$. We may assume that $\vec{u} \in U_\varepsilon(\vec{\Phi}_\omega)$ for $\varepsilon \le \varepsilon_0$, where ε_0 is determined in Lemma 2.5. Hence by Lemma 2.5, we can write

$$u = e^{i\theta}(\phi_{\lambda\omega} + \xi)(\cdot - y); \qquad u_t = e^{i\theta}(i\lambda\omega\phi_{\lambda\omega} + \eta)(\cdot - y) \tag{3.3}$$

with $\vec{\xi} = (\xi, \eta)$ satisfying the orthogonal conditions (2.25).

3.3. **Lower control of $\|u_t - i\omega u\|_{L^2}$.** In this subsection, we prove the following lemma.

Lemma 3.6. *Suppose that* $\vec{\xi} = (\xi, \eta)$ *defined in* (3.3) *satisfying the orthogonal conditions* (2.25)*, then*

$$\begin{aligned} \|u_t - i\omega u\|_{L^2}^2 =& (\lambda-1)^2\omega^2\|\phi_\omega\|_{L^2}^2 + \|\eta - i\omega\xi\|_{L^2}^2 \\ &+ O\Big(|\lambda-1|^3 + a|\lambda-1| + \|\vec{\xi}\|_{H^1\times L^2}^3\Big). \end{aligned}$$

Proof. By (3.3), we expand it as

$$\begin{aligned} \|u_t - i\omega u\|_{L^2}^2 =& \|i\lambda\omega\phi_{\lambda\omega} + \eta - i\omega(\phi_{\lambda\omega} + \xi)\|_{L^2}^2 \\ =& \|i(\lambda-1)\omega\phi_{\lambda\omega} + \eta - i\omega\xi\|_{L^2}^2 \\ =& (\lambda-1)^2\omega^2\|\phi_{\lambda\omega}\|_{L^2}^2 + 2(\lambda-1)\omega\big\langle \eta - i\omega\xi, i\phi_{\lambda\omega}\big\rangle + \|\eta - i\omega\xi\|_{L^2}^2. \end{aligned}$$

Noting that

$$\|\phi_{\lambda\omega}\|_{L^2}^2 = \|\phi_\omega\|_{L^2}^2 + O(|\lambda-1|),$$

then combining with the third orthogonal condition in (2.25), we further get

$$\begin{aligned} \|u_t - i\omega u\|_{L^2}^2 =& (\lambda-1)^2\omega^2\|\phi_\omega\|_{L^2}^2 + 2(\lambda-1)\omega\big\langle \eta, i\phi_{\lambda\omega}\big\rangle \\ &+ \|\eta - i\omega\xi\|_{L^2}^2 + O(|\lambda-1|^3). \end{aligned} \tag{3.4}$$

Now we consider the term $\langle \eta, i\phi_{\lambda\omega}\rangle$. First, we use the charge conservation law to obtain

$$\begin{aligned} Q(\vec{u}_0) - Q\left(\overrightarrow{\Phi_\omega}\right) &+ Q\left(\overrightarrow{\Phi_\omega}\right) - Q\left(\overrightarrow{\Phi_{\lambda\omega}}\right) \\ &= Q(\vec{u}) - Q\left(\overrightarrow{\Phi_{\lambda\omega}}\right) \\ &= -\big\langle \xi, \lambda\omega\phi_{\lambda\omega}\big\rangle - \big\langle \eta, i\phi_{\lambda\omega}\big\rangle + O\Big(\|\vec{\xi}\|^2_{H^1\times L^2}\Big). \end{aligned}$$

Then by the third orthogonal conditions in (2.25), we have

$$\big\langle \eta, i\phi_{\lambda\omega}\big\rangle = Q\left(\overrightarrow{\Phi_{\lambda\omega}}\right) - Q\left(\overrightarrow{\Phi_\omega}\right) - \Big[Q(\vec{u}_0) - Q\left(\overrightarrow{\Phi_\omega}\right)\Big] + O\Big(\|\vec{\xi}\|^2_{H^1\times L^2}\Big).$$

From Lemma 2.1, we have

$$Q\left(\overrightarrow{\Phi_{\lambda\omega}}\right) - Q\left(\overrightarrow{\Phi_\omega}\right) = O\big(|\lambda-1|^2\big),$$

and from Lemma 3.3, we have

$$Q(\vec{u}_0) - Q\left(\overrightarrow{\Phi_\omega}\right) = O(a).$$

Therefore, we obtain that

$$\big\langle \eta, i\phi_{\lambda\omega}\big\rangle = O\Big(a + |\lambda-1|^2 + \|\vec{\xi}\|^2_{H^1\times L^2}\Big). \tag{3.5}$$

Now together (3.4) with (3.5), we obtain the desirable result. □

3.4. **Upper control of $\|\vec{\xi}\|_{H^1\times L^2}$.** In this subsection, we give the following estimate on $\|\vec{\xi}\|_{H^1\times L^2}$.

Lemma 3.7. *Let $\vec{\xi} = (\xi, \eta)$ be defined in (3.3), then*

$$\|\vec{\xi}\|^2_{H^1\times L^2} = O(a|\lambda-1| + a^2) + o\big((\lambda-1)^2\big).$$

Proof. From the charge and energy conservation laws,

$$\begin{aligned} S_{\lambda\omega}(\vec{u}_0) &= S_{\lambda\omega}(\vec{u}) \\ &= S_{\lambda\omega}(\vec{u}) - S_{\lambda\omega}\left(\overrightarrow{\Phi_{\lambda\omega}}\right) + S_{\lambda\omega}\left(\overrightarrow{\Phi_{\lambda\omega}}\right) \\ &= \frac{1}{2}\left\langle S''_{\lambda\omega}\left(\overrightarrow{\Phi_{\lambda\omega}}\right)\vec{\xi}, \vec{\xi}\right\rangle + S_{\lambda\omega}\left(\overrightarrow{\Phi_{\lambda\omega}}\right) + o\big(\|\vec{\xi}\|^2_{H^1\times L^2}\big). \end{aligned}$$

Hence by Lemma 2.4,

$$\begin{aligned} \|\vec{\xi}\|^2_{H^1\times L^2} &\lesssim \frac{1}{2}\left\langle S''_{\lambda\omega}\left(\overrightarrow{\Phi_{\lambda\omega}}\right)\vec{\xi}, \vec{\xi}\right\rangle \\ &= \Big[S_{\lambda\omega}(\vec{u}_0) - S_{\lambda\omega}\left(\overrightarrow{\Phi_\omega}\right)\Big] - \Big[S_{\lambda\omega}\left(\overrightarrow{\Phi_{\lambda\omega}}\right) - S_{\lambda\omega}\left(\overrightarrow{\Phi_\omega}\right)\Big] + o\big(\|\vec{\xi}\|^2_{H^1\times L^2}\big). \end{aligned}$$

By Lemma 3.5,

$$S_{\lambda\omega}(\vec{u}_0) - S_{\lambda\omega}\left(\overrightarrow{\Phi_\omega}\right) = -2(\lambda-1)a\omega^2\|\phi_\omega\|^2_{L^2} + O(a^2),$$

and by Corollary 2.2,

$$S_{\lambda\omega}\big(\overrightarrow{\Phi_{\lambda\omega}}\big) - S_{\lambda\omega}\big(\overrightarrow{\Phi_\omega}\big) = o\big((\lambda-1)^2\big).$$

Therefore,

$$\|\vec{\xi}\|^2_{H^1\times L^2} = O(a|\lambda-1| + a^2) + o\big((\lambda-1)^2\big) + o\big(\|\vec{\xi}\|^2_{H^1\times L^2}\big).$$

Then, after absorbing the last term by the left-hand side one, we finish the proof of this lemma. □

3.5. **Proof of Theorem 1.2.** As discussed above, we assume that $\vec{u} \in U_\varepsilon(\vec{\Phi}_\omega)$, and thus $|\lambda - 1| \lesssim \varepsilon$. First, we note that from the definition of $I(t)$, we have the time uniform boundedness of $I(t)$,

$$\sup_{t\in\mathbb{R}} I(t) \lesssim R\Big(\|\overrightarrow{\Phi_\omega}\|^2_{H^1\times L^2} + 1\Big). \tag{3.6}$$

Now we consider the estimate on $I'(t)$. First, by (3.3), the exponential decaying of ϕ_ω and $\frac{1}{2} \le \lambda \le \frac{3}{2}$,

$$\begin{aligned}
&\int_{|x-y(t)|\ge R} \Big[|u_t|^2 + |u_x|^2 + |u|^2 + |u|^{p+1}\Big]\,dx \\
&\qquad \lesssim \int_{|x|\ge R} \Big[|\phi_{\lambda\omega}|^2 + |\partial_x\phi_{\lambda\omega}|^2 + |\xi|^2 + |\partial_x\xi|^2 + |\xi|^{p+1} + |\eta|^2\Big]\,dx \\
&\qquad = O\big(\|\vec{\xi}\|^2_{H^1\times L^2} + \frac{1}{R}\big).
\end{aligned}$$

Hence by Lemma 3.2,

$$\begin{aligned}
I'(t) =& -\frac{p+3}{p-1}\cdot 2E(u_0,u_1) - \frac{16\omega}{p-1}Q(u_0,u_1) \\
&\quad - 2\dot{y}P(u_0,u_1) + \frac{8}{p-1}\|u_t - i\omega u\|^2_{L^2} + O\Big(\|\vec{\xi}\|^2_{H^1\times L^2} + \frac{1}{R}\Big).
\end{aligned}$$

Now by Lemma 3.4, Lemma 3.3, and Lemma 3.6, we have

$$\begin{aligned}
I'(t) =& \frac{5-p}{p-1}\cdot 4a\omega^2\|\phi_\omega\|^2_{L^2} + (\lambda-1)^2\omega^2\|\phi_\omega\|^2_{L^2} + \|\eta - i\omega\xi\|^2_{L^2} \\
&\quad + O\Big(a^2 + a|\lambda - 1| + |\lambda-1|^3 + \|\vec{\xi}\|^2_{H^1\times L^2} + \frac{1}{R}\Big).
\end{aligned}$$

Setting $R = a^{-2}$ and applying Lemma 3.7, we further get

$$\begin{aligned}
I'(t) =& \frac{5-p}{p-1}\cdot 4a\omega^2\|\phi_\omega\|^2_{L^2} + (\lambda-1)^2\omega^2\|\phi_\omega\|^2_{L^2} + \|\eta - i\omega\xi\|^2_{L^2} \\
&\quad + O\big(a^2 + a|\lambda-1|\big) + o\big(|\lambda-1|^2\big).
\end{aligned}$$

Choosing ε and a_0 small enough, we obtain that for any $a \in (0, a_0)$,

$$I'(t) \ge \frac{5-p}{p-1}\cdot 2a\omega^2\|\phi_\omega\|^2_{L^2}.$$

This implies that $I(t) \to +\infty$ when $t \to +\infty$, which is contradicted with (3.6). Hence we prove the instability of the standing wave u_ω and thus give the proof of Theorem 1.2.

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