

# Precise Limit in Wasserstein Distance for Conditional Empirical Measures of Dirichlet Diffusion Processes\*

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November 24, 2021

## Abstract

Let  $M$  be a  $d$ -dimensional connected compact Riemannian manifold with boundary  $\partial M$ , let  $V \in C^2(M)$  such that  $\mu(dx) := e^{V(x)}dx$  is a probability measure, and let  $X_t$  be the diffusion process generated by  $L := \Delta + \nabla V$  with  $\tau := \inf\{t \geq 0 : X_t \in \partial M\}$ . Consider the conditional empirical measure  $\mu_t^\nu := \mathbb{E}^\nu(\frac{1}{t} \int_0^t \delta_{X_s} ds | t < \tau)$  for the diffusion process with initial distribution  $\nu$  such that  $\nu(\partial M) < 1$ . Then

$$\lim_{t \rightarrow \infty} \{t \mathbb{W}_2(\mu_t^\nu, \mu_0)\}^2 = \frac{1}{\{\mu(\phi_0)\nu(\phi_0)\}^2} \sum_{m=1}^{\infty} \frac{\{\nu(\phi_0)\mu(\phi_m) + \mu(\phi_0)\nu(\phi_m)\}^2}{(\lambda_m - \lambda_0)^3},$$

where  $\nu(f) := \int_M f d\nu$  for a measure  $\nu$  and  $f \in L^1(\nu)$ ,  $\mu_0 := \phi_0^2 \mu$ ,  $\{\phi_m\}_{m \geq 0}$  is the eigenbasis of  $-L$  in  $L^2(\mu)$  with the Dirichlet boundary,  $\{\lambda_m\}_{m \geq 0}$  are the corresponding Dirichlet eigenvalues, and  $\mathbb{W}_2$  is the  $L^2$ -Wasserstein distance induced by the Riemannian metric.

AMS subject Classification: 60D05, 58J65.

Keywords: Conditional empirical measure, Dirichlet diffusion process, Wasserstein distance, eigenvalues, eigenfunctions.

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\*Supported in part by the National Key R&D Program of China (No. 2020YFA0712900), NNSFC (11771326, 11831014, 11921001), and DFG through the CRC Taming uncertainty and profiting from randomness and low regularity in analysis, stochastics and their applications.

# 1 Introduction

Let  $M$  be a  $d$ -dimensional connected compact Riemannian manifold with a smooth boundary  $\partial M$ . Let  $V \in C^2(M)$  such that  $\mu(dx) = e^{V(x)}dx$  is a probability measure on  $M$ , where  $dx$  is the Riemannian volume measure. Let  $X_t$  be the diffusion process generated by  $L := \Delta + \nabla V$  with hitting time

$$\tau := \inf\{t \geq 0 : X_t \in \partial M\}.$$

Here, according to the convention in Riemannian geometry, the vector field  $\nabla V$  is regarded as a first-order differential operator with  $(\nabla V)f := \langle \nabla V, \nabla f \rangle$  for differentiable functions  $f$ . Denote by  $\mathcal{P}$  the set of all probability measures on  $M$ , and let  $\mathbb{E}^\nu$  be the expectation taken for the diffusion process with initial distribution  $\nu \in \mathcal{P}$ . Consider the conditional empirical measure

$$\mu_t^\nu := \mathbb{E}^\nu \left( \frac{1}{t} \int_0^t \delta_{X_s} ds \middle| t < \tau \right), \quad t > 0, \nu \in \mathcal{P}.$$

Since  $\tau = 0$  when  $X_0 \in \partial M$ , to ensure  $\mathbb{P}^\nu(\tau > t) > 0$  we only consider

$$\nu \in \mathcal{P}_0 := \{ \nu \in \mathcal{P} : \nu(M^\circ) > 0 \}, \quad M^\circ := M \setminus \partial M.$$

Let  $\{\phi_m\}_{m \geq 0}$  be the eigenbasis in  $L^2(\mu)$  of  $-L$  with the Dirichlet boundary such that  $\phi_0 > 0$  in  $M^\circ$ , and let  $\{\lambda_m\}_{m \geq 0}$  be the associated eigenvalues listed in the increasing order counting multiplicities; that is,  $\{\phi_m\}_{m \geq 0}$  is an orthonormal basis of  $L^2(\mu)$  such that

$$L\phi_m = -\lambda_m\phi_m, \quad m \geq 0.$$

Then  $\mu_0 := \phi_0^2\mu$  is a probability measure on  $M$ . It is easy to see from [5, Theorem 2.1] that for any probability measure  $\nu$  supported on  $M^\circ$ , we have

$$\lim_{t \rightarrow \infty} \|\mu_t^\nu - \mu_0\|_{var} = 0,$$

where  $\|\cdot\|_{var}$  is the total variational norm.

In this paper, we investigate the convergence of  $\mu_t^\nu$  to  $\mu_0$  under the Wasserstein distance  $\mathbb{W}_2$ :

$$\mathbb{W}_2(\mu_1, \mu_2) := \inf_{\pi \in \mathcal{C}(\mu_1, \mu_2)} \left( \int_{M \times M} \rho(x, y)^2 \pi(dx, dy) \right)^{\frac{1}{2}}, \quad \mu_1, \mu_2 \in \mathcal{P},$$

where  $\mathcal{C}(\mu_1, \mu_2)$  is the set of all probability measures on  $M \times M$  with marginal distributions  $\mu_1$  and  $\mu_2$ , and  $\rho(x, y)$  is the Riemannian distance between  $x$  and  $y$ , i.e. the length of the shortest curve on  $M$  linking  $x$  and  $y$ .

Recently, the convergence rate under  $\mathbb{W}_2$  has been characterized in [18] for the empirical measures of the  $L$ -diffusion processes without boundary (i.e.  $\partial M = \emptyset$ ) or with a reflecting boundary. Since in the present setting the diffusion process is killed at time  $\tau$ , it is reasonable to consider the conditional empirical measure  $\mu_t^\nu$  given  $t < \tau$ . This is a counterpart to the quasi-ergodicity for the convergence of the conditional distribution  $\tilde{\mu}_t$  of  $X_t$  given  $t < \tau$ . Unlike in the case without boundary or with a reflecting boundary where both the

distribution and the empirical measure of  $X_t$  converge to the unique invariant probability measure, in the present case the conditional distribution  $\tilde{\mu}_t$  of  $X_t$  given  $t < \tau$  converges to  $\tilde{\mu}_0 := \frac{\phi_0}{\mu(\phi_0)}\mu$  rather than  $\mu_0 := \phi_0^2\mu$ , and this convergence is called the quasi-ergodicity in the literature, see for instance [6] and references within.

Let  $\nu(f) := \int_M f d\nu$  for  $\nu \in \mathcal{P}$  and  $f \in L^1(\nu)$ . The main result of this paper is the following.

**Theorem 1.1.** *For any  $\nu \in \mathcal{P}_0$ ,*

$$\lim_{t \rightarrow \infty} \{t^2 \mathbb{W}_2(\mu_t^\nu, \mu_0)^2\} = I := \frac{1}{\{\mu(\phi_0)\nu(\phi_0)\}^2} \sum_{m=1}^{\infty} \frac{\{\nu(\phi_0)\mu(\phi_m) + \mu(\phi_0)\nu(\phi_m)\}^2}{(\lambda_m - \lambda_0)^3} > 0.$$

*If either  $d \leq 6$  or  $d \geq 7$  but  $\nu = h\mu$  with  $h \in L^{\frac{2d}{d+6}}(\mu)$ , then  $I < \infty$ .*

**Remark 1.1.** (1) Let  $X_t$  be the (reflecting) diffusion process generated by  $L$  on  $M$  where  $\partial M$  may be empty. We consider the mean empirical measure  $\hat{\mu}_t^\nu := \mathbb{E}(\frac{1}{t} \int_0^t \delta_{X_s} ds)$ , where  $\nu$  is the initial distribution of  $X_t$ . Then

$$(1.1) \quad \lim_{t \rightarrow \infty} \{t^2 \mathbb{W}_2(\hat{\mu}_t^\nu, \mu_0)^2\} = \sum_{m=1}^{\infty} \frac{\{\nu(\phi_m)\}^2}{\lambda_m^3} < \infty,$$

where  $\{\phi_m\}_{m \geq 1}$  is the eigenbasis of  $-L$  in  $L^2(\mu)$  with the Neumann boundary condition if  $\partial M$  exists,  $\{\lambda_m\}_{m \geq 1}$  are the corresponding non-trivial (Neumann) eigenvalues, and the limit is zero if and only if  $\nu = \mu$ . This can be confirmed by the proof of Theorem 1.1 with  $\phi_0 = 1, \lambda_0 = 0$  and  $\mu(\phi_m) = 0$  for  $m \geq 1$ . In this case,  $\mu$  is the unique invariant probability measure of  $X_t$ , so that  $\hat{\mu}_t^\mu = \mu$  for  $t \geq 0$  and hence the limit in (1.1) is zero for  $\nu = \mu$ . However, in the Dirichlet diffusion case, the conditional distribution of  $(X_s)_{0 \leq s \leq t}$  given  $t < \tau$  is no longer stationary, so that even starting from the limit distribution  $\mu_0$  we **do not have**  $\mu_t^{\mu_0} = \mu_0$  for  $t > 0$ . This leads to a non-zero limit in Theorem 1.1 even for  $\nu = \mu_0$ .

(2) It is also interesting to investigate the convergence of  $\mathbb{E}^\nu(\mathbb{W}_2(\mu_t, \mu_0)^2 | t < \tau)$  for  $\mu_t := \frac{1}{t} \int_0^t \delta_{X_s} ds$ , which is the counterpart to the study of [18] where the case without boundary or with a reflecting boundary is considered. According to [18], the convergence rate of  $\mathbb{E}^\nu(\mathbb{W}_2(\mu_t, \mu_0)^2 | t < \tau)$  will be at most  $t^{-1}$ , which is slower than the rate  $t^{-2}$  for  $\mathbb{W}_2(\mu_t^\nu, \mu_0)^2$  as shown in Theorem 1.1, see [15] for details, see also [16, 17] for extensions to diffusion processes on non-compact manifolds and SPDEs.

(3) Let  $\nu = h\mu$ . It is easy to see that  $I < \infty$  is equivalent to  $h \in \mathcal{D}((-L)^{-\frac{3}{2}})$ . By the Sobolev inequality, for any  $p \in [1, \frac{d}{3})$ , there exists a constant  $K > 0$  such that

$$(1.2) \quad \|(-L)^{-\frac{3}{2}} f\|_{L^{\frac{dp}{d-3p}}(\mu)} \leq K \|f\|_{L^p(\mu)}, \quad f \in L^p(\mu).$$

Taking  $p = \frac{2d}{d+6}$  which is large than 1 when  $d \geq 7$ , we see that  $h \in L^p(\mu)$  implies  $h \in \mathcal{D}((-L)^{-\frac{3}{2}})$  and hence  $I < \infty$ . So, the sharpness of the Sobolev inequality implies that of the condition  $h \in L^{\frac{2d}{d+6}}(\mu)$ .

In Section 2, we first recall some well known facts on the Dirichlet semigroup, then present an upper bound estimate on  $\|\nabla(\phi_m\phi_0^{-1})\|_\infty$ . The latter is non-trivial when  $\partial M$  is non-convex, and should be interesting by itself. With these preparations, we prove upper and lower bound estimates in Sections 3 and 4 respectively.

## 2 Some preparations

We first recall some well known facts on the Dirichlet semigroup, see for instances [4, 7, 8, 13]. Let  $\{\phi_m\}_{m \geq 0}$  be the eigenbasis of the Dirichlet operator  $L$  in  $L^2(\mu)$ , with Dirichlet eigenvalues  $\{\lambda_m\}_{m \geq 0}$  of  $-L$  listed in the increasing order counting multiplicities; that is,  $\{\phi_m\}_{m \geq 0}$  is an orthonormal basis of  $L^2(\mu)$  such that

$$L\phi_m = -\lambda_m\phi_m, \quad m \geq 0.$$

For simplicity, we denote  $a \preceq b$  for two positive functions  $a$  and  $b$  if  $a \leq cb$  holds for some constant  $c > 0$ . Then  $\lambda_0 > 0$  and

$$(2.1) \quad \|\phi_m\|_\infty \preceq \sqrt{m}, \quad m^{\frac{2}{d}} \preceq \lambda_m - \lambda_0 \preceq m^{\frac{2}{d}}, \quad m \geq 1.$$

Let  $\rho_\partial$  be the Riemannian distance function to the boundary  $\partial M$ . Then  $\phi_0^{-1}\rho_\partial$  is bounded such that

$$(2.2) \quad \|\phi_0^{-1}\|_{L^p(\mu_0)} < \infty, \quad p \in [1, 3].$$

The Dirichlet heat kernel has the representation

$$p_t^D(x, y) = \sum_{m=0}^{\infty} e^{-\lambda_m t} \phi_m(x) \phi_m(y), \quad t > 0, x, y \in M.$$

Let  $\mathbb{E}^x$  denote the expectation for the  $L$ -diffusion process starting at point  $x$ . Then Dirichlet diffusion semigroup generated by  $L$  is given by

$$(2.3) \quad \begin{aligned} P_t^D f(x) &:= \mathbb{E}^x[f(X_t)1_{\{t < \tau\}}] = \int_M p_t^D(x, y) f(y) \mu(dy) \\ &= \sum_{m=0}^{\infty} e^{-\lambda_m t} \mu(\phi_m f) \phi_m(x), \quad t > 0, f \in L^2(\mu). \end{aligned}$$

We have

$$(2.4) \quad \|P_t^D\|_{L^p(\mu) \rightarrow L^q(\mu)} := \sup_{\mu(|f|^p) \leq 1} \|P_t^D f\|_{L^q(\mu)} \preceq e^{-\lambda_0 t} (1 \wedge t)^{-\frac{d(q-p)}{2pq}}, \quad t > 0, q \geq p \geq 1.$$

Next, let  $L_0 = L + 2\nabla \log \phi_0$ . Then  $L_0$  is a self-adjoint operator in  $L^2(\mu_0)$  with semigroup  $P_t^0 := e^{tL_0}$  satisfying

$$(2.5) \quad P_t^0 f = e^{\lambda_0 t} \phi_0^{-1} P_t^D(f \phi_0), \quad f \in L^2(\mu_0), \quad t \geq 0.$$

So,  $\{\phi_0^{-1}\phi_m\}_{m \geq 0}$  is an eigenbasis of  $L_0$  in  $L^2(\mu_0)$  with

$$(2.6) \quad L_0(\phi_m\phi_0^{-1}) = -(\lambda_m - \lambda_0)\phi_m\phi_0^{-1}, \quad P_t^0(\phi_m\phi_0^{-1}) = e^{-(\lambda_m - \lambda_0)t}\phi_m\phi_0^{-1}, \quad m \geq 0, t \geq 0.$$

Consequently,

$$(2.7) \quad P_t^0 f = \sum_{m=0}^{\infty} \mu_0(f\phi_m\phi_0^{-1})e^{-(\lambda_m - \lambda_0)t}\phi_m\phi_0^{-1}, \quad f \in L^2(\mu_0),$$

and the heat kernel of  $P_t^0$  with respect to  $\mu_0$  is given by

$$(2.8) \quad p_t^0(x, y) = \sum_{m=0}^{\infty} (\phi_m\phi_0^{-1})(x)(\phi_m\phi_0^{-1})(y)e^{-(\lambda_m - \lambda_0)t}, \quad x, y \in M, t > 0.$$

By the intrinsic ultracontractivity, see for instance [9], we have

$$(2.9) \quad \|P_t^0 - \mu_0\|_{L^1(\mu_0) \rightarrow L^\infty(\mu_0)} := \sup_{\mu_0(|f|) \leq 1} \|P_t^0 f - \mu_0(f)\|_\infty \preceq \frac{e^{-(\lambda_1 - \lambda_0)t}}{(1 \wedge t)^{\frac{d+2}{2}}}, \quad t > 0.$$

Combining this with the semigroup property and the contraction of  $P_t^0$  in  $L^p(\mu)$  for any  $p \geq 1$ , we obtain

$$(2.10) \quad \|P_t^0 - \mu_0\|_{L^p(\mu_0)} := \sup_{\mu_0(|f|^p) \leq 1} \|P_t^0 f - \mu_0(f)\|_{L^p(\mu_0)} \preceq e^{-(\lambda_1 - \lambda_0)t}, \quad t \geq 0, p \geq 1.$$

By the interpolation theorem, (2.9) and (2.10) yield

$$(2.11) \quad \|P_t^0 - \mu_0\|_{L^p(\mu_0) \rightarrow L^q(\mu_0)} \preceq e^{-(\lambda_1 - \lambda_0)t} \{1 \wedge t\}^{-\frac{(d+2)(q-p)}{2pq}}, \quad t > 0, \infty \geq q > p \geq 1.$$

Since  $\mu_0(\phi_m^2\phi_0^{-2}) = 1$ , (2.11) for  $p = 2$  implies

$$\|\phi_m\phi_0^{-1}\|_\infty = e^{(\lambda_m - \lambda_0)t} \|P_t^0(\phi_m\phi_0^{-1})\|_\infty \preceq \frac{e^{(\lambda_m - \lambda_0)t}}{(1 \wedge t)^{\frac{d+2}{4}}}, \quad t > 0.$$

Taking  $t = (\lambda_m - \lambda_0)^{-1}$  and applying (2.1), we derive

$$(2.12) \quad \|\phi_m\phi_0^{-1}\|_\infty \preceq m^{\frac{d+2}{2d}}, \quad m \geq 1.$$

In the remainder of this section, we investigate gradient estimates on  $P_t^0$  and  $\phi_m\phi_0^{-1}$ , which will be used in Section 4 for the study of the lower bound estimate on  $\mathbb{W}_2(\mu_t^\nu, \mu_0)$ . To this end, we need to estimate the Hessian tensor of  $\log \phi_0$ .

Let  $N$  be the inward unit normal vector field of  $\partial M$ . We call  $M$  (or  $\partial M$ ) convex if

$$(2.13) \quad \langle \nabla_u N, u \rangle = \text{Hess}_{\rho_\partial}(u, u) \leq 0, \quad u \in T\partial M,$$

where  $\rho_\partial$  is the distance function to the boundary  $\partial M$ , and  $T\partial M$  is the tangent bundle of the  $(d-2)$ -dimensional manifold  $\partial M$ . When  $d = 1$ , the boundary  $\partial M$  degenerates to a set of two end points, such that  $\partial M = \emptyset$  and the condition (2.13) trivially holds; that is,  $M$  is convex for  $d = 1$ . Recall that  $M^\circ := M \setminus \partial M$  is the interior of  $M$ .

**Lemma 2.1.** *If  $\partial M$  is convex, then there exists a constant  $K_0 \geq 0$  such that*

$$\text{Hess}_{\log \phi_0}(u, u) \leq K_0 |u|^2, \quad u \in TM^\circ.$$

*Proof.* Since  $M$  is compact with smooth boundary, there exists a constant  $r_0 > 0$  such that  $\rho_\partial$  is smooth on the set

$$\partial_0 M := \{x \in M : \rho_\partial(x) \leq r_0\}.$$

Since  $\phi_0$  is smooth and satisfies  $\phi_0 \geq c\rho_\partial$  for some constant  $c > 0$ , we have  $\log(\phi_0 \rho_\partial^{-1}) \in C_b^2(\partial_0 M)$ . So, it suffices to find a constant  $c > 0$  such that

$$(2.14) \quad \text{Hess}_{\log \rho_\partial}(u, u) \leq c |u|^2, \quad u \in TM^\circ.$$

To this end, we first estimate  $\text{Hess}_{\rho_\partial}$  on the boundary  $\partial M$ . For any  $x \in \partial M$  and  $u \in T_x M$ , consider the orthogonal decomposition  $u = u_1 + u_2$ , where

$$u_1 = \langle N, u \rangle N, \quad u_2 := u - u_1 \in T\partial M.$$

Since  $|\nabla \rho_\partial| = 1$  on  $\partial_0 M$ , we have

$$(2.15) \quad \text{Hess}_{\rho_\partial}(X, N) = \text{Hess}_{\rho_\partial}(X, \nabla \rho_\partial) = \frac{1}{2} \langle X, \nabla |\nabla \rho_\partial|^2 \rangle = 0, \quad X \in T_x M.$$

On the other hand, since  $u_2 \in T\partial M$  and  $\nabla \rho_\partial = N$  on  $\partial M$ , (2.13) implies

$$\text{Hess}_{\rho_\partial}(u_2, u_2) = \langle \nabla_{u_2} N, u_2 \rangle \leq 0.$$

Combining this with (2.15) we obtain

$$\text{Hess}_{\rho_\partial}(u, u) = \langle N, u \rangle^2 \text{Hess}_{\rho_\partial}(N, N) + 2\langle N, u \rangle \text{Hess}_{\rho_\partial}(u_2, N) + \text{Hess}_{\rho_\partial}(u_2, u_2) \leq 0$$

for  $u \in \cup_{x \in \partial M} T_x M$ . Since  $\text{Hess}_{\rho_\partial}$  is smooth on the compact set  $\partial_0 M$ , this implies

$$\text{Hess}_{\rho_\partial}(u, u) \leq c |u|^2 \rho_\partial(x), \quad x \in M, u \in T_x M$$

for some constant  $c > 0$ . Then the desired estimate (2.14) follows from

$$\text{Hess}_{\log \rho_\partial}(u, u) = \rho_\partial^{-1} \text{Hess}_{\rho_\partial}(u, u) - \rho_\partial^{-2} \langle \nabla \rho_\partial, u \rangle^2 \leq c |u|^2, \quad u \in TM^\circ.$$

□

By Lemma 2.1, when  $\partial M$  is convex, there exists a constant  $K \geq 0$  such that

$$(2.16) \quad \text{Ric} - \text{Hess}_{V+2\log \phi_0} \geq -K.$$

Since the diffusion process generated by  $L_0 := \Delta + \nabla(V + 2\log \phi_0)$  is non-explosive in  $M^\circ$ , by (2.16) and Bakry-Emery's semigroup calculus, (see for instance [3] or [13, Theorem 2.3.3]), we have

$$(2.17) \quad |\nabla P_t^0 g| \leq e^{Kt} P_t^0 |\nabla g|, \quad t \geq 0, g \in C_b^1(M)$$

and for any  $p > 1$ , there exists a constant  $c(p) > 0$  such that

$$(2.18) \quad \begin{aligned} |\nabla P_t^0 g|^2 &\leq \frac{2K\{(P_t^0|g|^{p\wedge 2})(P_t^0|g|)^{(2-p)^+} - (P_t^0|g|)^2\}}{(p\wedge 2)(p\wedge 2-1)(1-e^{-2Kt})} \\ &\leq \frac{c(p)}{1\wedge t}(P_t^0|g|^p)^{\frac{2}{p}}, \quad t > 0, g \in \mathcal{B}_b(M). \end{aligned}$$

When  $\partial M$  is non-convex, we take as in [12] a conformal change of metric to make it convex under the new metric. More precisely, we have the following result.

**Lemma 2.2.** *There exists a function  $1 \leq \phi \in C_b^\infty(M)$  such that  $\partial M$  is convex under the metric  $\langle \cdot, \cdot \rangle_\phi := \phi^{-2} \langle \cdot, \cdot \rangle$ . Moreover, there exists a smooth vector field  $Z_\phi$  on  $M$  such that*

$$(2.19) \quad L_0 = \phi^{-2} \Delta^\phi + Z_\phi + 2\phi^{-1} \nabla^\phi \log \phi_0,$$

where  $\nabla^\phi$  and  $\Delta^\phi$  are the gradient and Laplace-Beltrami operators induced by  $\langle \cdot, \cdot \rangle_\phi$  respectively.

*Proof.* let  $\delta > 0$  such that the second fundamental form of  $\partial M$  is bounded below by  $-\delta$ . Take  $1 \leq \phi \in C_b^\infty(M)$  such that  $\phi = 1 + \delta \rho_\partial$  in a neighborhood of  $\partial M$  in which the distance function  $\rho_\partial$  to  $\partial M$  is smooth. By [14, Lemma 2.1] (see also [12]),  $\partial M$  is convex under the metric  $\langle \cdot, \cdot \rangle_\phi := \phi^{-2} \langle \cdot, \cdot \rangle$ . Next, according to the proof of [14, Lemma 2.2], there exists a smooth vector field  $Z_\phi$  on  $M$  such that (2.19) holds.  $\square$

Let  $1 \leq \phi \in C_b^\infty(M)$  be as in Lemma 2.2, and let  $P_t^\phi$  be the diffusion semigroup generated by

$$L^\phi := \phi L_0 = \phi^{-1} \Delta^\phi + \phi Z_\phi + 2\nabla^\phi \log \phi_0.$$

We have the following result.

**Lemma 2.3.** *Let  $1 \leq \phi \in C_b^\infty(M)$  be as in Lemma 2.2.*

(1) *For any  $q \in (1, \infty]$ , there exists a constant  $c(q) > 0$  such that*

$$(2.20) \quad |\nabla^\phi P_t^\phi f|_\phi \leq \frac{c(q)}{\sqrt{t}} (P_t^\phi |f|^q)^{\frac{1}{q}}, \quad t > 0, f \in C_b^1(M).$$

*Moreover, there exists a constant  $K > 0$  such that*

$$(2.21) \quad |\nabla^\phi P_t^\phi f|_\phi \leq e^{Kt} P_t^\phi |\nabla^\phi f|_\phi, \quad t > 0, f \in C_b^1(M).$$

(2) *There exists a constant  $c > 0$  such that*

$$(2.22) \quad \|P_t^\phi\|_{L^p(\mu_0) \rightarrow L^\infty(\mu_0)} \leq c(1 \wedge t)^{-\frac{d+2}{2p}}, \quad t > 0, p \in [1, \infty].$$

*Proof.* (1) Since  $\partial M$  is convex under the metric  $\langle \cdot, \cdot \rangle_\phi$ , by Lemma 2.1, we find a constant  $K_0^\phi > 0$  such that

$$(2.23) \quad 2\text{Hess}_{\log \phi_0}^\phi(u, u) \leq K_0^\phi |u|_\phi^2, \quad u \in TM^\circ,$$

where  $\text{Hess}^\phi$  is the Hessian tensor induced by the metric  $\langle \cdot, \cdot \rangle_\phi$ . Since the operator  $A^\phi := \phi^{-1}\Delta^\phi + \phi Z_\phi$  is a  $C^2$ -smooth strictly elliptic second order differential operator on the compact manifold  $M$ , it has bounded below Bakry-Emery curvature; that is, there exists a constant  $K_1^\phi > 0$  such that

$$A^\phi |\nabla^\phi f|_\phi^2 - 2\langle \nabla^\phi A^\phi f, \nabla^\phi f \rangle_\phi \geq -K_1^\phi |\nabla^\phi f|_\phi^2, \quad f \in C^\infty(M).$$

Combining this with (2.23) we obtain

$$L^\phi |\nabla^\phi f|_\phi^2 - 2\langle \nabla^\phi L^\phi f, \nabla^\phi f \rangle_\phi \geq -(K_0^\phi + K_1^\phi) |\nabla^\phi f|_\phi^2 =: -K^\phi |\nabla^\phi f|_\phi^2, \quad f \in C^\infty(M^\circ),$$

which means that the Bakry-Emery curvature of  $L^\phi$  is bounded below by  $-K^\phi$ . By the same reason leading to (2.17) and (2.18), this implies (2.20) and (2.21).

(2) To estimate  $\|P_t^\phi\|_{L^p(\mu_0) \rightarrow L^\infty(\mu_0)}$ , we make use of [10, Theorem 4.5(b)] or [11, Theorem 3.3.15(2)], which says that (2.9) implies the super Poincaré inequality

$$\mu_0(f^2) \leq r\mu_0(|\nabla f|^2) + \beta(1 + r^{-\frac{d+2}{2}})\mu_0(|f|)^2, \quad r > 0, f \in C_b^1(M)$$

for some constant  $\beta > 0$ . Let  $\mu^\phi = \frac{\phi^{-1}\mu_0}{\mu_0(\phi^{-1})}$ . By  $L^\phi = \phi L_0$  we obtain

$$\mathcal{E}^\phi(f, g) := - \int_M f L^\phi g d\mu^\phi = - \frac{1}{\mu_0(\phi^{-1})} \int_M f L_0 g d\mu_0 = \frac{1}{\mu(\phi^{-1})} \mu_0(\langle \nabla f, \nabla g \rangle), \quad f, g \in C_b^2(M).$$

Then the above super Poincaré inequality implies

$$\mu^\phi(f^2) \leq r\mathcal{E}^\phi(f, f) + \beta'(1 + r^{-\frac{d+2}{2}})\mu^\phi(|f|)^2, \quad f \in C_b^1(M)$$

for some constant  $\beta' > 0$ . Using [10, Theorem 4.5(b)] or [11, Theorem 3.3.15(2)] again, this implies

$$\|P_t^\phi\|_{L^p(\mu^\phi) \rightarrow L^\infty(\mu^\phi)} \leq \kappa(1 \wedge t)^{-\frac{d+2}{2p}}, \quad t > 0, p \in [1, \infty]$$

for some constant  $\kappa > 0$ . Noting that

$$\|\phi\|_\infty^{-1}\mu_0 \leq \mu^\phi \leq \|\phi\|_\infty\mu_0,$$

we find a constant  $c > 0$  such that (2.22) holds.  $\square$

**Lemma 2.4.** *For any  $p \in (1, \infty]$  and  $q \in (1, p)$ , there exists a constant  $c > 0$  such that for any  $f \in D(L_0)$ ,*

$$(2.24) \quad \|\nabla P_t^0 f\|_\infty \leq ce^{-\lambda_0 t} \left\{ (1 \wedge t)^{-\frac{1}{2} - \frac{d+2}{2p}} \|f\|_{L^p(\mu_0)} + (1 \wedge t)^{\frac{1}{2} - \frac{q(d+2)}{2p}} \|L_0 f\|_{L^p(\mu_0)} \right\}, \quad t > 0.$$

*Consequently, there exists a constant  $c > 0$  such that*

$$(2.25) \quad \|\nabla(\phi_m \phi_0^{-1})\|_\infty \leq cm^{\frac{d+4}{2d}}, \quad m \geq 1.$$

*Proof.* (a) By the semigroup property and the  $L^p(\mu_0)$  contraction of  $P_t^0$ , for the proof of (2.24) it suffices to consider  $t \in (0, 1]$ . Since  $1 \leq \phi \in C_b^\infty(M)$ , we have  $\mathcal{D}(L_0) = \mathcal{D}(L^\phi)$  and

$$(2.26) \quad P_t^\phi f = P_t^\phi f - \int_0^t P_s^\phi \{(\phi - 1)P_{t-s}^0 L_0 f\} ds, \quad t \geq 0, f \in \mathcal{D}(L_0).$$

Next, by (2.20) and (2.22), we obtain

$$(2.27) \quad \begin{aligned} \|\nabla P_t^\phi f\|_\infty &= \|\nabla P_{t/2}^\phi(P_{t/2}^\phi f)\|_\infty \\ &\leq t^{-\frac{1}{2}} \|P_{t/2}^\phi f\|_\infty \leq t^{-\frac{1}{2} - \frac{d+2}{2p}} \|f\|_{L^p(\mu_0)}, \quad t \in (0, 1]. \end{aligned}$$

Combining this with (2.11) and (2.20) leads to

$$\begin{aligned} \int_0^t \|\nabla P_s^\phi \{(\phi - 1)P_{t-s}^0 L_0 f\}\|_\infty ds &\leq \int_0^t s^{-\frac{1}{2}} \|\{P_s^\phi |P_{t-s}^0 L_0 f|^q\}^{\frac{1}{q}}\|_\infty ds \\ &\leq \int_0^{\frac{t}{2}} s^{-\frac{1}{2}} \|P_{t-s}^0 L_0 f\|_\infty ds + \int_{\frac{t}{2}}^t s^{-\frac{1}{2}} \|\{P_s^\phi |P_{t-s}^0 L_0 f|^q\}^{\frac{1}{q}}\|_\infty ds \\ &\leq \int_0^{\frac{t}{2}} s^{-\frac{1}{2}} \|P_{t-s}^0\|_{L^p(\mu_0) \rightarrow L^\infty(\mu_0)} \|L_0 f\|_{L^p(\mu_0)} ds + \int_{\frac{t}{2}}^t s^{-\frac{1}{2}} \|P_s^\phi\|_{L^{\frac{p}{q}}(\mu_0) \rightarrow L^\infty(\mu_0)} \|L_0 f\|_{L^p(\mu_0)} ds \\ &\leq t^{\frac{1}{2} - \frac{q(d+2)}{2p}} \|L_0 f\|_{L^p(\mu_0)}. \end{aligned}$$

Substituting this and (2.27) into (2.26), we prove (2.24).

(b) Applying (2.24) to  $p = \infty$ ,  $f = \phi_m \phi_0^{-1}$ ,  $t = (\lambda_m - \lambda_0)^{-1}$  and using (2.6), we obtain

$$e^{-1} \|\nabla(\phi_m \phi_0^{-1})\|_\infty \leq (\lambda_m - \lambda_0)^{\frac{1}{2}} \|\phi_m \phi_0^{-1}\|_\infty, \quad m \geq 1.$$

This together with (2.1) and (2.12) implies (2.25) for some constant  $c > 0$ . □

### 3 Upper bound estimate

According to [18, Lemma 2.3], we have

$$(3.1) \quad \mathbb{W}_2(\mu_t^\nu, \mu_0)^2 \leq \int_M \frac{|\nabla L_0^{-1}(h_t^\nu - 1)|^2}{\mathcal{M}(h_t^\nu, 1)} d\mu_0 a,$$

where

$$h_t^\nu := \frac{d\mu_t^\nu}{d\mu_0}, \quad \mathcal{M}(a, b) := 1_{\{a \wedge b > 0\}} \frac{a - b}{\log a - \log b}.$$

So, to investigate the upper bound estimate, we first calculate  $h_t^\nu$ .

By (2.8), we have

$$(3.2) \quad \psi_s^\nu := \int_M \phi_0(x) p_s^0(x, \cdot) \nu(dx) = \nu(\phi_0) + \sum_{m=1}^{\infty} \nu(\phi_m) e^{-(\lambda_m - \lambda_0)s} \phi_m \phi_0^{-1}, \quad s > 0.$$

Next, (2.5) and (2.8) imply

$$(3.3) \quad \nu(P_s^D f) = e^{-\lambda_0 s} \nu(\phi_0 P_s^0(f \phi_0^{-1})) = e^{-\lambda_0 s} \int_M \psi_s^\nu \phi_0^{-1} f d\mu_0, \quad f \in \mathcal{B}^+(M),$$

where  $\mathcal{B}^+(M)$  is the class of nonnegative measurable functions on  $M$ . Moreover, for any  $t \geq s > 0$ , by the Markov property, (2.3), (2.5) and (3.3), we obtain

$$\begin{aligned} \int_M f d\mathbb{E}^\nu[\delta_{X_s} 1_{\{t < \tau\}}] &= \mathbb{E}^\nu[f(X_s) 1_{\{s < \tau\}}(P_{t-s}^D 1)(X_s)] = \nu(P_s^D \{f P_{t-s}^D 1\}) \\ &= e^{-\lambda_0 t} \int_M (\psi_s^\nu P_{t-s}^0 \phi_0^{-1}) f d\mu_0, \quad f \in \mathcal{B}^+(M). \end{aligned}$$

Then

$$\frac{d\mathbb{E}^\nu[\delta_{X_s} 1_{\{t < \tau\}}]}{d\mu_0} = e^{-\lambda_0 t} \psi_s^\nu P_{t-s}^0 \phi_0^{-1}.$$

Noting that (3.3) implies

$$\mathbb{E}^\nu[1_{\{t < \tau\}}] = \nu(P_t^D 1) = e^{-\lambda_0 t} \mu_0(\psi_t^\nu \phi_0^{-1}) = e^{-\lambda_0 t} \nu(\phi_0 P_t^0 \phi_0^{-1}),$$

we arrive at

$$\begin{aligned} (3.4) \quad h_t^\nu &:= \frac{d\mu_t^\nu}{d\mu_0} = \frac{1}{t \mathbb{E}^\nu[1_{\{t < \tau\}}]} \int_0^t \frac{d\mathbb{E}^\nu[\delta_{X_s} 1_{\{t < \tau\}}]}{d\mu_0} ds = 1 + \rho_t^\nu, \\ \rho_t^\nu &:= \frac{1}{t \nu(\phi_0 P_t^0 \phi_0^{-1})} \int_0^t \{\psi_s^\nu P_{t-s}^0 \phi_0^{-1} - \nu(\phi_0 P_t^0 \phi_0^{-1})\} ds. \end{aligned}$$

By (2.11),  $\|\phi_0\|_\infty < \infty$  and  $\|\phi_0^{-1}\|_{L^2(\mu_0)} = 1$ , we find a constant  $c > 0$  such that

$$\begin{aligned} (3.5) \quad |\nu(\phi_0 P_t^0 \phi_0^{-1}) - \nu(\phi_0) \mu(\phi_0)| &\leq \nu(\phi_0) \|P_t^0 \phi_0^{-1} - \mu_0(\phi_0^{-1})\|_\infty \\ &\leq c e^{-(\lambda_1 - \lambda_0)t}, \quad t \geq 1, \nu \in \mathcal{P}_0. \end{aligned}$$

Due to the lack of simple representation of the product  $\psi_s^\nu P_{t-s}^0 \phi_0^{-1}$  in terms of the eigenbasis  $\{\phi_m \phi_0^{-1}\}_{m \geq 0}$ , it is inconvenient to estimate the upper bound in (3.1). To this end, below we reduce this product to a linear combination of  $\psi_s^\nu$  and  $P_{t-s}^0 \phi_0^{-1}$ , for which the spectral representation works. Write

$$\begin{aligned} (3.6) \quad \psi_s^\nu P_{t-s}^0 \phi_0^{-1} - \nu(\phi_0 P_t^0 \phi_0^{-1}) &= I_1(s) + I_2(s), \\ I_1(s) &:= \{\psi_s^\nu - \nu(\phi_0)\} \cdot \{P_{t-s}^0 \phi_0^{-1} - \mu(\phi_0)\} + \nu(\phi_0 \{\mu(\phi_0) - P_t^0 \phi_0^{-1}\}), \\ I_2(s) &:= \mu(\phi_0) \{\psi_s^\nu - \nu(\phi_0)\} + \nu(\phi_0) \{P_{t-s}^0 \phi_0^{-1} - \mu(\phi_0)\}. \end{aligned}$$

By (2.7), (2.8) and (3.2), we have

$$\begin{aligned} (3.7) \quad P_{t-s}^0 \phi_0^{-1} - \mu(\phi_0) &= \sum_{m=1}^{\infty} \mu(\phi_m) e^{-(\lambda_m - \lambda_0)(t-s)} \phi_m \phi_0^{-1}, \\ \psi_s^\nu - \nu(\phi_0) &= \sum_{m=1}^{\infty} \nu(\phi_m) e^{-(\lambda_m - \lambda_0)s} \phi_m \phi_0^{-1}, \quad t > s > 0. \end{aligned}$$

Then

$$\begin{aligned}
\rho_t^\nu &= \tilde{\rho}_t^\nu + \frac{1}{t\nu(\phi_0 P_t^0 \phi_0^{-1})} \int_0^t I_1(s) ds - A_t, \\
(3.8) \quad \tilde{\rho}_t^\nu &:= \frac{1}{t\nu(\phi_0 P_t^0 \phi_0^{-1})} \sum_{m=1}^{\infty} \frac{\mu(\phi_0)\nu(\phi_m) + \nu(\phi_0)\mu(\phi_m)}{\lambda_m - \lambda_0} \phi_m \phi_0^{-1} \\
A_t &:= \frac{1}{t\nu(\phi_0 P_t^0 \phi_0^{-1})} \sum_{m=1}^{\infty} \frac{\{\mu(\phi_0)\nu(\phi_m) + \nu(\phi_0)\mu(\phi_m)\} e^{-(\lambda_m - \lambda_0)t}}{\lambda_m - \lambda_0} \phi_m \phi_0^{-1}.
\end{aligned}$$

Since  $\rho_t^\nu \in L^1(\mu_0)$ , the following lemma implies  $\tilde{\rho}_t^\nu \in L^1(\mu_0)$  for  $t > 0$ .

**Lemma 3.1.** *For any  $t_0 > 0$ , there exists a constant  $c > 0$  such that*

$$\mu_0(|\rho_t^\nu - \tilde{\rho}_t^\nu|) \leq c \|h\|_{L^2(\mu)} e^{-(\lambda_1 - \lambda_0)t}, \quad t \geq t_0, \nu = h\mu \in \mathcal{P}_0.$$

*Proof.* By (2.1) and (2.12), for any  $t_0 > 0$  we have

$$(3.9) \quad \sum_{m=1}^{\infty} \|\phi_m\|_{\infty} e^{-(\lambda_m - \lambda_0)t} \preceq e^{-(\lambda_1 - \lambda_0)t}, \quad t \geq t_0.$$

Combining this with (3.8) and (3.5), and noting that  $\|h\phi_0^{-1}\|_{L^2(\mu_0)} = \|h\|_{L^2(\mu)}$ , it suffices to show that

$$(3.10) \quad B := \frac{1}{t} \int_0^t \left\| \{\psi_s^\nu - \nu(\phi_0)\} \cdot \{P_{t-s}^0 \phi_0^{-1} - \mu(\phi_0)\} \right\|_{L^1(\mu_0)} ds \preceq \|h\|_{L^2(\mu)} e^{-(\lambda_1 - \lambda_0)t}, \quad t \geq t_0.$$

Since  $\|\phi_0^{-1}\|_{L^2(\mu_0)} = 1$  and  $\psi_s^\nu = P_0^s(h\phi_0^{-1})$  for  $\nu = h\mu$ , (2.10) yields that

$$\begin{aligned}
B &\leq \frac{1}{t} \int_0^t \|P_{t-s}^0 \phi_0^{-1} - \mu_0(\phi_0^{-1})\|_{L^2(\mu_0)} \|P_s^0(h\phi_0^{-1}) - \mu_0(h\phi_0^{-1})\|_{L^2(\mu_0)} ds \\
&\leq \frac{1}{t} \int_0^t \|P_{t-s}^0 - \mu_0\|_{L^2(\mu_0)} \|P_s^0 - \mu_0\|_{L^2(\mu_0)} \|h\|_{L^2(\mu)} ds \\
&\preceq \|h\|_{L^2(\mu)} e^{-(\lambda_1 - \lambda_0)t}, \quad t \geq t_0.
\end{aligned}$$

□

**Lemma 3.2.** *For any  $\alpha > 0$ , there exist constants  $c_0, t_0 > 0$  such that*

$$(3.11) \quad \tilde{\rho}_t^\nu \geq -\frac{c_0}{\nu(\phi_0)t}, \quad t \geq t_0, \quad \nu \in \mathcal{P}_0, \nu \in \mathcal{P}_0.$$

Consequently, if  $\nu = h\mu$  with  $h \in L^2(\mu)$ , then  $\tilde{\mu}_t^\nu := (1 + \tilde{\rho}_t^\nu)\mu_0$  is a probability measure for  $t > t_0(1 + c_0)$ .

*Proof.* By Lemma 3.1, if  $\nu = h\mu$  with  $h \in L^2(\mu)$ , we have  $\tilde{\rho}_t^\nu \in L^1(\mu_0)$  for  $t > 0$ , and it is easy to see that  $\mu_0(\tilde{\rho}_t^\nu) = 0$ . Since (3.11) implies  $1 + \tilde{\rho}_t^\nu > 0$  for  $t > t_0(1 + c_0)$ ,  $\tilde{\mu}_t^\nu$  is a probability measure. It remains to prove (3.11).

By (3.5) and (3.8), it suffices to find a constant  $c_1 > 0$  such that

$$(3.12) \quad g := \sum_{m=1}^{\infty} \frac{\mu(\phi_0)\nu(\phi_m) + \nu(\phi_0)\mu(\phi_m)}{\lambda_m - \lambda_0} \phi_m \phi_0^{-1} \geq -c_1.$$

By (2.1) and (2.12), we have

$$(3.13) \quad \|P_1^0 g\|_\infty \leq c_2 := \sum_{m=1}^{\infty} \frac{2\|\phi_0\|_\infty \|\phi_m\|_\infty \|\phi_m \phi_0^{-1}\|_\infty}{(\lambda_m - \lambda_0)e^{\lambda_m - \lambda_0}} < \infty.$$

Next, by (3.7) and the same formula for  $\mu = \nu$ , we obtain

$$(3.14) \quad P_s^0 g = (-L_0)^{-1} \{ \mu(\phi_0)(\psi_s^\nu - \nu(\phi_0)) + \nu(\phi_0)(\psi_s^\mu - \mu(\phi_0)) \} = (-L_0)^{-1} g_s, \quad s > 0,$$

where by  $\phi_0, \psi_s^\nu, \psi_s^\mu \geq 0$ ,

$$g_s := \mu(\phi_0)(\psi_s^\nu - \nu(\phi_0)) + \nu(\phi_0)(\psi_s^\mu - \mu(\phi_0)) \geq -2\mu(\phi_0)\nu(\phi_0) \geq -2\nu(\phi_0), \quad s > 0.$$

This together with (3.14) yields

$$-L_0 P_s^0 g \geq -2\nu(\phi_0), \quad s > 0.$$

Therefore, it follows from (3.13) that

$$g = P_1^0 g - \int_0^1 L_0 P_r^0 g dr \geq -c_2 - 2\nu(\phi_0) \geq -c_2 - 2\|\phi_0\|_\infty.$$

So, (3.12) holds for  $c_1 = c_2 + 2\|\phi_0\|_\infty$ . □

**Lemma 3.3.** *There exist constants  $c, t_0 > 0$  such that for any  $t \geq t_0$ , and any  $\nu \in \mathcal{P}_0$  with  $\nu = h\mu$  such that  $h \in L^2(\mu)$ , we have  $\tilde{\mu}_t^\nu \in \mathcal{P}_0$  and*

$$(3.15) \quad t^2 \mathbb{W}_2(\tilde{\mu}_t^\nu, \mu_0)^2 \leq \frac{1 + ct^{-1}}{\{\mu(\phi_0)\nu(\phi_0)\}^2} \sum_{m=1}^{\infty} \frac{\{\nu(\phi_0)\mu_0(\phi_m) + \mu(\phi_0)\nu(\phi_m)\}^2}{(\lambda_m - \lambda_0)^3}.$$

*Proof.* By Lemma 3.2, there exist constants  $c, t_0 > 0$  such that  $\tilde{\mu}_t^\nu \in \mathcal{P}_0$  for  $t \geq t_0$ , and

$$\mathcal{M}(1 + \tilde{\rho}_t^\nu, 1) \geq 1 \wedge (1 + \tilde{\rho}_t^\nu) \geq \frac{1}{1 + ct^{-1}}, \quad t \geq t_0.$$

So, [18, Lemma 2.3] implies

$$(3.16) \quad \mathbb{W}_2(\tilde{\mu}_t^\nu, \mu_0)^2 \leq \int_M \frac{|\nabla L_0^{-1} \tilde{\rho}_t^\nu|^2}{\mathcal{M}(1 + \tilde{\rho}_t^\nu, 1)} d\mu_0 \leq (1 + ct^{-1}) \mu_0(|\nabla L_0^{-1} \tilde{\rho}_t^\nu|^2), \quad t \geq t_0.$$

Next, (2.6) and (3.8) yield

$$t^2 \mu_0(|\nabla L_0^{-1} \tilde{\rho}_t^\nu|^2) = \frac{1}{\{\nu(\phi_0)P_t^0 \phi_0^{-1}\}^2} \sum_{m=1}^{\infty} \frac{\{\mu(\phi_0)\nu(\phi_m) + \nu(\phi_0)\mu(\phi_m)\}^2}{(\lambda_m - \lambda_0)^3}.$$

Combining this with (3.5) and (3.16), we finish the proof. □

We are now ready to prove the following result.

**Proposition 3.4.** *For any  $\nu \in \mathcal{P}_0$ ,*

$$(3.17) \quad \limsup_{t \rightarrow \infty} \{t^2 \mathbb{W}_2(\mu_t^\nu, \mu_0)^2\} \leq I.$$

*Proof.* (1) We first consider  $\nu = h\mu$  with  $h \in L^2(\mu)$ . Let  $D$  be the diameter of  $M$ . By Lemma 3.1, there exist constants  $c_1, t_0 > 0$  such that  $\tilde{\mu}_t^\nu$  is probability measure for  $t \geq t_0$  and

$$(3.18) \quad \mathbb{W}_2(\mu_t^\nu, \tilde{\mu}_t^\nu)^2 \leq D^2 \|\mu_t^\nu - \tilde{\mu}_t^\nu\|_{var} = D^2 \mu_0(|\rho_t^\nu - \tilde{\rho}_t^\nu|) \leq c_1 \|h\|_{L^2(\mu)} e^{-(\lambda_1 - \lambda_0)t}, \quad t \geq t_0.$$

Combining this with Lemma 3.3 and the triangle inequality of  $\mathbb{W}_2$ , we obtain

$$(3.19) \quad t^2 \mathbb{W}_2(\mu_t^\nu, \mu_0)^2 \leq (1 + \delta^{-1}) c_1 t^2 e^{-(\lambda_1 - \lambda_0)t} \|h\|_{L^2(\mu)} + (1 + \delta)(1 + ct^{-1})I, \quad \delta > 0.$$

(2) In general, we may go back to the first situation by shifting a small time  $\varepsilon > 0$ . More precisely, by the Markov property, (2.3), (2.5) and (3.2), for any  $f \in \mathcal{B}_b(M)$  and  $t \geq s \geq \varepsilon > 0$ , we have

$$\begin{aligned} \mathbb{E}^\nu[f(X_s)1_{\{t < \tau\}}] &= \mathbb{E}^\nu[1_{\{\varepsilon < \tau\}} \mathbb{E}^{X_\varepsilon}(f(X_{s-\varepsilon})1_{\{t-\varepsilon < \tau\}})] \\ &= \int_M p_\varepsilon^D(x, y) \mathbb{E}^y[f(X_{s-\varepsilon})1_{\{t-\varepsilon < \tau\}}] \nu(dx) \mu(dy) \\ &= e^{-\lambda_0 \varepsilon} \int_M (\psi_\varepsilon^\nu \phi_0)(y) \mathbb{E}^y[f(X_{s-\varepsilon})1_{\{t-\varepsilon < \tau\}}] \nu(dx) \mu(dy). \end{aligned}$$

With  $f = 1$  this implies

$$\mathbb{P}^\nu(t < \tau) = e^{-\lambda_0 \varepsilon} \int_M (\psi_\varepsilon^\nu \phi_0)(y) \mathbb{P}^y(t - \varepsilon < \tau) \mu(dy) \mu(dy).$$

So, letting

$$\nu_\varepsilon = \frac{\psi_\varepsilon^\nu \phi_0}{\mu(\psi_\varepsilon^\nu \phi_0)} =: h_\varepsilon \mu,$$

we arrive at

$$\mathbb{E}^\nu[f(X_s)|t < \tau] = \frac{\mathbb{E}^\nu[f(X_s)1_{\{t < \tau\}}]}{\mathbb{P}^\nu(t < \tau)} = \frac{\mathbb{E}^{\nu_\varepsilon}[f(X_{s-\varepsilon})1_{\{t-\varepsilon < \tau\}}]}{\mathbb{P}^{\nu_\varepsilon}(t - \varepsilon < \tau)} = \mathbb{E}^{\nu_\varepsilon}[f(X_{s-\varepsilon})|t - \varepsilon < \tau].$$

Therefore,

$$(3.20) \quad \mu_{t,\varepsilon}^\nu := \frac{1}{t - \varepsilon} \int_\varepsilon^t \mathbb{E}^\nu(\delta_{X_s} | t < \tau) ds = \mu_{t-\varepsilon}^{\nu_\varepsilon}, \quad t > \varepsilon.$$

Since

$$\mu(\psi_\varepsilon^\nu \phi_0) = \int_M p_\varepsilon^0(x, y) \phi_0(x) \phi_0(y) \nu(dx) \mu(dy) = \nu(\phi_0 P_\varepsilon^0 \phi_0^{-1}) \geq \nu(\phi_0) \|\phi_0\|_\infty^{-1} =: \alpha > 0,$$

by (2.9) we find a constant  $c_2 > 0$  such that

$$(3.21) \quad \|h_\varepsilon \phi_0^{-1}\|_{L^2(\mu_0)} \leq \alpha^{-1} \|\psi_\varepsilon^\nu\|_{L^2(\mu_0)} \leq \alpha^{-1} \|\phi_0\|_\infty \|p_\varepsilon^0\|_{L^\infty(\mu_0)} \leq c_2 \varepsilon^{-\frac{d+2}{2}}, \quad \varepsilon \in (0, 1).$$

Then (3.19) and (3.20) yield

$$(3.22) \quad \begin{aligned} & t^2 \mathbb{W}_2(\mu_{t,\varepsilon}^\nu, \mu_0)^2 \\ & \leq (1 + \delta^{-1}) c_1 c_2 \alpha^{-1} t^2 e^{-(\lambda_1 - \lambda_0)t} \varepsilon^{-\frac{d+2}{2}} + (1 + \delta)(1 + ct^{-1}) I_\varepsilon, \quad \delta > 0, \varepsilon \in (0, 1), \end{aligned}$$

where

$$I_\varepsilon := \frac{1}{\{\mu(\phi_0) \nu_\varepsilon(\phi_0)\}^2} \sum_{m=1}^{\infty} \frac{\{\nu_\varepsilon(\phi_0) \mu(\phi_m) + \mu(\phi_0) \nu_\varepsilon(\phi_m)\}^2}{(\lambda_m - \lambda_0)^3}.$$

By (2.5), (2.6) and (3.2), we have

$$\begin{aligned} \mu(\psi_\varepsilon^\nu \phi_0) &= \nu(\phi_0 P_\varepsilon^{-1} \phi_0^{-1}) = e^{\lambda_0 \varepsilon} \nu(P_\varepsilon^D 1), \\ \mu(\psi_\nu \phi_0) &= \nu(\phi_0 P_\varepsilon^0(\phi_m \phi_0^{-1})) = e^{-(\lambda_m - \lambda_0) \varepsilon} \nu(\phi_m), \end{aligned}$$

so that

$$\nu_\varepsilon(\phi_m) = \frac{e^{-\lambda_m \varepsilon} \nu(\phi_m)}{\nu(P_\varepsilon^D 1)}, \quad m \geq 0.$$

Thus,  $\lim_{\varepsilon \rightarrow 0} \nu_\varepsilon(\phi_0) = \nu(\phi_0)$  and there exists a constant  $C > 1$  such that

$$(3.23) \quad C^{-1} e^{-\lambda_m \varepsilon} |\nu(\phi_m)| \leq |\nu_\varepsilon(\phi_m)| \leq C |\nu(\phi_m)|, \quad m \geq 1, \varepsilon \in (0, 1).$$

Therefore, if  $I < \infty$ , by this and

$$(3.24) \quad \sum_{m=1}^{\infty} \mu(\phi_m)^2 \leq \mu(1) = 1,$$

we may apply the dominated convergence theorem to derive  $\lim_{\varepsilon \rightarrow 0} I_\varepsilon = I$ . On the other hand, if  $I = \infty$ , which is equivalent to

$$\sum_{m=1}^{\infty} \frac{\nu(\phi_m)^2}{(\lambda_m - \lambda_0)^3} = \infty,$$

then by (3.23) and the monotone convergence theorem we get

$$\liminf_{\varepsilon \rightarrow 0} \sum_{m=1}^{\infty} \frac{\nu_\varepsilon(\phi_m)^2}{(\lambda_m - \lambda_0)^3} \geq C^{-2} \liminf_{\varepsilon \rightarrow 0} \sum_{m=1}^{\infty} \frac{e^{-2\lambda_m \varepsilon} \nu(\phi_m)^2}{(\lambda_m - \lambda_0)^3} = \infty,$$

which together with (3.24) and  $\nu_\varepsilon(\phi_0) \rightarrow \nu(\phi_0)$  implies

$$\liminf_{\varepsilon \rightarrow 0} I_\varepsilon = \frac{1}{\{\mu(\phi_0) \nu(\phi_0)\}^2} \liminf_{\varepsilon \rightarrow 0} \sum_{m=1}^{\infty} \frac{\{\nu_\varepsilon(\phi_0) \mu(\phi_m) + \mu(\phi_0) \nu_\varepsilon(\phi_m)\}^2}{(\lambda_m - \lambda_0)^3}$$

$$\geq \frac{1}{\{\mu(\phi_0)\nu(\phi_0)\}^2} \liminf_{\varepsilon \rightarrow 0} \frac{\frac{1}{2}\{\mu(\phi_0)\nu_\varepsilon(\phi_m)\}^2 - \|\phi_0\|_\infty^2 \mu(\phi_m)^2}{(\lambda_m - \lambda_0)^3} = \infty.$$

In conclusion, we have

$$(3.25) \quad \lim_{\varepsilon \rightarrow 0} I_\varepsilon = I.$$

This together with (3.22) for  $\varepsilon = t^{-2}$  gives

$$(3.26) \quad \limsup_{t \rightarrow \infty} \{t^2 \mathbb{W}_2(\mu_{t,t-2}^\nu, \mu_0)^2\} \leq I.$$

On the other hand, it is easy to see that

$$\|\mu_{t,\varepsilon}^\nu - \mu_t^\nu\|_{var} \leq \frac{2\varepsilon}{t}, \quad 0 < \varepsilon < t,$$

so that

$$(3.27) \quad \mathbb{W}_2(\mu_t^\nu, \mu_{t,t-2}^\nu)^2 \leq D^2 \|\mu_{t,t-2}^\nu - \mu_t^\nu\|_{var} \leq 2D^2 t^{-3}, \quad t > 1.$$

Combining this with (3.26), we prove (3.17).  $\square$

## 4 Lower bound estimate and the finiteness of the limit

We will follow the idea of [1, 18], for which we need to modify  $\tilde{\mu}_t^\nu$  as follows. For any  $\beta > 0$ , consider

$$\tilde{\mu}_{t,\beta}^\nu := (1 + \tilde{\rho}_{t,\beta}^\nu) \mu_0, \quad \tilde{\rho}_{t,\beta}^\nu := P_{t-\beta}^0 \tilde{\rho}_t^\nu, \quad t > 0.$$

According to Lemma 3.2, there exists  $t_0 > 0$  such that

$$(4.1) \quad \tilde{h}_t^\nu := 1 + \tilde{\rho}_t^\nu \geq \frac{1}{2}, \quad \tilde{h}_{t,\beta}^\nu := 1 + \tilde{\rho}_{t,\beta}^\nu \geq \frac{1}{2}, \quad \beta > 0, t \geq t_0.$$

Consequently,  $\tilde{\mu}_{t,\beta}^\nu$  and  $\tilde{\mu}_t^\nu$  are probability measures for any  $\beta > 0, t \geq t_0$ .

**Lemma 4.1.** *For any  $\beta > 0$ , there exists a constant  $c > 0$  such that  $f_{t,\beta} := L_0^{-1} \tilde{\rho}_{t,\beta}^\nu$  satisfies*

$$\|f_{t,\beta}\|_\infty + \|L_0 f_{t,\beta}\|_\infty + \|\nabla f_{t,\beta}\|_\infty \leq c t^{\frac{5\beta d}{4}-1}, \quad t \geq 1.$$

*Proof.* By (2.6) and (3.8), we have

$$\begin{aligned} f_{t,\beta} &= - \sum_{m=1}^{\infty} \frac{\{\mu(\phi_0)\nu(\phi_m) + \nu(\phi_0)\mu(\phi_m)\} e^{-(\lambda_m - \lambda_0)t^{-\beta}}}{t(\lambda_m - \lambda_0)^2 \nu(\phi_0 P_t^0 \phi_0^{-1})} (\phi_m \phi_0^{-1}), \\ L_0 f_{t,\beta} &= \sum_{m=1}^{\infty} \frac{\{\mu(\phi_0)\nu(\phi_m) + \nu(\phi_0)\mu(\phi_m)\} e^{-(\lambda_m - \lambda_0)t^{-\beta}}}{t(\lambda_m - \lambda_0) \nu(\phi_0 P_t^0 \phi_0^{-1})} (\phi_m \phi_0^{-1}). \end{aligned}$$

Combining these with (2.1), (2.12), (3.5), and

$$|\mu(\phi_0)\nu(\phi_m) + \nu(\phi_0)\mu(\phi_m)| \leq \|\phi_0\|_\infty + \|\phi_m\|_\infty \preceq m, \quad m \geq 1,$$

we find a constant  $c_1 > 0$  such that

$$\begin{aligned} t\{\|f_{t,\beta}\|_\infty + \|L_0 f_{t,\beta}\|_\infty\} &\preceq \sum_{m=1}^{\infty} \frac{e^{-(\lambda_m - \lambda_0)t^{-\beta}} m^{\frac{3d+2}{2d}}}{\lambda_m - \lambda_0} \\ &\preceq \sum_{m=1}^{\infty} e^{-c_1 m^{\frac{2}{d}} t^{-\beta}} m^{\frac{3d-2}{2d}} \preceq \int_0^\infty e^{-c_1 s^{\frac{2}{d}} t^{-\beta}} s^{\frac{3d-2}{2d}} ds \preceq t^{\frac{\beta(5d-2)}{4}}, \quad t \geq 1. \end{aligned}$$

Similarly, (2.25) implies

$$\begin{aligned} t\|\nabla f_{t,\beta}\|_\infty &\preceq \sum_{m=1}^{\infty} \frac{e^{-(\lambda_m - \lambda_0)t^{-\beta}} m^{\frac{3d+4}{2d}}}{(\lambda_m - \lambda_0)^2} \\ &\preceq \sum_{m=1}^{\infty} e^{-c_1 m^{\frac{2}{d}} t^{-\beta}} m^{\frac{3d-4}{2d}} \preceq t^{\frac{\beta(5d-4)}{4}}, \quad t \geq 1. \end{aligned}$$

Then the proof is finished. □

**Lemma 4.2.** *For any  $\beta \in (0, \frac{1}{20d}]$ , there exists a constant  $c > 0$  such that*

$$t^2 \mathbb{W}_2(\tilde{\mu}_{t,\beta}^\nu, \mu_0)^2 \geq \frac{1 - ct^{-1}}{\{\mu(\phi_0)\nu(\phi_0)\}^2} \sum_{m=1}^{\infty} \frac{\{\mu(h\phi_0)\mu_0(\phi_m) + \mu(\phi_0)\nu(\phi_m)\}^2}{(\lambda_m - \lambda_0)^3} - ct^{-\frac{1}{4}}.$$

*Proof.* To estimate  $\mathbb{W}_2(\tilde{\mu}_{t,\beta}^\nu, \mu_0)$  from below by using the argument in [1, 18], we take

$$\varphi_\theta^\varepsilon := -\varepsilon \log P_{\varepsilon\theta}^0 e^{-\varepsilon^{-1} f_{t,\beta}}, \quad \theta \in [0, 1], \varepsilon > 0.$$

We have  $\varphi_0^\varepsilon = f_{t,\beta}$ ,  $\|\varphi_\theta^\varepsilon\|_\infty \leq \|f_{t,\beta}\|_\infty$ , and by [18, Lemma 2.9], there exists a constant  $c_1 > 0$  such that for any  $\varepsilon \in (0, 1)$ ,

$$\begin{aligned} \varphi_1^\varepsilon(y) - \varphi_0^\varepsilon(x) &\leq \frac{1}{2} \{\rho(x, y)^2 + \varepsilon \|(L_0 f_{t,\beta})^+\|_\infty + c_1 \sqrt{\varepsilon} \|\nabla f_{t,\beta}\|_\infty^2\}, \quad x, y \in M, \\ \int_M (\varphi_0^\varepsilon - \varphi_1^\varepsilon) d\mu_0 &\leq \frac{1}{2} \int_M |\nabla f_{t,\beta}|^2 d\mu_0 + c_1 \varepsilon^{-1} \|\nabla f_{t,\beta}\|_\infty^4. \end{aligned}$$

Therefore, by the Kantorovich dual formula,  $\varphi_0^\varepsilon = f_{t,\beta}$  and the integration by parts formula

$$\int_M f_{t,\beta} \tilde{\rho}_{t,\beta}^\nu d\mu_0 = \int_M f_{t,\beta} L_0 f_{t,\beta} d\mu_0 = - \int_M |\nabla f_{t,\beta}|^2 d\mu_0,$$

we find a constant  $c > 0$  such that

$$\begin{aligned} c(\varepsilon \|L_0 f_{t,\beta}\|_\infty + \varepsilon^{\frac{1}{2}} \|\nabla f_{t,\beta}\|_\infty^2) + \frac{1}{2} \mathbb{W}_2(\tilde{\mu}_{t,\beta}^\nu, \mu_0)^2 &\geq \int_M \varphi_1^\varepsilon d\mu_0 - \int_M \varphi_0^\varepsilon d\tilde{\mu}_{t,\beta}^\nu \\ (4.2) \quad &= \int_M (\varphi_1^\varepsilon - \varphi_0^\varepsilon) d\mu_0 - \int_M f_{t,\beta} \tilde{\rho}_{t,\beta}^\nu d\mu_0 = \int_M (\varphi_1^\varepsilon - \varphi_0^\varepsilon) d\mu_0 - \int_M f_{t,\beta} L_0 f_{t,\beta} d\mu_0 \\ &\geq \frac{1}{2} \int_M |\nabla f_{t,\beta}|^2 d\mu_0 - c\varepsilon^{-1} \|\nabla f_{t,\beta}\|_\infty^4. \end{aligned}$$

Taking  $\varepsilon = t^{-\frac{3}{2}}$  and applying Lemma 4.1, when  $\beta \leq \frac{1}{20d}$  we find a constant  $c' > 0$  such that

$$(4.3) \quad t^2 \mathbb{W}_2(\tilde{\mu}_{t,\beta}^\nu, \mu_0)^2 \geq t^2 \mu_0(|\nabla f_{t,\beta}|^2) - c't^{-\frac{1}{4}}, \quad t \geq t_0.$$

Combining this with (3.5) and (4.3), we complete the proof.  $\square$

**Lemma 4.3.** *There exist constants  $c, t_0 > 0$  such that for any  $\nu = h\mu \in \mathcal{P}_0$  with  $h \in L^2(\mu)$ ,  $\tilde{\mu}_{t,\beta}^\nu, \tilde{\mu}_t^\nu \in \mathcal{P}_0$  for  $t \geq t_0$  and*

$$t \mathbb{W}_2(\tilde{\mu}_{t,\beta}^\nu, \tilde{\mu}_t^\nu) \leq c \|h\|_{L^2(\mu)} t^{-\beta}, \quad t \geq t_0.$$

*Proof.*  $\tilde{\mu}_{t,\beta}^\nu, \tilde{\mu}_t^\nu \in \mathcal{P}_0$  for large  $t$  is implied by Lemma 3.2. Next, by (4.1), we have

$$\mathcal{M}(\tilde{h}_t^\nu, \tilde{h}_{t,\beta}^\nu) \geq \frac{1}{2},$$

so that [18, Lemma 2.3] implies

$$(4.4) \quad \mathbb{W}_2(\tilde{\mu}_{t,\beta}^\nu, \tilde{\mu}_t^\nu)^2 \leq \int_M \frac{|\nabla L_0^{-1}(\tilde{h}_t^\nu - \tilde{h}_{t,\beta}^\nu)|^2}{\mathcal{M}(\tilde{h}_t^\nu, \tilde{h}_{t,\beta}^\nu)} d\mu_0 \leq 2\mu_0(|\nabla L_0^{-1}(\tilde{\rho}_t^\nu - \tilde{\rho}_{t,\beta}^\nu)|^2).$$

To estimate the upper bound in this inequality, we first observe that by (3.7) and (3.8), when  $\nu = h\mu$  we have

$$(4.5) \quad \begin{aligned} L_0^{-1}(\tilde{\rho}_{t,\beta}^\nu - \tilde{\rho}_t^\nu) &= L_0^{-1}(P_{t^{-\beta}}^0 \tilde{\rho}_t^\nu - \tilde{\rho}_t^\nu) = \int_0^{t^{-\beta}} P_r^0 \tilde{\rho}_t^\nu dr \\ &= \frac{1}{t\nu(\phi_0 P_t^0 \phi_0^{-1})} \int_0^{t^{-\beta}} (-L_0)^{-1}(P_r^0 - \mu_0)g dr, \end{aligned}$$

where

$$g := \mu(\phi_0)h\phi_0^{-1} + \nu(\phi_0)\phi_0^{-1}.$$

Since  $\|h\|_{L^2(\mu)} \geq \mu(h) = 1$ ,

$$(4.6) \quad \|g\|_{L^2(\mu_0)} \leq \|\phi_0\|_\infty(1 + \|h\|_{L^2(\mu)}) \leq 2\|\phi_0\|_\infty\|h\|_{L^2(\mu)}.$$

By (2.10), (4.6) and the fact that  $(-L_0)^{-\frac{1}{2}} = c \int_0^\infty P_{s^2}^0 ds$  for some constant  $c > 0$ , we find a constants  $c_1, c_2 > 0$  such that

$$\begin{aligned} \|\nabla L_0^{-1}(P_r^0 - \mu_0)g\|_{L^2(\mu_0)} &= \|L_0^{-\frac{1}{2}}(P_r^0 - \mu_0)g\|_{L^2(\mu_0)} \leq \int_0^\infty \|(P_{r+s^2}^0 - \mu_0)g\|_{L^2(\mu_0)} ds \\ &\leq c_1 \|h\|_{L^2(\mu)} \int_1^\infty e^{-(\lambda_1 - \lambda_0)(s^2 + r)} ds \leq c_2 \|h\|_{L^2(\mu)}, \quad r \in [0, 1]. \end{aligned}$$

Therefore, by (3.5) and (4.5), we obtain

$$\|\nabla L_0^{-1}(\tilde{\rho}_{t,\beta}^\nu - \tilde{\rho}_t^\nu)\|_{L^2(\mu_0)} \leq \frac{1}{t} \int_0^{t^{-\beta}} \|\nabla L_0^{-1}(P_r^0 - \mu_0)g\|_{L^2(\mu_0)} dr \leq t^{-(1+\beta)} \|h\|_{L^2(\mu)}, \quad t \geq t_0.$$

Combining this with (4.4) we finish the proof.  $\square$

We are now ready to prove the following result.

**Proposition 4.4.** *For any  $\nu \in \mathcal{P}_0$ ,*

$$(4.7) \quad \liminf_{t \rightarrow \infty} \{t^2 \mathbb{W}_2(\mu_t^\nu, \mu_0)^2\} \geq I > 0,$$

and  $I < \infty$  provided either  $d \leq 7$ , or  $d \geq 7$  but  $\nu = h\mu$  with  $h \in L^{\frac{2d}{d+6}}$ .

*Proof.* Let  $\beta \in (0, \frac{1}{20d}]$ . By (3.18), Lemma 4.2 and Lemma 4.3, there exist constants  $c, t_0 > 0$  such that for  $\nu = h\mu \in \mathcal{P}_0$  and  $t \geq t_0$ ,

$$\begin{aligned} t\mathbb{W}_2(\mu_t^\nu, \tilde{\mu}_t^\nu) &\leq c\|h\|_{L^2(\mu)}t^{-\beta t}, \\ t\mathbb{W}_2(\tilde{\mu}_{t,\beta}^\nu, \mu_0) &\geq \left(\{(1-ct^{-1})I - ct^{-\frac{1}{4}}\}^+\right)^{\frac{1}{2}}, \\ t\mathbb{W}_2(\mu_t^\nu, \tilde{\mu}_t^\nu) &\leq cte^{-(\lambda_1-\lambda_0)t/2}\|h\|_{L^2(\mu)}^{\frac{1}{2}}. \end{aligned}$$

Then

$$(4.8) \quad t\mathbb{W}_2(\mu_t^\nu, \mu_0) \geq \left(\{(1-ct^{-1})I - ct^{-\frac{1}{4}}\}^+\right)^{\frac{1}{2}} - c\|h\|_{L^2(\mu)}t^{-\beta t} - cte^{-(\lambda_1-\lambda_0)t/2}\|h\|_{L^2(\mu)}^{\frac{1}{2}}, \quad t \geq t_0.$$

In general, let  $\mu_{t,\varepsilon}^\nu = \mu_{t-\varepsilon}^\nu$  be as in the proof of Proposition 3.4. Applying (4.8) to  $\mu_{t,t-2}^\nu$  replacing  $\mu_y^\nu$  and using (3.21), (3.25), we obtain

$$\liminf_{t \rightarrow \infty} \{t\mathbb{W}_2(\mu_{t,t-2}^\nu, \mu_0)\} \geq \sqrt{I},$$

which together with (3.27) proves (4.7).

It remains to prove  $I > 0$  and  $I < \infty$  under given conditions, where due to (3.24),  $I < \infty$  is equivalent to

$$(4.9) \quad I' := \sum_{m=1}^{\infty} \frac{\nu(\phi_m)^2}{\lambda_m^3} < \infty.$$

Below we first prove  $I > 0$  then show  $I' < \infty$  under the given conditions.

(a)  $I > 0$ . If this is not true, then

$$\mu(h\phi_0)\mu(\phi_m) = -\mu(\phi_0)\mu(h\phi_m), \quad m \geq 1.$$

Combining this with the representation in  $L^2(\mu)$

$$f = \sum_{m=0}^{\infty} \mu(f\phi_m)\phi_m, \quad f \in L^2(\mu),$$

where the equation holds point-wisely if  $f \in C_b(M)$  by the continuity, we obtain

$$\mu(\phi_0)\nu(f) = \sum_{m=0}^{\infty} \mu(f\phi_m)\mu(\phi_0)\nu(\phi_m) = 2\mu(f\phi_0)\nu(\phi_0)\mu(\phi_0) - \sum_{m=0}^{\infty} \mu(f\phi_m)\mu(\phi_m)\nu(\phi_0)$$

$$= 2\mu(f\phi_0)\nu(\phi_0)\mu(\phi_0) - \nu(\phi_0)\mu(f), \quad f \in C_b(M).$$

Consequently,

$$0 \leq \mu(\phi_0) \frac{d\nu}{d\mu} = 2\phi_0\nu(\phi_0)\mu(\phi_0) - \nu(\phi_0),$$

which is however impossible since the upper bound is negative in a neighborhood of  $\partial M$ , because  $\nu(M^\circ) > 0$  implies  $\nu(\phi_0) > 0$  for  $\phi_0 > 0$  in  $M^\circ$ , and  $\phi_0$  is continuous with  $\phi_0|_{\partial M} = 0$ . Therefore, we must have  $I > 0$ .

(b)  $I' < \infty$ . Let  $\{h_n\}_{n \geq 1}$  be a sequence of probability density functions with respect to  $\mu$  such that

$$(4.10) \quad \nu_n := h_n \mu \rightarrow \nu \text{ weakly as } n \rightarrow \infty.$$

By the spectral representation for  $(-L)^{-\frac{3}{2}}$ , and applying the Sobolev inequality (1.2) with  $p = \frac{2d}{d+6} \vee 1$ , we obtain

$$(4.11) \quad I'_n := \sum_{m=1}^{\infty} \frac{\nu_n(\phi_m)^2}{\lambda_m^3} \leq \|(-L)^{-\frac{3}{2}} h_n\|_{L^2(\mu)}^2 \leq K^2 \|h_n\|_{L^{\frac{2d}{d+6} \vee 1}(\mu)}^2, \quad n \geq 1.$$

It is easy to see that for  $d \leq 6$  we have  $\frac{2d}{d+6} \leq 1$ , so that  $\|h_n\|_{L^{\frac{2d}{d+6} \vee 1}(\mu)} = \mu(h_n) = 1$ . Combining this with (4.10), (4.11) and applying Fatou's lemma, we derive

$$I' = \sum_{m=1}^{\infty} \liminf_{n \rightarrow \infty} \frac{\nu_n(\phi_m)^2}{\lambda_m^3} \leq \liminf_{n \rightarrow \infty} I'_n \leq K^2 < \infty, \quad d \leq 6.$$

Finally, when  $d \geq 7$  and  $\nu = h\mu$  with  $h \in L^{\frac{2d}{d+6}}(\mu)$ , by applying (4.11) to  $h_n = h$  we prove  $I' < \infty$ .  $\square$

**Acknowledgement.** The author would like to thank the referee for helpful comments and corrections.

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