INSTABILITY OF THE SOLITARY WAVE SOLUTIONS FOR THE GENERALIZED DERIVATIVE NONLINEAR SCHRÖDINGER EQUATION IN THE CRITICAL FREQUENCY CASE

ZIHUA GUO, CUI NING, AND YIFEI WU

Abstract. We study the stability theory of solitary wave solutions for the generalized derivative nonlinear Schrödinger equation

$$i\partial_t u + \partial_x^2 u + i|u|^{2\sigma}\partial_x u = 0.$$

The equation has a two-parameter family of solitary wave solutions of the form

$$\phi_{\omega,c}(x) = \varphi_{\omega,c}(x)\exp\{i\frac{c}{2}x - \frac{i}{2\sigma+2}\int_{-\infty}^{x}\varphi_{\omega,c}^{2\sigma}(y)dy\}.$$

Here $\varphi_{\omega,c}$ is some real-valued function. It was proved in [29] that the solitary wave solutions are stable if $-2\sqrt{\omega} < c < 2z_0\sqrt{\omega}$, and unstable if $2z_0\sqrt{\omega} < c < 2\sqrt{\omega}$ for some $z_0 \in (0,1)$. We prove the instability at the borderline case $c = 2z_0\sqrt{\omega}$ for $1 < \sigma < 2$, improving the previous results in [7] where $7/6 < \sigma < 2$.

1. Introduction

In this paper, we study the stability theory of the solitary wave solutions for the generalized derivative nonlinear Schrödinger equation:

$$i\partial_t u + \partial_x^2 u + i|u|^{2\sigma}\partial_x u = 0, \qquad t \in \mathbb{R},\ x \in \mathbb{R} \tag{1.1}$$

for $\sigma > 0$. It describes an Alfvén wave and appears in plasma physics, nonlinear optics, and so on (see [34, 35]). In the case of $\sigma = 1$, by a suitable gauge transformation, (1.1) is transformed to the standard derivative nonlinear Schrödinger equation:

$$i\partial_t u + \partial_x^2 u + i\partial_x(|u|^2 u) = 0. \tag{1.2}$$

This equation (1.2) was widely studied. The local well-posedness was proved by Hayashi and Ozawa [18, 19] in the energy space $H^1(\mathbb{R})$ and by Guo and Tan [10] in the smooth space. In the paper of [18], the authors proved the global well-posedness in the energy space when the initial data u_0 satisfies the mass condition $\|u_0\|_{L^2} < \sqrt{2\pi}$. This condition seems natural for global well-posedness in view of the mass critical nonlinear Schrödinger equation and generalized KdV equation, as it ensures a priori estimate of H^1-norm from mass and energy conservations. However, recently, the third author extended the condition to $\|u_0\|_{L^2} < 2\sqrt{\pi}$ in [45, 46], in which the key ingredient in the proof is the use of the momentum conservation. A simplified proof was later given by the first and third authors in their paper [15], where the global well-posedness in $H^{\frac{1}{2}}(\mathbb{R})$ was also proved under the same mass constraint. The problems for large mass are still unclear at the moment. In [6], Fukaya, Hayashi and Inui constructed a class of large global solution with high oscillation. In the papers of Cher, Simpson and Sulem [3], Jenkins, Liu, Perry, Sulem

Key words and phrases. derivative Schrödinger equation, solitary wave solutions, orbital instability.

[22, 28, 26, 27], Pelinovsky and Shimabukuro [39, 40], the authors constructed a class of global solution by using the inverse scattering method. On the long-time behavior and modified scattering theory, see [13] and references therein. On the low regularity theory, see [1, 4, 5, 8, 9, 14, 21, 31, 36, 37, 42, 43] and the references therein.

In the case of $\sigma \neq 1$, the Cauchy problems of (1.1) have been investigated by many researchers. In the case of $\sigma > 1$, local well-posedness in energy spaces $H^1(\mathbb{R})$ was studied by Hayashi and Ozawa [20] for any $\sigma > 1$, by Hao [16] in $H^{\frac{1}{2}}(\mathbb{R})$ for any $\sigma > \frac{5}{2}$, and by Santos [41] in $H^{\frac{1}{2}}(\mathbb{R})$ for any $\sigma > 1$ and small data. In the case of $\frac{1}{2} \leq \sigma < 1$, local well-posedness in $H^2(\mathbb{R})$ was studied by Hayashi and Ozawa [20], see also Santos [41] in the weight Sobolev spaces. In the case of $0 < \sigma < \frac{1}{2}$, local well-posedness in some weighted spaces was studied by Linares, Ponce and Santos [25]. Note that in this case, the nonlinear term is not regular enough, appropriate construction of the working space is needed to handle nonlinearity. Global well-posedness was studied in [6, 20, 33]. In particular, in the case of $0 < \sigma < 1$, the global existence (without uniqueness) of the solution in $H^1(\mathbb{R})$ was shown by Hayashi and Ozawa [20]; while in the case of $\sigma > 1$, the global well-posedness of the solution in $H^1(\mathbb{R})$ was shown by Fukaya, Hayashi, Inui [6] with some suitable size restriction on the initial datum.

Also, the stability theory was widely studied. The equation (1.1) has a two-parameter family of solitary waves (see [29]),

$$u_{\omega,c}(t) = e^{i\omega t}\phi_{\omega,c}(x - ct),$$

where $\phi_{\omega,c}$ is the solution of the form

$$\phi_{\omega,c}(x) = \varphi_{\omega,c}(x) \exp\Big\{\frac{c}{2}ix - \frac{i}{2\sigma+2}\int_{-\infty}^{x} \varphi_{\omega,c}^{2\sigma}(y)dy\Big\}, \tag{1.3}$$

with

$$\varphi_{\omega,c}(x) = \Big\{\frac{(\sigma+1)(4\omega - c^2)}{2\sqrt{\omega}\cosh(\sigma\sqrt{4\omega - c^2}\, x) - c}\Big\}^{\frac{1}{2\sigma}}.$$

Actually $\phi_{\omega,c}$ is the solution of the following equation

$$-\partial_x^2\phi + \omega\phi + ci\partial_x\phi - i|\phi|^{2\sigma}\partial_x\phi = 0. \tag{1.4}$$

Note that the equation (1.4) can be solved in $H^1(\mathbb{R})$ when $4\omega > c^2, c \in \mathbb{R}$ or $4\omega = c^2, c > 0$.

When $\sigma = 1$, Colin and Ohta [2] proved the stability of the soliton waves when $c^2 < 4\omega$, and see also [11] for previous result in the case of $c > 0$. The endpoint case $c^2 = 4\omega, c > 0$ was studied in [23]. Further more, Le Coz and Wu [24], Miao, Tang and Xu [32] proved the stability of the multi-solitary wave solutions. A consequence of these results is the existence of a class of global solutions with arbitrary large mass.

In the case of $0 < \sigma < 1$, Liu, Simpson and Sulem [29] proved that the solitary wave solution $u_{\omega,c}$ is stable for any $-2\sqrt{\omega} < c < 2\sqrt{\omega}$, and Guo [12] further proved the stability of the solitary wave solutions in the endpoint case $0 < c = 2\sqrt{\omega}$. In the case of $\sigma \geq 2$, the solitary wave solution $u_{\omega,c}$ is unstable for any $-2\sqrt{\omega} < c < 2\sqrt{\omega}$ (see [29]).

The case $1 < \sigma < 2$ is more complicated. It was proved by Liu, Simpson and Sulem [29] that there exists $z_0(\sigma) \in (0,1)$, which solves the equation $F_\sigma(z) = 0$ with

$$F_\sigma(z) = (\sigma-1)^2\left[\int_0^\infty (\cosh y - z)^{-\frac{1}{\sigma}} dy\right]^2 - \left[\int_0^\infty (\cosh y - z)^{-\frac{1}{\sigma}-1}(z\cosh y - 1)dy\right]^2,$$

such that when $-2\sqrt{\omega} < c < 2z_0\sqrt{\omega}$ the solitary wave solution $u_{\omega,c}$ is stable, and when $2z_0\sqrt{\omega} < c < 2\sqrt{\omega}$ the solitary wave solution $u_{\omega,c}$ is unstable. See also Tang and Xu [44] for the stability of the sum of two solitary waves. Fukaya [7] proved that the solitary waves solution is unstable when $\frac{7}{6} < \sigma < 2$, $c = 2z_0\sqrt{\omega}$. The case $1 < \sigma \le \frac{7}{6}$, $c = 2z_0\sqrt{\omega}$ is still unknown. However, as pointed out by the referee, in view of the form of the linearized action $S''_{\omega,c}$ defined later, the energy seems in fact C^3-functional at least when $1 < \sigma < 2$, so the proof in [7] may be applicable for all case of $1 < \sigma < 2$ and $c = 2z_0\sqrt{\omega}$. In this paper, we aim to solve this case by a different approach which does not require more regularity of the energy than C^2.

Before stating our theorem, we adopt some notations. For $\varepsilon > 0$, we define

$$U_\varepsilon(\phi_{\omega,c}) = \{u \in H^1(\mathbb{R}) : \inf_{(\theta,y)\in\mathbb{R}^2} \|u - e^{i\theta}\phi_{\omega,c}(\cdot - y)\|_{H^1} < \varepsilon\}.$$

Definition 1. *We say that the solitary wave solution $u_{\omega,c}$ of (1.1) is stable if for any $\varepsilon > 0$ there exists $\delta > 0$ such that if $u_0 \in U_\delta(\phi_{\omega,c})$, then the solution $u(t)$ of (1.1) with $u(0) = u_0$ exists for all $t > 0$, and $u(t) \in U_\varepsilon(\phi_{\omega,c})$ for all $t > 0$. Otherwise, $u_{\omega,c}$ is said to be unstable.*

The main result in the present paper is

Theorem 1. *Let $1 < \sigma < 2$ and $z_0 = z_0(\sigma) \in (-1,1)$ satisfy $F_\sigma(z_0) = 0$. Then the solitary wave solutions $e^{i\omega t}\phi_{\omega,c}(x-ct)$ of (1.1) is unstable if $c = 2z_0\sqrt{\omega}$.*

In this paper, we use the same ideas as in [47]. It relies on the modulation theory and construction of the virial identities. Compared to [7], the idea is to utilize virial identities to replace the Lyapunov functional, to obtain the lower bound on modulations. This can be used to avoid the requirement of the high-order regularities of the energy. However, the construction in the present paper is much more delicate, due to the complicated structure of the equation. In addition, the construction of virial quantities in this paper possibly gives a hint to analyze global behaviors of the solutions to (1.1) such as blowup.

This paper is organized as follows. In Section 2, we give the definitions of some important functionals and some useful lemmas. In Section 3, we obtain the modulation result and show the coercivity for the second variation. In Section 4, we prove the main theorem.

Remark 1. *We note that the same result in Theorem 1 was obtained independently by Miao-Tang-Xu in [30] (appear on arXiv on March 20, 2018) by different method. They used the third derivative of the energy around the solitary wave.*

2. Preliminaries

2.1. **Notations.** We use $A \lesssim B$ or $A = O(B)$ to denote an estimate of the form $A \le CB$ for some constant $C > 0$. Similarly, we use $A \sim B$ to mean $A \lesssim B$ and $B \lesssim A$. We denote $\dot{f} = \partial_t f$.

For $u, v \in L^2(\mathbb{R}) = L^2(\mathbb{R}, \mathbb{C})$, we define

$$\langle u, v\rangle = \mathrm{Re}\int_{\mathbb{R}} u(x)\overline{v(x)}\,dx$$

and regard $L^2(\mathbb{R})$ as a real Hilbert space.

For a function $f(x)$, we define its L^q-norm $\|f\|_{L^q} = \Big(\int_{\mathbb{R}} |f(x)|^q dx\Big)^{\frac{1}{q}}$ and its H^1-norm $\|f\|_{H^1} = (\|f\|_{L^2}^2 + \|\partial_x f\|_{L^2}^2)^{\frac{1}{2}}$.

2.2. **Conservation laws.** The strong solution $u(t)$ of (1.1) satisfies three conservation laws,

$$E(u(t)) = E(u_0), \quad P(u(t)) = P(u_0), \quad M(u(t)) = M(u_0)$$

for all $t \in [0, T_{\max})$, where $T_{\max}$ denotes the maximal existence time of $u(t)$, and

$$\begin{aligned} E(u) =& \frac{1}{2}\|\partial_x u\|_{L^2}^2 - \frac{1}{2(\sigma+1)}\mathrm{Im}\int_{\mathbb{R}} |u|^{2\sigma} u\,\overline{\partial_x u}\,dx,\\ P(u) =& \frac{1}{2}(i\partial_x u, u)_{L^2} = \frac{1}{2}\mathrm{Im}\int_{\mathbb{R}} u\,\overline{\partial_x u}\,dx,\\ M(u) =& \frac{1}{2}\|u\|_{L^2}^2. \end{aligned}$$

2.3. **Some functionals.** From the definitions of E, P and M, we have

$$\begin{aligned} E'(u) =& -\partial_x^2 u - i|u|^{2\sigma}\partial_x u,\\ P'(u) =& i\partial_x u,\\ M'(u) =& u. \end{aligned}$$

Let

$$S_{\omega,c}(u) = E(u) + \omega M(u) + cP(u),$$

then we have

$$\begin{aligned} S'_{\omega,c}(u) =& E'(u) + \omega M'(u) + cP'(u)\\ =& -\partial_x^2 u - i|u|^{2\sigma}\partial_x u + \omega u + ic\partial_x u. \end{aligned} \tag{2.5}$$

Hence, (1.4) is equivalent to $S'_{\omega,c}(\phi) = 0$. Hence for the solution $\phi_{\omega,c}$ to (1.4), we have

$$S'_{\omega,c}(\phi_{\omega,c}) = 0. \tag{2.6}$$

Moreover, by (2.5), we obtain

$$\begin{aligned} S''_{\omega,c}(\phi_{\omega,c})f =& -\partial_x^2 f + \omega f + ic\partial_x f - i\sigma|\phi_{\omega,c}|^{2\sigma-2}\overline{\phi_{\omega,c}}\,\partial_x\phi_{\omega,c}\,f\\ & - i\sigma|\phi_{\omega,c}|^{2\sigma-2}\phi_{\omega,c}\partial_x\phi_{\omega,c}\,\overline{f} - i|\phi_{\omega,c}|^{2\sigma}\partial_x f. \end{aligned} \tag{2.7}$$

2.4. **Useful Lemma.** In this subsection, we give some lemmas which are useful in the following sections. First, we have following formulas.

Lemma 1. *Let $1<\sigma<2$ and $(\omega,c)\in\mathbb{R}^2$ satisfy $c^2<4\omega$, we have*

$$\begin{aligned} S''_{\omega,c}(\phi_{\omega,c})\phi_{\omega,c} =& -2\sigma i|\phi_{\omega,c}|^{2\sigma}\partial_x\phi_{\omega,c}, \\ S''_{\omega,c}(\phi_{\omega,c})(i\partial_x\phi_{\omega,c}) =& -2\sigma\omega|\phi_{\omega,c}|^{2\sigma}\phi_{\omega,c}. \end{aligned} \tag{2.8}$$

Proof. First, using (2.6) and (2.7), we get

$$\begin{aligned} S''_{\omega,c}(\phi_{\omega,c})\phi_{\omega,c} =& -\partial_x^2\phi_{\omega,c}+\omega\phi_{\omega,c}+ic\partial_x\phi_{\omega,c}-i\sigma|\phi_{\omega,c}|^{2\sigma-2}|\phi_{\omega,c}|^2\,\partial_x\phi_{\omega,c} \\ & -i\sigma|\phi_{\omega,c}|^{2\sigma-2}|\phi_{\omega,c}|^2\partial_x\phi_{\omega,c}-i|\phi_{\omega,c}|^{2\sigma}\partial_x\phi_{\omega,c} \\ =& -\partial_x^2\phi_{\omega,c}-(2\sigma+1)i|\phi_{\omega,c}|^{2\sigma}\partial_x\phi_{\omega,c}+\omega\phi_{\omega,c}+ic\partial_x\phi_{\omega,c} \\ =& -2\sigma i|\phi_{\omega,c}|^{2\sigma}\partial_x\phi_{\omega,c}. \end{aligned}$$

Similarly, using (2.6) and (2.7), we obtain

$$\begin{aligned} S''_{\omega,c}(\phi_{\omega,c})(i\partial_x\phi_{\omega,c}) =& i\partial_x\big[-\partial_x^2\phi_{\omega,c}+\omega\phi_{\omega,c}+ic\partial_x\phi_{\omega,c}-i\sigma|\phi_{\omega,c}|^{2\sigma}\,\partial_x\phi_{\omega,c}\big] \\ & -2\sigma|\phi_{\omega,c}|^{2\sigma-2}\,\phi_{\omega,c}|\partial_x\phi_{\omega,c}|^2 \\ =& -2\sigma\omega|\phi_{\omega,c}|^{2\sigma}\phi_{\omega,c}. \end{aligned}$$

This concludes the proof of Lemma 1. □

Let

$$J(u)=\operatorname{Im}\int_{\mathbb{R}}|u|^{2\sigma}u\,\overline{\partial_x u}dx.$$

Then we have

$$J'(u)=2(\sigma+1)i|u|^{2\sigma}\partial_x u. \tag{2.9}$$

Moreover, we have the following lemma.

Lemma 2. *Let $1<\sigma<2$ and $(\omega,c)\in\mathbb{R}^2$ satisfy $c^2<4\omega$, then*

$$\|\partial_x\phi_{\omega,c}\|_{L^2}^2=\omega\|\phi_{\omega,c}\|_{L^2}^2. \tag{2.10}$$

Moreover,

$$J(\phi_{\omega,c})=4\omega M(\phi_{\omega,c})+2cP(\phi_{\omega,c}), \tag{2.11}$$

$$\frac{\sigma-1}{\sigma+1}J(\phi_{\omega,c})=2cP(\phi_{\omega,c})+4E(\phi_{\omega,c}), \tag{2.12}$$

and

$$J'(\phi_{\omega,c})=-\frac{\sigma+1}{\sigma}S''_{\omega,c}(\phi_{\omega,c})\phi_{\omega,c}. \tag{2.13}$$

Proof. From the equation $S'_{\omega,c}(\phi_{\omega,c})=0$, by multiplying on both sides with $\overline{x\partial_x\phi_{\omega,c}}$ and $\overline{\phi_{\omega,c}}$ respectively, and integrating over x, we obtain

$$\|\partial_x\phi_{\omega,c}\|_{L^2}^2=\omega\|\phi_{\omega,c}\|_{L^2}^2,$$

and

$$\|\partial_x\phi_{\omega,c}\|_{L^2}^2+\omega\|\phi_{\omega,c}\|_{L^2}^2+c\operatorname{Im}\int_{\mathbb{R}}\phi_{\omega,c}\,\overline{\partial_x\phi_{\omega,c}}dx-J(\phi_{\omega,c})=0.$$

Therefore, we have

$$J(\phi_{\omega,c})=4\omega M(\phi_{\omega,c})+2cP(\phi_{\omega,c}).$$

Combining the definition of E and (2.10), we have

$$\begin{aligned} E(\phi_{\omega,c}) =& \frac{1}{2}\|\partial_x\phi_{\omega,c}\|_{L^2}^2 - \frac{1}{2(\sigma+1)}\mathrm{Im}\int_{\mathbb{R}} |\phi_{\omega,c}|^{2\sigma}\phi_{\omega,c}\,\overline{\partial_x\phi_{\omega,c}}\,dx \\ =& \omega M(\phi_{\omega,c}) - \frac{1}{2(\sigma+1)}J(\phi_{\omega,c}). \end{aligned}$$

Then, we get

$$\omega M(\phi_{\omega,c}) = E(\phi_{\omega,c}) + \frac{1}{2(\sigma+1)}J(\phi_{\omega,c}).$$

Hence, we obtain

$$\begin{aligned} J(\phi_{\omega,c}) =& 4[E(\phi_{\omega,c}) + \frac{1}{2(\sigma+1)}J(\phi_{\omega,c})] + 2cP(\phi_{\omega,c}) \\ =& 2cP(\phi_{\omega,c}) + 4E(\phi_{\omega,c}) + \frac{2}{\sigma+1}J(\phi_{\omega,c}). \end{aligned}$$

That is,

$$\frac{\sigma-1}{\sigma+1}J(\phi_{\omega,c}) = 2cP(\phi_{\omega,c}) + 4E(\phi_{\omega,c}).$$

Moreover, from (2.8) and (2.9), we have

$$\begin{aligned} J'(\phi_{\omega,c}) =& 2(\sigma+1)i|\phi_{\omega,c}|^{2\sigma}\partial_x\phi_{\omega,c} \\ =& -\frac{\sigma+1}{\sigma}S''_{\omega,c}(\phi_{\omega,c})\phi_{\omega,c}. \end{aligned}$$

This completes the proof. □

Lemma 3. *Let $1 < \sigma < 2$ and $(\omega, c) \in \mathbb{R}^2$ satisfy $c^2 < 4\omega$, then*

$$\|\phi_{\omega,c}\|_{L^{2\sigma+2}}^{2\sigma+2} = 4(\sigma+1)\big[\frac{c}{2}M(\phi_{\omega,c}) + P(\phi_{\omega,c})\big], \tag{2.14}$$

and

$$\partial_c M(\phi_{\omega,c}) = \partial_\omega P(\phi_{\omega,c}), \quad \partial_c P(\phi_{\omega,c}) = \omega\partial_\omega M(\phi_{\omega,c}).$$

Proof. The details are given in Appendix. □

For any $(\omega, c) \in \mathbb{R}^2$ satisfying $c^2 < 4\omega$, we define a function $d(\omega, c)$ by

$$d(\omega, c) = S_{\omega,c}(\phi_{\omega,c}).$$

Thus, we have

$$d'(\omega, c) = \big(\partial_\omega d(\omega, c), \partial_c d(\omega, c)\big) = \big(M(\phi_{\omega,c}), P(\phi_{\omega,c})\big),$$

and the Hessian matrix $d''(\omega, c)$ of $d(\omega, c)$ is given by

$$d''(\omega, c) = \begin{bmatrix} \partial_{\omega\omega} d(\omega, c) & \partial_{\omega c} d(\omega, c) \\ \partial_{c\omega} d(\omega, c) & \partial_{cc} d(\omega, c) \end{bmatrix} = \begin{bmatrix} \partial_\omega M(\phi_{\omega,c}) & \partial_\omega P(\phi_{\omega,c}) \\ \partial_c M(\phi_{\omega,c}) & \partial_c P(\phi_{\omega,c}) \end{bmatrix}.$$

For general exponents $1 < \sigma < 2$, Liu, Simpson and Sulem [29] proved that $c = 2z_0\sqrt{\omega}$ is the unique solution of $\det[d''(\omega, c)] = 0$. Let (μ, ν) to be the eigenvector associated to zero eigenvalue of the Hessian matrix $d''(\omega, c)$. Since zero is a simple eigenvalue, (μ, ν) is unique up to a constant. That is,

$$\begin{cases} \mu\,\partial_\omega M(\phi_{\omega,c}) + \nu\,\partial_\omega P(\phi_{\omega,c}) = 0, \\ \mu\,\partial_c M(\phi_{\omega,c}) + \nu\,\partial_c P(\phi_{\omega,c}) = 0. \end{cases} \tag{2.15}$$

Together with $\partial_c M(\phi_{\omega,c}) = \partial_\omega P(\phi_{\omega,c})$, (2.15) is equivalent to

$$\begin{cases} \mu\,\partial_\omega M(\phi_{\omega,c}) + \nu\,\partial_c M(\phi_{\omega,c}) = 0, \\ \mu\,\partial_\omega P(\phi_{\omega,c}) + \nu\,\partial_c P(\phi_{\omega,c}) = 0. \end{cases} \tag{2.16}$$

Now we have the following lemma.

Lemma 4. *Let $1 < \sigma < 2$ and $(\omega, c) \in \mathbb{R}^2$ satisfy $c = 2z_0\sqrt{\omega}$, then*

$$P(\phi_{\omega,c}) = a_0 M(\phi_{\omega,c}), \quad \frac{\mu}{\nu} = \sqrt{\omega},$$

where $a_0 = (\sigma - 1)\sqrt{\omega} > 0$. Moreover, there exists $\kappa_0 > 0$, such that

$$\mu^2\partial_{\omega\omega} M(\phi_{\omega,c}) + 2\mu\nu\partial_{\omega c} M(\phi_{\omega,c}) + \nu^2\partial_{cc} M(\phi_{\omega,c}) = \kappa_0 z_0, \tag{2.17}$$

$$\mu^2\partial_{\omega\omega} P(\phi_{\omega,c}) + 2\mu\nu\partial_{\omega c} P(\phi_{\omega,c}) + \nu^2\partial_{cc} P(\phi_{\omega,c}) = -\kappa_0\sqrt{\omega}. \tag{2.18}$$

Proof. The proof of Lemma 4 is postponed to Appendix. □

For convenience, we denote the quality $Q_{\mu,\nu}$ to be:

$$Q_{\mu,\nu}(f) = \mu M(f) + \nu P(f).$$

Moreover, we let $\psi = \partial_\lambda \phi_{\omega+\lambda\mu,c+\lambda\nu}|_{\lambda=0} = \mu\partial_\omega\phi_{\omega,c} + \nu\partial_c\phi_{\omega,c}$.

Lemma 5. *Let $1 < \sigma < 2$. If $c = 2z_0\sqrt{\omega}$, then*

$$\langle M'(\phi_{\omega,c}), \psi\rangle = \langle P'(\phi_{\omega,c}), \psi\rangle = 0, \tag{2.19}$$

and

$$S''_{\omega,c}(\phi_{\omega,c})\psi = -Q'_{\mu,\nu}(\phi_{\omega,c}), \quad \left\langle S''_{\omega,c}(\phi_{\omega,c})\psi, \psi\right\rangle = 0. \tag{2.20}$$

Proof. By (2.16), we get

$$\begin{cases} \langle M'(\phi_{\omega,c}), \mu\partial_\omega\phi_{\omega,c} + \nu\partial_c\phi_{\omega,c}\rangle = 0, \\ \langle P'(\phi_{\omega,c}), \mu\partial_\omega\phi_{\omega,c} + \nu\partial_c\phi_{\omega,c})\rangle = 0, \end{cases}$$

that is,

$$\langle M'(\phi_{\omega,c}), \psi\rangle = \langle P'(\phi_{\omega,c}), \psi\rangle = 0.$$

From (2.15), we have

$$\begin{cases} \langle \mu M'(\phi_{\omega,c}) + \nu P'(\phi_{\omega,c}), \partial_\omega\phi_{\omega,c}\rangle = 0, \\ \langle \mu M'(\phi_{\omega,c}) + \nu P'(\phi_{\omega,c}), \partial_c\phi_{\omega,c}\rangle = 0. \end{cases}$$

Therefore, we have

$$\langle Q'_{\mu,\nu}(\phi_{\omega,c}), \psi\rangle = 0.$$

On the other hand, differentiating $S'_{\omega+\lambda\mu,c+\lambda\nu}(\phi_{\omega+\lambda\mu,c+\lambda\nu}) = 0$ with respect to λ, we have

$$\begin{aligned} &S''_{\omega+\lambda\mu,c+\lambda\nu}(\phi_{\omega+\lambda\mu,c+\lambda\nu})\partial_\lambda\phi_{\omega+\lambda\mu,c+\lambda\nu}\big|_{\lambda=0} \\ &\qquad = -\left[\mu M'(\phi_{\omega+\lambda\mu,c+\lambda\nu}) + \nu P'(\phi_{\omega+\lambda\mu,c+\lambda\nu})\right]\big|_{\lambda=0}. \end{aligned}$$

That is

$$S''_{\omega,c}(\phi_{\omega,c})\psi = -Q'_{\mu,\nu}(\phi_{\omega,c}).$$

Thus, we have

$$\left\langle S''_{\omega,c}(\phi_{\omega,c})\psi, \psi\right\rangle = -\langle Q'_{\mu,\nu}(\phi_{\omega,c}), \psi\rangle = 0.$$

This proves the lemma. □

Lemma 6. *Let $1<\sigma<2$. If $c=2z_0\sqrt{\omega}$, then*

$$\langle J'(\phi_{\omega,c}),\psi\rangle = 4\mu M(\phi_{\omega,c})+2\nu P(\phi_{\omega,c})\neq 0.$$

Proof. Note that $\psi=\partial_\lambda\phi_{\omega+\lambda\mu,c+\lambda\nu}|_{\lambda=0}$, using (2.11), we can write

$$\begin{aligned}\langle J'(\phi_{\omega,c}),\psi\rangle =&\partial_\lambda J(\phi_{\omega+\lambda\mu,c+\lambda\nu})|_{\lambda=0}\\ =&\partial_\lambda\big[4(\omega+\lambda\mu)M(\phi_{\omega+\lambda\mu,c+\lambda\nu})+2(c+\lambda\nu)P(\phi_{\omega+\lambda\mu,c+\lambda\nu})\big]\big|_{\lambda=0}\\ =&4\mu M(\phi_{\omega,c})+2\nu P(\phi_{\omega,c})+4\omega\langle M'(\phi_{\omega,c}),\partial_\lambda\phi_{\omega+\lambda\mu,c+\lambda\nu}|_{\lambda=0}\rangle\\ &+2c\langle P'(\phi_{\omega,c}),\partial_\lambda\phi_{\omega+\lambda\mu,c+\lambda\nu}|_{\lambda=0}\rangle.\end{aligned}$$

When $c=2z_0\sqrt{\omega}$, together with (2.19), we obtain

$$\begin{aligned}\langle J'(\phi_{\omega,c}),\psi\rangle =&4\mu M(\phi_{\omega,c})+2\nu P(\phi_{\omega,c})+4\omega\langle M'(\phi_{\omega,c}),\psi\rangle+2c\langle P'(\phi_{\omega,c}),\psi\rangle\\ =&4\mu M(\phi_{\omega,c})+2\nu P(\phi_{\omega,c}).\end{aligned}$$

Using Lemma 4, we find

$$4\mu M(\phi_{\omega,c})+2\nu P(\phi_{\omega,c})=\nu\big(4\sqrt{\omega}+2a_0\big)M(\phi_{\omega,c})\neq 0.$$

Hence, $\langle J'(\phi_{\omega,c}),\psi\rangle\neq 0$. This finishes the proof. □

3. Modulation and the coercivity property

Proposition 1. *There exists $\delta_0>0$, such that for any $\delta\in(0,\delta_0)$, $u\in U_\delta(\phi_{\omega,c})$, the following properties is verified. There exist C^1-functions*

$$(\theta,y,\lambda):U_\delta(\phi_{\omega,c})\to\mathbb{R}\times\mathbb{R}\times\mathbb{R}^+,$$

such that if we define $\varepsilon(t)$ by

$$\varepsilon(t)=e^{-i\theta(t)}u(t,\cdot+y(t))-\phi_{\omega+\lambda(t)\mu,c+\lambda(t)\nu},\tag{3.21}$$

then ε satisfies the following orthogonality conditions for any $t\in\mathbb{R}$,

$$\langle\varepsilon,i\phi_{\omega+\lambda\mu,c+\lambda\nu}\rangle=\langle\varepsilon,\partial_x\phi_{\omega+\lambda\mu,c+\lambda\nu}\rangle=\langle\varepsilon,J'(\phi_{\omega+\lambda\mu,c+\lambda\nu})\rangle=0.$$

Moreover,

$$\big\langle S''_{\omega+\lambda\mu,c+\lambda\nu}(\phi_{\omega+\lambda\mu,c+\lambda\nu})\varepsilon,\varepsilon\big\rangle\gtrsim\|\varepsilon\|^2_{H^1(\mathbb{R})}.\tag{3.22}$$

Proof. The proof of the proposition can be splitted into the following three steps.

Step 1: modulation for fixed time. Fixing $t\in\mathbb{R}$, let $\overrightarrow{F}(\theta,y,\lambda;u)=(F_1,F_2,F_3)$ with

$$\begin{aligned}F_1(\theta,y,\lambda;u)=&\langle\varepsilon,i\phi_{\omega+\lambda\mu,c+\lambda\nu}\rangle,\\ F_2(\theta,y,\lambda;u)=&\langle\varepsilon,\partial_x\phi_{\omega+\lambda\mu,c+\lambda\nu}\rangle,\\ F_3(\theta,y,\lambda;u)=&\langle\varepsilon,J'(\phi_{\omega+\lambda\mu,c+\lambda\nu})\rangle.\end{aligned}$$

Note that

$$\partial_\theta\varepsilon\big|_{(0,0,0;\phi_{\omega,c})}=-i\phi_{\omega,c};\quad \partial_y\varepsilon\big|_{(0,0,0;\phi_{\omega,c})}=\partial_x\phi_{\omega,c};\quad \partial_\lambda\varepsilon\big|_{(0,0,0;\phi_{\omega,c})}=-\psi.$$

Then, the Jacobian matrix of the derivative of the function $(\theta, y, \lambda; u) \mapsto \overrightarrow{F}$ with respect to (θ, y, λ) is as follows.

$$D\overrightarrow{F}\Big|_{(0,0,0;\phi_{\omega,c})} = \begin{pmatrix} \partial_\theta F_1 & \partial_y F_1 & \partial_\lambda F_1 \\ \partial_\theta F_2 & \partial_y F_2 & \partial_\lambda F_2 \\ \partial_\theta F_3 & \partial_y F_3 & \partial_\lambda F_3 \end{pmatrix}\Bigg|_{(0,0,0;\phi_{\omega,c})}$$
$$= \begin{pmatrix} -\|\phi_{\omega,c}\|_{L^2}^2 & -2P(\phi_{\omega,c}) & -\langle \psi, i\phi_{\omega,c}\rangle \\ 2P(\phi_{\omega,c}) & \|\partial_x \phi_{\omega,c}\|_{L^2}^2 & -\langle \psi, \partial_x \phi_{\omega,c}\rangle \\ 0 & 0 & -\langle J'(\phi_{\omega,c}), \psi\rangle \end{pmatrix}.$$

Thus, we can get

$$\det(D\overrightarrow{F})\Big|_{(0,0,0;\phi_{\omega,c})} = 4\sigma(2-\sigma)\omega[M(\phi_{\omega,c})]^2\langle J'(\phi_{\omega,c}), \psi\rangle.$$

From Lemma 6, we have

$$\det(D\overrightarrow{F})\Big|_{(0,0,0;\phi_{\omega,c})} \neq 0.$$

Therefore, the implicit function theorem implies that there exists $\delta_0 > 0$, such that for any $\delta \in (0, \delta_0)$, $u \in U_\delta(\phi_{\omega,c})$, the following properties is verified. There exist continuity functions

$$(\theta, y, \lambda) : U_\delta(\phi_{\omega,c}) \to \mathbb{R} \times \mathbb{R} \times \mathbb{R}^+,$$

such that $F_j(\theta, y, \lambda; u) = 0$, $j = 1, 2, 3$.

Step 2: the regularity of the parameters in time. It follows from the regularization arguments that the parameters $(\theta, y, \lambda) \in C_t^1$.

Step 3: the coercivity property of $S''_{\omega,c}(\phi_{\omega,c})$. In this step, we shall prove that for any (ω, c) satisfying $c^2 < 4\omega$, and any $\varepsilon \in H^1(\mathbb{R})$ satisfying

$$\langle \varepsilon, i\phi_{\omega,c}\rangle = \langle \varepsilon, \partial_x \phi_{\omega,c}\rangle = \langle \varepsilon, J'(\phi_{\omega,c})\rangle = 0,$$

the following estimate holds,

$$\left\langle S''_{\omega,c}(\phi_{\omega,c})\varepsilon, \varepsilon\right\rangle \gtrsim \|\varepsilon\|_{H^1(\mathbb{R})}^2.$$

From [29] Theorem 3.1, we obtain that $S''_{\omega,c}(\phi_{\omega,c})$ has exactly one negative eigenvalue. Hence there exists only one $\lambda_{-1} < 0$, such that,

$$S''_{\omega,c}(\phi_{\omega,c})g_{-1} = \lambda_{-1} g_{-1}, \quad \|g_{-1}\|_{L^2} = 1.$$

Moreover, we have the following decomposition,

$$\varepsilon = a_{-1} g_{-1} + a_1 i\phi_{\omega,c} + a_2 \partial_x \phi_{\omega,c} + h_1,$$

with

$$\langle h_1, g_{-1}\rangle = \langle h_1, i\phi_{\omega,c}\rangle = \langle h_1, \partial_x \phi_{\omega,c}\rangle = 0,$$

and

$$\langle S''_{\omega,c}(\phi_{\omega,c})h_1, h_1\rangle \gtrsim \|h_1\|_{H^1(\mathbb{R})}^2.$$

Since $\langle \varepsilon, i\phi_{\omega,c}\rangle = \langle \varepsilon, \partial_x \phi_{\omega,c}\rangle = 0$, we have $a_1 = a_2 = 0$. Then, we can write

$$\varepsilon = a_{-1} g_{-1} + h_1.$$

Next, using (2.13), we have

$$J'(\phi_{\omega,c}) = S''_{\omega,c}(\phi_{\omega,c})(-\frac{\sigma+1}{\sigma}\phi_{\omega,c}).$$

For convenience, we put

$$h=-\frac{\sigma+1}{\sigma}\phi_{\omega,c}.$$

Note that $\langle h, i\phi_{\omega,c}\rangle = \langle h, \partial_x\phi_{\omega,c}\rangle = 0$, we can also write that

$$h = b_{-1}g_{-1}+h_2$$

with

$$\langle h_2, g_{-1}\rangle = \langle h_2, i\phi_{\omega,c}\rangle = \langle h_2, \partial_x\phi_{\omega,c}\rangle = 0,$$

and

$$\langle S''_{\omega,c}(\phi_{\omega,c})h_2, h_2\rangle \gtrsim \|h_2\|^2_{H^1(\mathbb{R})}.$$

For simplicity, we denote

$$\gamma = -\langle S''_{\omega,c}(\phi_{\omega,c})h, h\rangle = -\langle J'(\phi_{\omega,c}), h\rangle = \frac{2(\sigma+1)^2}{\sigma}J(\phi_{\omega,c}).$$

Then, from (2.11) and Lemma 4, we know $\gamma>0$.

Moreover, we have

$$\left\langle S''_{\omega,c}(\phi_{\omega,c})\varepsilon, \varepsilon\right\rangle = \lambda_{-1}a^2_{-1} + \langle S''_{\omega,c}(\phi_{\omega,c})h_1, h_1\rangle, \tag{3.23}$$

$$\left\langle S''_{\omega,c}(\phi_{\omega,c})h, h\right\rangle = \lambda_{-1}b^2_{-1} + \langle S''_{\omega,c}(\phi_{\omega,c})h_2, h_2\rangle = -\gamma<0. \tag{3.24}$$

According to the orthogonality condition $\langle \varepsilon, J'(\phi_{\omega,c})\rangle = 0$ and some direct computations, we have

$$\lambda_{-1}a_{-1}b_{-1} + \langle S''_{\omega,c}(\phi_{\omega,c})h_1, h_2\rangle = 0. \tag{3.25}$$

Together with (3.23), (3.24), (3.25) and the Cauchy-Schwartz inequality, we obtain that

$$\begin{aligned}-\lambda_{-1}a^2_{-1} =& \frac{\lambda^2_{-1}a^2_{-1}b^2_{-1}}{-\lambda_{-1}b^2_{-1}} = \frac{\langle S''_{\omega,c}(\phi_{\omega,c})h_1, h_2\rangle^2}{-\lambda_{-1}b^2_{-1}}\\ \le& \frac{\langle S''_{\omega,c}(\phi_{\omega,c})h_1, h_2\rangle^2}{\gamma+\langle S''_{\omega,c}(\phi_{\omega,c})h_2, h_2\rangle} \le \frac{\langle S''_{\omega,c}(\phi_{\omega,c})h_1, h_1\rangle\langle S''_{\omega,c}(\phi_{\omega,c})h_2, h_2\rangle}{\gamma+\langle S''_{\omega,c}(\phi_{\omega,c})h_2, h_2\rangle}.\end{aligned} \tag{3.26}$$

Thus, from (3.23) we get

$$\begin{aligned}\left\langle S''_{\omega,c}(\phi_{\omega,c})\varepsilon, \varepsilon\right\rangle \ge& -\frac{\langle S''_{\omega,c}(\phi_{\omega,c})h_1, h_1\rangle\langle S''_{\omega,c}(\phi_{\omega,c})h_2, h_2\rangle}{\gamma+\langle S''_{\omega,c}(\phi_{\omega,c})h_2, h_2\rangle} + \langle S''_{\omega,c}(\phi_{\omega,c})h_1, h_1\rangle\\ =& \frac{\gamma}{\gamma+\langle S''_{\omega,c}(\phi_{\omega,c})h_2, h_2\rangle}\langle S''_{\omega,c}(\phi_{\omega,c})h_1, h_1\rangle\\ \gtrsim& \|h_1\|^2_{H^1(\mathbb{R})}.\end{aligned}$$

By (3.26) and Hölder's inequality, we have

$$a^2_{-1} \lesssim \|h_1\|^2_{H^1(\mathbb{R})}.$$

Hence, by $\varepsilon = a_{-1}g_{-1}+h_1$, we have

$$\|\varepsilon\|^2_{L^2} \lesssim a^2_{-1} + \|h_1\|^2_{H^1(\mathbb{R})} \lesssim \|h_1\|^2_{H^1(\mathbb{R})}.$$

Since,

$$\left\langle S''_{\omega,c}(\phi_{\omega,c})\varepsilon, \varepsilon\right\rangle \gtrsim \|h_1\|^2_{H^1(\mathbb{R})} \gtrsim \|\varepsilon\|^2_{L^2(\mathbb{R})}.$$

From the definition of $S''_{\omega,c}(\phi_{\omega,c})$ in (2.7), we have

$$\|\varepsilon\|^2_{\dot{H}^1} \lesssim \left\langle S''_{\omega,c}(\phi_{\omega,c})\varepsilon, \varepsilon\right\rangle + \|\varepsilon\|^2_{L^2(\mathbb{R})}.$$

Therefore, we get that

$$\langle S''_{\omega,c}(\phi_{\omega,c})\varepsilon,\varepsilon\rangle \gtrsim \|\varepsilon\|^2_{H^1(\mathbb{R})}.$$

This finishes the proof of the proposition. □

Remark 2. *The choosing of the orthogonal condition $\langle \varepsilon, J'(\phi_{\omega+\lambda\mu,c+\lambda\nu})\rangle = 0$ is based on the following two reasons. The first, it guarantees the reversibility of the corresponding Jacobian matrix. The second, it brings convenience to obtain the desired estimate of the dynamic of the parameter $\dot{y}$ as presented in the following lemma.*

Lemma 7. *There exists $C_{\omega,c} \in \mathbb{R}$, such that*

$$\dot{y} - c - \lambda\nu = C_{\omega,c}\dot{\lambda} + \langle S''_{\omega,c}(\phi_{\omega,c})\frac{-a_0\phi_{\omega,c} + i\partial_x\phi_{\omega,c}}{2(a_0^2-\omega)M(\phi_{\omega,c})}, \varepsilon\rangle + O(\lambda\|\varepsilon\|_{H^1(\mathbb{R})} + \|\varepsilon\|^2_{H^1(\mathbb{R})}), \tag{3.27}$$

and

$$\dot{\theta} - \omega - \lambda\mu = O(\|\varepsilon\|_{H^1(\mathbb{R})}), \quad \dot{\lambda} = O(\|\varepsilon\|_{H^1(\mathbb{R})}). \tag{3.28}$$

Here the parameters θ, y, λ are given by Proposition 1.

Proof. Now, we consider the dynamic of the parameters. From (3.21), we have

$$u = e^{i\theta(t)}\big(\phi_{\omega+\lambda(t)\mu,c+\lambda(t)\nu} + \varepsilon(t)\big)(x - y(t)).$$

Using (1.1), we obtain

$$\begin{aligned} &i\dot{\varepsilon} - (\dot{\theta} - \omega - \lambda\mu)(\phi_{\omega+\lambda\mu,c+\lambda\nu} + \varepsilon) - (\dot{y} - c - \lambda\nu)(i\partial_x\phi_{\omega+\lambda\mu,c+\lambda\nu} + i\partial_x\varepsilon) \\ &\quad + \dot{\lambda}\cdot i\partial_\lambda\phi_{\omega+\lambda\mu,c+\lambda\nu} = S''_{\omega,c}(\phi_{\omega,c})\varepsilon + O(\lambda\varepsilon + \varepsilon^2), \end{aligned} \tag{3.29}$$

where $O(\cdot)$ is a functional with the order equal or more than one.

First, multiplying the equation (3.29) on both sides with $-a_0\phi_{\omega+\lambda\mu,c+\lambda\nu}+i\partial_x\phi_{\omega+\lambda\mu,c+\lambda\nu}$ and integrating over x, we get

$$\begin{aligned} &-(\dot{\theta} - \omega - \lambda\mu)\big[-2a_0M(\phi_{\omega+\lambda\mu,c+\lambda\nu}) + 2P(\phi_{\omega+\lambda\mu,c+\lambda\nu}) + O(\|\varepsilon\|_{H^1(\mathbb{R})})\big] \\ &-(\dot{y} - c - \lambda\nu)\big[-2a_0P(\phi_{\omega+\lambda\mu,c+\lambda\nu}) + \|\partial_x\phi_{\omega+\lambda\mu,c+\lambda\nu}\|^2_{L^2} + O(\|\varepsilon\|_{H^1(\mathbb{R})})\big] \\ &+\dot{\lambda}\big[\langle i\partial_\lambda\phi_{\omega+\lambda\mu,c+\lambda\nu}, -a_0\phi_{\omega+\lambda\mu,c+\lambda\nu} + i\partial_x\phi_{\omega+\lambda\mu,c+\lambda\nu}\rangle - \langle\varepsilon, ia_0\partial_\lambda\phi_{\omega+\lambda\mu,c+\lambda\nu} + \partial_\lambda\partial_x\phi_{\omega+\lambda\mu,c+\lambda\nu}\rangle\big] \\ &=\big\langle S''_{\omega,c}(\phi_{\omega,c})(-a_0\phi_{\omega+\lambda\mu,c+\lambda\nu} + i\partial_x\phi_{\omega+\lambda\mu,c+\lambda\nu}), \varepsilon\big\rangle + O(\lambda\|\varepsilon\|_{H^1(\mathbb{R})} + \|\varepsilon\|^2_{H^1(\mathbb{R})}) \end{aligned}$$

By a direct expansion, we have

$$\phi_{\omega+\lambda\mu,c+\lambda\nu} = \phi_{\omega,c} + \lambda\psi + O(\lambda^2), \tag{3.30}$$

and

$$\partial_\lambda\phi_{\omega+\lambda\mu,c+\lambda\nu} = \psi + O(\lambda).$$

Moreover, together with (2.19), we have

$$\begin{aligned} M(\phi_{\omega+\lambda\mu,c+\lambda\nu}) &= M(\phi_{\omega,c}) + \lambda\langle M'(\phi_{\omega,c}),\psi\rangle + O(\lambda^2) = M(\phi_{\omega,c}) + O(\lambda^2), \\ P(\phi_{\omega+\lambda\mu,c+\lambda\nu}) &= P(\phi_{\omega,c}) + \lambda\langle P'(\phi_{\omega,c}),\psi\rangle + O(\lambda^2) = P(\phi_{\omega,c}) + O(\lambda^2). \end{aligned}$$

From (2.10), we get

$$\begin{aligned} \|\partial_x\phi_{\omega+\lambda\mu,c+\lambda\nu}\|^2_{L^2} &= (\omega+\lambda\mu)\|\phi_{\omega+\lambda\mu,c+\lambda\nu}\|^2_{L^2} \\ &= 2\omega M(\phi_{\omega,c}) + O(\lambda). \end{aligned}$$

We collect the above computations and obtain

$$\begin{aligned}
&-(\dot\theta-\omega-\lambda\mu)\big[-2a_0M(\phi_{\omega,c})+2P(\phi_{\omega,c})+O(\lambda^2+\|\varepsilon\|_{H^1(\mathbb{R})})\big]\\
&-(\dot y-c-\lambda\nu)\big[-2a_0P(\phi_{\omega,c})+2\omega M(\phi_{\omega,c})+O(\lambda+\|\varepsilon\|_{H^1(\mathbb{R})})\big]\\
&+\dot\lambda\big[\langle i\psi,-a_0\phi_{\omega,c}+i\partial_x\phi_{\omega,c}\rangle+O(\lambda+\|\varepsilon\|_{H^1(\mathbb{R})})\big]\\
&=\big\langle S''_{\omega,c}(\phi_{\omega,c})(-a_0\phi_{\omega,c}+i\partial_x\phi_{\omega,c}+O(\lambda)),\varepsilon\big\rangle+O(\lambda\|\varepsilon\|_{H^1(\mathbb{R})}+\|\varepsilon\|^2_{H^1(\mathbb{R})}).
\end{aligned}$$

By Lemma 4, we know that

$$-a_0M(\phi_{\omega,c})+P(\phi_{\omega,c})=0,\quad -a_0P(\phi_{\omega,c})+\omega M(\phi_{\omega,c})=(\omega-a_0^2)M(\phi_{\omega,c})\neq 0.$$

Then, we get

$$\begin{aligned}
&-(\dot\theta-\omega-\lambda\mu)\big[O(\lambda^2+\|\varepsilon\|_{H^1(\mathbb{R})})\big]\\
&-(\dot y-c-\lambda\nu)\big[2(\omega-a_0^2)M(\phi_{\omega,c})+O(\lambda+\|\varepsilon\|_{H^1(\mathbb{R})})\big]\\
&+\dot\lambda\big[\langle i\psi,-a_0\phi_{\omega,c}+i\partial_x\phi_{\omega,c}\rangle+O(\lambda+\|\varepsilon\|_{H^1(\mathbb{R})})\big]\\
&=\big\langle S''_{\omega,c}(\phi_{\omega,c})(-a_0\phi_{\omega,c}+i\partial_x\phi_{\omega,c}),\varepsilon\big\rangle+O(\lambda\|\varepsilon\|_{H^1(\mathbb{R})}+\|\varepsilon\|^2_{H^1(\mathbb{R})}).
\end{aligned}\tag{3.31}$$

Next, multiplying $\phi_{\omega+\lambda\mu,c+\lambda\nu}$ on both sides of the equation (3.29) and treating as above, we get

$$\begin{aligned}
&-(\dot\theta-\omega-\lambda\mu)[2M(\phi_{\omega,c})+O(\lambda^2+\|\varepsilon\|_{H^1(\mathbb{R})})]\\
&-(\dot y-c-\lambda\nu)[2P(\phi_{\omega,c})+O(\lambda^2+\|\varepsilon\|_{H^1(\mathbb{R})})]\\
&+\dot\lambda[(i\psi,\phi_{\omega,c})+O(\lambda+\|\varepsilon\|_{H^1(\mathbb{R})})]\\
&=O(\|\varepsilon\|_{H^1(\mathbb{R})}).
\end{aligned}\tag{3.32}$$

Finally, multiplying $iJ'(\phi_{\omega+\lambda\mu,c+\lambda\nu})$ on both sides of the equation (3.29) and treating as above, we obtain

$$\begin{aligned}
&-(\dot\theta-\omega-\lambda\mu)\cdot O(\lambda+\|\varepsilon\|_{H^1(\mathbb{R})})-(\dot y-c-\lambda\nu)\cdot O(\lambda+\|\varepsilon\|_{H^1(\mathbb{R})})\\
&+\dot\lambda[\langle J'(\phi_{\omega,c}),\psi\rangle+O(\lambda+\|\varepsilon\|_{H^1(\mathbb{R})})]\\
&=O(\|\varepsilon\|_{H^1(\mathbb{R})}).
\end{aligned}\tag{3.33}$$

Using (3.31), (3.32), (3.33) and Lemma 6, we obtain

$$\dot y-c-\lambda\nu=C_{\omega,c}\dot\lambda+\Big\langle S''_{\omega,c}(\phi_{\omega,c})\frac{-a_0\phi_{\omega,c}+i\partial_x\phi_{\omega,c}}{2(a_0^2-\omega)M(\phi_{\omega,c})},\varepsilon\Big\rangle+O(\lambda\|\varepsilon\|_{H^1(\mathbb{R})}+\|\varepsilon\|^2_{H^1(\mathbb{R})}),$$

and

$$\dot\theta-\omega-\lambda\mu=O(\|\varepsilon\|_{H^1(\mathbb{R})}),\quad \dot\lambda=O(\|\varepsilon\|_{H^1(\mathbb{R})}).$$

Here

$$C_{\omega,c}=\frac{-1}{2(a_0^2-\omega)M(\phi_{\omega,c})}\langle i\psi,-a_0\phi_{\omega,c}+i\partial_x\phi_{\omega,c}\rangle.$$

This completes the proof. □

4. Instability of the Solitary Wave Solutions

4.1. **Virial Estimates.** To prove the main theorem, first, we need the localized virial identities.

Lemma 8. *Let $\varphi \in C^\infty(\mathbb{R})$, bounded and u be a strong solution to (1.1), then*

$$\frac{d}{dt}\int_{\mathbb{R}} \varphi|u|^2dx = -2Im\int_{\mathbb{R}} \varphi' u\,\overline{\partial_x u}dx + \frac{1}{\sigma+1}\int_{\mathbb{R}} \varphi'|u|^{2\sigma+2}dx,$$
$$\frac{d}{dt}\int_{\mathbb{R}} \varphi Im(u\,\overline{\partial_x u}) = -2\int_{\mathbb{R}} \varphi'|\partial_x u|^2dx + \frac{1}{2}\int_{\mathbb{R}} \varphi'''|u|^2dx + Im\int_{\mathbb{R}} \varphi'|u|^{2\sigma}u\,\overline{\partial_x u}.$$

Proof. Combining the equation (1.1) and integration by parts, we have

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}} \varphi|u|^2dx =& 2\mathrm{Re}\int_{\mathbb{R}} \varphi\partial_t u\,\overline{u}\\ =& 2\mathrm{Re}\int_{\mathbb{R}} \varphi(i\partial_x^2 u - |u|^{2\sigma}\partial_x u)\overline{u}\\ =& -2\mathrm{Re}\int_{\mathbb{R}} i\varphi'\partial_x u\,\overline{u}dx + \frac{1}{\sigma+1}\mathrm{Re}\int_{\mathbb{R}} \varphi'|u|^{2\sigma+2}dx\\ =& -2\mathrm{Im}\int_{\mathbb{R}} \varphi' u\,\overline{\partial_x u}dx + \frac{1}{\sigma+1}\int_{\mathbb{R}} \varphi'|u|^{2\sigma+2}dx.\end{aligned}$$

By the same way, we obtain

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}} \varphi\mathrm{Im}(u\,\overline{\partial_x u}) =& \mathrm{Im}\int_{\mathbb{R}} \varphi\partial_t u\,\overline{\partial_x u}dx + \mathrm{Im}\int_{\mathbb{R}} \varphi u\,\overline{\partial_{tx} u}dx\\ =& 2\mathrm{Im}\int_{\mathbb{R}} \varphi\partial_t u\,\overline{\partial_x u}dx + \mathrm{Im}\int_{\mathbb{R}} \varphi'\partial_t u\,\overline{u}dx\\ =& 2\mathrm{Im}\int_{\mathbb{R}} \varphi(i\partial_x^2 u - |u|^{2\sigma}\partial_x u)\overline{\partial_x u}dx + \mathrm{Im}\int_{\mathbb{R}} \varphi'(i\partial_x^2 u - |u|^{2\sigma}\partial_x u)\overline{u}dx\\ =& -2\int_{\mathbb{R}} \varphi'|\partial_x u|^2dx + \frac{1}{2}\int_{\mathbb{R}} \varphi'''|u|^2dx + \mathrm{Im}\int_{\mathbb{R}} \varphi'|u|^{2\sigma}u\,\overline{\partial_x u}.\end{aligned}$$

This proves the lemma. □

Now we define $\varphi_R \in C^\infty(\mathbb{R})$ satisfying

$$\varphi_R(x) = \begin{cases} x, & |x| \le R,\\ 2R, & |x| \ge 2R,\end{cases}$$

and $0 \le |\varphi_R'| \le 1$ for any $x \in \mathbb{R}$. Moreover, we denote

$$I_1(t) = \int_{\mathbb{R}} \varphi_R(x - y(t))|u|^2dx,$$
$$I_2(t) = \int_{\mathbb{R}} \varphi_R(x - y(t))\mathrm{Im}(u\,\overline{\partial_x u})dx.$$

To prove the main theorem, we define the key functional $I(t)$ as

$$I(t) = -\sqrt{\omega}I_1(t) + I_2(t) + \widetilde{C}_{\omega,c}\lambda,$$

where $\widetilde{C}_{\omega,c} = 2C_{\omega,c}(M(\phi_{\omega,c}) + P(\phi_{\omega,c}))$.

Hence, by Lemma 8 we can obtain the following localized virial estimates.

Lemma 9. *Assume that*

$$u = e^{i\theta(t)}\big(\phi_{\omega+\lambda(t)\mu,c+\lambda(t)\nu} + \varepsilon(t)\big)(x - y(t)), \tag{4.34}$$

with the parameters obeying the estimates in Lemma 7. Then the following estimates hold:

$$\begin{aligned} I_1'(t) =& -2C_{\omega.c}\dot\lambda M(\phi_{\omega,c}) - 2c[M(u_0) - M(\phi_{\omega,c})] - 4[P(u_0) - P(\phi_{\omega,c})] \\ &- \frac{\sigma-1}{\sigma(2-\sigma)\omega}\big\langle S''_{\omega,c}(\phi_{\omega,c})(\sqrt{\omega}\phi_{\omega,c} - i\partial_x\phi_{\omega,c}), \varepsilon\big\rangle \\ &+ \frac{1}{2(\sigma+1)}\lambda^2\frac{d^2}{d\lambda^2}\|\phi_{\omega+\lambda\mu,c+\lambda\nu}\|^{2\sigma+2}_{L^{2\sigma+2}}\Big|_{\lambda=0} \\ &+ O(\lambda\|\varepsilon\|_{H^1(\mathbb{R})} + \|\varepsilon\|^2_{H^1(\mathbb{R})} + \frac{1}{R}) + o(\lambda^2), \end{aligned}$$

and

$$\begin{aligned} I_2'(t) =& -2C_{\omega.c}\dot\lambda P(\phi_{\omega,c}) - 2[P(u_0) - P(\phi_{\omega,c})] - 4[E(u_0) - E(\phi_{\omega,c})] \\ &- \frac{\sigma-1}{\sigma(2-\sigma)\sqrt{\omega}}\big\langle S''_{\omega,c}(\phi_{\omega,c})(\sqrt{\omega}\phi_{\omega,c} - i\partial_x\phi_{\omega,c}), \varepsilon\big\rangle \\ &+ \frac{\sigma-1}{2(\sigma+1)}\lambda^2\frac{d^2}{d\lambda^2}J(\phi_{\omega+\lambda\mu,c+\lambda\nu})\Big|_{\lambda=0} \\ &+ O(\lambda\|\varepsilon\|_{H^1(\mathbb{R})} + \|\varepsilon\|^2_{H^1(\mathbb{R})} + \frac{1}{R}) + o(\lambda^2). \end{aligned}$$

Proof. By Lemma 8 and some direct computation, we have the following formulas.

$$\begin{aligned} I_1'(t) =& -\dot y\int_{\mathbb{R}}\varphi_R'(x-y(t))|u|^2 - 2\mathrm{Im}\int_{\mathbb{R}}\varphi_R'(x-y(t))u\,\overline{\partial_x u}dx \\ &+ \frac{1}{\sigma+1}\int_{\mathbb{R}}\varphi_R'(x-y(t))|u|^{2\sigma+2}dx, \\ I_2'(t) =& -\dot y\int_{\mathbb{R}}\varphi_R'(x-y(t))\mathrm{Im}(u\,\overline{\partial_x u})dx - \int_{\mathbb{R}}\varphi_R'(x-y(t))[2|\partial_x u|^2 - \mathrm{Im}(|u|^{2\sigma}u\,\overline{\partial_x u})]dx \\ &+ \frac{1}{2}\int_{\mathbb{R}}\varphi_R'''(x-y(t))|u|^2dx. \end{aligned}$$

First, with the definitions of M and P yields

$$\begin{aligned} I_1'(t) =& -2\dot y M(u) - 2\dot y\int_{\mathbb{R}}\big[\varphi_R'(x-y(t)) - 1\big]|u|^2dx - 4P(u) \qquad (4.35)\\ &- 2\mathrm{Im}\int_{\mathbb{R}}\big[\varphi_R'(x-y(t)) - 1\big]u\,\overline{\partial_x u}\,dx \\ &+ \frac{1}{\sigma+1}\|u\|^{2\sigma+2}_{L^{2\sigma+2}} + \frac{1}{\sigma+1}\int_{\mathbb{R}}\big[\varphi_R'(x-y(t)) - 1\big]|u|^{2\sigma+2}dx \\ =& -2(\dot y - c - \lambda\nu)M(u) - 2(c+\lambda\nu)M(u) - 4P(u) + \frac{1}{\sigma+1}\|u\|^{2\sigma+2}_{L^{2\sigma+2}} \\ &+ O\Big(\int_{\mathbb{R}}\big[\varphi_R'(x-y(t)) - 1\big]\big(\dot y|u|^2 + \mathrm{Im}(u\,\overline{\partial_x u}) + |u|^{2\sigma+2}\big)dx\Big). \qquad (4.36) \end{aligned}$$

In fact, $\mathrm{supp}[\varphi_R'(x-y(t)) - 1] \subset \{x : |x - y(t)| \geq R\}$, $0 \leq |\varphi_R'| \leq 1$. From Lemma 7, we know that $|\dot y| \lesssim 1$. Then, after using (4.34) and the exponential decaying of

$\phi_{\omega+\lambda\mu,c+\lambda\nu}$, we have

$$\begin{aligned}&\int_{\mathbb{R}}\big[\varphi_R'(x-y(t))-1\big]\big(\dot{y}|u|^2+\mathrm{Im}(u\,\overline{\partial_x u})+|u|^{2\sigma+2}\big)dx\\&\lesssim\int_{|x|\geq R}\Big[|\phi_{\omega+\lambda\mu,c+\lambda\nu}|^2+|\partial_x^2\phi_{\omega+\lambda\mu,c+\lambda\nu}|+|\varepsilon|^2+|\partial_x\varepsilon|^2+|\varepsilon|^{2\sigma+2}\Big]\\&=O\big(\|\varepsilon\|_{H^1(\mathbb{R})}^2+\frac{1}{R}\big).\end{aligned}\tag{4.37}$$

Mergering (3.27) and (4.37) into (4.35), we obtain

$$\begin{aligned}I_1'(t)=&-2\big[C_{\omega,c}\dot{\lambda}+\big\langle S_{\omega,c}''(\phi_{\omega,c})\frac{-a_0\phi_{\omega,c}+i\partial_x\phi_{\omega,c}}{2(a_0^2-\omega)M(\phi_{\omega,c})},\varepsilon\big\rangle+O(\lambda\|\varepsilon\|_{H^1(\mathbb{R})}+\|\varepsilon\|_{H^1(\mathbb{R})}^2)\big]M(u)\\&-2(c+\lambda\nu)M(u)-4P(u)+\frac{1}{\sigma+1}\|u\|_{L^{2\sigma+2}}^{2\sigma+2}+O\big(\|\varepsilon\|_{H^1(\mathbb{R})}^2+\frac{1}{R}\big).\end{aligned}$$

Now, using (2.19), (3.30) and (4.34), we get

$$M(u)=M(\phi_{\omega,c})+\langle M'(\phi_{\omega,c}),\varepsilon\rangle+O(\lambda\|\varepsilon\|_{H^1(\mathbb{R})}+\|\varepsilon\|_{H^1(\mathbb{R})}^2),\tag{4.38}$$

and

$$P(u)=P(\phi_{\omega,c})+\langle P'(\phi_{\omega,c}),\varepsilon\rangle+O(\lambda\|\varepsilon\|_{H^1(\mathbb{R})}+\|\varepsilon\|_{H^1(\mathbb{R})}^2).\tag{4.39}$$

Moreover,

$$\begin{aligned}\|u\|_{L^{2\sigma+2}}^{2\sigma+2}=&\|\phi_{\omega,c}\|_{L^{2\sigma+2}}^{2\sigma+2}+2(\sigma+1)\lambda\langle|\phi_{\omega,c}|^{2\sigma}\phi_{\omega,c},\psi\rangle+2(\sigma+1)\langle|\phi_{\omega,c}|^{2\sigma}\phi_{\omega,c},\varepsilon\rangle\\&+\frac{1}{2}\lambda^2\frac{d^2}{d\lambda^2}\|\phi_{\omega+\lambda\mu,c+\lambda\nu}\|_{L^{2\sigma+2}}^{2\sigma+2}\Big|_{\lambda=0}+O\big(\lambda\|\varepsilon\|_{H^1(\mathbb{R})}+\|\varepsilon\|_{H^1(\mathbb{R})}^2\big)+o(\lambda^2),\end{aligned}\tag{4.40}$$

and

$$\begin{aligned}J(u)=&J(\phi_{\omega,c})+\lambda\langle J'(\phi_{\omega,c}),\psi\rangle+\frac{1}{2}\lambda^2\frac{d^2}{d\lambda^2}J(\phi_{\omega+\lambda\mu,c+\lambda\nu})\big|_{\lambda=0}\\&+\frac{\sigma-1}{\sigma+1}\langle J'(\phi_{\omega,c}),\epsilon\rangle+O\big(\lambda\|\varepsilon\|_{H^1(\mathbb{R})}+\|\varepsilon\|_{H^1(\mathbb{R})}^2\big)+o(\lambda^2).\end{aligned}\tag{4.41}$$

Combining (4.38) and (4.40) yields

$$\begin{aligned}I_1'(t)=&-2C_{\omega,c}\dot{\lambda}M(\phi_{\omega,c})-2cM(u)-2\lambda\nu M(\phi_{\omega,c})-4P(u)+\frac{1}{\sigma+1}\|\phi_{\omega,c}\|_{L^{2\sigma+2}}^{2\sigma+2}\\&+\big\langle-S_{\omega,c}''(\phi_{\omega,c})\frac{-a_0\phi_{\omega,c}+i\partial_x\phi_{\omega,c}}{a_0^2-\omega}+2|\phi_{\omega,c}|^{2\sigma}\phi_{\omega,c},\varepsilon\big\rangle+2\lambda\langle|\phi_{\omega,c}|^{2\sigma}\phi_{\omega,c},\psi\rangle\\&+\frac{1}{2(\sigma+1)}\lambda^2\frac{d^2}{d\lambda^2}\|\phi_{\omega+\lambda\mu,c+\lambda\nu}\|_{L^{2\sigma+2}}^{2\sigma+2}\Big|_{\lambda=0}+O\big(\lambda\|\varepsilon\|_{H^1(\mathbb{R})}+\|\varepsilon\|_{H^1(\mathbb{R})}^2+\frac{1}{R}\big)+o(\lambda^2).\end{aligned}$$

From the conservation laws and (2.14), we get

$$\begin{aligned}I_1'(t)=&-2C_{\omega,c}\dot{\lambda}M(\phi_{\omega,c})-2cM(u_0)-4P(u_0)-2\lambda\nu M(\phi_{\omega,c})+2cM(\phi_{\omega,c})+4P(\phi_{\omega,c})\\&+\big\langle-S_{\omega,c}''(\phi_{\omega,c})\frac{-a_0\phi_{\omega,c}+i\partial_x\phi_{\omega,c}}{a_0^2-\omega}+2|\phi_{\omega,c}|^{2\sigma}\phi_{\omega,c},\varepsilon\big\rangle+2\lambda\langle|\phi_{\omega,c}|^{2\sigma}\phi_{\omega,c},\psi\rangle\\&+\frac{1}{2(\sigma+1)}\lambda^2\frac{d^2}{d\lambda^2}\|\phi_{\omega+\lambda\mu,c+\lambda\nu}\|_{L^{2\sigma+2}}^{2\sigma+2}\Big|_{\lambda=0}+O\big(\lambda\|\varepsilon\|_{H^1(\mathbb{R})}+\|\varepsilon\|_{H^1(\mathbb{R})}^2+\frac{1}{R}\big)+o(\lambda^2).\end{aligned}$$

Observe that

$$\langle|\phi_{\omega,c}|^{2\sigma}\phi_{\omega,c},\psi\rangle=\frac{1}{2(\sigma+1)}\partial_\lambda\Big(\|\phi_{\omega+\lambda\mu,c+\lambda\nu}\|_{L^{2\sigma+2}}^{2\sigma+2}\Big)\Big|_{\lambda=0}.$$

Moreover, using the equality (2.14) again yields

$$\langle|\phi_{\omega,c}|^{2\sigma}\phi_{\omega,c},\psi\rangle=\frac{1}{2(\sigma+1)}\partial_\lambda\Big(4(\sigma+1)\big(\frac{c+\lambda\nu}{2}M(\phi_{\omega+\lambda\mu,c+\lambda\nu})+P(\phi_{\omega+\lambda\mu,c+\lambda\nu})\big)\Big)\Big|_{\lambda=0}.$$

Using Lemma 5 again, we obtain that

$$\langle|\phi_{\omega,c}|^{2\sigma}\phi_{\omega,c},\psi\rangle=\nu M(\phi_{\omega,c}).$$

By Lemma 1, we have

$$-S''_{\omega,c}(\phi_{\omega,c})\frac{-a_0\phi_{\omega,c}+i\partial_x\phi_{\omega,c}}{a_0^2-\omega}+2|\phi_{\omega,c}|^{2\sigma}\phi_{\omega,c}=-\frac{\sigma-1}{\sigma(2-\sigma)\omega}S''_{\omega,c}(\phi_{\omega,c})\big(\sqrt{\omega}\phi_{\omega,c}-i\partial_x\phi_{\omega,c}\big).$$

Finally, we collect the above equalities and obtain

$$\begin{aligned}I_1'(t)=&-2C_{\omega,c}\dot\lambda M(\phi_{\omega,c})-2c[M(u_0)-M(\phi_{\omega,c})]-4[P(u_0)-P(\phi_{\omega,c})]\\&-\frac{\sigma-1}{\sigma(2-\sigma)\omega}\big\langle S''_{\omega,c}(\phi_{\omega,c})\big(\sqrt{\omega}\phi_{\omega,c}-i\partial_x\phi_{\omega,c}\big),\varepsilon\big\rangle+\frac{1}{2(\sigma+1)}\lambda^2\frac{d^2}{d\lambda^2}\|\phi_{\omega+\lambda\mu,c+\lambda\nu}\|_{L^{2\sigma+2}}^{2\sigma+2}\Big|_{\lambda=0}\\&+O\big(\lambda\|\varepsilon\|_{H^1(\mathbb{R})}+\|\varepsilon\|_{H^1(\mathbb{R})}^2+\frac{1}{R}\big)+o(\lambda^2).\end{aligned}$$

Similarly, from the definitions of P, E and J, we have

$$\begin{aligned}I_2'(t)=&-2\dot yP(u)-4E(u)+\frac{\sigma-1}{\sigma+1}J(u)\\&-\dot y\int_{\mathbb{R}}[\varphi_R'(x-y(t))-1][\mathrm{Im}(u\,\overline{\partial_x u})-2|\partial_x u|^2+\mathrm{Im}(|u|^{2\sigma}u\,\overline{\partial_x u})]dx\\=&-2(\dot y-c-\lambda\nu)P(u)-2(c+\lambda\nu)P(u)-4E(u)+\frac{\sigma-1}{\sigma+1}J(u)+O\big(\|\varepsilon\|_{H^1(\mathbb{R})}^2+\frac{1}{R}\big).\end{aligned}$$

Using (3.27), we get

$$\begin{aligned}I_2'(t)=&-2\big[C_{\omega,c}\dot\lambda+\big\langle S''_{\omega,c}(\phi_{\omega,c})\frac{-a_0\phi_{\omega,c}+i\partial_x\phi_{\omega,c}}{2(a_0^2-\omega)M(\phi_{\omega,c})},\varepsilon\big\rangle+O(\lambda\|\varepsilon\|_{H^1(\mathbb{R})}+\|\varepsilon\|_{H^1(\mathbb{R})}^2)\big]P(u)\\&-2(c+\lambda\nu)P(u)-4E(u)+\frac{\sigma-1}{\sigma+1}J(u)+O\big(\|\varepsilon\|_{H^1(\mathbb{R})}^2+\frac{1}{R}\big).\end{aligned}$$

Combining (4.39) and (4.41), we obtain

$$\begin{aligned}I_2'(t)=&-2C_{\omega.c}\dot\lambda P(\phi_{\omega,c})-2cP(u)-2\lambda\nu P(\phi_{\omega,c})-4E(u)+\frac{\sigma-1}{\sigma+1}J(\phi_{\omega,c})\\&-\frac{P(\phi_{\omega,c})}{M(\phi_{\omega,c})}\big\langle S''_{\omega,c}(\phi_{\omega,c})\frac{-a_0\phi_{\omega,c}+i\partial_x\phi_{\omega,c}}{a_0^2-\omega},\varepsilon\big\rangle+\frac{\sigma-1}{\sigma+1}\langle J'(\phi_{\omega,c}),\varepsilon\rangle\\&+\frac{\sigma-1}{\sigma+1}\lambda\langle J'(\phi_{\omega,c}),\psi\rangle+\frac{\sigma-1}{2(\sigma+1)}\lambda^2\frac{d^2}{d\lambda^2}J(\phi_{\omega+\lambda\mu,c+\lambda\nu})\big|_{\lambda=0}\\&+O\big(\lambda\|\varepsilon\|_{H^1(\mathbb{R})}+\|\varepsilon\|_{H^1(\mathbb{R})}^2+\frac{1}{R}\big)+o(\lambda^2).\end{aligned}$$

Using (2.11), (2.12), (2.13), Lemma 4 and the conservation laws, we have

$$\begin{aligned}I_2'(t) =& -2C_{\omega,c}\dot{\lambda}P(\phi_{\omega,c}) - 2c[P(u_0) - P(\phi_{\omega,c})] - 4[E(u_0) - E(\phi_{\omega,c})]\\ &- \frac{\sigma-1}{\sigma(2-\sigma)\sqrt{\omega}}\langle S''_{\omega,c}(\phi_{\omega,c})(\sqrt{\omega}\phi_{\omega,c} - i\partial_x\phi_{\omega,c}), \varepsilon\rangle + \frac{\sigma-1}{2(\sigma+1)}\lambda^2\frac{d^2}{d\lambda^2}J(\phi_{\omega+\lambda\mu,c+\lambda\nu})\Big|_{\lambda=0}\\ &+ O(\lambda\|\varepsilon\|_{H^1(\mathbb{R})} + \|\varepsilon\|^2_{H^1(\mathbb{R})} + \frac{1}{R}) + o(\lambda^2).\end{aligned}$$

This completes the proof of the lemma. □

According to Lemma 9, we have following result.

Lemma 10. *Under the assumptions of Lemma 9, we have*

$$I'(t) = A(u_0) + B(\lambda) + O\big(\lambda\|\varepsilon\|_{H^1(\mathbb{R})} + \|\varepsilon\|^2_{H^1(\mathbb{R})} + \frac{1}{R}\big) + o(\lambda^2),$$

with $A(u_0), B(\lambda)$ verifying

$$\begin{aligned}A(u_0) =&(2c\sqrt{\omega} + 4\omega)\big[(M(u_0) - M(\phi_{\omega,c}))\big] + (4\sqrt{\omega} - 2c)\big[P(u_0) - P(\phi_{\omega,c})\big]\\ &- 4\big[S_{\omega,c}(u_0) - S_{\omega,c}(\phi_{\omega,c})\big],\end{aligned}$$

and

$$B(\lambda) = b_1\lambda^2,$$

for some $b_1 > 0$.

Remark 3. *The form of $I(t)$ removes the linear term of ε in $I'(t)$. The key observation is that the linear terms of ε from $I_1'(t)$ and $I_2'(t)$ are almost the same.*

Proof of Lemma 10. From the definition of $I(t)$, we have

$$I'(t) = -\sqrt{\omega}I_1'(t) + I_2'(t) + \widetilde{C}_{\omega,c}\dot{\lambda}.$$

By Lemma 9, we obtain

$$\begin{aligned}I'(t) =&\sqrt{\omega}\big[2c(M(u_0) - M(\phi_{\omega,c})) + 4(P(u_0) - P(\phi_{\omega,c}))\big]\\ &- 2c\big[P(u_0) - P(\phi_{\omega,c})\big] - 4\big[E(u_0) - E(\phi_{\omega,c})\big]\\ &+ \frac{1}{2(\sigma+1)}\frac{d^2}{d\lambda^2}\Big[-\sqrt{\omega}\|\phi_{\omega+\lambda\mu,c+\lambda\nu}\|^{2\sigma+2}_{L^{2\sigma+2}} + (\sigma-1)J(\phi_{\omega+\lambda\mu,c+\lambda\nu})\Big]\Big|_{\lambda=0}\\ &+ O\big(\lambda\|\varepsilon\|_{H^1(\mathbb{R})} + \|\varepsilon\|^2_{H^1(\mathbb{R})} + \frac{1}{R}\big) + o(\lambda^2).\end{aligned}$$

We denote

$$\begin{aligned}A(u_0) =&\sqrt{\omega}\big[2c(M(u_0) - M(\phi_{\omega,c})) + 4(P(u_0) - P(\phi_{\omega,c}))\big]\\ &- 2c\big[P(u_0) - P(\phi_{\omega,c})\big] - 4\big[E(u_0) - E(\phi_{\omega,c})\big],\end{aligned}$$

and

$$B(\lambda) = \frac{1}{2(\sigma+1)}\frac{d^2}{d\lambda^2}\Big[-\sqrt{\omega}\|\phi_{\omega+\lambda\mu,c+\lambda\nu}\|^{2\sigma+2}_{L^{2\sigma+2}} + (\sigma-1)J(\phi_{\omega+\lambda\mu,c+\lambda\nu})\Big]\Big|_{\lambda=0}.$$

Then, we have

$$I'(t) = A(u_0) + B(\lambda) + O\big(\lambda\|\varepsilon\|_{H^1(\mathbb{R})} + \|\varepsilon\|^2_{H^1(\mathbb{R})} + \frac{1}{R}\big) + o(\lambda^2).$$

By the definition of $S_{\omega,c}$, we have

$$\begin{aligned}A(u_0) =&(2c\sqrt{\omega}+4\omega)\big[(M(u_0)-M(\phi_{\omega,c}))\big]+(4\sqrt{\omega}-2c)\big[P(u_0)-P(\phi_{\omega,c})\big]\\&-4\big[S_{\omega,c}(u_0)-S_{\omega,c}(\phi_{\omega,c})\big].\end{aligned}$$

Now we consider $B(\lambda)$. Observe, from (2.11) and (2.14), that

$$\begin{aligned}&-\sqrt{\omega}\|\phi_{\omega+\lambda\mu,c+\lambda\nu}\|_{L^{2\sigma+2}}^{2\sigma+2}+(\sigma-1)J(\phi_{\omega+\lambda\mu,c+\lambda\nu})\\&\quad=-\sqrt{\omega}\cdot 4(\sigma+1)[\frac{c+\lambda\nu}{2}M(\phi_{\omega+\lambda\mu,c+\lambda\nu})+P(\phi_{\omega+\lambda\mu,c+\lambda\nu})]\\&\qquad+(\sigma-1)[4(\omega+\lambda\mu)M(\phi_{\omega+\lambda\mu,c+\lambda\nu})+2(c+\lambda\nu)P(\phi_{\omega+\lambda\mu,c+\lambda\nu})]\\&\quad=[4\omega(\sigma-1)-2c\sqrt{\omega}(\sigma+1)]M(\phi_{\omega+\lambda\mu,c+\lambda\nu})+[2c(\sigma-1)-4\sqrt{\omega}(\sigma+1)]P(\phi_{\omega+\lambda\mu,c+\lambda\nu})\\&\qquad+[4\mu(\sigma-1)-2\nu\sqrt{\omega}(\sigma+1)]\lambda M(\phi_{\omega+\lambda\mu,c+\lambda\nu})+2\lambda\nu P(\phi_{\omega+\lambda\mu,c+\lambda\nu}).\end{aligned}$$

Next, using (2.17) and (2.18), we calculate the terms above separately:

$$\begin{aligned}\frac{d^2}{d\lambda^2}M(\phi_{\omega+\lambda\mu,c+\lambda\nu})\Big|_{\lambda=0}=&\mu^2\partial_{\omega\omega}M(\phi_{\omega,c})+2\mu\nu\partial_{\omega c}M(\phi_{\omega,c})+\nu^2\partial_{cc}M(\phi_{\omega,c})=\kappa_0 z_0,\\\frac{d^2}{d\lambda^2}P(\phi_{\omega+\lambda\mu,c+\lambda\nu})\Big|_{\lambda=0}=&\mu^2\partial_{\omega\omega}P(\phi_{\omega,c})+2\mu\nu\partial_{\omega c}P(\phi_{\omega,c})+\nu^2\partial_{cc}P(\phi_{\omega,c})=-\kappa_0\sqrt{\omega}.\end{aligned}$$

Finally, together with (2.19), and the three estimates above, we get

$$\begin{aligned}&\frac{1}{2(\sigma+1)}\frac{d^2}{d\lambda^2}\Big[-\sqrt{\omega}\|\phi_{\omega+\lambda\mu,c+\lambda\nu}\|_{L^{2\sigma+2}}^{2\sigma+2}+(\sigma-1)J(\phi_{\omega+\lambda\mu,c+\lambda\nu})\Big]\Big|_{\lambda=0}\\&\quad=\frac{1}{\sigma+1}[2\omega(\sigma-1)-\sqrt{\omega}c(\sigma+1)]\frac{d^2}{d\lambda^2}M(\phi_{\omega+\lambda\mu,c+\lambda\nu})\Big|_{\lambda=0}\\&\qquad+\frac{1}{\sigma+1}[c(\sigma-1)-2\sqrt{\omega}(\sigma+1)]\frac{d^2}{d\lambda^2}P(\phi_{\omega+\lambda\mu,c+\lambda\nu})\Big|_{\lambda=0}\\&\quad=\frac{1}{\sigma+1}[2\omega(\sigma-1)-\sqrt{\omega}c(\sigma+1)]\cdot\kappa_0 z_0\\&\qquad+\frac{1}{\sigma+1}[c(\sigma-1)-2\sqrt{\omega}(\sigma+1)]\cdot-\kappa_0\sqrt{\omega}\\&\quad=2\kappa_0\omega(1-z_0^2).\end{aligned}$$

Let $b_1=2\kappa_0\omega(1-z_0^2)$, then $b_1>0$. Hence, we obtain that

$$B(\lambda)=b_1\lambda^2.$$

This concludes the proof of Lemma 10. □

4.2. **Proof of Theorem 1.** Now we give the proof of Theorem 1. Suppose that $e^{i\omega t}\phi_{\omega,c}(x-ct)$ of (1.1) is stable. Choose

$$u_0=\phi_{\omega,c}+\delta_1(-a_0\phi_{\omega,c}+i\partial_x\phi_{\omega,c}),\quad \delta_1>0.$$

Here δ_1 is small enough such that $u_0\in U_\delta(\phi_{\omega,c})$ which is given by Proposition 1.

Let u be the corresponding solution of (1.1) with the initial data u_0. Then, we can write

$$u=e^{i\theta}\big(\phi_{\omega+\lambda\mu,c+\lambda\nu}+\varepsilon\big)(x-y),$$

with (θ,y,λ) obtained in Proposition 1, and $|\lambda|\ll 1$.

Lemma 11. *There exists $b_2 > 0$, such that*

$$A(u_0) \geq b_2\delta_1.$$

Proof. Recalling that $c = 2z_0\sqrt{\omega}$, $a_0 = (\sigma-1)\sqrt{\omega}$ and the choose of u_0, we have

$$\begin{aligned} M(u_0) - M(\phi_{\omega,c}) =&\delta_1\langle M'(\phi_{\omega,c}), -a_0\phi_{\omega,c} + i\partial_x\phi_{\omega,c}\rangle + o(\delta_1)\\ =&\delta_1[-2a_0M(\phi_{\omega,c}) + 2P(\phi_{\omega,c})] + o(\delta_1)\\ =&o(\delta_1), \end{aligned} \tag{4.42}$$

and

$$\begin{aligned} P(u_0) - P(\phi_{\omega,c}) =&\delta_1\langle P'(\phi_{\omega,c}), -a_0\phi_{\omega,c} + i\partial_x\phi_{\omega,c}\rangle + o(\delta_1)\\ =&2(\omega - a_0^2)M(\phi_{\omega,c})\delta_1 + o(\delta_1)\\ =&2\omega\sigma(2-\sigma)M(\phi_{\omega,c})\delta_1 + o(\delta_1). \end{aligned} \tag{4.43}$$

Moreover, using $S'_{\omega,c}(\phi_{\omega,c}) = 0$, we get

$$\begin{aligned} S_{\omega,c}(u_0) - S_{\omega,c}(\phi_{\omega,c}) =&\delta_1\langle S'_{\omega,c}(\phi_{\omega,c}), -a_0\phi_{\omega,c} + i\partial_x\phi_{\omega,c}\rangle + o(\delta_1)\\ =&o(\delta_1). \end{aligned} \tag{4.44}$$

Now, we collect the above computations and obtain

$$\begin{aligned} A(u_0) =&(2c\sqrt{\omega} + 4\omega)\cdot o(\delta_1) + (4\sqrt{\omega} - 2c)\cdot 2\omega\sigma(2-\sigma)M(\phi_{\omega,c})\delta_1 + o(\delta_1) - 4\cdot o(\delta_1)\\ =&8\omega\sqrt{\omega}\,\sigma(2-\sigma)M(\phi_{\omega,c})\delta_1 + o(\delta_1)\\ \geq&b_2\delta_1, \end{aligned}$$

where $b_2 = 4\omega\sqrt{\omega}\,\sigma(2-\sigma)M(\phi_{\omega,c}) > 0$. This proves the lemma. □

We further give the estimate on $\|\varepsilon\|^2_{H^1(\mathbb{R})}$.

Lemma 12. *Let ε be defined in (3.21), there exists $b_3 > 0$, then*

$$\|\varepsilon\|^2_{H^1(\mathbb{R})} \leq b_3\lambda\delta_1.$$

Proof. Without loss of generality, we may assume that $\nu > 0$. From the conservation laws, we have

$$\begin{aligned} S_{\omega+\lambda\mu,c+\lambda\nu}(u_0) =&S_{\omega+\lambda\mu,c+\lambda\nu}(u)\\ =&S_{\omega+\lambda\mu,c+\lambda\nu}(\phi_{\omega+\lambda\mu,c+\lambda\nu}) + \frac{1}{2}\big\langle S''_{\omega+\lambda\mu,c+\lambda\nu}(\phi_{\omega+\lambda\mu,c+\lambda\nu})\varepsilon,\varepsilon\big\rangle + o(\|\varepsilon\|^2_{H^1(\mathbb{R})}). \end{aligned}$$

Combining (2.6) and Lemma 5 yields

$$\begin{aligned} S_{\omega+\lambda\mu,c+\lambda\nu}(\phi_{\omega+\lambda\mu,c+\lambda\nu}) =&S_{\omega+\lambda\mu,c+\lambda\nu}(\phi_{\omega,c}) + \lambda\big\langle S'_{\omega+\lambda\mu,c+\lambda\nu}(\phi_{\omega,c}),\psi\big\rangle\\ &+ \frac{1}{2}\lambda^2\big\langle S''_{\omega+\lambda\mu,c+\lambda\nu}(\phi_{\omega,c})\psi,\psi\big\rangle + o(\lambda^2)\\ =&S_{\omega+\lambda\mu,c+\lambda\nu}(\phi_{\omega,c}) + \lambda\langle S'_{\omega,c}(\phi_{\omega,c}),\psi\rangle + \lambda\mu\langle M'(\phi_{\omega,c}),\psi\rangle\\ &+ \lambda\nu\langle P'(\phi_{\omega,c}),\psi\rangle + \frac{1}{2}\lambda^2\big\langle S''_{\omega,c}(\phi_{\omega,c})\psi,\psi\big\rangle + o(\lambda^2)\\ =&S_{\omega+\lambda\mu,c+\lambda\nu}(\phi_{\omega,c}) + o(\lambda^2). \end{aligned}$$

Then, we have

$$S_{\omega+\lambda\mu,c+\lambda\nu}(u_0) = S_{\omega+\lambda\mu,c+\lambda\nu}(\phi_{\omega,c}) + \frac{1}{2}\big\langle S''_{\omega+\lambda\mu,c+\lambda\nu}(\phi_{\omega+\lambda\mu,c+\lambda\nu})\varepsilon,\varepsilon\big\rangle + o(\lambda^2 + \|\varepsilon\|^2_{H^1(\mathbb{R})}).$$

Together with (4.42), (4.43) and (4.44), we have

$$\begin{aligned}S_{\omega+\lambda\mu,c+\lambda\nu}(u_0)-&S_{\omega+\lambda\mu,c+\lambda\nu}(\phi_{\omega,c})\\ =&S_{\omega,c}(u_0)-S_{\omega,c}(\phi_{\omega,c})+\lambda\mu(M(u_0)-M(\phi_{\omega,c}))+\lambda\nu(P(u_0)-P(\phi_{\omega,c}))\\ =&2\nu\omega\sigma(2-\sigma)M(\phi_{\omega,c})\lambda\delta_1+o(\delta_1). \end{aligned}\tag{4.45}$$

Therefore, by (3.22), (3.28) and (4.45), there exists $C>0$, such that

$$\begin{aligned}\|\varepsilon\|^2_{H^1(\mathbb{R})} \le& C\big\langle S''_{\omega+\lambda\mu,c+\lambda\nu}(\phi_{\omega+\lambda\mu,c+\lambda\nu})\varepsilon,\varepsilon\big\rangle\\ =&C\big[S_{\omega+\lambda\mu,c+\lambda\nu}(u_0)-S_{\omega+\lambda\mu,c+\lambda\nu}(\phi_{\omega,c})\big]+o(\lambda^2+\|\varepsilon\|^2_{H^1(\mathbb{R})})\\ =&2C\nu\omega\sigma(2-\sigma)M(\phi_{\omega,c})\lambda\delta_1+o(\delta_1)+o(\lambda^2+\|\varepsilon\|^2_{H^1(\mathbb{R})})\\ \le&2b_3\lambda\delta_1+o(\|\varepsilon\|^2_{H^1(\mathbb{R})}),\end{aligned}$$

where $b_3=2C\nu\omega\sigma(2-\sigma)M(\phi_{\omega,c})>0$. Then we obtain

$$\|\varepsilon\|^2_{H^1(\mathbb{R})}\le b_3\lambda\delta_1.$$

This completes the proof. □

Proof of Theorem 1. On one hand, we note that from the definition of $I(t)$, we have the time uniform boundedness of $I(t)$. That is, if $|\lambda|\lesssim 1$, then

$$\sup_{t\in\mathbb{R}} I(t)\lesssim R(\|\phi_{\omega,c}\|^2_{H^1(\mathbb{R})}+1).\tag{4.46}$$

On the other hand, using (3.28) and Lemmas 10, 11, we get

$$\begin{aligned}I'(t)=&A(u_0)+B(\lambda)+O\big(\lambda\|\varepsilon\|_{H^1(\mathbb{R})}+\|\varepsilon\|^2_{H^1(\mathbb{R})}+\frac{1}{R}\big)+o(\lambda^2)\\ \ge&b_2\delta_1+b_1\lambda^2+O\big(\|\varepsilon\|^2_{H^1(\mathbb{R})}+\frac{1}{R}\big)+o(\lambda^2)\\ \ge&\frac{1}{2}b_2\delta_1+\frac{1}{2}b_1\lambda^2+O(\|\varepsilon\|^2_{H^1(\mathbb{R})}),\end{aligned}$$

by choosing $R\ge 10(b_2\delta_1)^{-1}$.

Moreover, combining Lemma 12 yields

$$I'(t)\ge\frac{1}{4}b_2\delta_1+\frac{1}{2}b_1\lambda^2>0,$$

when $|\lambda|\ll 1$.

This implies that $I(t)\to+\infty$ when $t\to+\infty$, which contradicts to (4.46). Hence we prove the instability of the solitary wave solutions $e^{i\omega t}\phi_{\omega,c}(x-ct)$ of (1.1). This completes the proof of Theorem 1. □

Appendix: Proof of Lemma 3 and Lemma 4

Throughout this appendix, let $1<\sigma<2$ and $z_0=z_0(\sigma)\in(-1,1)$ satisfy $F_\sigma(z_0)=0$.

Now we adopt some notations from [29]. More precisely, for any $(\omega,c)\in\mathbb{R}^2$ satisfying $c^2<4\omega$, we denote

$$\kappa=\sqrt{4\omega-c^2}>0,\quad \tilde{\kappa}=2^{\frac{1}{\sigma}-2}\sigma^{-1}(1+\sigma)^{\frac{1}{\sigma}}\kappa^{\frac{2}{\sigma}-2}\omega^{-\frac{1}{2\sigma}-\frac{1}{2}},\quad f(\omega,c)=\frac{(\sigma+1)\kappa^2}{2\sqrt{\omega}},$$

$$h(x;\sigma;\omega,c)=\cosh(\sigma\kappa x)-\frac{c}{2\sqrt{\omega}},\qquad \alpha_n=\int_0^\infty h^{-\frac{1}{\sigma}-n}dx,\ n\in\mathbb{Z}^+.$$

Proof of Lemma 3. For any $(\omega, c) \in \mathbb{R}^2$ satisfying $c^2 < 4\omega$, by (1.3), we have

$$\partial_x \phi_{\omega,c}(x) = e^{i\theta}\big[(i\frac{c}{2} - \frac{i}{2\sigma+2}\varphi_{\omega,c}^{2\sigma})\varphi_{\omega,c} + \partial_x \varphi_{\omega,c}\big].$$

Therefore, we have

$$\begin{aligned} P(\phi_{\omega,c}) =& \frac{1}{2}\mathrm{Im}\int_{\mathbb{R}} \phi_{\omega,c}\overline{\partial_x \phi_{\omega,c}}dx \\ =& \frac{1}{2}\mathrm{Im}\int_{\mathbb{R}} -\big[i\frac{c}{2} - \frac{i}{2\sigma+2}\varphi_{\omega,c}^{2\sigma}\big]\varphi_{\omega,c}^2 dx \\ =& -\frac{c}{4}\|\varphi_{\omega,c}\|_{L^2}^2 + \frac{1}{4(\sigma+1)}\|\varphi_{\omega,c}\|_{L^{2\sigma+2}}^{2\sigma+2}. \end{aligned}$$

Finally, we obtain

$$\|\phi_{\omega,c}\|_{L^{2\sigma+2}}^{2\sigma+2} = 4(\sigma+1)\big[\frac{c}{2}M(\phi_{\omega,c}) + P(\phi_{\omega,c})\big].$$

According to [29] Appendix Lemma A.3, we have that

$$\partial_c M(\phi_{\omega,c}) = \partial_\omega P(\phi_{\omega,c}), \quad \partial_c P(\phi_{\omega,c}) = \omega\partial_\omega M(\phi_{\omega,c}).$$

This completes the proof. □

Now, we focus on the critical case $c = 2z_0\sqrt{\omega}$.

Proof of Lemma 4. From [29] Lemma 4.2, $\det[d''(\omega, c)] = 0$ is equivalent to

$$\big[(\sigma-1)\sqrt{\omega}M(\phi_{\omega,c})\big]^2 = P(\phi_{\omega,c})^2.$$

When $c = 2z_0\sqrt{\omega}$, we have $P(\phi_{\omega,c}) > 0$. Indeed, if $P(\phi_{\omega,c}) < 0$, since $\frac{P(\phi_{\omega,c})}{M(\phi_{\omega,c})} \to +\infty$, when $c \to -2\sqrt{\omega}$. Then there exist two solutions $c_1 = c_1(\sqrt{\omega})$, $c_2 = c_2(\sqrt{\omega})$, such that

$$\big|\frac{P(\phi_{\omega,c})}{M(\phi_{\omega,c})}\big| = (\sigma-1)\sqrt{\omega}.$$

This contradicts to the fact that $c = 2z_0\sqrt{\omega}$ is the unique solution of $\det[d''(\omega, c)] = 0$. Hence,

$$P(\phi_{\omega,c}) > 0, \quad P(\phi_{\omega,c}) = (\sigma-1)\sqrt{\omega}M(\phi_{\omega,c}).$$

From [29] Appendix (A.2) Lemmas A.1 and A.2, we know that $M(\phi_{\omega,c}) = f^{\frac{1}{\sigma}}\alpha_0$ and $P(\phi_{\omega,c}) = \frac{1}{4\sqrt{\omega}}f^{\frac{1}{\sigma}}(-2\omega^{\frac{1}{2}}c\alpha_0 + \kappa^2\alpha_1)$. Since $P(\phi_{\omega,c}) > 0$, then we have

$$\kappa^2\alpha_1 > 2\omega^{\frac{1}{2}}c\alpha_0.$$

Together with [29] Appendix Lemma A.3, we obtain

$$\begin{aligned} \partial_\omega M(\phi_{\omega,c}) =& \tilde{\kappa}\omega^{-1}[-8(\sigma-1)\omega^{\frac{2}{3}}\alpha_0 + c(2\omega^{\frac{1}{2}}c\alpha_0 - \kappa^2\alpha_1)] < 0, \\ \partial_\omega P(\phi_{\omega,c}) =& 2\tilde{\kappa}[2c\,\omega^{\frac{1}{2}}\alpha_0(\sigma-1) - 2c\,\omega^{\frac{1}{2}}\alpha_0 + \kappa^2\alpha_1] > 0. \end{aligned}$$

On one hand, by $\partial_c P(\phi_{\omega,c}) = \omega\partial_\omega M(\phi_{\omega,c})$ and $\partial_\omega P(\phi_{\omega,c}) = \partial_c M(\phi_{\omega,c})$, we have

$$\frac{\mu}{\nu} = -\frac{\partial_c M(\phi_{\omega,c})}{\partial_\omega M(\phi_{\omega,c})} = -\frac{\partial_\omega P(\phi_{\omega,c})}{\partial_\omega M(\phi_{\omega,c})} > 0.$$

On the other hand,

$$\frac{\mu}{\nu} = -\frac{\partial_c P(\phi_{\omega,c})}{\partial_\omega P(\phi_{\omega,c})} = -\frac{\omega\partial_\omega M(\phi_{\omega,c})}{\partial_\omega P(\phi_{\omega,c})}.$$

Hence, combining with the two equations above, we get

$$\big(\frac{\mu}{\nu}\big)^2 = -\frac{\partial_\omega P(\phi_{\omega,c})}{\partial_\omega M(\phi_{\omega,c})} \cdot -\frac{\omega\partial_\omega M(\phi_{\omega,c})}{\partial_\omega P(\phi_{\omega,c})} = \omega.$$

Then we obtain

$$\frac{\mu}{\nu} = \sqrt{\omega}.$$

Differentiating $M(\phi_{\omega,c})$ and $P(\phi_{\omega,c})$ with respect to ω and c, we have the following relations:

$$\begin{aligned}
\partial_{\omega\omega} M(\phi_{\omega,c}) =& \frac{1}{\omega}\big(\partial_{\omega c} P(\phi_{\omega,c}) - \partial_\omega M(\phi_{\omega,c})\big), \qquad \partial_{\omega c} M(\phi_{\omega,c}) = \partial_{\omega\omega} P(\phi_{\omega,c}),\\
\partial_{cc} M(\phi_{\omega,c}) =& \partial_{\omega c} P(\phi_{\omega,c}), \qquad\qquad\qquad\qquad \partial_{cc} P(\phi_{\omega,c}) = \omega\partial_{\omega\omega} P(\phi_{\omega,c}).
\end{aligned}$$

Since $\partial_\omega P(\phi_{\omega,c}) = \sqrt{\omega}\partial_\omega M(\phi_{\omega,c})$, we obtain $(z_0^2-1)\alpha_1 = (1-z_0-\sigma)\alpha_0$. From [7] Appendix Lemma 10, we have

$$\partial_\omega M(\phi_{\omega,c}) = 8\sqrt{\omega}\tilde{\kappa}\alpha_0(z_0^2-\sigma+1) + 8\sqrt{\omega}\tilde{\kappa}\alpha_1 z_0(z_0^2-1) = 8\sqrt{\omega}\tilde{\kappa}\alpha_0(1-\sigma)(1+z_0),$$

$$2\sqrt{\omega}\partial_{\omega\omega} P(\phi_{\omega,c}) + 2\partial_{\omega c} P(\phi_{\omega,c}) - \frac{1}{2}\partial_\omega M(\phi_{\omega,c}) = -4\sqrt{\omega}\tilde{\kappa}\alpha_0(\sigma-1)(1-z_0),$$

and

$$\mu^2\partial_{\omega\omega} P(\phi_{\omega,c}) + 2\mu\nu\partial_{\omega c} P(\phi_{\omega,c}) + \nu^2\partial_{cc} P(\phi_{\omega,c}) = -8\nu^2\omega\tilde{\kappa}_\omega\alpha_0(\sigma-1).$$

Moreover, we have

$$\begin{aligned}
&\mu^2\partial_{\omega\omega} M(\phi_{\omega,c}) + 2\mu\nu\partial_{\omega c} M(\phi_{\omega,c}) + \nu^2\partial_{cc} M(\phi_{\omega,c})\\
&\quad= \nu^2\big[\omega\partial_{\omega\omega} M(\phi_{\omega,c}) + 2\sqrt{\omega}\partial_{\omega c} M(\phi_{\omega,c}) + \partial_{cc} M(\phi_{\omega,c})\big]\\
&\quad= \nu^2\big[-\partial_\omega M(\phi_{\omega,c}) + 2\partial_{\omega c} P(\phi_{\omega,c}) + 2\sqrt{\omega}\partial_{\omega\omega} P(\phi_{\omega,c})\big]\\
&\quad= \nu^2[2\sqrt{\omega}\partial_{\omega\omega} P(\phi_{\omega,c}) + 2\partial_{\omega c} P(\phi_{\omega,c}) - \frac{1}{2}\partial_\omega M(\phi_{\omega,c}) - \frac{1}{2}\partial_\omega M(\phi_{\omega,c})]\\
&\quad= \nu^2[-4\sqrt{\omega}\tilde{\kappa}\alpha_0(\sigma-1)(1-z_0) + 4\sqrt{\omega}\tilde{\kappa}_\omega\alpha_0(1-\sigma)(z_0+1)]\\
&\quad= 8\nu^2\sqrt{\omega}\tilde{\kappa}\alpha_0(\sigma-1)z_0.
\end{aligned}$$

Take $\kappa_0 = 8\nu^2\sqrt{\omega}\tilde{\kappa}_\omega\alpha_0(\sigma-1)$, then $\kappa_0 > 0$. This completes the proof. □

Acknowledgements

Part of this work was done while C. Ning and Y.Wu were visiting Monash University. The authors thank the institutions for their hospitality. Z. Guo is supported by ARC DP170101060. C. Ning and Y.Wu are partially supported by NSFC 11771325 and 11571118. The authors are grateful to the anonymous referee who carefully read the paper and gave many helpful comments and suggestions.

References

[1] Biagioni, H., Linares, F., Ill-posedness for the derivative Schrödinger and generalized Benjamin-Ono equations, Trans. Amer. Math. Soc., 353 (9), 3649–3659 (2001).
[2] Colin, M., Ohta, M., Stability of solitary waves for derivative nonlinear Schrödinger equation, Ann. I. H. Poincaré-AN, 23, 753–764 (2006).
[3] Cher, Y., Simpson, G., Sulem, C., Local structure of singular profiles for a Derivative Nonlinear Schrödinger Equation, arXiv:1602.02381.
[4] Colliander, J., Keel, M., Staffilani, G., Takaoka, H., Tao, T., Global well-posedness result for Schrödinger equations with derivative, SIAM J. Math. Anal., 33 (2), 649–669 (2001).

[5] Colliander, J., Keel, M., Staffilani, G., Takaoka, H., Tao, T., A refined global well-posedness result for Schrödinger equations with derivatives, SIAM J. Math. Anal., 34, 64-86 (2002).
[6] Fukaya, N., Hayashi, M., Inui, T., A sufficient condition for global existence of solutions to a generalized derivative nonlinear Schrödinger equation, Analysis & PDE, 10 (5), 1149–1167 (2017).
[7] Fukaya, N., Instability of solitary waves for a generalized derivative nonlinear Schrödinger equation in a borderline case, Kodai Math. J., 40, 450–467 (2017).
[8] Grünrock, A., Bi- and trilinear Schrödinger estimates in one space dimension with applications to cubic NLS and DNLS. Int. Math. Res. Not. no. 41, 2525–2558 (2005).
[9] Grünrock, A., Herr, S., Low regularity local well-posedness of the derivative nonlinear Schrödinger equation with periodic initial data, SIAM J. Math. Anal. 39 (6), 1890–1920 (2008).
[10] Guo, B., Tan, S., On smooth solution to the initial value problem for the mixed nonlinear Schrödinger equations, Proc. Roy. Soc. Edinburgh, 119, 31–45 (1991).
[11] Guo, B., Wu, Y., Orbital stability of solitary waves for the nonlinear derivative Schrödinger equation, J. Differential Equations, 123, 35–55 (1995).
[12] Guo, Q., Orbital stability of solitary waves for generalized derivative nonlinear Schrödinger equations in the endpoint case, arXiv:1705.04458.
[13] Guo, Z., Hayashi, N., Lin, Y., Naumkin, P., Modified scattering operator for the derivative nonlinear Schrödinger equation. SIAM J. Math. Anal. 45, no. 6, 3854–3871 (2013).
[14] Guo, S., Ren, X., Wang, B., Local well-posedness for the derivative nonlinear Schrödinger equations with L^2 subcritical data. arXiv:1608.03136.
[15] Guo, Z., Wu, Y., Global well-posedness for the derivative nonlinear Schrödinger equation in $H^{\frac{1}{2}}(\mathbb{R})$, Disc. Cont. Dyn. Sys., 37(1), 257–264 (2017).
[16] Hao, C., Well-posedness for one-dimensional derivative nonlinear Schrödinger equations, Comm. Pure and Applied Analysis, 6(4), 997–1021 (2007).
[17] Hayashi, N., The initial value problem for the derivative nonlinear Schrödinger equation in the energy space, Nonl. Anal., 20, 823–833 (1993).
[18] Hayashi, N., Ozawa, T., On the derivative nonlinear Schrödinger equation, Physica D., 55, 14–36 (1992).
[19] Hayashi, N., Ozawa, T., Finite energy solution of nonlinear Schrödinger equations of derivative type, SIAM J. Math. Anal., 25, 1488–1503 (1994).
[20] Hayashi, M., Ozawa, T., Well-posedness for a generalized derivative nonlinear Schrödinger equation, J. Differential Equations 261, no. 10, 5424–5445 (2016).
[21] Herr, S., On the Cauchy problem for the derivative nonlinear Schrödinger equation with periodic boundary condition, Int. Math. Res. Not. Art. ID 96763, 33 (2006).
[22] Jenkins, R., Liu, J., Perry, P., Sulem. C., Global well-posedness and soliton resolution for the derivative nonlinear Schrödinger equation. Preprint, arXiv: 1706.06252, 2017.
[23] Kwon, S., Wu, Y., Orbital stability of solitary waves for derivative nonlinear Schrödinger equation, to appear in J. d'Anal Math.
[24] Le Coz, S., Wu, Y., Stability of multi-solitons for the derivative nonlinear schrödinger equation, to appear Intern. Math. Res. Notices, (2017).
[25] Linares, L., Ponce, G., Santos, G. N., On a class of solutions to the generalized derivative schrödinger equations, preprint.
[26] Liu, J., Perry, P., Sulem, C., Global existence for the derivative nonlinear Schrödinger equation by the method of inverse scattering. Comm. Part. Diff. Eqs 41(11), 1692–1760 (2016).
[27] Liu, J., Perry, P., Sulem, C., Long-time behavior of solutions to the derivative nonlinear Schrödinger equation by the method of inverse scattering. to appear in Ann. Inst. H. Poincaré C-Analyse non-linéaire.
[28] Liu, J., Perry, P., Sulem, C., Long-time behavior of solutions to the derivative nonlinear Schrödinger equation for soliton-free initial data, arXiv:1608.07659.
[29] Liu, X., Simpson, G., Sulem, C., Stability of solitary waves for a generalized derivative nonlinear Schrödinger equation, J. Nonlinear Science, 23(4), 557–583 (2013).
[30] Miao, C., Tang, X., Xu, G., Instability of the solitary waves for the generalized derivative nonlinear Schrödinger equation in the degenerate case, arXiv:1803.06451v1.

[31] Miao, C.; Wu, Y., Xu, G., Global well-posedness for Schrödinger equation with derivative in $H^{\frac{1}{2}}(\mathbb{R})$, J. Diff. Eq., 251, 2164–2195 (2011).
[32] Miao, C., Tang X., Xu G., Stability of the traveling waves for the derivative Schrödinger equation in the energy space. Calc. Var. Partial Differential Equations, 56(2), Art. 45, 48 pp (2017).
[33] Miao, C., Tang X., Xu G., Solitary waves for nonlinear Schrödinger equation with derivative, preprint arXiv:1702.07856.
[34] Mio, W., Ogino, T., Minami, K., Takeda, S., Modified nonlinear Schrödinger for Alfvén waves propagating along the magnetic field in cold plasma, J. Phys. Soc. Japan, 41, 265–271 (1976).
[35] Mjolhus, E., On the modulational instability of hydromagnetic waves parallel to the magnetic field, J. Plasma Physc., 16, 321–334 (1976).
[36] Mosincat, R., Global well-posedness of the derivative nonlinear Schrödinger equation with periodic boundary condition in $H^{\frac{1}{2}}$. J. Differential Equations, 263 (8), 4658–4722 (2017).
[37] Mosincat, R., Oh, T., A remark on global well-posedness of the derivative nonlinear Schrödinger equation on the circle. C. R. Math. Acad. Sci. Paris, 353(9), 837–841 (2015).
[38] Ozawa, T., On the nonlinear Schrödinger equations of derivative type, Indiana Univ. Math. J., 45, 137–163 (1996).
[39] Pelinovsky D., Shimabukuro, Y., Existence of global solutions to the derivative NLS equation with the inverse scattering transform method, Intern. Math. Res. Notices, (2017).
[40] Pelinovsky, D., Saalmann, A., and Shimabukuro, Y., The derivative NLS equation: global existence with solitons, Dynamics of PDEs, 14(3), 217–294 (2017).
[41] Santos G. N., Existence and uniqueness of solutions for a generalized Nonlinear Derivative Schrödinger equation, J. Diff. Eqs., 259, 2030–2060 (2015).
[42] Takaoka, H., Well-posedness for the one dimensional Schrödinger equation with the derivative nonlinearity, Adv. Diff. Eq., 4 , 561–680 (1999).
[43] Takaoka, H., A priori estimates and weak solutions for the derivative nonlinear Schrödinger equation on torus below $H^{1/2}$. J. Diff. Eqns., 260(1), 818–859 (2016).
[44] Tang, X., Xu, G., Stability of the sum of two solitary waves for (gDNLS) in the energy space, preprint arXiv:1702.07858.
[45] Wu, Y., Global well-posedness of the derivative nonlinear Schrödinger equations in energy space, Analysis & PDE, 6(8), 1989–2002 (2013).
[46] Wu, Y., Global well-posedness on the derivative nonlinear Schrödinger equation, Analysis & PDE, 8(5), 1101–1113 (2015).
[47] Wu, Y., Instability of the standing waves for the nonlinear Klein-Gordon equations in one dimension, arXiv:1705.04216.

School of Mathematical Sciences, Monash University, VIC 3800, Australia
E-mail address: zihua.guo@monash.edu

School of Mathematics, South China University of Technology, Guangzhou, Guangdong 510640, P.R.China
E-mail address: cuiningmath@gmail.com

Center for Applied Mathematics, Tianjin University, Tianjin 300072, P.R.China
E-mail address: yerfmath@gmail.com