Dynamic Bifurcation from Infinity of Nonlinear Evolution Equations*

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Abstract. This paper is concerned with dynamic bifurcation from infinity and multiplicity of stationary solutions for nonlinear evolution equations near resonance. First, we prove some new global continuation results and establish a general theorem on dynamic bifurcation from infinity in the framework of local semiflows on metric spaces. Then, by applying these abstract results, we derive more precise descriptions on the dynamic bifurcation from infinity of evolution equations in Banach spaces. Finally, we focus our attention on the parabolic equation $u_t - \Delta u = \lambda u + f(x, u)$ associated with the Dirichlet boundary condition, where f satisfies appropriate Landesman-Laser type condition. A detailed discussion on the dynamical behavior and the multiplicity of stationary solutions of the equation near resonance will be presented.

Key words. Conley index, nonlinear evolution equation, bifurcation from infinity, parabolic equation, resonance

AMS subject classifications. 37B30, 35B32, 35K55, 34C23

1. Introduction.

This paper is concerned with the nonlinear evolution equation

$$\frac{du}{dt} + Au = \lambda u + f(u, \lambda) \tag{1}$$

on a Banach space X, where A is a sectorial operator on X with compact resolvent, $\lambda \in \mathbb{R}$ is the bifurcation parameter, and $f(u, \lambda)$ is a locally Lipschitz continuous mapping from $X^\alpha \times \mathbb{R}$ $(0 \le \alpha < 1)$ to X which is sublinear as $||u||_\alpha \to \infty$ uniformly on bounded λ-intervals. We are basically interested in the dynamic bifurcation from infinity of the equation and its applications.

This topic can be traced back to the work of Rabinowitz [30], in which the author studied the bifurcation from infinity of stationary solutions of the equation in a general setting of operator equations of the following form:

$$u = \lambda Lu + K(u, \lambda), \tag{2}$$

where L is a compact operator, $\lambda \in \mathbb{R}$, and $K(u, \lambda) = o(||u||)$ as $||u|| \to \infty$ uniformly on bounded λ-intervals. It was shown that if μ^{-1} is a real eigenvalue of L of odd multiplicity, then (∞, μ) is a bifurcation point. Furthermore, there is a continuum of solutions of (2) which goes to infinity as $\lambda \to \mu$. This result was partially extended by Toland [38], Dias and Hernandez [10] and Schmitt and Wang [36] to potential operator equations to cover the case of even multiplicity. The interested reader is referred to [1, 2, 5, 6, 13, 16, 22, 23, 27, 28, 31, 33],

*Submitted to the editors DATE.

Funding: This work is supported by the grant of China (11471240, 11071185, 11571295).

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etc. for concrete examples on the bifurcation from infinity and multiplicity near resonance for differential equations under various boundary conditions.

For a nonlinear system as in (1), stationary solutions may be far from being adequate for understanding its dynamics. This is because that the dynamics of the system is usually determined not only by its stationary solutions, but also by all other bounded full ones. In fact, it is often the case that a system may have no stationary solutions. It is therefore of great importance to develop appropriate theories to analyze the bifurcation of bounded full solutions. A fundamental one in this line is the well-known Hopf's bifurcation theory, which was first developed in the very early work of Poincaré [29] around 1892. Actually it forms the central part of the classical dynamic bifurcation theory. The Hopf's bifurcation theory focuses on the case when there are exactly a pair of conjugate eigenvalues of the linearized equation crossing the imaginary axis, and was fully developed in the 20-th century. One can find a vast body of literature on how to determine Hopf bifurcations for nonlinear systems arising from applications. To deal with the general case, some other dynamic bifurcation theories need to be developed, and the Conley index theory, attractor theory and so on allow us to take a step; see e.g. [12, 18, 21, 34, 39, 40], etc.

Very recently, Li and Wang [18] established some new local and global bifurcation results in terms of invariant sets via the Conley index theory, completely extending the well-known Rabinowitz's global bifurcation theorem to the dynamic bifurcation of nonlinear evolution equations without requiring the "crossing odd-multiplicity" condition. Inspired by this work and some other ones mentioned above, in this paper we consider the dynamic bifurcation from infinity of (1). This problem was actually addressed in Ward [40]. The author first established a global continuation theorem (see Remark 3.2). Then he proved the following interesting result: For any real numbers $c < d$ such that the interval $[c, d]$ contains exactly one number $\mu \in \operatorname{Re}\sigma(A)$ with $c < \mu < d$, there exists a continuum $\mathscr{C} \subset X^\alpha \times [c, d]$ meeting $X^\alpha \times \{c, d\}$ such that

(1) for $\lambda \neq c, d$, $\mathscr{C}[\lambda]$ consists of bounded full solutions, where

$$\mathscr{C}[\lambda] = \{x : \ (x, \lambda) \in \mathscr{C}\};$$

(2) there is a sequence $\lambda_n \to \mu$ such that $\mathscr{C}[\lambda_n]$ is unbounded as $\lambda_n \to \mu$.

(See [40, Theorem 3.2].) Note that the continuum $\mathscr{C}$ in the above result may contain either all the connected branches of bounded full solutions of the equation meeting $X^\alpha \times \{c\}$, or all the connected branches of bounded full solutions meeting $X^\alpha \times \{d\}$, according to which side $\mathscr{C}$ will meet. Here, by using the techniques in [18] we will prove some new continuation results and establish an abstract theorem on bifurcation from infinity in terms of local semiflows on metric spaces. Then based on these theoretical results, we give some more precise descriptions on the dynamic bifurcation from infinity for (1).

As an example, we consider the parabolic equation

$$(3) \qquad u_t - \Delta u = \lambda u + f(x, u), \qquad x \in \Omega$$

associated with the homogeneous Dirichlet boundary condition, where Ω is a bounded domain in $\mathbb{R}^n$, and f is a bounded function satisfying the following Landesman-Laser type condition:

(4)
$$\liminf_{s\to+\infty} f(x,s) \ge \overline{f} > 0, \qquad \limsup_{s\to-\infty} f(x,s) \le -\underline{f} < 0$$

uniformly for $x \in \overline{\Omega}$ (where $\overline{f}$ and $\underline{f}$ are independent of x). First, we give a detailed discussion on the dynamic bifurcation from infinity of the equation near any eigenvalue μ_k of the operator $-\Delta$ (in $H_0^1(\Omega)$). Specifically, we prove that there exists $\delta > 0$ such that for each $\lambda \in \Lambda_- = [\mu_k - \delta, \mu_k)$, the maximal compact invariant set S_λ of the equation has a Morse decomposition $\mathcal{M} = \{M_\lambda^\infty,\, M_\lambda^1\}$ with M_λ^1 being uniformly bounded on Λ_- while

$$\lim_{\lambda\to\mu_k^-} \min_{v\in M_\lambda^\infty} ||v|| = \infty.$$

Besides, there is at least one connecting trajectory γ between M_λ^∞ and M_λ^1. More interestingly, it will be shown that each of the following two sets

$$\mathscr{K}^1 = \overline{\bigcup_{\lambda\in\Lambda_-} (M_\lambda^1 \times \{\lambda\})}, \quad \mathscr{K}^\infty = \overline{\bigcup_{\lambda\in\Lambda_-} (M_\lambda^\infty \times \{\lambda\})}$$

contains a connected component Γ with

$$\Gamma[\lambda] := \{u : \ (u,\lambda) \in \Gamma\} \ne \emptyset, \qquad \forall\, \lambda \in \Lambda_-.$$

The bifurcation and multiplicity of elliptic equations near resonance is always an interesting topic and has attracted much attention in the past decades. As a byproduct of our dynamical argument, we can naturally derive some bifurcation and multiplicity results on the stationary problem:

(5)
$$\begin{cases} -\Delta u = \lambda u + f(x,u), & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega. \end{cases}$$

This problem was first studied by Mawhin and Schmitt [22], where the authors considered the case when λ crosses an eigenvalue of odd multiplicity. Later Schmitt and Wang [36] developed a theory on bifurcation from infinity for potential operators, through which they extended the results in [22] to the case when λ crosses any eigenvalue μ_k. More specifically, under an abstract Landesman-Laser type condition on the Nemitski operator $\tilde{f} : H_0^1(\Omega) \to L^2(\Omega)$ corresponding to the function $f(x,s)$, the authors proved the following result and its "dual" version: there exists $\delta > 0$ such that for each $\lambda \in (\mu_k, \mu_k + \delta]$ the equation (5) has at least two solutions with one of which approaching ∞ as $\lambda \to \mu_k$, and for each $\lambda \in [\mu_k - \delta, \mu_k]$ it has at least one. (Some further development and extensions can be found in [4, 6, 9, 11, 27, 37], etc.) As an application of our dynamical bifurcation results, we show that the "dual" version of the above result holds true under the hypothesis (4). Moreover, there exists an open dense subset $\mathcal{D}$ of $\mathbb{R}$ such that for $\lambda \in \Lambda_- \cap \mathcal{D}$, where $\Lambda_- = [\mu_k - \delta, \mu_k)$, the problem has at least three distinct solutions.

Special attention will also be paid to the case where $f(x,s) = o(|s|)$ as $|s| \to 0$ uniformly for $x \in \overline{\Omega}$, in which we can say a little more on the multiplicity of nontrivial solutions of (5). Such a case was studied in Chiappinelli, Mawhin and Nugari [6], where the authors considered the multiplicity of solutions of the problem near the first eigenvalue μ_1. Under appropriate Landesman-Laser type conditions, it was proved, among other things, that the problem has at least two distinct nontrivial solutions as $\lambda \to \mu_1^+$. (We mention that the nonlinearity in [6] was allowed to be unbounded.) Here we present some more precise information on the multiplicity of solutions for the problem near any eigenvalue μ_k under the condition (4). Roughly speaking, we show that (5) has at least two distinct nontrivial solutions for $\lambda \in \Lambda_-$, provided δ is sufficiently small. Furthermore, there is always a one-sided neighborhood Λ_1 of μ_k such that the problem has at least three distinct nontrivial stationary solutions for $\lambda \in \Lambda_1 \setminus \{\mu_k\}$.

It is worth mentioning that "dual" versions of all our results on (3) and (5) mentioned above hold true if, instead of (4), we assume

$$\limsup_{s\to+\infty} f(x,s) \le -\overline{f} < 0, \qquad \liminf_{s\to-\infty} f(x,s) \ge \underline{f} > 0 \tag{6}$$

uniformly for $x \in \overline{\Omega}$.

This work is organized as follows. In section 2 we make some preliminaries. In section 3 we first prove some new global continuation results by utilizing the theory of Conley index. Then we apply these results to establish a general dynamical bifurcation theorem from infinity for infinite dynamical systems. In section 4 we use the abstract results to prove some bifurcation theorems from infinity for nonlinear evolution equations. Finally in section 5, we discuss the dynamic bifurcation from infinity and multiplicity of stationary solutions for the parabolic equation mentioned above.

2. Preliminaries.

In this section we make some preliminaries.

2.1. Basic topological notions and results.

Let X be a complete metric space with metric $d(\cdot,\cdot)$.

Let A and B be nonempty subsets of X. The *distance* $d(A,B)$ between A and B is defined as

$$d(A,B) = \inf\{d(x,y) : \ x \in A, \ y \in B\},$$

and the *Hausdorff semi-distance* and *Hausdorff distance* of A and B are defined, respectively, as

$$d_{\mathrm{H}}(A,B) = \sup_{x\in A} d(x,B), \quad \delta_{\mathrm{H}}(A,B) = \max\{d_{\mathrm{H}}(A,B), d_{\mathrm{H}}(B,A)\}.$$

The closure, interior and boundary of A in X are denoted, respectively, by $\overline{A}$, $\operatorname{int} A$ and ∂A. Sometimes we also write $\overline{A}$, $\operatorname{int} A$ and ∂A as $\overline{A}^X$, $\operatorname{int}_X A$ and $\partial_X A$, respectively, to emphasize in which space these operations are taken.

The *ε-neighborhood* of A, denoted by $\mathrm{B}(A,\varepsilon)$ or $\mathrm{B}_X(A,\varepsilon)$, is defined to be the set $\{y \in X : d(y,A) < \varepsilon\}$.

A subset U of X is called a *neighborhood* of A, if $\overline{A} \subset \operatorname{int} U$.

Lemma 2.1. *[30] Let X be a compact metric space, and let A and B be two disjoint closed subsets of X. Then either there exists a subcontinuum C of X such that*

$$A \cap C \ne \emptyset \ne B \cap C,$$

or $X = X_A \cup X_B$, where X_A and X_B are disjoint compact subsets of X containing A and B, respectively.

Lemma 2.2. *([3, pp. 41]), Let X be a compact metric space. Denote $\mathcal{C}(X)$ the family of compact subsets of X which is equipped with the Hausdorff metric $\delta_H(\cdot,\cdot)$. Then $\mathcal{C}(X)$ is a compact metric space.*

2.2. Wedge/smash product of pointed spaces. Let (X, x_0) and (Y, y_0) be two pointed spaces. The *wedge product* $(X, x_0) \vee (Y, y_0)$ and *smash product* $(X, x_0) \wedge (Y, y_0)$ are defined, respectively, as follows:

$$(X, x_0) \vee (Y, y_0) = (\mathcal{W}, (x_0, y_0)), \quad (X, x_0) \wedge (Y, y_0) = ((X \times Y)/\mathcal{W}, [\mathcal{W}]),$$

where $\mathcal{W} = X \times \{y_0\} \cup \{x_0\} \times Y$.

Denote $[(X, x_0)]$ the *homotopy type* of a pointed space (X, x_0). Since the operations "$\vee$" and "$\wedge$" preserve homotopy equivalence relations, they can be naturally extended to the homotopy types of pointed spaces.

Let $\overline{0}$ be the homotopy type of the one-point space $(\{p\}, p)$. Denote Σ^m $(m \geq 0)$ the homotopy type of a pointed m-dimensional sphere. Then

$$[(X, x_0)] \vee \overline{0} = [(X, x_0)], \quad \text{and } \Sigma^m \wedge \Sigma^n = \Sigma^{m+n} \ (\forall\, m, n \geq 0).$$

2.3. Local semiflows on metric spaces. For completeness and the reader's convenience, let us first collect some fundamental notions and facts on local semiflows.

2.3.1. Local semiflows. Let X be a complete metric space.

A *local semiflow* Φ on X is a continuous mapping from an open set $\mathcal{D}(\Phi) \subset \mathbb{R}_+ \times X$ to X that enjoys the following properties:

(1) for each $x \in X$, there exists $0 < T_x \leq \infty$, called the *escape time* of $\Phi(t, x)$, such that

$$(t, x) \in \mathcal{D}(\Phi) \Longleftrightarrow t \in [0, T_x);$$

(2) $\Phi(0, \cdot) = id_X$, and

$$\Phi(t + s, x) = \Phi(t, \Phi(s, x))$$

for all $x \in X$ and $t, s \in \mathbb{R}_+$ with $t + s \leq T_x$.

Let Φ be a given local semiflow on X. For simplicity, we usually rewrite $\Phi(t, x)$ as $\Phi(t)x$.

Let $I \subset \mathbb{R}$ be an interval. A *trajectory* (or *solution*) of Φ on I is a continuous mapping $\gamma : I \to X$ such that

$$\gamma(t) = \Phi(t - s)\gamma(s), \quad \forall t, s \in I,\ t \geq s.$$

A trajectory γ on $\mathbb{R}$ is called a *full trajectory*.

The *orbit* of a trajectory γ on I is the set

$$\text{orb}(\gamma) = \{\gamma(t) : \ t \in I\}.$$

The orbit of a full trajectory is simply called a *full orbit*.

The *ω-limit set* $\omega(\gamma)$ and ω^*-limit set of a full trajectory γ are defined as

$$\omega(\gamma) = \{y \in X : \ \text{ there exists } t_n \to \infty \text{ such that } \gamma(t_n) \to y\},$$

$$\omega^*(\gamma) = \{y \in X : \text{ there exists } t_n \to -\infty \text{ such that } \gamma(t_n) \to y\}.$$

Given $U \subset X$, denote $K_\infty(\Phi, U)$ the *union of all bounded full orbits* in U. In the case where $U = X$, we will simply write

$$K_\infty(\Phi, X) = K_\infty(\Phi).$$

Let $N \subset X$. We say that Φ *does not explode* in N, if

$$\Phi([0, T_x))x \subset N \Longrightarrow T_x = \infty.$$

Definition 2.1. *[32] $N \subset X$ is said to be admissible, if for any sequences $x_n \in N$ and $t_n \to \infty$ with $\Phi([0, t_n])x_n \subset N$ for all n, the sequence $\Phi(t_n)x_n$ has a convergent subsequence. N is said to be strongly admissible, if in addition, Φ does not explode in N.*

Definition 2.2. *Φ is said to be asymptotically compact on X, if each bounded set $B \subset X$ is strongly admissible.*

Let $S \subset X$. S is said to be *positively invariant* (resp. *invariant*), if $\Phi(t)S \subset S$ (resp. $\Phi(t)S = S$) for all $t \geq 0$.

A compact invariant set $\mathcal{A} \subset X$ is called an *attractor* of Φ, if it attracts a neighborhood U of itself, namely, $\lim_{t\to+\infty} d_H(\Phi(t)U, \mathcal{A}) = 0$.

Let S be a compact invariant set of Φ. An ordered collection $\mathcal{M} = \{M_1, \cdots, M_l\}$ of disjointed compact invariant subsets of S is called a *Morse decomposition of* S, if for any full trajectory γ contained in $S \setminus \left(\bigcup_{1\leq k\leq l} M_k\right)$, there exist i and j with $i < j$ such that

$$\omega^*(\gamma) \subset M_j, \quad \omega(\gamma) \subset M_i. \tag{1}$$

Remark 2.1. *A full trajectory satisfying (1) will be referred to as a* connecting trajectory *between M_i and M_j.*

Remark 2.2. *One may use equivalent definitions of Morse decompositions; see e.g. [32, Chap. III].*

2.4. Conley index. In this subsection we briefly recall the definition of Conley index. The interested reader is referred to [7, 25] and [32], etc. for details.

Let Φ be a local semiflow on X. Since X may be an infinite dimensional space, we always assume Φ is asymptotically compact, hence each bounded subset of X is *strongly admissible.*

A compact invariant set S of Φ is said to be *isolated*, if there is a neighborhood N of S such that S is the maximal compact invariant set in $\overline{N}$. Correspondingly, N is called an *isolating neighborhood* of S.

Remark 2.3. *Note that we do not require an isolating neighborhood to be bounded, although the bounded ones are always of particular interest.*

An important example for isolating neighborhoods is the so called *isolating block*, which plays a crucial role in the computation of Conley index.

Let $B \subset X$ be a bounded closed set. $x \in \partial B$ is called a *strict egress* (resp. *strict ingress*, *bounce-off*) point of B, if for every trajectory $\gamma : [-\tau, s] \to X$ with $\gamma(0) = x$, where $\tau \geq 0$, $s > 0$, the following properties hold:

(1) there exists $0 < \varepsilon < s$ such that

$$\gamma(t) \notin B \ \ (\text{resp. } \ \gamma(t) \in \text{int}B, \ \ \text{resp. } \ \gamma(t) \notin B), \qquad \forall\, t \in (0, \varepsilon);$$

(2) if $\tau > 0$, then there exists $0 < \delta < \tau$ such that

$$\gamma(t) \in \text{int}B \ \ (\text{resp. } \ \gamma(t) \notin B, \ \ \text{resp. } \ \gamma(t) \notin B), \qquad \forall\, t \in (-\delta, 0).$$

Denote B^e (resp. B^i, B^b) the set of all strict egress (resp. ingress, bounce-off) points of the closed set B, and set $B^- = B^e \cup B^b$.

B is called an *isolating block* [32], if B^- is closed and $\partial B = B^i \cup B^-$.

Let N, E be two closed subsets of X. E is called an *exit set* of N, if (1) E is *N-positively invariant*, that is, for any $x \in E$ and $t \geq 0$,

$$\Phi([0,t])x \subset N \Longrightarrow \Phi([0,t])x \subset E;$$

and (2) for any $x \in N$, if $\Phi(t_1)x \notin N$ for some $t_1 > 0$, then there exists $t_0 \in [0, t_1]$ such that $\Phi(t_0) \in E$.

Let S be a compact isolated invariant set. A pair of bounded closed subsets (N, E) is called an *index pair* of S, if (1) $N \setminus E$ is an isolating neighborhood of S; and (2) E is an exit set of N. We infer from [32] that if B is a bounded isolating block, then (B, B^-) is an index pair of the maximal compact invariant set $S = K_\infty(\Phi, B)$ in B.

Definition 2.3. *The homotopy Conley index of S, denoted by $h(\Phi, S)$, is defined to be the homotopy type $[(N/E, [E])]$ of the pointed space $(N/E, [E])$ for any index pair (N, E) of S.*

Remark 2.4. *For convenience, if U is an isolating neighborhood of a compact invariant set S (U need not to be bounded), we also write*

$$h(\Phi, U) = h(\Phi, S),$$

hoping that this will not cause any confusion.

Example 2.1. As an example (and also for later use), let us compute the Conley index of an asymptotically stable equilibrium e (e is an attractor of Φ).

Let $L(x)$ be a Lyapunov function of e defined on an open neighborhood U of e which is strictly decreasing along each trajectory of Φ in U outside e (see e.g. [15, pp. 226] for the construction of such a function). We may assume $L(e) = 0$ (hence $L(x) > 0$ for $x \in U \setminus \{e\}$). Take a $\delta > 0$ sufficiently small so that $B = \{x : \ L(x) \leq \delta\} \subset U$ and is a closed neighborhood of e. Then one easily sees that B is an isolating block with $B^- = \emptyset$.

We claim that B is contractible. Indeed, set

$$H(s, x) = \begin{cases} \Phi(s/(1-s))x, & x \in B, \ s \in [0, 1); \\ x, & x \in B, \ s = 1. \end{cases}$$

Then H is a strong deformation retraction.

Now by the definition of Conley index, we have

$$h(\Phi, \{e\}) = [(B/B^-, [B^-])] = \Sigma^0.$$

Let S be a compact isolated invariant set of Φ. Denote H_* and H^* the singular homology and cohomology theories with coefficient group $\mathbb{Z}$, respectively. Applying H_* and H^* to $h(\Phi, S)$ one immediately obtains the *homology* and *cohomology Conley indices* of S.

The *Poincaré polynomial* of S, denoted by $p(t, S)$, is the *formal polynomial*

$$p(t, S) = \sum_{q=0}^{\infty} \beta_q \, t^q$$

with $\beta_q = \operatorname{rank} H_q(h(\Phi, S))$. If S has a Morse decomposition $\mathcal{M} = \{M_1, \cdots, M_l\}$, then the following *Morse equation*

$$p(t, M_1) + \cdots + p(t, M_l) = p(t, S) + (1 + t)Q(t)$$

holds for some formal polynomial $Q(t) = \sum_{q=0}^{\infty} d_q \, t^q$ with $d_q \in \mathbb{Z}_+$.

Let us also recall briefly the basic continuation property of the Conley index.

Let Φ_λ ($\lambda \in \Lambda$) be a family of semiflows on X, where Λ is a metric space. We say that Φ_λ *depends on* λ *continuously*, if $\Phi_\lambda(t)x$ is defined at the point (t, x, λ), then for any sequence (t_n, x_n, λ_n) converging to (t, x, λ), $\Phi_{\lambda_n}(t_n)x_n$ is defined as well for all n sufficiently large, furthermore,

$$\Phi_{\lambda_n}(t_n)x_n \to \Phi_\lambda(t)x \quad \text{as } n \to \infty.$$

Suppose Φ_λ depends on λ continuously. Define

$$\Pi(t)(x, \lambda) = (\Phi_\lambda(t)x, \lambda), \qquad (x, \lambda) \in \mathscr{X} = X \times \Lambda.$$

Then Π is a local semiflow on the product space $\mathscr{X}$, which will be called the *skew-product flow* of the family Φ_λ ($\lambda \in \Lambda$).

We say that Φ_λ ($\lambda \in \Lambda$) is λ-*locally uniformly asymptotically compact* (λ-l.u.a.c. in short), if its skew-product flow Π is asymptotically compact.

Remark 2.5. *It is trivial to see that if Φ_λ ($\lambda \in \Lambda$) is λ-l.u.a.c., then Λ is necessarily locally compact.*

For convenience, given $K \subset \mathscr{X}$ and $\lambda \in \Lambda$, we will write

$$K[\lambda] = \{x : \ (x, \lambda) \in \mathscr{X}\}.$$

$K[\lambda]$ is called the λ-*section of* K. The following continuation result is actually a particular case of [32, Chap. I, Theorem 12.2].

Theorem 2.1. *Let Φ_λ ($\lambda \in \Lambda$) be a family of semiflows on X, where Λ is a connected compact metric space. Suppose Φ_λ depends on λ continuously and is λ-l.u.a.c.*

Let K be a compact isolated invariant set of the skew-product flow Π of Φ_λ ($\lambda \in \Lambda$). Then

$$h(\Phi_\lambda, K[\lambda]) \equiv const., \qquad \lambda \in \Lambda.$$

Proof. Take a bounded closed isolating neighborhood $\mathcal{U}$ of K in $\mathscr{X}$. Then for each $\lambda \in \Lambda$, the λ-section $\mathcal{U}_\lambda$ of $\mathcal{U}$ is an isolating neighborhood of $K[\lambda]$. By the compactness of K one can easily verify that $K[\lambda]$ is upper semicontinuous in λ. Consequently $\mathcal{U}_\lambda$ is also an isolating neighborhood of $K[\lambda']$ for λ' near λ. The conclusion then directly follows from [32, Chap. I, Theorem 12.2.]. ■

Remark 2.6. *We emphasize that in the above theorem, we allow $K[\lambda'] = \emptyset$ for some $\lambda' \in \Lambda$. Note also that when such a case occurs, one necessarily has $h(\Phi_\lambda, K[\lambda]) = \overline{0}$ for all $\lambda \in \Lambda$.*

2.5. Sectorial operators. For the readers' convenience, we finally recall some basic notions concerning sectorial operators.

Let X be a Banach space. A closed and densely defined linear operator $A : \mathrm{D}(A) \subset X \to X$ is called a *sectorial operator*, if there exist real numbers $\phi \in (0, \pi/2)$, $a \in \mathbb{R}$ and $M \geq 1$ such that the sector

$$S_{a,\phi} = \{\lambda : \phi \leq |\arg(\lambda - a)| \leq \pi, \quad \lambda \neq a\}$$

is contained in the resolvent set of A, moreover,

$$\|(\lambda I - A)^{-1}\| \leq M/|\lambda - a|$$

for all $\lambda \in S_{a,\phi}$, where I denotes the identity on X.

Let A be a sectorial operator in X. Denote $\sigma(A)$ the spectral of A. If $\min_{z\in\sigma(A)} \mathrm{Re}\, z > 0$, then A generates an analytic semigroup $T(t) = e^{-At}$ with

$$||T(t)|| \leq Ce^{-\beta t}, \qquad t \geq 0$$

for some $C, \beta > 0$. This allows us to define the fractional powers of A as follows: for each $\alpha > 0$,

$$A^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} e^{-At} dt,$$

where $\Gamma(s) = \int_0^\infty t^{s-1} e^{-t} dt$ is the Gamma function, and let A^α be the inverse of $A^{-\alpha}$ with $\mathrm{D}(A^\alpha) = \mathrm{R}(A^{-\alpha})$; see Henry [14, Chap. I] for details. We also assign $A^0 = I$.

Note that in general we may not have $\min_{z\in\sigma(A)} \mathrm{Re}\, z > 0$. However, one can always find a real number a such that $\min_{z\in\sigma(A_1)} \mathrm{Re}\, z > 0$, where $A_1 = A + aI$. Hence we can define the fractional powers of A_1 as above. For each $\alpha \geq 0$, denote $X^\alpha = \mathrm{D}(A_1^\alpha)$. We equip X^α with the norm $\|\cdot\|_\alpha$ defined as

$$\|u\|_\alpha = \|A_1^\alpha u\|, \quad u \in X^\alpha.$$

Then X^α is a Banach space, which is called the *fractional power* of X. It is well known that the definition of X^α is independent of the choice of the number a, and different choices of a give equivalent norms on X^α [14, Chap. I].

3. Continuation Theorems and Bifurcation from Infinity of Local Semiflows. In this section, we establish some abstract continuation theorems on invariant sets and prove a general result on bifurcation from infinity in the framework of local semiflows by using Conley index.

Let X be a complete metric space with metric $d(\cdot,\cdot)$, and set

$$\mathscr{X} = X \times \mathbb{R}, \quad \mathscr{X}_\pm = X \times \mathbb{R}_\pm.$$

$\mathscr{X}$ is equipped with the metric defined by

$$\varrho((u_1, \lambda_1), (u_2, \lambda_2)) = d(u_1, u_2) + |\lambda_1 - \lambda_2|, \qquad (u_1, \lambda_1), (u_2, \lambda_2) \in \mathscr{X}.$$

3.1. Global continuation theorem. Let Φ_λ ($\lambda \in \mathbb{R}$) be a family of local semiflows on X. Henceforth we always assume that Φ_λ depends on λ continuously and is λ-l.u.a.c.

Given $\Lambda \subset \mathbb{R}$ and $U \subset X$, denote

$$\mathscr{K}(\Lambda, U) = \overline{\bigcup_{\lambda\in\Lambda} \left(K_\infty(\Phi_\lambda, U) \times \{\lambda\}\right)}. \tag{1}$$

For simplicity, we will write

$$\mathscr{K}(\Lambda, X) = \mathscr{K}(\Lambda).$$

Remark 3.1. *By the λ-l.u.a.c. property of Φ_λ ($\lambda \in \mathbb{R}$) and the invariance property of $\mathscr{K}(\Lambda, U)$, one can easily verify that if Λ and U are bounded then $\mathscr{K}(\Lambda, U)$ is compact.*

Theorem 3.1. *Let S be a compact isolated invariant set of Φ_0, and U an isolating neighborhood of S. Denote $\mathscr{F}_\pm$ the family of components of $\mathscr{K}(\mathbb{R}_\pm)$ meeting $S \times \{0\}$.*

Suppose $h(\Phi_0, S) \neq \overline{0}$. Then there is a $\Gamma \in \mathscr{F}_\pm$ such that either $\Gamma[0] \setminus U \neq \emptyset$, or Γ is unbounded in the space $\mathscr{X}_\pm$.

Proof. We only consider the case of $\mathscr{F}_+$. The argument for that of $\mathscr{F}_-$ is parallel.

We argue by contradiction and suppose the assertion in the theorem was false. Then each $\Gamma \in \mathscr{F}_+$ would be bounded in $\mathscr{X}_+$. Furthermore, $\Gamma[0] \subset U$ (hence $\Gamma[0] \subset S$).

Denote $\mathscr{C}(S)$ the family of all components of S. For each $Z \in \mathscr{C}(S)$, there is a (unique) $\Gamma_Z \in \mathscr{F}_+$ such that $Z \subset \Gamma_Z[0]$ (note that $\Gamma_Z[0]$ may not be connected). It can be easily seen that for any $Z_1, Z_2 \in \mathscr{C}(S)$, one has

$$\text{either } \Gamma_{Z_1} = \Gamma_{Z_2}, \text{ or } \Gamma_{Z_1} \cap \Gamma_{Z_2} = \emptyset. \tag{2}$$

Let $Z \in \mathscr{C}(S)$. Pick a number δ with $0 < \delta < d(Z, \partial U)$, and let $\mathcal{V}_\delta = \mathrm{B}_{\mathscr{X}_+}(\Gamma_Z, \delta)$ be the δ-neighborhood of Γ_Z in $\mathscr{X}_+$. Set

$$\mathcal{K} = \overline{\mathcal{V}}_\delta \cap \mathscr{K}(\mathbb{R}_+), \quad \mathcal{K}_\delta = \partial_+\mathcal{V}_\delta \cap \mathscr{K}(\mathbb{R}_+),$$

where $\partial_+\mathcal{V} = \partial_{\mathscr{X}_+}\mathcal{V}$ denotes the boundary of $\mathcal{V}$ in $\mathscr{X}_+$ for any $\mathcal{V} \subset \mathscr{X}_+$. Then by the boundedness of $\mathcal{V}_\delta$ and Remark 3.1 we easily deduce that both $\mathcal{K}$ and $\mathcal{K}_\delta$ are compact. Because Γ is a component of $\mathscr{K}(\mathbb{R}_+)$ and $\Gamma_Z \cap \mathcal{K}_\delta = \emptyset$, by virtue of Lemma 2.1 there exist two disjoint closed subsets $\mathcal{K}_1, \mathcal{K}_2$ of $\mathcal{K}$ with $\mathcal{K} = \mathcal{K}_1 \cup \mathcal{K}_2$ such that

$$\Gamma_Z \subset \mathcal{K}_1, \quad \mathcal{K}_\delta \subset \mathcal{K}_2\,.$$

Note that $\mathcal{K}_1$ is contained in the interior of $\mathcal{V}_\delta$ in $\mathscr{X}_+$.

Take a number $\delta_Z > 0$ with

$$\delta_Z < \frac{1}{4}\min\{\varrho(\mathcal{K}_1, \mathcal{K}_2), \varrho(\mathcal{K}_1, \partial_+\mathcal{V}_\delta)\}.$$

Let $\mathcal{V}_Z = \mathrm{B}_{\mathscr{X}_+}(\mathcal{K}_1, 2\delta_Z)$. Then by the choice of δ_Z we have

$$\mathrm{B}_{\mathscr{X}_+}(\partial_+\mathcal{V}_Z, \delta_Z) \cap \mathscr{K}(\mathbb{R}_+) = \emptyset. \tag{3}$$

Let $\lambda_1 = \max\{\lambda : \ \mathcal{V}_Z[\lambda] \neq \emptyset\}$. Thanks to Theorem 2.1, one deduces that

$$h(\Phi_\lambda, \mathcal{V}_Z[\lambda]) \equiv const., \qquad \lambda \in [0, \lambda_1). \tag{4}$$

But $K_\infty(\Phi_\lambda, \overline{\mathcal{V}}_Z[\lambda]) = \emptyset$ if λ is close to λ_1; see Fig. 3.1.

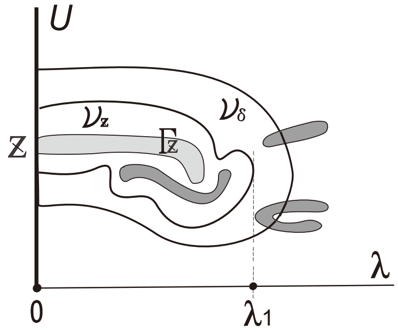

Fig. 3.1: $\mathcal{V}_Z$ is an isolating neighborhood.

By (4) it follows that

$$h(\Phi_\lambda, \mathcal{V}_Z[\lambda]) = \bar{0}, \qquad \lambda \in [0, \lambda_1).$$

In particular, we have

$$h(\Phi_0, \Omega_Z) = \bar{0}, \quad \text{where } \Omega_Z = \mathcal{V}_Z[0]. \tag{5}$$

Note that $\Omega_Z \subset U$. We also infer from (3) that

$$\mathrm{B}_X(\partial\Omega_Z, \delta_Z) \cap S = \emptyset. \tag{6}$$

(Here $\partial\Omega_Z$ is the boundary of Ω_Z in X.) As S is the maximal compact invariant set of Φ_0 in U, (6) implies that Ω_Z is an isolating neighborhood of Φ_0.

Since S is compact, there exist a finite number of components $Z_1, \cdots, Z_l$ of S such that $S \subset \bigcup_{i=1}^l \Omega_{Z_i}$. Let $W_1 = \Omega_{Z_1}$, and

$$W_k = \Omega_{Z_k} \setminus (\overline{\Omega}_{Z_1} \cup \cdots \cup \overline{\Omega}_{Z_{k-1}}), \qquad k = 2, \cdots, l.$$

Then $W_k{}'s$ are disjoint open sets in X, and

$$\partial W_k \subset \bigcup_{i=1}^k \partial\Omega_{Z_i}. \tag{7}$$

As $S \bigcap \left(\bigcup_{i=1}^l \partial\Omega_{Z_i} \right) = \emptyset$ (see (6)), one finds that

$$S \subset \left(\bigcup_{i=1}^l \Omega_{Z_i} \right) \setminus \left(\bigcup_{i=1}^l \partial\Omega_{Z_i} \right) = \bigcup_{i=1}^l W_i.$$

Set $S_k = S \cap W_k$. We observe that if $w \in S_k$, then by (6),

$$d(w, \partial\Omega_{Z_i}) \geq \delta_{Z_i} \geq \min_{1\leq i\leq l} \delta_{Z_i} > 0, \qquad 1 \leq i \leq l.$$

Thus by (7) it holds that

$$d(S_k, \partial W_k) > 0, \tag{8}$$

which implies that S_k is compact. We also infer from (8) that W_k is an isolating neighborhood of S_k (with respect to Φ_0). We claim that

$$h(\Phi_0, S_k) = \overline{0}. \tag{9}$$

Indeed, let $M_k = K_\infty(\Phi_0, \Omega_{Z_k}) \setminus S_k$. Then $M_k \subset \Omega_{Z_k} \setminus W_k$. Therefore by (8) we deduce that

$$d(S_k, M_k) > 0,$$

from which one can easily see that M_k is compact. (5) then asserts that

$$\begin{aligned} \overline{0} = h(\Phi_0, \Omega_{Z_k}) &= h\left(\Phi_0, K_\infty(\Phi_0, \Omega_{Z_k})\right) \\ &= h(\Phi_0, S_k \cup M_k) = h(\Phi_0, S_k) \vee h(\Phi_0, M_k). \end{aligned}$$

By the basic knowledge in the theory of Conley index (see e.g. [32, pp. 52]) one immediately concludes the validity of (9).

Now since S_k are disjoint isolated invariant sets of Φ_0 and $S = \bigcup_{1 \le k \le l} S_k$, we have

$$h(\Phi_0, S) = h(\Phi_0, S_1) \vee \cdots \vee h(\Phi_0, S_l) = \overline{0},$$

which leads to a contradiction. ■

Remark 3.2. *In [40], Ward gave a continuation theorem asserting that* $\mathscr{S}_\pm = \bigcup_{\Gamma \in \mathscr{F}_\pm} \Gamma$ *either meets* $(X \setminus U) \times \{0\}$*, or is unbounded. Theorem 3.1 significantly improves this result.*

Theorem 3.2. *Let* S *be an isolated invariant set of* Φ_0 *with* $h(\Phi_0, S) \neq \overline{0}$*, and* U *an isolating neighborhood of* S*. Let* $0 < d \le \infty$*, and denote* Λ *either the interval* $[0, d)$ *or the one* $(-d, 0]$*. Denote* $\mathscr{F}$ *the family of components of* $\mathscr{K}(\Lambda, U)$ *meeting* $S \times \{0\}$*.*

Then there exists $\Gamma \in \mathscr{F}$ *such that one of the alternatives below holds:*

(1) Γ *is unbounded; see Fig. 3.2.*
(2) Γ *meeting* $\partial U \times \Lambda$*; see Fig. 3.3.*
(3) $\Gamma[\lambda] \neq \emptyset$ *for all* $\lambda \in \Lambda$*; see Fig. 3.4.*

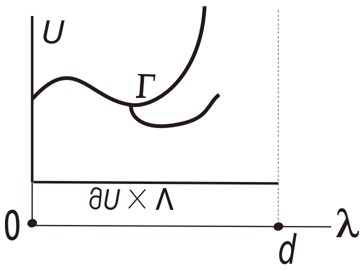

Fig. 3.2

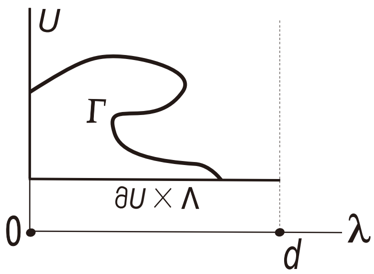

Fig. 3.3

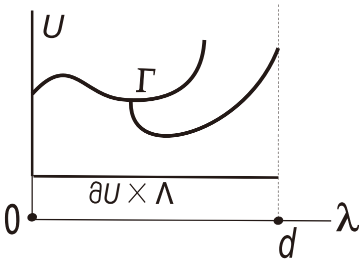

Fig. 3.4

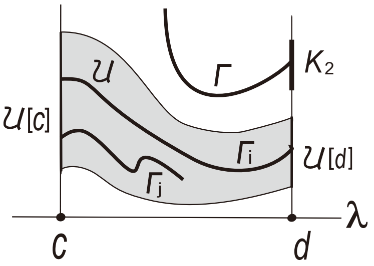

Fig. 3.5: $K_\infty(\Phi_c) \subset \mathcal{U}[c]$

Proof. The proof can be easily obtained by slightly modifying the one of Theorem 3.1. We omit the details. ■

3.2. An abstract theorem on bifurcation from infinity. We now establish a new abstract theorem on dynamic bifurcation from infinity.

Let Φ_λ $(\lambda \in \mathbb{R})$ be as in subsection 3.1.

Theorem 3.3. *Let $\Lambda = [c, d]$ be a compact interval. Suppose both $K_\infty(\Phi_c)$ and $K_\infty(\Phi_d)$ are compact, furthermore,*

$$h\left(\Phi_c,\, K_\infty(\Phi_c)\right) \neq h\left(\Phi_d,\, K_\infty(\Phi_d)\right). \tag{10}$$

Then the set $\mathscr{K}(\Lambda, X)$ has an unbounded component Γ meeting $X \times \{c, d\}$.

Proof. Denote $\mathscr{T}$ the family of connected components of $\mathscr{K}(\Lambda, X)$, and let

$$\mathscr{T}_c = \{\Gamma \in \mathscr{T} :\ \Gamma[c] \neq \emptyset\}, \quad \mathscr{T}_d = \{\Gamma \in \mathscr{T} :\ \Gamma[d] \neq \emptyset\}.$$

In the following we prove that if every $\Gamma \in \mathscr{T}_c$ is bounded, then there is a $\Gamma \in \mathscr{T}_d$ such that Γ is unbounded.

Let $\mathcal{H} = X \times [c, d]$. Denote $\partial_{\mathcal{H}}\mathcal{V}$ the boundary of $\mathcal{V}$ in $\mathcal{H}$ for any $\mathcal{V} \subset \mathcal{H}$.

Let $\Gamma \in \mathscr{T}_c$. Since Γ is bounded, as in Remark 3.1 one easily deduces by the λ-l.u.a.c. property of Φ_λ that Γ is compact. Take a number $\varepsilon > 0$, and let

$$\mathcal{V}_\varepsilon = \mathrm{B}_{\mathcal{H}}(\Gamma, \varepsilon) := \{(x, \lambda) \in \mathcal{H} :\ \varrho((x, \lambda), \Gamma) < \varepsilon\}$$

be the ε-neighborhood of Γ in $\mathcal{H}$. Set

$$\mathcal{C} = \overline{\mathcal{V}}_\varepsilon \cap \mathscr{K}(\Lambda, X), \quad \mathcal{C}_\varepsilon = \partial_{\mathcal{H}}\mathcal{V}_\varepsilon \cap \mathcal{C}.$$

By Remark 3.1 we see that both $\mathcal{C}$ and $\mathcal{C}_\varepsilon$ are compact. Since Γ does not intersect any other component of $\mathcal{C}$, by Lemma 2.1 there exist two disjoint closed subsets $\mathcal{C}_1$ and $\mathcal{C}_2$ of $\mathcal{C}$ with $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2$ such that

$$\Gamma \subset \mathcal{C}_1, \quad \mathcal{C}_\varepsilon \subset \mathcal{C}_2.$$

Clearly $\mathcal{C}_1$ is contained in the interior of $\mathcal{V}_\varepsilon$ in $\mathcal{H}$.

Pick a number $\varepsilon_\Gamma > 0$ with

$$\varepsilon_\Gamma < \frac{1}{4}\min\{\varrho(\mathcal{C}_1, \mathcal{C}_2),\ \varrho(\mathcal{C}_1, \partial_{\mathcal{H}}\mathcal{V}_\varepsilon)\}.$$

Let $\mathcal{U}_\Gamma = \mathrm{B}_{\mathcal{H}}(\mathcal{C}_1, 2\varepsilon_\Gamma)$ be the $2\varepsilon_\Gamma$-neighborhood of $\mathcal{C}_1$ in $\mathcal{H}$. Then by the choice of ε_Γ we see that $\mathcal{U}_\Gamma \subset \mathcal{V}_\varepsilon$, and moreover,

$$\mathrm{B}_{\mathcal{H}}(\partial_{\mathcal{H}}\mathcal{U}_\Gamma, \varepsilon_\Gamma) \cap \mathscr{K}(\Lambda, X) = \emptyset. \tag{11}$$

Now we observe that $\mathscr{U} = \{\mathcal{U}_\Gamma[c]\}_{\Gamma\in\mathscr{T}_c}$ forms an open covering of $K_\infty(\Phi_c)$ in X. Thus there exist $\Gamma_1, \cdots, \Gamma_n \in \mathscr{T}_c$ such that

$$K_\infty(\Phi_c) \subset \textstyle\bigcup_{1\le i\le n} \mathcal{U}_{\Gamma_i}[c].$$

Let $\mathcal{U} = \bigcup_{1\le i\le n} \mathcal{U}_{\Gamma_i}$. We infer from (11) that $\mathcal{U}$ is an isolating neighborhood of the skew-product flow Π of $\{\Phi_\lambda\}_{\lambda\in\Lambda}$ in $\mathcal{H}$ with $K_\infty(\Phi_c) \subset \mathcal{U}[c]$; see Fig. 3.5. Therefore by Theorem 2.1 one concludes that

$$h\left(\Phi_c,\, K_\infty(\Phi_c)\right) = h\left(\Phi_c,\, \mathcal{U}[c]\right) = h\left(\Phi_d,\, \mathcal{U}[d]\right) = h\left(\Phi_d,\, K_1\right), \tag{12}$$

where $K_1 = K_\infty\left(\Phi_d,\, \mathcal{U}[d]\right)$.

For any component Γ of $\mathscr{K}(\Lambda, X)$, by (11) we have $\Gamma \cap \partial_{\mathcal{H}}\mathcal{U}_{\Gamma_i} = \emptyset$ for all $1 \le i \le n$. Hence one finds that

$$\text{either } \Gamma \subset \mathcal{U}, \quad \text{or } \Gamma \cap \overline{\mathcal{U}} = \emptyset.$$

Consequently, for any component C of $K_\infty(\Phi_d)$, we have

$$\text{either } C \subset \mathcal{U}[d], \quad \text{or } C \cap \overline{\mathcal{U}}[d] = \emptyset.$$

Thus we deduce that $K_\infty(\Phi_d) = K_1 \cup K_2$, where

$$K_2 = \textstyle\bigcup\{C:\ C \text{ is a component of } K_\infty(\Phi_d) \text{ with } C \cap \overline{\mathcal{U}}[d] = \emptyset\}.$$

As K_1 is isolated with $\mathcal{U}[d]$ being an isolating neighborhood, it is trivial to check that K_2 is isolated as well. Thereby

$$h\left(\Phi_d,\, K_\infty(\Phi_d)\right) = h\left(\Phi_d,\, K_1\right) \vee h\left(\Phi_d,\, K_2\right). \tag{13}$$

This, along with (10) and (12), yields that

$$h\left(\Phi_d,\, K_2\right) \ne \overline{0}.$$

Now by virtue of Theorem 3.1, one immediately concludes that there is a $\Gamma \in \mathscr{T}_d$ with $\Gamma[d] \subset K_2$ such that Γ is unbounded; see Fig. 3.5. ∎

3.3. Two examples. In this subsection we give two simple illustrating examples by considering ODE systems, which may help the reader have a better understanding to the abstract results given above.

Example 3.1. Consider the planar system

$$\begin{cases} \dot{x} = x - \lambda x(x^2+y^2), & x = x(t) \in \mathbb{R}, \\ \dot{y} = y - \lambda y(x^2+y^2), & y = y(t) \in \mathbb{R}, \end{cases} \tag{14}$$

where λ is the bifurcation parameter.

Denote Φ_λ the semiflow on $X = \mathbb{R}^2$ generated by the system. Multiplying the first equation in (14) by x and the second one by y, summing the results we obtain that

$$\frac{d}{dt} r^2 = 2r^2(1 - \lambda r^2), \tag{15}$$

where $r^2 = x^2 + y^2$. Let $\lambda \le 0$. Then by (15) we have

$$\frac{d}{dt} r^2 = 2r^2(1 - \lambda r^2) \ge 2r^2, \tag{16}$$

by which we deduce that $K_\infty(\Phi_\lambda) = \{(0,0)\}$ and is a repeller of the system. Let $B = \overline{\mathrm{B}}(2)$, where $\mathrm{B}(r)$ denotes the ball in X centered at $(0,0)$ with radius r. By (16) it is clear that B is an isolating block of $K_\infty(\Phi_\lambda)$ with $B^- = \partial \mathrm{B}(2)$. Hence

$$h(\Phi_\lambda, K_\infty(\Phi_\lambda)) = [(B/B^-, [B^-])] = \Sigma^2.$$

Now assume $\lambda > 0$. By (15) we find that

$$\frac{d}{dt} r^2 \le -r^2 \tag{17}$$

as long as $r(t) \ge \sqrt{2/\lambda}$, from which it can be easily seen that the system is dissipative with $K_\infty(\Phi_\lambda)$ being the global attractor. Let $\lambda = 1$. Then we infer from (17) that $B = \overline{\mathrm{B}}(2)$ is an isolating block of $K_\infty(\Phi_1)$ with $B^- = \emptyset$. Since B is contractible, one has

$$h(\Phi_1, K_\infty(\Phi_1)) = [(B/B^-, [B^-])] = \Sigma^0.$$

Let $\Lambda = [-1, 1]$. Then $h(\Phi_{-1}, K_\infty(\Phi_{-1})) \ne h(\Phi_1, K_\infty(\Phi_1))$. By Theorem 3.3 one immediately concludes that the set $\mathscr{K}(\Lambda, X)$ has an unbounded component Γ meeting $X \times \{\pm 1\}$.

One can also discuss the bifurcation phenomena of the system by choosing appropriate isolating neighborhoods of the system and applying Theorem 3.2. For instance, take $U = X \setminus \mathrm{B}\left(\frac{1}{2}\right)$. Then for $\lambda \in [0,1]$, we have by (15) that

$$\frac{d}{dt} r^2 = 2r^2(1 - \lambda r^2) > 0, \quad \text{if } (x(t), y(t)) \in \partial U, \tag{18}$$

from which one easily deduces that

$$K_\infty(\Phi_\lambda, U) \cap \partial U = \emptyset, \qquad \forall\, \lambda \in [0,1]. \tag{19}$$

Since $K_\infty(\Phi_\lambda, U) \subset K_\infty(\Phi_\lambda)$ and hence is compact for all λ, by (19) we find that U is an isolating neighborhood of Φ_λ for each $\lambda \in [0,1]$.

Set $S = K_\infty(\Phi_1, U)$. We infer from the above argument that $S \subset B := \overline{\mathrm{B}}(2) \setminus \mathrm{B}\left(\frac{1}{2}\right)$; furthermore, B is an isolating block of S with $\mathrm{B}^- = \emptyset$. We have

$$h(\Phi_1, S) = [(B/\emptyset, [\emptyset])] = [(B \cup \{q\}, q)] \ne \overline{0},$$

where q is an element with $q \notin B$. By virtue of Theorem 3.2 one concludes that $\mathscr{K}((0,1],U)$ has a component Γ_U meeting $S \times \{1\}$ such that one of the alternatives (1)-(3) in the theorem holds true. We claim that Γ_U is unbounded. To see this, we first observe that $K_\infty(\Phi_0, U) = \emptyset$. Now we argue by contradiction and suppose the contrary. Then one can easily verify that $\Gamma_U[\lambda] \subset K_\infty(\Phi_\lambda, U)$ for all $\lambda \in (0,1]$. It follows by (19) that the second alternative (2) in Theorem 3.2 does not occur. Thus we necessarily have $\Gamma_U[\lambda] \neq \emptyset$ for all $\lambda \in (0,1]$. But this and the boundedness of Γ_U then imply that $\Gamma_U[0] \neq \emptyset$, which leads to a contradiction and proves our claim.

Now let us give a simple observation that justifies our theoretical results obtained above. By (15) we see that the circle

$$C_\lambda : \, r = r_\lambda := 1/\sqrt{\lambda}$$

is a closed orbit of the system for each $\lambda > 0$, which depends on λ continuously. Clearly $r_\lambda \to \infty$ as $\lambda \to 0$.

It is also worth mentioning that the bifurcating branches Γ and Γ_U given above may be different. In fact, it is easy to check that for $\lambda \in (0,1]$, $\Gamma_U[\lambda]$ consists of exactly the closed orbit C_λ, whereas $\Gamma[\lambda]$ may contain C_λ and the equilibrium $(0,0)$ and also the connecting orbits between them.

Example 3.2. Consider the following non-autonomous scalar equation

$$\dot{x} = -(\lambda + h(t))x + e^{-x^2} \tag{20}$$

on $\mathbb{R}$, where $h \in C(\mathbb{R})$ is a T-periodic function ($T > 0$). To have a better understanding of the dynamics of the equation, as usual we embed the equation into a cocycle system below:

$$\dot{x} = -(\lambda + p(t))x + e^{-x^2}, \qquad p \in \mathcal{H}, \tag{21}$$

where $\mathcal{H} = \{h(\tau + \cdot) : \ \tau \in \mathbb{R}\}$, which is equipped with the topology of uniform convergence on $[0,T]$ (and hence on $\mathbb{R}$). It is a basic knowledge that due to the periodicity of h, $\mathcal{H}$ is homeomorphic to the unit circle (or, one-dimensional sphere) S^1.

Let $X = \mathbb{R} \times \mathcal{H}$, and denote $\phi_\lambda(t,p)x_0$ the unique solution of (21) with $x(0) = x_0$. Set

$$\Phi_\lambda(t)(x,p) = (\phi_\lambda(t,p)x, \theta_t p), \qquad (x,p) \in X,$$

where θ_t is the translation group on $\mathcal{H}$,

$$(\theta_t p)(\cdot) = p(t + \cdot), \qquad \forall\, p \in \mathcal{H}, \ t \in \mathbb{R}.$$

Then Φ_λ is a flow on X, called the *skew-product flow* of (21).

For the sake of simplicity, we may assume $\max_{\mathbb{R}} |h(t)| \leq 1$. Let $\Lambda = [-2,2]$. For $\lambda = -2$, multiplying the equation (21) by x we find that

$$\begin{aligned} \tfrac{1}{2}\tfrac{d}{dt}x^2 &= (2 - p(t))x^2 + xe^{-x^2} \\ &\geq x^2 - |x| = |x|(|x| - 1), \end{aligned} \tag{22}$$

from which it is clear that for any solution $x(t)$ of (21), if $|x(t_0)| > 1$ for some $t_0 \in \mathbb{R}$ then $|x(t)| > 1$ for all $t \geq t_0$; moreover, $|x(t)| \to \infty$ as $t \to +\infty$. It follows that

$$K_\infty(\Phi_{-2}) \subset [-1,1] \times \mathcal{H}.$$

Let $B_1 = [-2,2] \times \mathcal{H}$. Making use of (22) it is trivial to check that B_1 is an isolating block of $K_\infty(\Phi_{-2})$ with $B_1^- = \{\pm 2\} \times \mathcal{H}$. Thus

$$h\left(\Phi_{-2}, K_\infty(\Phi_{-2})\right) = [(B_1/B_1^-, [B_1^-])].$$

Since B_1 is pass-connected, one can easily verify that the quotient space B_1/B_1^- is pass-connected as well. Hence

$$H_0\left(h\left(\Phi_{-2}, K_\infty(\Phi_{-2})\right)\right) = H_0\left((B_1/B_1^-, [B_1^-])\right) = 0. \tag{23}$$

Now we consider the case where $\lambda = 2$. A fully analogous argument as above applies to show that $K_\infty(\Phi_2) \subset [-1,1] \times \mathcal{H}$ with $B_2 = [-2,2] \times \mathcal{H}$ being an isolating block with $B_2^- = \emptyset$. ($K_\infty(\Phi_2)$ is actually the global attractor of Φ_2.) Thus we have

$$\begin{aligned} h\left(\Phi_2, K_\infty(\Phi_2)\right) &= [(B_2/\emptyset, [\emptyset])] = [(([-2,2] \times \mathcal{H})/\emptyset, [\emptyset])] \\ &= [(S^1/\emptyset, [\emptyset])] = [(S^1 \cup \{q\}, q)], \end{aligned}$$

where q is an element with $q \notin S^1$. Therefore

$$H_0\left(h\left(\Phi_2, K_\infty(\Phi_2)\right)\right) = H_0\left((S^1 \cup \{q\}, q)\right) = \mathbb{Z}. \tag{24}$$

(23) and (24) indicate that $h\left(\Phi_{-2}, K_\infty(\Phi_{-2})\right) \neq h\left(\Phi_2, K_\infty(\Phi_2)\right)$. Applying Theorem 3.3 one immediately concludes that the system Φ_λ undergoes a dynamic bifurcation from infinity as λ varies in the interval Λ, although we know little about where and how this bifurcation occurs.

4. Bifurcation from Infinity of Nonlinear Evolution Equations.

In this section we use our general results in section 3 to discuss the bifurcation phenomena from infinity of the nonlinear evolution equation

$$\frac{du}{dt} + Au - \lambda u - f(u,\lambda) = 0 \tag{1}$$

on a Banach space X, where A is a sectorial operator on X with compact resolvent, $\lambda \in \mathbb{R}$, and $f(u,\lambda)$ is a locally Lipschitz continuous mapping from $X^\alpha \times \mathbb{R}$ to X for some $0 \leq \alpha < 1$. Our main goal is to present some more precise descriptions on the dynamic bifurcation from infinity.

Denote $\|\cdot\|$ and $\|\cdot\|_\alpha$ the norms of X and X^α, respectively.

4.1. Existence of unbounded bifurcating branch. It is well known (see e.g. [14, Theorem 3.3.3]) that the Cauchy problem of (1) is well-posed in X^α, that is, for any $u_0 \in X^\alpha$, there exist $T > 0$ and a (unique) continuous function $u : [0,T) \to X^\alpha$ with $u(0) = u_0$, called the *strong solution* of the problem, such that $u(t) \in \mathrm{D}(A)$ and $\frac{d}{dt}u(t)$ exists for $t \in (0,T)$, moreover, the differential equation (1) is satisfied on $(0,T)$.

Denote Φ_λ the local semiflow generated by the equation. By the continuity property of f in λ, one can easily verify that Φ_λ depends on λ continuously. Also, by very standard argument (see e.g. [32, Chap. I, Theorem 4.4]), it can be shown that the family Φ_λ $(\lambda \in \mathbb{R})$ is λ-l.u.a.c.

We always assume f satisfies the following *sublinear condition*:

(A) $\lim_{\|u\|_\alpha \to \infty} \|f(u,\lambda)\|/\|u\|_\alpha = 0$ uniformly on compact λ-intervals.

Hence Φ_λ is actually a global semiflow on X^α for each λ.

Definition 4.1. *We say that* (1) *bifurcates from infinity at* $\lambda = \mu$ *(or,* (∞,μ) *is a bifurcation point), if for any* $\varepsilon > 0$, *there exist* $\lambda \in \mathbb{R}$ *with* $|\lambda - \mu| < \varepsilon$ *and a bounded full solution* $u_\lambda = u_\lambda(t)$ *of* (1) *such that*

$$\|u_\lambda\|_\infty > 1/\varepsilon,$$

where $\|u_\lambda\|_\infty = \sup_{t \in \mathbb{R}} ||u_\lambda(t)||_\alpha$.

Denote $\sigma(A)$ the spectral of A, and write

$$\mathrm{Re}\,\sigma(A) = \{\mathrm{Re}\,z : \ z \in \sigma(A)\}.$$

Theorem 4.1. *Let* $\mu \in \mathrm{Re}\,\sigma(A)$. *Then* (∞,μ) *is a bifurcation point of* (1). *Specifically, for any* $c,d \in \mathbb{R}$ *with* $c < \mu < d$ *and* $\mathrm{Re}\,\sigma(A) \cap [c,d] = \{\mu\}$, *the set* $\mathscr{K}([c,d])$ *(see* (1) *in section 3 for the definition) has a component* Γ *meeting* $X^\alpha \times \{c,d\}$ *such that for some sequence* $\lambda_n \to \mu$,

$$\sup\{\|x\|_\alpha : \ x \in \Gamma[\lambda_n]\} \to \infty \quad \textit{as} \ n \to \infty. \tag{2}$$

Proof. Let us begin with the following linear equation

$$\frac{du}{dt} + Au - \lambda u = 0. \tag{3}$$

Let c,d be the numbers given in the theorem. Then if $\lambda = c,d$, the set $\{0\}$ is an isolated invariant set for the semiflow ϕ_λ in X^α generated by (3). By [32] (see Chap. I, Corollary 11.2) there exist two nonnegative integers p and q with $q - p > 0$ such that

$$h(\phi_c,\{0\}) = \Sigma^p, \quad h(\phi_d,\{0\}) = \Sigma^q. \tag{4}$$

($q-p$ is actually the total algebraic multiplicity of all the eigenvalues z of the operator A with $\mathrm{Re}\,z = \mu$.)

Now consider the nonlinear equation

$$\frac{du}{dt} + Au - \lambda u - \nu f(u,\lambda) = 0, \tag{5}$$

where $\nu \in [0,1]$ is the homotopy parameter. By appropriately modifying the argument in the proof of [32, Chap. II, Theorem 5.1] (see also the proof of [40, Theorem 3.2]), it can be shown that for any $\varepsilon > 0$ with

$$c < \mu - \varepsilon < \mu + \varepsilon < d,$$

there exists $R_\varepsilon > 0$ such that for any bounded full solution $u = u(t)$ of (5) with $\lambda \in [c, \mu - \varepsilon] \cup [\mu + \varepsilon, d]$ and $\nu \in [0,1]$, we have

$$\|u\|_\infty < R_\varepsilon. \tag{6}$$

Denote ϕ_λ^ν the semiflow generated by (5). By virtue of the continuation property of Conley index, we conclude that

$$\begin{aligned} h\left(\Phi_\lambda, K_\infty(\Phi_\lambda)\right) &= h\left(\phi_\lambda^1, K_\infty(\phi_\lambda^1)\right) \\ &= h\left(\phi_\lambda^0, K_\infty(\phi_\lambda^0)\right) = h(\phi_\lambda, \{0\}) = \Sigma^p \end{aligned} \tag{7}$$

for $\lambda \in [c, \mu - \varepsilon]$, and

$$\begin{aligned} h\left(\Phi_\lambda, K_\infty(\Phi_\lambda)\right) &= h\left(\phi_\lambda^1, K_\infty(\phi_\lambda^1)\right) \\ &= h\left(\phi_\lambda^0, K_\infty(\phi_\lambda^0)\right) = h(\phi_\lambda, \{0\}) = \Sigma^q \end{aligned} \tag{8}$$

for $\lambda \in [\mu + \varepsilon, d]$. Thanks to Theorem 3.3, one immediately concludes that $\mathscr{K}([c,d])$ has an unbounded connected component Γ meeting $X^\alpha \times \{c, d\}$. On the other hand, (6) implies that for any $\varepsilon > 0$,

$$\Gamma[\lambda] \subset \mathrm{B}_{X^\alpha}(R_\varepsilon), \qquad \forall\, \lambda \in [c, \mu - \varepsilon] \cup [\mu + \varepsilon, d],$$

where $\mathrm{B}_{X^\alpha}(R_\varepsilon)$ denotes the ball in X^α centered at 0 with radius R_ε. Thus there exists a sequence $\lambda_n \to \mu$ such that (2) holds true. ■

Remark 4.1. *In Theorem 4.1 one should distinguish two cases of the bifurcation. One is that $K_\infty(\Phi_\mu)$ is unbounded. When this occurs we say that (1) undergoes a* vertical bifurcation from infinity *at $\lambda = \mu$. The other is that $K_\infty(\Phi_\mu)$ is bounded, in which case we deduce that there is a sequence $\lambda_n \to \mu$ ($\lambda_n \neq \mu$ for all n) such that $\Gamma[\lambda_n]$ is unbounded, where Γ is the connected bifurcating branch given in the theorem. Note that both cases may occur. This can be seen from the following two simple examples.*

Example 4.1. Consider the linear equation

$$\dot{u} + u = \lambda u, \qquad u = u(t) \in \mathbb{R}, \tag{9}$$

where $\lambda \in \mathbb{R}$ is the bifurcation parameter. Then we can see that $\mu = 1$ is a bifurcation value, at which each constant function $u(t) = c$ ($c \in \mathbb{R}$) is a bounded full solution of the equation. Hence the equation undergoes a vertical bifurcation from infinity at $\lambda = 1$.

It is also interesting to note that for each $\lambda \neq 1$, the equation has no bounded full solutions other than the trivial one.

Example 4.2. Consider the non-homogenous equation

$$\dot{u} + u = \lambda u + 1, \qquad u \in \mathbb{R}, \tag{10}$$

where $\lambda \in \mathbb{R}$ is the bifurcation parameter. Again $\mu = 1$ is a bifurcation value, at which each solution of (10) is given by $u = t + c$ ($c \in \mathbb{R}$). Clearly $K_\infty(\Phi_\mu) = \emptyset$.

On the other hand, if we let $[c, d] = [0, 2]$, then by Theorem 4.1 we see that $\mathscr{K}([0, 2])$ has an unbounded connected component Γ in the space $\mathbb{R} \times [0, 2]$ with $\Gamma \cap (\mathbb{R} \times \{0, 2\}) \neq \emptyset$. Actually, for $\lambda \neq 1$, the unique bounded full solution of the equation is the stationary one $u_\lambda(t) = (1 - \lambda)^{-1}$. Hence

$$\Gamma = \{(u_\lambda, \lambda) : \ 0 \le \lambda < 1\}$$

is a component of $\mathscr{K}([0, 2])$ fulfilling all the requirements in the theorem.

4.2. Further results on dynamic bifurcation from infinity. We infer from Theorem 4.1 that there is a sequence $\lambda_n \to \mu$ such that for each $\lambda = \lambda_n$, (1) has a bounded full solution $u_n = u_n(t)$ with $\|u_n\|_\infty \to \infty$. In what follows we give another result on the bifurcation of the equation from infinity, which seems to be more precise in some aspects.

Theorem 4.2. *Assume f satisfies the sublinear condition* (**A**) *in Theorem 4.1. Let $\mu \in \mathrm{Re}\,\sigma(A)$. Then one of the following alternatives holds.*

(1) There is a sequence u_n of bounded full solutions of (1) at $\lambda = \mu$ such that $\lim_{n\to\infty} \|u_n\|_\infty = \infty$.

(2) There is a one-sided neighborhood Λ_1 of μ such that for each $\lambda \in \Lambda_1 \setminus \{\mu\}$, (1) has two distinct bounded full solutions u_λ and v_λ such that

$$\lim_{\lambda\to\mu} \|u_\lambda\|_\infty = \infty, \tag{11}$$

whereas $\|v_\lambda\|_\infty$ remains bounded on the λ-interval Λ_1.

(3) There is a two-sided neighborhood Λ of μ such that for each $\lambda \in \Lambda \setminus \{\mu\}$, the equation (1) has a bounded full solution u_λ satisfying (11).

Proof. If (1) holds true then we are done. Thus we assume the contrary, and hence S_μ is a bounded set, where (and below) $S_\lambda = K_\infty(\Phi_\lambda)$.

Take two numbers $c, d \in \mathbb{R}$ as in Theorem 4.1. Since the number ε in (7) and (8) is arbitrary, we infer from (7) and (8) that

$$h(\Phi_\lambda, S_\lambda) = \Sigma^p \ (\lambda \in [c, \mu)\,), \quad h(\Phi_\lambda, S_\lambda) = \Sigma^q \ (\lambda \in (\mu, d]) \tag{12}$$

for some nonnegative integers p and q with $p < q$.

Pick a bounded closed isolating neighborhood U of S_μ. Choose a $\delta > 0$ sufficiently small so that U is an isolating neighborhood of Φ_λ for all $\lambda \in \Lambda = [\mu - \delta, \mu + \delta]$. Then

$$h\,(\Phi_\lambda, U) \equiv const. \tag{13}$$

Two possibilities may occur.

Case 1) $h(\Phi_\mu, S_\mu) \neq \bar{0}$. In such a case we show that the second assertion (2) holds true. It is obvious that

$$\text{either } h(\Phi_\mu, S_\mu) \neq \Sigma^p, \quad \text{or} \quad h(\Phi_\mu, S_\mu) \neq \Sigma^q.$$

Let us first consider the case where $h(\Phi_\mu, S_\mu) \neq \Sigma^p$. By (12) we have

$$(14) \qquad h(\Phi_\lambda, S_\lambda) \neq h(\Phi_\mu, S_\mu), \qquad \lambda \in [c, \mu).$$

We claim that

$$(15) \qquad S_\lambda \setminus U \neq \emptyset, \qquad \forall\, \lambda \in \Lambda_- := [\mu - \delta, \mu).$$

Indeed, if $S_\lambda \subset U$ for some $\lambda \in \Lambda_-$, then by (12) and (13) one finds that

$$h\,(\Phi_\mu, S_\mu) = h\,(\Phi_\mu, U) = h\,(\Phi_\lambda, U) = h\,(\Phi_\lambda, S_\lambda) = \Sigma^p,$$

which leads to a contradiction.

For each $\lambda \in \Lambda_-$, pick an $x_\lambda \in S_\lambda \setminus U$. Let u_λ be a full trajectory of Φ_λ contained in S_λ with $u_\lambda(0) = x_\lambda$. We show that u_λ fulfills (11).

Suppose the contrary. Then there would exist a sequence $\lambda_n \to \mu$ ($\lambda_n \neq \mu$) such that the sequence $u_n = u_{\lambda_n}$ is uniformly bounded on $\mathbb{R}$. By very standard argument it can be shown that u_n has a subsequence converging to a bounded full trajectory u_0 of Φ_μ uniformly on any compact interval of $\mathbb{R}$. u_0 is necessarily contained in S_μ. On the other hand, since $u_n(0) = x_{\lambda_n} \notin U$, we deduce that $u_0(0) \notin \operatorname{int} U$. This leads to a contradiction.

Now assume that $h(\Phi_\mu, S_\mu) \neq \Sigma^q$. Then by a fully analogous argument as above, one concludes that for each $\lambda \in \Lambda_+ = (\mu, \mu + \delta]$, the equation has a bounded full solution u_λ satisfying (11).

Since $h(\Phi_\mu, S_\mu) \neq \overline{0}$, by (13) we have

$$h\,(\Phi_\lambda, U) = h\,(\Phi_\mu, U) = h(\Phi_\mu, S_\mu) \neq \overline{0}, \qquad \lambda \in \Lambda.$$

It follows that $K_\infty(\Phi_\lambda, U) \neq \emptyset$. For each $\lambda \in \Lambda$, pick a full solution v_λ in $K_\infty(\Phi_\lambda, U)$. Then $||v_\lambda||_\infty$ remains bounded on Λ.

Case 2) $h(\Phi_\mu, S_\mu) = \overline{0}$. In this case, we have

$$\Sigma^p \neq h(\Phi_\mu, S_\mu) \neq \Sigma^q.$$

The same argument as in Case 1) applies to show that for each $\lambda \in \Lambda_- \cup \Lambda_+$, the equation has a bounded full solution u_λ satisfying (11). Hence the assertion (3) holds. ∎

5. Dynamic Bifurcation and Multiplicity for Parabolic Equations.

In this section we consider the following boundary value problem:

$$(1) \qquad \begin{cases} u_t - \Delta u = \lambda u + f(x,u), & x \in \Omega; \\ u(x,t) = 0, & x \in \partial\Omega, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^n$, $\lambda \in \mathbb{R}$, and $f \in C^1(\overline{\Omega} \times \mathbb{R})$.

Let $H = L^2(\Omega)$ and $V = H_0^1(\Omega)$. By $(\cdot,\cdot)$ and $|\cdot|$ we denote the usual inner product and norm on H, respectively. The norm $\|\cdot\|$ on V is defined by

$$\|u\| = \left(\int_\Omega |\nabla u|^2 \mathrm{d}x\right)^{1/2}, \qquad u \in V.$$

Denote A the operator $-\Delta$ associated with the homogenous Dirichlet boundary condition. A is a sectorial operator and has a compact resolvent. Denote

$$0 < \mu_1 < \mu_2 < \cdots < \mu_k < \cdots$$

the eigenvalues of A.

We may convert (1) into an abstract equation on V:

$$u_t + Au = \lambda u + \tilde{f}(u), \quad u = u(t) \in V, \tag{2}$$

where $\tilde{f}(u)$ is the Nemitski operator from V to H given by

$$\tilde{f}(u)(x) = f(x, u(x)), \qquad u \in V.$$

If we assume that

$$f(x, s) = o(|s|) \quad \text{as } |s| \to \infty \tag{3}$$

uniformly with respect to $x \in \overline{\Omega}$, then one can trivially verify that the Nemitski operator $\tilde{f}$ in (2) satisfies the sublinear condition (**A**) in section 4. Thus applying the abstract results in section 4, one can immediately obtain some interesting information on the bifurcation of the equation. For instance, we have

Theorem 5.1. *Let μ_k be an eigenvalue of A. Then one of the following alternatives holds.*

(1) There is a sequence u_n of bounded full solutions of (2) at $\lambda = \mu_k$ such that

$$\lim_{n\to\infty} \|u_n\|_\infty = \infty.$$

(2) There is a one-sided neighborhood Λ_1 of μ_k such that for $\lambda \in \Lambda_1 \setminus \{\mu_k\}$, the equation (2) has at least two distinct bounded full solutions u_λ and v_λ such that

$$\lim_{\lambda\to\mu_k} \|u_\lambda\|_\infty = \infty, \tag{4}$$

whereas $\|v_\lambda\|$ remains bounded on Λ_1.

(3) There is a two-sided neighborhood Λ of μ_k such that for each $\lambda \in \Lambda \setminus \{\mu_k\}$, (2) has at least one bounded full solution u_λ satisfying (4).

In this present work, we are basically interested in a particular but very important case, namely, the case where f satisfies the Landesman-Laser type condition (4) in section 1. We will give some precise descriptions on the bifurcation of the equation and discuss the multiplicity of stationary solutions of the equation.

Henceforth we always assume

(H) f satisfies the Landesman-Laser type condition (4) in section 1.

Denote Φ_λ the semiflow associated with (2), namely, for each $u_0 \in X^\alpha$,

$$u(t) = \Phi_\lambda(t)u_0$$

is the solution of the equation on $\mathbb{R}_+$ with initial value $u(0) = u_0$.

5.1. Preliminaries. Let us begin with a fundamental result on f.

Given a function w on Ω, we use $w_\pm$ to denote the positive and negative parts of w, respectively,

$$w_\pm = \max\{\pm w(x), 0\}, \qquad x \in \Omega.$$

Then $w = w_+ - w_-$. We have

Lemma 5.1. *For any $R, \varepsilon > 0$, there exists $s_0 > 0$ such that*

$$\int_\Omega f(x, v + sw) w dx \geq \int_\Omega (\overline{f} w_+ + \underline{f} w_-) dx - \varepsilon$$

for all $s \geq s_0$, $v \in \overline{\mathrm{B}}_H(R)$ and $w \in \overline{\mathrm{B}}_H(1)$, where $B_H(r)$ denotes the ball in H centered at 0 with radius r.

Proof. This is a slightly modified version of [17, Lemma 6.7]. Here we give the details of the proof for completeness and the reader's convenience.

Let

$$I = \int_\Omega f(x, v + sw) w dx - \int_\Omega (\overline{f} w_+ + \underline{f} w_-) dx.$$

Since $w = w_+ - w_-$, we can rewrite I as $I_+ - I_-$, where

$$I_+ = \int_\Omega (f(x, v + sw) - \overline{f}) w_+ dx, \quad I_- = \int_\Omega (f(x, v + sw) + \underline{f}) w_- dx.$$

In what follows, let us estimate I_+ for $v \in \overline{\mathrm{B}}_H(R)$ and $w \in \overline{\mathrm{B}}_H(1)$.

We observe that

$$R^2 \geq \int_\Omega |v|^2 dx \geq \int_{\{|v| \geq \sigma\}} |v|^2 dx \geq \sigma^2 |\{|v| \geq \sigma\}|,$$

from which it can be easily seen that $|\{|v| \geq \sigma\}| \to 0$ as $\sigma \to +\infty$ uniformly with respect to $v \in \overline{\mathrm{B}}_H(R)$. (Here and below $|E|$ denotes the Lebesgue measure for any measurable subset E of $\mathbb{R}^n$.) Therefore there exists $\sigma > 0$ such that

$$|\{|v| \geq \sigma\}|^{1/2} < \delta := \varepsilon / 8\|f\|(|\Omega| + 1), \qquad v \in \overline{\mathrm{B}}_H(R), \tag{5}$$

where $\|f\| = \sup_{x \in \overline{\Omega},\, s \in \mathbb{R}} |f(x, s)|$.

For each $v \in \overline{\mathrm{B}}_H(R)$ and $w \in \overline{\mathrm{B}}_H(1)$, let

$$D = D_{v,w} := \{|v| < \sigma\} \cap \{w_+ > \delta\}.$$

Then $\Omega = D \cup \{|v| \geq \sigma\} \cup \{w_+ \leq \delta\}$. Hence

$$\begin{aligned} I_+ \quad &\geq \textstyle\int_D (f(x, v + sw) - \overline{f}) w_+ dx - \int_{\{|v| \geq \sigma\}} |f(x, v + sw) - \overline{f}| w_+ dx \\ &\quad - \textstyle\int_{\{w_+ \leq \delta\}} |f(x, v + sw) - \overline{f}| w_+ dx \\ &\geq \textstyle\int_D (f(x, v + sw) - \overline{f}) w_+ dx - 2\|f\| \left(\int_{\{|v| \geq \sigma\}} w_+ dx + \int_{\{w_+ \leq \delta\}} w_+ dx \right). \end{aligned}$$

Note that

$$\begin{aligned}\textstyle\int_{\{|v|\geq\sigma\}} w_+dx &\leq \left(\textstyle\int_{\{|v|\geq\sigma\}} w_+^2dx\right)^{1/2} |\{|v|\geq\sigma\}|^{1/2}\\ &\leq (\text{by } (5)) \leq |w|\delta \leq \delta.\end{aligned}$$

It is obvious that

$$\int_{\{w_+\leq\delta\}} w_+dx \leq |\Omega|\delta.$$

Thereby

$$\begin{aligned}I_+ &\geq \textstyle\int_D (f(x,v+sw)-\overline{f})w_+dx - 2\|f\|(|\Omega|+1)\delta\\ &= \textstyle\int_D (f(x,v+sw)-\overline{f})w_+dx - \frac{\varepsilon}{4}.\end{aligned} \tag{6}$$

Since $z+s\eta\to+\infty$ (as $s\to+\infty$) uniformly for $z\in[-\sigma,\sigma]$ and $\eta\geq\delta$, there exists $s_1>0$ (independent of v and w) such that if $s>s_1$ then

$$f(x,z+s\eta)-\overline{f}\geq -\frac{\varepsilon}{4|\Omega|^{1/2}},\qquad \forall z\in[-\sigma,\sigma],\ \eta\geq\delta.$$

Now suppose that $s>s_1$. Then by the definition of D, we have (note that $w=w_+$ on D)

$$\begin{aligned}\textstyle\int_D (f(x,v+sw)-\overline{f})w_+dx &\geq -\frac{\varepsilon}{4|\Omega|^{1/2}}\textstyle\int_D w_+dx\\ &\geq -\frac{\varepsilon}{4|\Omega|^{1/2}}|D|^{1/2}(\textstyle\int_D |w|^2dx)^{1/2}\geq -\frac{\varepsilon}{4}.\end{aligned}$$

Thus by (6) we see that

$$I_+\geq \int_D (f(x,v+sw)-\overline{f})w_+dx-\frac{\varepsilon}{4}>-\frac{\varepsilon}{2}.$$

Similarly it can be shown that there exists $s_2>0$ (independent of v and w) such that $I_-<\frac{\varepsilon}{2}$, provided $s>s_2$.

Set $s_0=\max\{s_1,s_2\}$. Then if $s>s_0$, we have

$$I\geq I_+-I_->-\frac{\varepsilon}{2}-\frac{\varepsilon}{2}=-\varepsilon$$

for all $v\in\overline{\mathrm{B}}_H(R)$ and $w\in\overline{\mathrm{B}}_H(1)$. This completes the proof of the lemma. ■

Now we prove some basic facts concerning the dynamical behavior of the equation (2).

Let $L=A-\mu_k$, where μ_k is an eigenvalue of A. The space H can be decomposed into the orthogonal direct sum of its subspaces H^-, H^0 and H^+ corresponding to the negative, zero and positive eigenvalues of L, respectively. Note that both H^- and H^0 are of finite-dimensional. Denote P^σ ($\sigma\in\{0,\pm\}$) the projection from H to H^σ. Set

$$V^\sigma=V\cap H^\sigma,\qquad \sigma\in\{0,\pm\}.$$

By the finite dimensionality of H^- and H^0, one finds that V^- and V^0 coincide with H^- and H^0, respectively. We also have

$$V=V^-\oplus V^0\oplus V^+.$$

Lemma 5.2. *Assume $\lambda \le \mu_k + \eta$, where $\eta = (\mu_{k+1} - \mu_k)/2$. Then there exists $\rho_0 > 0$ (independent of λ) such that for any solution $u = u(t)$ of (2) on $\mathbb{R}_+$,*

$$\|u^+(t)\|^2 \le \|u_0^+\|^2\, e^{-\eta t} + \rho_0^2(1 - e^{-\eta t}), \qquad \forall\, t \ge 0.$$

Here $u^+ = P^+ u$.

Proof. Taking the inner product of the equation with Au^+ in H, it yields

$$\begin{aligned} \tfrac{1}{2}\tfrac{d}{dt}\|u^+\|^2 + |Au^+|^2 \ &= \lambda\|u^+\|^2 + (Au^+, \tilde{f}(u)) \\ &\le \lambda\|u^+\|^2 + \varepsilon|Au^+|^2 + C_\varepsilon \end{aligned}$$

for any $\varepsilon > 0$, where C_ε is a positive constant depending only upon ε and the upper bound of $|\tilde{f}(s)|$. Hence

$$\frac{1}{2}\frac{d}{dt}\|u^+\|^2 + (1-\varepsilon)|Au^+|^2 \le \lambda\|u^+\|^2 + C_\varepsilon. \tag{7}$$

Note that $|Au^+|^2 \ge \mu_{k+1}\|u^+\|^2$. Therefore by (7) we have

$$\frac{1}{2}\frac{d}{dt}\|u^+\|^2 \le -\left((1-\varepsilon)\mu_{k+1} - \lambda\right)\|u^+\|^2 + C_\varepsilon. \tag{8}$$

Fix an $\varepsilon > 0$ sufficiently small so that $(1-\varepsilon)\mu_{k+1} > \mu_k + \frac{3}{2}\eta$. Then for $\lambda \le \mu_k + \eta$, one has

$$(1-\varepsilon)\mu_{k+1} - \lambda > \left(\mu_k + \frac{3}{2}\eta\right) - (\mu_k + \eta) = \eta/2.$$

Now the conclusion follows from (8) and the classical Gronwall Lemma. ∎

Denote

$$\Xi_\rho = \{v \in V : \ \|P^+ v\| \le \rho\}, \qquad \rho > 0.$$

As a direct consequence of Lemma 5.2, we have

Corollary 5.1. *Assume $\lambda \le \mu_k + \eta$. Then*

$$K_\infty(\Phi_\lambda) \subset \Xi_{\rho_0}.$$

Furthermore, Ξ_ρ is positively invariant under Φ_λ for any $\rho > \rho_0$.

Set $W = V^- \oplus V^0$, and let

$$P_W = P^- + P^0$$

be the projection from V to W. Given $0 \le a < b \le \infty$ and $\rho > 0$, denote

$$\Xi_\rho[a,b] = \{u \in \Xi_\rho : \ a \le |P_W u| \le b\}. \tag{9}$$

Lemma 5.3. *Let η and ρ_0 be as in Lemma 5.2, and $\rho > \rho_0$. Then there exist $R_0, c_0 > 0$ such that the following assertions hold.*

(1) If $\lambda \in [\mu_k, \mu_k + \eta]$, then for any solution $u(t)$ of the equation (2) in $\Xi_\rho[R_0, \infty]$, we have

$$\frac{d}{dt}|w(t)|^2 \ge c_0|w(t)|, \tag{10}$$

where $w(t) = P_W u(t)$.

(2) For any $R > R_0$, there exists $0 < \theta \le \eta$ such that if $\lambda \in [\mu_k - \theta, \mu_k)$, then (10) holds true for any solution $u(t)$ of (2) in $\Xi_\rho[R_0, R]$.

Proof. Let $\lambda \in [\mu_k - \eta, \mu_k + \eta]$, and $u = u(t)$ a solution of (2) in Ξ_ρ. Taking the inner product of (2) with $w = P_W u$ in H, it yields

$$\frac{1}{2}\frac{d}{dt}|w|^2 + \|w\|^2 = \lambda|w|^2 + (\tilde{f}(u), w). \tag{11}$$

Because $\|w\|^2 \le \mu_k|w|^2$, by (11) we have

$$\frac{1}{2}\frac{d}{dt}|w|^2 \ge (\lambda - \mu_k)|w|^2 + (\tilde{f}(u), w). \tag{12}$$

Let us first estimate the last term in (12).

As the norm $\|\cdot\|_{L^1(\Omega)}$ of $L^1(\Omega)$ and that of $H = L^2(\Omega)$ are equivalent on W, one easily sees that

$$\min\{\|v\|_{L^1(\Omega)} :\ v \in W,\ |v| = 1\} := m > 0. \tag{13}$$

Pick a number $\delta > 0$ with $\delta \le \min\{\overline{f}, \underline{f}\}$. By virtue of Lemma 5.1 there exists $s_0 > 0$ (depending only upon ρ) such that if $s \ge s_0$, then

$$(\widetilde{f}(h + sv), v) = \int_\Omega f(x, h + sv)v\,dx \ge \int_\Omega \left(\overline{f}v_+ + \underline{f}v_-\right) dx - \frac{1}{2}m\delta \tag{14}$$

for all $h \in \overline{\mathrm{B}}_H(\rho)$ and $v \in \overline{\mathrm{B}}_H(1)$.

Now we rewrite

$$w = sv, \ \text{ where } s = |w|.$$

Then $|v| = 1$. Suppose $s \ge s_0$. Noticing that $||u^+|| \le \rho$, by (14) one finds that

$$(\tilde{f}(u), w) = s(f(x, u^+ + sv), v) \ge s\left(\int_\Omega \left(\overline{f}v_+ + \underline{f}v_-\right) dx - \frac{1}{2}m\delta\right).$$

Observing that

$$\begin{aligned} &\textstyle\int_\Omega \left(\overline{f}v_+ + \underline{f}v_-\right) dx - \frac{1}{2}m\delta \\ \ge\ & \textstyle\delta \int_\Omega |v|dx - \frac{1}{2}m\delta \ge (\text{by } (13)) \ge \frac{1}{2}m\delta, \end{aligned}$$

we conclude that

$$(\tilde{f}(u), w) \ge \frac{1}{2}m\delta s = \frac{1}{2}m\delta|w|. \tag{15}$$

Now we combine (15) and (12) together to obtain that

$$\frac{d}{dt}|w(t)|^2 \geq 2\left(\lambda-\mu_k\right)|w|^2 + m\delta|w(t)| \tag{16}$$

as long as $|w(t)| \geq s_0$.

Set $R_0 = s_0$, $c_0 = m\delta/2$. Assume $\lambda \in [\mu_k, \mu_k+\eta]$. Then $\lambda-\mu_k \geq 0$, and we infer from (16) that

$$\frac{d}{dt}|w(t)|^2 \geq m\delta|w(t)| > c_0|w(t)|$$

at any point t where $|w(t)| \geq R_0$. Hence the assertion (1) holds.

Now assume $\lambda < \mu_k$. Let $R > R_0$. Choose a $\theta > 0$ with $\theta R^2 < m\delta s_0/4$. Then if $\lambda \in [\mu_k-\theta, \mu_k)$, for any solution $u(t)$ of (2) in $\Xi_\rho[R_0, R]$, by (16) we conclude that

$$\begin{aligned}\tfrac{d}{dt}|w(t)|^2 &\geq -2|\lambda-\mu_k|\,R^2 + m\delta|w(t)| \\ &\geq c_0|w(t)| + \left(c_0|w(t)| - 2\theta R^2\right) \\ &\geq c_0|w(t)| + \left(c_0 s_0 - 2\theta R^2\right) \geq c_0|w(t)|,\end{aligned}$$

which justifies the second assertion (2). ∎

5.2. Dynamic bifurcation from infinity.

We are now ready to discuss the bifurcation of the equation (2) near $\lambda = \mu_k$.

Let Φ_λ be the semiflow generated by (2). First, as a consequence of Lemma 5.3 we have the following basic fact.

Proposition 5.1. *Assume the hypothesis (**H**). Then $K_\infty(\Phi_\lambda)$ is uniformly bounded in V for $\lambda \in [\mu_k, \mu_k+\eta]$, and*

$$h\left(\Phi_{\mu_k}, K_\infty(\Phi_{\mu_k})\right) = h\left(\Phi_\lambda, K_\infty(\Phi_\lambda)\right) = \Sigma^{p+r}, \tag{17}$$

where p is the sum of the multiplicities of the eigenvalues μ_i $(0 \leq i \leq k-1)$ of A, and r the multiplicity of μ_k.

Proof. Let η and ρ_0 be the numbers given in Lemma 5.2. Fix a number $\rho > \rho_0$. Then there exist $R_0, c_0 > 0$ such that the first assertion (1) in Lemma 5.3 holds true, by which one easily deduces that

$$K_\infty(\Phi_\lambda) \subset \Xi_\rho[0, R_0], \qquad \forall\,\lambda \in [\mu_k, \mu_k+\eta]. \tag{18}$$

On the other hand, as in (12) in section 4 it can be shown that

$$h(\Phi_\lambda, K_\infty(\Phi_\lambda)) = \begin{cases} \Sigma^{p+r}, & \lambda \in (\mu_k, \mu_{k+1}); \\ \Sigma^{p}, & \lambda \in (\mu_{k-1}, \mu_k). \end{cases} \tag{19}$$

By (18) and the continuation property of Conley index we immediately conclude that $h\left(\Phi_{\mu_k}, K_\infty(\Phi_{\mu_k})\right) = \Sigma^{p+r}$. ∎

Now we state and prove the main result in this subsection on the dynamic bifurcation from infinity of the equation near each eigenvalue μ_k.

Theorem 5.2. *Assume the hypothesis* (**H**). *Then* $S_\lambda := K_\infty(\Phi_\lambda)$ *is nonvoid for all* $\lambda \in \mathbb{R}$, *and there exists* $\delta > 0$ *such that the following assertions hold.*

(1) For each $\lambda \in \Lambda_- := [\mu_k - \delta, \mu_k)$, S_λ *has a Morse decomposition* $\mathcal{M} = \{M_\lambda^\infty, M_\lambda^1\}$. *Furthermore, there is at least one connecting trajectory* γ *between* M_λ^1 *and* M_λ^∞.

(2) M_λ^1 *remains uniformly bounded on* Λ_-, *whereas*

$$\lim_{\lambda\to\mu_k^-}\ \min_{v\in M_\lambda^\infty} ||v|| = \infty. \tag{20}$$

(3) Each of the sets $\mathscr{K}^1$ *and* $\mathscr{K}^\infty$ *has a component* Γ *with* $\Gamma[\lambda] \neq \emptyset$ *for all* $\lambda \in \Lambda_-$, *where*

$$\mathscr{K}^1 = \overline{\bigcup_{\lambda\in\Lambda_-} (M_\lambda^1 \times \{\lambda\})}, \quad \mathscr{K}^\infty = \overline{\bigcup_{\lambda\in\Lambda_-} (M_\lambda^\infty \times \{\lambda\})}.$$

Proof. (i) We infer from the proof of Theorem 4.1 (see (6) in section 4) and Proposition 5.1 that S_λ is a compact subset of V for all $\lambda \in \mathbb{R}$. Since the Conley index of S_λ is nontrivial (see (17) and (19)), one concludes that $S_\lambda \neq \emptyset$.

(ii) Corollary 5.1 asserts that

$$S_\lambda \subset \Xi_{\rho_0}, \qquad \forall\, \lambda \le \mu_k + \eta,$$

where ρ_0 is the number in Lemma 5.2. Fix a $\rho > \rho_0$, and let R_0 and c_0 be the numbers given by Lemma 5.3. Pick a bounded isolating neighborhood N_1 of S_{μ_k} with

$$\Xi_\rho[0, R_0] \subset N_1. \tag{21}$$

Then one can restrict $\delta > 0$ sufficiently small so that N_1 is also an isolating neighborhood of Φ_λ for all $\lambda \in \Lambda := [\mu_k - \delta, \mu_k + \delta]$. Hence

$$h(\Phi_\lambda, M_\lambda^1) \equiv const., \qquad \lambda \in \Lambda, \tag{22}$$

where $M_\lambda^1 = K_\infty(\Phi_\lambda, N_1)$. Further by Proposition 5.1 we deduce that

$$h(\Phi_\lambda, M_\lambda^1) = h(\Phi_{\mu_k}, M_{\mu_k}^1) = h(\Phi_{\mu_k}, S_{\mu_k}) = \Sigma^{p+r}, \qquad \lambda \in \Lambda. \tag{23}$$

It is clear that $M_\lambda^1 \subset S_\lambda \subset \Xi_\rho$. Therefore

$$M_\lambda^1 \subset N_1 \cap \Xi_\rho := \widetilde{N}_1, \qquad \lambda \in \Lambda. \tag{24}$$

Because $\widetilde{N}_1$ is bounded, one can find a number $R_1 > 0$ such that

$$\widetilde{N}_1 \subset \Xi_\rho[0, R_1/2]. \tag{25}$$

By Lemma 5.3 (2), there exists $\theta_1 > 0$ such that if $\lambda \in [\mu_k - \theta_1, \mu_k)$, then for any solution $u(t)$ of (2) in $\Xi_\rho[R_0, R_1]$, one has

$$\frac{d}{dt}|w(t)|^2 \ge c_0|w(t)| \ge c_0 R_0 > 0, \tag{26}$$

where $w(t) = P_W u(t)$. We may assume $\delta \le \theta_1$. Let $\lambda \in \Lambda_- := [\mu_k - \delta, \mu_k)$. Then for any bounded full solution $u(t)$ of (2) in Ξ_ρ with $u(t_0) \in \Xi_\rho[R_0, R_1]$ for some t_0, by (26) one easily deduces that there exists $T > 0$ such that

$$u(t) \in \Xi_\rho[0, R_0] \ \ (t < -T), \quad \text{and} \ \ u(t) \in \Xi_\rho[R_1, \infty] \ \ (t > T). \tag{27}$$

Combining (24), (25) and (27) together, it yields that

$$M^1_\lambda \subset \Xi_\rho[0, R_0], \qquad \forall\, \lambda \in \Lambda_-. \tag{28}$$

As M^1_λ is the maximal compact invariant set of Φ_λ in N_1, (21) and (28) imply that M^1_λ is the maximal compact invariant set in $\Xi_\rho[0, R_0]$.

Set

$$M^\infty_\lambda = K_\infty\left(\Phi_\lambda,\, \Xi_\rho[R_1, \infty]\right).$$

Then $M^\infty_\lambda \subset K_\infty(\Phi_\lambda) = S_\lambda$. We prove that $\mathcal{M} = \{M^\infty_\lambda, M^1_\lambda\}$ forms a Morse decomposition of S_λ. For this purpose, let us first show that if $u = u(t)$ is a full solution in $S_\lambda \setminus (M^1_\lambda \cup M^\infty_\lambda)$, then

$$\omega^*(u) \subset M^1_\lambda, \quad \omega(u) \subset M^\infty_\lambda. \tag{29}$$

Indeed, let u be such a solution. Then since $S_\lambda \subset \Xi_\rho$ and M^1_λ and M^∞_λ are the maximal compact invariant sets in $\Xi_\rho[0, R_0]$ and $\Xi_\rho[R_1, \infty]$, respectively, there exists $t_0 \in \mathbb{R}$ such that $u(t_0) \in \Xi_\rho[R_0, R_1]$. Hence (29) directly follows from (27).

Now we check that $M^\infty_\lambda \ne \emptyset$. Thus $\mathcal{M}$ is a Morse decomposition of S_λ. Suppose the contrary. Then by (29) we find that $S_\lambda = M^1_\lambda$. Hence

$$h(\Phi_\lambda, M^1_\lambda) = h(\Phi_\lambda, S_\lambda) = (\text{by } (19)) = \Sigma^p,$$

which contradicts (23).

To complete the proof of (1), there remains to check the existence of a connecting trajectory between M^1_λ and M^∞_λ. To this end, we consider the Morse equation of $\mathcal{M}$:

$$p(t, M^1_\lambda) + p(t, M^\infty_\lambda) = p(t, S_\lambda) + (1 + t)Q(t). \tag{30}$$

Recalling that $h(\Phi_\lambda, M^1_\lambda) = \Sigma^{p+r}$ and $h(\Phi_\lambda, S_\lambda) = \Sigma^p$, we have

$$p(t, M^1_\lambda) = t^{p+r}, \quad p(t, S_\lambda) = t^p.$$

Thus (30) reads

$$t^{p+r} + p(t, M^\infty_\lambda) = t^p + (1 + t)Q(t), \tag{31}$$

which implies that $Q(t) \ne 0$. By the basic knowledge in the Morse theory of invariant sets (see [32, Chap. III, Theorem 3.5]), one immediately concludes that there is at least one connecting trajectory between M^1_λ and M^∞_λ.

We also infer from (31) that $p(t, M_\lambda^\infty) \neq 0$. Consequently

$$h(\Phi_\lambda, M_\lambda^\infty) \neq \bar{0}, \qquad \forall\, \lambda \in \Lambda_-. \tag{32}$$

(iii) Clearly M_λ^1 remains uniformly bounded on Λ.

For any $R > R_1$, by Lemma 5.3 there exists $0 < \theta < \theta_1$ such that when $\lambda \in [\mu_k - \theta, \mu_k)$, the differential inequality (26) holds true for any solution $u(t)$ of (2) in $\Xi_\rho[R_0, 2R]$. Using this basic fact, it can be easily seen that if $\lambda \in [\mu_k - \theta, \mu_k)$, any bounded full solution in $\Xi_\rho[R_1, \infty]$ is necessarily contained in $\Xi_\rho[R, \infty]$. Hence

$$M_\lambda^\infty \subset \Xi_\rho[R, \infty], \tag{33}$$

which implies what we desired in (20) and completes the proof of (2).

(iv) Finally, let us verify the validity of (3).

Let $U = \Xi_\rho[R_1, \infty]$. Then $\partial U = C_1 \cup C_2$, where

$$C_1 = \{v : \|v^+\| = \rho,\ |w| \geq R_1\}, \quad C_2 = \{v : \|v^+\| \leq \rho,\ |w| = R_1\}.$$

Here $v^+ = P^+v$, and $w = P_W v$. Let $\lambda \in \Lambda_-$. By the choice of ρ and Lemma 5.2, we see that $M_\lambda^\infty \cap C_1 = \emptyset$. Fix an $R > R_1$. Then we infer from the above argument in (iii) that one can restrict $\delta > 0$ to be sufficiently small so that (33) holds. Consequently $M_\lambda^\infty \cap C_2 = \emptyset$. Thus

$$M_\lambda^\infty \cap \partial U = \emptyset,$$

namely, U is an isolating neighborhood of M_λ^∞.

Because $h\,(\Phi_{\mu_k - \delta}, U) \neq \bar{0}$ (by (32)), $\mathscr{K}^\infty$ has a connected component Γ with $\Gamma[\mu_k - \delta] \neq \emptyset$ such that one of the alternatives (1)-(3) in Theorem 3.2 holds true. As $\Gamma[\lambda] \subset \operatorname{int} U$ for all $\lambda \in \Lambda_-$, we conclude that either Γ is unbounded, or $\Gamma[\mu_k] \neq \emptyset$. Because $\Gamma[\lambda]$ is uniformly bounded on $[\mu_k - \delta, \mu_k - \varepsilon]$ for any $\varepsilon \in (0, \delta)$, in any case we deduce that $\Gamma[\lambda] \neq \emptyset$ for all $\lambda \in \Lambda_-$.

The argument for $\mathscr{K}^1$ is similar. We omit the details. ■

5.3. Bifurcation and multiplicity of stationary solutions. We now turn to the static bifurcation and multiplicity of stationary solutions of (2). Since the equation has a natural Lyapunov function $J(u)$ defined by

$$J(u) = \frac{1}{2}(\|u\|^2 - \lambda |u|^2) - \int_\Omega F(x, u)dx, \qquad u \in V,$$

where $F(x, s) = \int_0^s f(x, t)dt$, this problem can be treated in the framework of dynamical systems.

Theorem 5.3. *Assume the hypothesis* (**H**). *Let $\delta > 0$ be the same as in Theorem 5.2. Then*

(1) Φ_λ has at least one equilibrium e_λ for all $\lambda \in \mathbb{R}$;

(2) there exists $\delta > 0$ such that Φ_λ has at least two distinct equilibria e_λ^∞ and e_λ^c for each $\lambda \in \Lambda_- = [\mu_k - \delta, \mu_k)$, and

$$\lim_{\lambda \to \mu_k^-} \|e_\lambda^\infty\| = \infty, \tag{34}$$

whereas e_λ^c *remains bounded on* Λ_-*; and*

(3) there is an open dense subset $\mathcal{D}$ *of* $\mathbb{R}$ *such that for each* $\lambda \in \Lambda_- \cap \mathcal{D}$*,* Φ_λ *has at least three distinct equilibria.*

Proof. (1) Since each nonempty compact invariant set contains at least one stationary solution, the conclusion (1) directly follows from Theorem 5.2.

(2) Let N_1 be the isolating neighborhood of S_{μ_k} given in the proof of Theorem 5.2, and let $\mathcal{M} = \{M_\lambda^\infty, M_\lambda^1\}$ be the Morse decomposition of S_λ for $\lambda \in \Lambda_-$. Then $M_\lambda^1 \subset N_1$. By (25) we have

$$N_1 \cap \Xi_\rho = \widetilde{N}_1 \subset \Xi_\rho[0, R_1/2], \qquad \lambda \in \Lambda_-. \tag{35}$$

As $M_\lambda^\infty \subset \Xi_\rho[R_1, \infty]$, by (35) we find that $M_\lambda^\infty \cap N_1 = \emptyset$. Pick two stationary solutions e_λ and e_λ^∞ from M_λ^1 and M_λ^∞, respectively. Then e_λ and e_λ^∞ fulfill the requirements in (2).

(3) By slightly modifying the proof of [35, Theorem 2.1], it can be shown that there is an open dense subset $\mathcal{D}$ of $\mathbb{R}$ such that all the equilibria of Φ_λ are hyperbolic if $\lambda \in \mathcal{D}$. Now assume $\lambda \in \Lambda_- \cap \mathcal{D}$. We show that there is another equilibrium $z_\lambda^\infty \in M_\lambda^\infty$ with $z_\lambda^\infty \neq e_\lambda^\infty$. Consequently Φ_λ has at least three distinct equilibria.

We argue by contradiction and suppose M_λ^∞ consists of exactly one hyperbolic stationary solution e_λ^∞. Then $p(t, M_\lambda^\infty) = t^m$ for some $m \geq 0$. Accordingly the Morse equation (30) reads

$$t^{p+r} + t^m = t^p + (1+t)Q(t).$$

But this is impossible for any formal polynomial $Q(t)$ with coefficients in $\mathbb{Z}_+$, as the sum of the coefficients of the left-hand side does not equal that of the right-hand side.

The proof of the theorem is finished. ∎

Remark 5.1. *We infer from the above argument that for each* $\lambda \in \Lambda_- \cap \mathcal{D}$*,* Φ_λ *has at least two distinct equilibria outside the domain* N_1*.*

Finally, we pay some special attention to the particular case where

(F) $f(x,s) = o(|s|)$ as $|s| \to 0$ uniformly for $x \in \overline{\Omega}$.

We prove some new multiplicity results on stationary solutions for the equation (2) near each eigenvalue μ_k. The main results are summarized in the following theorem.

Theorem 5.4. *Assume* f *satisfies the hypotheses* (**H**) *and* (**F**)*. Denote* $W_{loc}^c(0)$ *the local center manifold of* Φ_{μ_k} *at the trivial equilibrium* 0*, and let* ϕ *be the restriction of* Φ_{μ_k} *on* $W_{loc}^c(0)$*. Suppose* 0 *is an isolated equilibrium of* Φ_{μ_k}*. Then there exists* $\delta > 0$ *such that one of the following assertions holds:*

(1) 0 *is an attractor of* ϕ*. In this case, the system* Φ_λ *has at least two distinct nontrivial equilibria* e_λ^c *and* e_λ^∞ *for* $\lambda \in \Lambda_- = [\mu_k - \delta, \mu_k)$*, whereas it has at least three distinct ones* e_λ^1*,* e_λ^2 *and* e_λ^c *for* $\lambda \in \Lambda_+ = (\mu_k, \mu_k + \delta]$*.*

(2) 0 *is a repeller of* ϕ *(i.e., an attractor of the inverse flow* ϕ^{-1}*). In this case,* Φ_λ *has at least three distinct nontrivial equilibria* e_λ^1*,* e_λ^2 *and* e_λ^∞ *for each* $\lambda \in \Lambda_-$*.*

(3) 0 is neither an attractor nor a repeller of ϕ. When this occurs, Φ_λ has at least three nontrivial equilibria e_λ^1, e_λ^c and e_λ^∞ for $\lambda \in \Lambda_-$, whereas it has at least two distinct ones e_λ^1 and e_λ^c for $\lambda \in \Lambda_+$.

Furthermore, we have

$$\lim_{\lambda\to\mu_k} \|e_\lambda^\infty\| = \infty, \qquad \lim_{\lambda\to\mu_k} \|e_\lambda^i\| = 0 \ \ (i = 1, 2), \tag{36}$$

and

$$0 < \liminf_{\lambda\to\mu_k} \|e_\lambda^c\| \le \limsup_{\lambda\to\mu_k} \|e_\lambda^c\| < \infty. \tag{37}$$

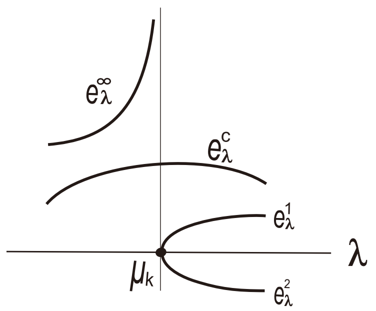

Fig. 5.1

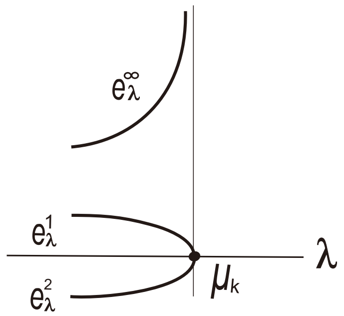

Fig. 5.2

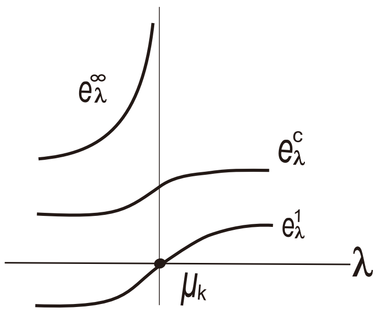

Fig. 5.3

Proof. In the following argument, we always assume that $\delta > 0$ is sufficiently small so that the conclusions in Theorem 5.2 and Theorem 5.3 are valid.

(1) The case where 0 is an attractor of ϕ.

Let N_1 be the isolating neighborhood of S_{μ_k} in the proof of Theorem 5.3. Then by Theorem 5.3, for each $\lambda \in \Lambda_-$ the system Φ_λ always has an equilibrium e_λ^∞ outside N_1 satisfying the first equation in (36).

Pick a number $\beta > 0$ and an isolating neighborhood N_0 of 0 with

$$N_0 \subset \mathrm{B}_V(\beta) \subset N_1.$$

We may restrict δ so that both N_0 and N_1 are isolating neighborhoods of Φ_λ for all $\lambda \in \Lambda := [\mu_k - \delta, \mu_k + \delta]$. Let

$$K_\lambda^i = K_\infty(\Phi_\lambda, N_i), \qquad i = 0, 1.$$

Then for each i,

$$h(\Phi_\lambda, K_\lambda^i) \equiv const., \qquad \lambda \in \Lambda. \tag{38}$$

It is trivial to check that

$$d_H\left(K_\lambda^0, \{0\}\right) \to 0 \ \text{ as } \ \lambda \to \mu_k. \tag{39}$$

It can be assumed that N_0 is sufficiently small so that the product formula of Conley index given in [32, Chap. II, Theorem 3.1] holds true, hence

$$h(\Phi_{\mu_k}, \{0\}) = \Sigma^p \wedge h(\phi, \{0\}).$$

We infer from Example 2.1 that $h(\phi, \{0\}) = \Sigma^0$. Therefore

$$h(\Phi_{\mu_k}, \{0\}) = \Sigma^p \wedge \Sigma^0 = \Sigma^p.$$

It then follows by (38) that

$$h(\Phi_\lambda, K^0_\lambda) = h(\Phi_{\mu_k}, K^0_{\mu_k}) = h(\Phi_{\mu_k}, \{0\}) = \Sigma^p, \qquad \lambda \in \Lambda. \tag{40}$$

By (38) and Proposition 5.1 we also deduce that

$$h(\Phi_\lambda, K^1_\lambda) = h(\Phi_{\mu_k}, K^1_{\mu_k}) = h(\Phi_{\mu_k}, S_{\mu_k}) = \Sigma^{p+r}, \qquad \lambda \in \Lambda. \tag{41}$$

Thus we see that $K^1_\lambda \neq K^0_\lambda$. As K^0_λ is the maximal invariant set in N_0, one concludes that

$$K^1_\lambda \setminus N_0 \neq \emptyset, \qquad \lambda \in \Lambda. \tag{42}$$

For each $\lambda \in \Lambda$, pick a $v_\lambda \in K^1_\lambda \setminus N_0$. Let $u_\lambda(t)$ be a bounded full trajectory of Φ_λ in K^1_λ with $u_\lambda(0) = v_\lambda$. We claim that if δ is small enough then

$$\text{either } \omega(u_\lambda) \setminus N_0 \neq \emptyset, \quad \text{or } \omega^*(u_\lambda) \setminus N_0 \neq \emptyset. \tag{43}$$

Indeed, if this was false, there would exist a sequence $\lambda_n \to \mu_k$ (as $n \to \infty$) such that both $\omega(u_n)$ and $\omega^*(u_n)$ are contained in N_0 and hence in $K^0_{\lambda_n}$, where $u_n = u_{\lambda_n}$. Thus by (39) we deduce that

$$\lim_{n\to\infty} \max_{v\in\omega(u_n)} |J(v)| = 0 = \lim_{n\to\infty} \max_{v\in\omega^*(u_n)} |J(v)|. \tag{44}$$

Set

$$\Gamma_n = \overline{\mathrm{orb}(u_n)} = \mathrm{orb}(u_n) \cup \omega(u_n) \cup \omega^*(u_n).$$

Then

$$\min_{v\in\Gamma_n} J(v) = \min_{v\in\omega(u_n)} J(v), \quad \max_{v\in\Gamma_n} J(v) = \max_{v\in\omega^*(u_n)} J(v).$$

It follows by (44) that

$$\max_{v\in\Gamma_n} |J(v)| \to 0 \quad \text{as } n \to \infty. \tag{45}$$

On the other hand, since $\Gamma_n \subset K^1_{\lambda_n} \subset N_1$ and Φ_λ is λ-l.u.a.c., it is easy to verify that $\bigcup_{\lambda_n\in\Lambda} \Gamma_n$ is precompact. Hence by Lemma 2.2 it can be assumed that Γ_n converges to a nonempty compact invariant set K of Φ_{μ_k} (in the sense of Hausdorff distance). Noticing that $\Gamma_n \cap K^0_{\lambda_n} \neq \emptyset$, by (39) we find that $0 \in K$. Because each Γ_n is connected, K is connected as well. (45) implies that $J(v) \equiv 0$ on K. Thereby each point in K is an equilibrium of Φ_{μ_k}. As

$$u_n(0) \in \Gamma_n \setminus N_0 \tag{46}$$

for all n, we deduce that $K \setminus \operatorname{int} N_0 \neq \emptyset$. Further by the connectedness of K one concludes that $K \cap \partial V \neq \emptyset$ for any small neighborhood V of 0, which leads to a contradiction and completes the proof of our claim.

In view of (43), for each $\lambda \in \Lambda$ we can pick an equilibrium e_λ^c of Φ_λ with

$$e_\lambda^c \in (\omega(u_\lambda) \cup \omega^*(u_\lambda)) \setminus N_0 \subset N_1 \setminus N_0. \tag{47}$$

Hence if $\lambda \in \Lambda_-$, the system Φ_λ has at least two distinct nontrivial equilibria e_λ^c and e_λ^∞.

We infer from the attractor bifurcation theory (see e.g. Ma and Wang [20, Theorem 4.3], [19, Theorem 6.1] or Li and Wang [18, Theorem 4.2]) that K_λ^0 contains at least two distinct equilibrium points e_λ^1 and e_λ^2 for $\lambda \in \Lambda_+$, provided δ is sufficiently small. By (47) one concludes that Φ_λ has at least three distinct nontrivial equilibria for $\lambda \in \Lambda_+$.

(2) 0 is a repeller of ϕ. In this case, as in (1), by applying the attractor bifurcation theory we deduce that K_λ^0 contains at least two distinct equilibria e_λ^1 and e_λ^2 for $\lambda \in \Lambda_-$. Since Φ_λ has a nontrivial equilibrium e_λ^∞ outside N_1 for each $\lambda \in \Lambda_-$, it has at least three distinct ones for $\lambda \in \Lambda_-$.

(3) Finally, let us consider the case where 0 is neither an attractor nor a repeller of ϕ. By Li and Wang [18, Theorem 4.4] we deduce that the system bifurcates at each side of μ_k a nonempty compact invariant set $M_\lambda \subset N_0$ with $0 \notin M_\lambda$ and

$$d_H(M_\lambda, \{0\}) \to 0 \quad \text{as} \quad \lambda \to \mu_k. \tag{48}$$

M_λ contains at least one nontrivial equilibrium e_λ^1.

We show that

$$h(\Phi_{\mu_k}, \{0\}) \neq \Sigma^{p+r}, \tag{49}$$

which fact will yield another equilibrium $e_\lambda^c \in N_1 \setminus N_0$ at both sides of μ_k.

Consider the local center-unstable manifold $W_{loc}^{cu}(0)$ of Φ_{μ_k} at 0. Denote ψ the restriction of Φ_{μ_k} on $W_{loc}^{cu}(0)$. Then

$$h(\Phi_{\mu_k}, \{0\}) = h(\psi, \{0\}). \tag{50}$$

Thus to prove (49), it suffices to check that

$$H_*(h(\psi, \{0\})) \neq H_*(\Sigma^{p+r}). \tag{51}$$

We argue by contradiction and suppose the contrary. Then

$$H_{p+r}(h(\psi, \{0\})) = H_{p+r}(\Sigma^{p+r}) = \mathbb{Z}.$$

Therefore by the Poincaré-Lefschetz duality theory of the Conley index (see McCord [24, Theorem 2.1] and Mrozek and Srzednicki [26, pp. 164]),

$$H^0(h(\psi^{-1}, \{0\})) = H_{p+r}(h(\psi, \{0\})) = \mathbb{Z}.$$

On the other hand, pick a pass-connected isolating block $B \subset W^{cu}_{loc}(0)$ of $S_0 = \{0\}$ with respect to the inverse flow ψ^{-1}. (Such an isolating block is always available due to [8, Theorem 1.5].) Since S_0 is not an attractor of ψ^{-1} (note that S_0 is not an attractor of ϕ^{-1} on $W^{c}_{loc}(0)$), we necessarily have

$$B^- \neq \emptyset.$$

Thus by the basic knowledge in the theory of algebraic topology, one easily deduces that $H^0(B, B^-) = 0$. Consequently

$$H^0(h(\psi^{-1}, \{0\})) = H^0(B, B^-) = 0,$$

which leads to a contradiction and justifies the validity of (49).

Recall that (see (41))

$$h(\Phi_\lambda, K^1_\lambda) = \Sigma^{p+r}, \qquad \lambda \in \Lambda = [\mu_k - \delta, \mu_k + \delta].$$

Noticing that

$$h(\Phi_\lambda, K^0_\lambda) = h(\Phi_{\mu_k}, K^0_{\mu_k}) = h(\Phi_{\mu_k}, \{0\}) \neq \Sigma^{p+r}, \qquad \forall \lambda \in \Lambda,$$

we conclude that $K^1_\lambda \neq K^0_\lambda$. As K^0_λ is the maximal invariant set in N_0, one finds that

$$K^1_\lambda \setminus N_0 \neq \emptyset, \qquad \lambda \in \Lambda.$$

We are now in a quite similar situation as in (42). Repeating the same argument below (42), it can be easily shown that the system has an equilibrium e^c_λ in $N_1 \setminus N_0$.

In conclusion, there are at least two distinct nontrivial equilibria in N_1 for $\lambda \in \Lambda \setminus \{\mu_k\}$. Because Φ_λ has an equilibrium e^∞_λ outside N_1 for $\lambda \in \Lambda_-$, the system has at least three distinct nontrivial equilibria as $\lambda \in \Lambda_-$. This completes the proof of (3).

The second equation in (36) follows from (39) and (48). (37) is a direct consequence of the choice that $e^c_\lambda \in N_1 \setminus N_0$. ■

Remark 5.2. *It is interesting to note that there is always a one-sided neighborhood Λ_1 of μ_k such that the equation has at least three distinct nontrivial stationary solutions for $\lambda \in \Lambda_1$.*

Remark 5.3. *Dual versions of all the results in this section hold true if, instead of (**H**), we assume that (6) in section 1 is fulfilled.*

Acknowledgements We would like to express our gratitude to the anonymous referees for their stimulating comments and suggestions, which helped us greatly improve the presentation of this paper.

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