# Self-normalized deviation inequalities with application to $t$-statistic

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**Abstract**

Let $(\xi_i)_{i\geq 1}$ be a sequence of independent and symmetric random variables. We obtain some upper bounds on tail probabilities of self-normalized deviations

$$\mathbf{P}\Big(\max_{1\leq k\leq n}\sum_{i=1}^{k}\xi_i\Big/(\sum_{i=1}^{n}|\xi_i|^{\beta})^{1/\beta}\geq x\Big)$$

for $x>0$ and $\beta>1$. Our bound is the best that can be obtained from the Bernstein inequality under the present assumption. An application to Student's $t$-statistic is also given.

*Keywords:* Self-normalized deviations; Student's $t$-statistic; exponential inequalities

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## 1. Introduction

Let $(\xi_i)_{i\geq 1}$ be a sequence of independent, centered and nondegenerate real-valued random variables (r.v.s). Denote by

$$S_n=\sum_{i=1}^{n}\xi_i \quad \text{and} \quad V_n(\beta)=\Big(\sum_{i=1}^{n}|\xi_i|^{\beta}\Big)^{1/\beta},\ \beta>1.$$

The study of the tail probabilities $\mathbf{P}(S_n/V_n(\beta)\geq x)$ certainly has attracted some particular attentions. In the case where r.v.s $(\xi_i)_{i\geq 1}$ are identically distributed and $\mathbf{E}|\xi_1|^{\beta}=\infty, \beta>1$, Shao [9] proved the following deep large deviation principle (LDP) result: for any $x>0$,

$$\lim_{n\to\infty}\mathbf{P}\Big(\frac{S_n}{V_n(\beta)\,n^{1-1/\beta}}\geq x\Big)^{1/n}=\sup_{c\geq 0}\inf_{t\geq 0}\mathbf{E}\Big[\exp\Big\{t\Big(cX-x\Big(\frac{1}{\beta}|X|^{\beta}+\frac{\beta-1}{\beta}c^{\beta/(\beta-1)}\Big)\Big)\Big\}\Big].$$

The related moderate deviation principles (MDP) are also given by Shao [9] and Jing, Liang and Zhou [6]. However, the LDP and MDP results do not diminish the need for tail probability inequalities valid for given $n$. Such inequalities have been obtained in particular by Wang and Jing [10]. For instance, they proved that if the r.v.s $(\xi_i)_{i\geq 1}$ are symmetric (around 0), then for all $x>0$,

$$\mathbf{P}\Big(\frac{S_n}{V_n(2)}\geq x\Big)\leq\exp\Big\{-\frac{x^2}{2}\Big\}. \tag{1}$$

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This bound is rather tight for moderate $x$'s. Indeed, as showed by the MDP result of Shao [9] (cf. Theorem 3.1), for certain class of r.v.s it holds that

$$x_n^{-2} \ln \mathbf{P}\left( \frac{S_n}{V_n(2)} \geq x_n \right) = -\frac{1}{2}, \tag{2}$$

for $x_n \to \infty$ and $x_n = o(\sqrt{n}\,)$ as $n \to \infty$. See also Theorem 2.1 of Jing, Liang and Zhou [6] for non identically distributed r.v.s. In Fan, Grama and Liu [3], inequality (1) has been further extended to the case of partial maximum: for all $x > 0$,

$$\mathbf{P}\left( \max_{1\leq k\leq n} \frac{S_k}{V_n(2)} \geq x \right) \leq \exp\left\{ -\frac{x^2}{2} \right\}. \tag{3}$$

On the other hand, by the Cauchy-Schwarz inequality, it is easy to see that $S_n^2 \leq n\,(V_n(2))^2$. Therefore, for all $x > \sqrt{n}$,

$$\mathbf{P}\left( \frac{S_n}{V_n(2)} \geq x \right) \leq \mathbf{P}\left( \max_{1\leq k\leq n} \frac{S_k}{V_n(2)} \geq x \right) = 0, \tag{4}$$

which cannot be deduced from (1) and (3). Hence, the inequalities (1) and (3) are not tight enough for large $x$'s.

In this paper we give an improvement on inequality (3); see inequality (5). Our inequality coincides with (4). More general, we establish an upper bound on tail probabilities $\mathbf{P}(\max_{1\leq k\leq n} S_k/V_n(\beta) \geq x), x > 0$, for symmetric r.v.s $(\xi_i)_{i\geq 1}$. In particular, we show that our inequality is the best that can be obtained from the classical Bernstein inequality: $\mathbf{P}(X > x) \leq \inf_{\lambda>0} \mathbf{E}[e^{\lambda(X-x)}]$. An application to Student's $t$-statistic is also given.

The paper is organized as follows. Our main results and applications are stated and discussed in Section 2. Proofs are deferred to Section 3.

## 2. Main results

In the following theorem, we give a self-normalized deviation inequality for independent and symmetric random variables.

**Theorem 2.1.** *Assume that $(\xi_i)_{i\geq 1}$ is a sequence of independent, symmetric and nondegenerate random variables. Denote by*

$$V_n(\beta) = \Big( \sum_{i=1}^n |\xi_i|^\beta \Big)^{1/\beta}, \qquad \beta \in (1, \infty).$$

*Then for all $x > 0$,*

$$\mathbf{P}\left( \max_{1\leq k\leq n} \frac{S_k}{V_n(\beta)} \geq x \right) \leq B_n(\beta, x) := \frac{1}{2^n} \left( \sqrt{t} + \frac{1}{\sqrt{t}} \right)^n t^{-\frac{1}{2} n^{1/\beta} x}\, \mathbf{1}_{\{x \leq n^{(\beta-1)/\beta}\}}, \tag{5}$$

*where*

$$t = \frac{n^{(\beta-1)/\beta} + x}{n^{(\beta-1)/\beta} - x}$$

*with the convention that* $B_n(\beta, n^{(\beta-1)/\beta}) = 2^{-n}$. *Moreover,* $B_n(2,x)$ *is increasing in* $n$ *and for any* $x > 0$,

$$\lim_{n\to\infty} B_n(2,x) = \exp\left\{-\frac{x^2}{2}\right\}. \tag{6}$$

Hölder's inequality implies that $S_n \le V_n(\beta) n^{(\beta-1)/\beta}$. Thus when $x > n^{(\beta-1)/\beta}$, it holds that

$$\mathbf{P}\Big(\max_{1\le k\le n} \frac{S_k}{V_n(\beta)} \ge x\Big) = 0. \tag{7}$$

This feature coincides with the fact that $B_n(\beta, x) = 0$ for all $x > n^{(\beta-1)/\beta}$.

Notice that bound (5) is the best that can be obtained from the following Bernstein inequality

$$\mathbf{P}\left(\frac{S_n}{V_n(\beta)} \ge x\right) \le \inf_{\lambda\ge 0} \mathbf{E}\Big[e^{\lambda\left(\frac{S_n}{V_n(\beta)} - x\right)}\Big]. \tag{8}$$

Indeed, if $\xi_i = \pm a, a > 0$, with probabilities $1/2$, then it holds for all $0 < x < \sqrt{n}$,

$$\inf_{\lambda\ge 0} \mathbf{E}\Big[e^{\lambda\left(\frac{S_n}{V_n(\beta)} - x\right)}\Big] = \inf_{\lambda\ge 0} \mathbf{E}\Big[e^{\lambda\left(\frac{S_n}{an^{1/\beta}} - x\right)}\Big] = \inf_{\lambda\ge 0} e^{-\lambda x}\Big(\cosh(\frac{\lambda}{n^{1/\beta}})\Big)^n = B_n(\beta, x).$$

Moreover, when $x \nearrow n^{(\beta-1)/\beta}$, bound (5) tends to $2^{-n}$, which is the best possible at $x = n^{(\beta-1)/\beta}$. Indeed, for the $\xi_i$'s mentioned above, it holds

$$\mathbf{P}\Big(\max_{1\le k\le n} \frac{S_k}{V_n(\beta)} \ge n^{(\beta-1)/\beta}\Big) = \mathbf{P}\Big(\xi_i = a \text{ for all } i \in [1,n]\Big) = \frac{1}{2^n}.$$

By (6), it holds

$$\lim_{n\to\infty} B_n(2,x) = e^{-x^2/2} \to e^{-n/2}, \qquad x \to n^{1/2}, \tag{9}$$

which seems to be contracted with $B_n(2,x) \to 2^{-n}$ as $x \to n^{1/2}$. In fact, (6) holds for any fixed $x > 0$, but it doesn't hold for $x = c\,n^{1/2}$, where $c$ is a positive absolute constant. For example, consider the following function

$$B_n(2,x) = \exp\Big\{-\frac{x^2}{2} - (\ln 2 - \frac{1}{2})\frac{x^4}{n}\Big\}, \quad x > 0.$$

Then the last function satisfies (9) and $\lim_{x\to n^{1/2}} B_n(2,x) = 2^{-n}$.

Since the r.v.s $(\xi_i)_{i\ge 1}$ are symmetric, it is obvious that for all $x > 0$,

$$\mathbf{P}\left(\max_{1\le k\le n} \frac{S_k}{V_n(\beta)} \le -x\right) \le B_n(\beta, x),$$

where $B_n(\beta, x)$ is defined by (5).

When $\beta \in (1,2]$, inequality (5) implies the following bound.

**Corollary 2.1.** *Assume condition of Theorem 2.1. If $\beta \in (1,2]$, then for all $x > 0$,*

$$\mathbf{P}\Big( \max_{1\le k\le n} \frac{S_k}{V_n(\beta)} \ge x \Big) \le \exp\left\{ -\frac{x^2}{2} n^{\frac{2}{\beta}-1} \right\}. \tag{10}$$

*In particular, the last inequality implies that for any $\beta \in (1,2)$,*

$$\frac{S_n}{V_n(\beta)} \to 0, \qquad n \to \infty,$$

*in probability.*

For $\beta \in (1,2]$, inequality (10) implies the following upper bound of LDP:

$$\limsup_{n\to\infty} \frac{1}{n} \ln \mathbf{P}\Big( \max_{1\le k\le n} \frac{S_k}{V_n(\beta) n^{(\beta-1)/\beta}} \ge x \Big) \le -\frac{x^2}{2}, \qquad x \in (0,1]. \tag{11}$$

It also implies the following upper bound of MDP: for any $\alpha \in (\frac{\beta-2}{2\beta}, \frac{\beta-1}{\beta})$,

$$\limsup_{n\to\infty} \frac{1}{n^{2\alpha+\frac{2}{\beta}-1}} \ln \mathbf{P}\left( \max_{1\le k\le n} \frac{S_k}{V_n(\beta) n^{\alpha}} \ge x \right) \le -\frac{x^2}{2}, \qquad x \in (0,\infty). \tag{12}$$

With certain regularity conditions on tail probabilities of $\xi_i$, the LDP and MDP results are allowed to be established. We refer to Shao [9] and Jing, Liang and Zhou [6].

Wang and Jing [10] proved that for all $x > 0$,

$$\mathbf{P}\left( \frac{S_n}{V_n(2)} \ge x \right) \le \exp\left\{ -\frac{x^2}{2} \right\}. \tag{13}$$

An earlier result similar to (13) can be found in [5], where Hitczenko has obtained the same upper bound on tail probabilities $\mathbf{P}(S_n \ge x\,||V_n(2)||_\infty)$. When $\beta = 2$, inequality (10) reduces to the following inequality of Fan *et al.* [3]: for all $x > 0$,

$$\mathbf{P}\Big( \max_{1\le k\le n} \frac{S_k}{V_n(2)} \ge x \Big) \le \exp\left\{ -\frac{x^2}{2} \right\}. \tag{14}$$

Thus inequality (10) can be regarded as a generalization of (13) and (14). Moreover, since bound (5) is less than bound (14), our inequality (5) improves on (14).

If $(\xi_i)_{i\ge1}$ have $(2+\delta)$th moments with $0 < \delta \le 1$, the inequalities (13) and (14) can be further improved. For instance, Jing, Shao and Wang [7] proved the following Cramér type large deviations for i.i.d. (not necessarily symmetric) r.v.s:

$$\mathbf{P}\Big( S_n \ge x V_n(2) \Big) = \Big( 1 - \Phi(x) \Big) \Big( 1 + o(1) \Big), \quad n \to \infty, \tag{15}$$

uniformly for all $0 \le x = o(n^{\delta/(4+2\delta)})$. Similarly, Liu, Shao and Wang [8] proved the following result for the maximum of sums:

$$\mathbf{P}\Big( \max_{1\le k\le n} S_k \ge x V_n(2) \Big) = 2\Big( 1 - \Phi(x) \Big) \Big( 1 + o(1) \Big), \quad n \to \infty, \tag{16}$$

uniformly for all $0 \le x = o(n^{\delta/(4+2\delta)})$. Moreover, these asymptotic estimations are also more precise than (5) for moderate $x$'s.

Let $(Y_i)_{i\ge1}$ be a sequence of independent nondegenerate r.v.s, and $(d_i)_{i\ge1}$ be a sequence of independent Rademacher r.v.s, i.e. $\mathbf{P}(d_i = \pm1) = \frac{1}{2}$. Let $\xi_i = d_iY_i$. Assume that $(Y_i)_{i\ge1}$ and $(d_i)_{i\ge1}$ are independent. Then we now have

$$S_n = \sum_{i=1}^n d_iY_i, \qquad V_n(\beta) = \Big(\sum_{i=1}^n |Y_i|^\beta\Big)^{1/\beta}, \qquad \beta > 1.$$

The following result easily follows from Theorem 2.1 and Corollary 2.1.

**Corollary 2.2.** *Let $\xi_i = d_iY_i$ for all $i \ge 1$. If $\beta \in (1, 2]$, then for all $0 < x \le n^{(\beta-1)/\beta}$,*

$$\begin{aligned}\mathbf{P}\left(\max_{1\le k\le n}\frac{S_k}{V_n(\beta)} \ge x\right) &\le B_n(\beta, x)\\ &\le \exp\left\{-\frac{x^2}{2}n^{\frac{2}{\beta}-1}\right\}.\end{aligned}$$

*In particular, the last inequality implies that for any $\beta \in (1, 2)$,*

$$\frac{S_n}{V_n(\beta)} \to 0, \qquad n \to \infty,$$

*in probability.*

Consider Student's $t$-statistic $T_n$ defined by

$$T_n = \sqrt{n}\,\overline{\xi}_n/\widehat{\sigma},$$

where

$$\overline{\xi}_n = \frac{S_n}{n} \quad \text{and} \quad \widehat{\sigma}^2 = \sum_{i=1}^n \frac{(\xi_i - \overline{\xi}_n)^2}{n-1}.$$

It is known that for all $x > 0$,

$$\mathbf{P}\Big(T_n \ge x\Big) = \mathbf{P}\Big(\frac{S_n}{\sqrt{[S]_n}} \ge x\Big(\frac{n}{n+x^2-1}\Big)^{1/2}\Big);$$

see Efron [1]. Notice that for all $x > 0$, it holds that $0 < x\Big(\frac{n}{n+x^2-1}\Big)^{1/2} < n^{1/2}$. Using inequality (5), we have the following exponential bound for Student's $t$-statistic.

**Theorem 2.2.** *Assume that $(\xi_i)_{i\ge1}$ is a sequence of independent, symmetric and nondegenerate random variables. Then for all $x > 0$,*

$$\mathbf{P}\Big(T_n \ge x\Big) \;\le\; B_n\Big(2,\; x\Big(\frac{n}{n+x^2-1}\Big)^{1/2}\Big), \tag{17}$$

*where $B_n(2, x)$ is defined by (5).*

## 3. Proofs of Theorem 2.1 and Corollary 2.1

The proof of Theorem 2.1 is based on a method called change of probability measure for martingales. The method is developed by Grama and Haeusler [4]. See also Fan, Grama and Liu [2].

*Proof of Theorem 2.1.* For any $i \geq 1$, set

$$\eta_i = \frac{\xi_i}{V_n(\beta)}, \quad \mathcal{F}_0 = \sigma\Big(|\xi_j|, 1 \leq j \leq n\Big) \text{ and } \mathcal{F}_i = \sigma\Big(\xi_k, 1 \leq k \leq i, \ |\xi_j|, 1 \leq j \leq n\Big). \tag{18}$$

Since $(\xi_i)_{i\geq 1}$ are independent and symmetric, then

$$\mathbf{E}[\xi_i > y \,|\, \mathcal{F}_{i-1}] = \mathbf{E}\Big[\xi_i > y \,\Big|\, |\xi_i|\Big] = \mathbf{E}\Big[ -\xi_i > y \,\Big|\, |-\xi_i|\Big] = \mathbf{E}[-\xi_i > y \,|\, \mathcal{F}_{i-1}].$$

Thus $(\eta_i, \mathcal{F}_i)_{i=1,\ldots,n}$ is a sequence of conditionally symmetric martingale differences, i.e. $\mathbf{E}[\eta_i > y|\, \mathcal{F}_{i-1}] = \mathbf{E}[-\eta_i > y|\, \mathcal{F}_{i-1}]$. It is easy to see that

$$\frac{S_n}{\mathrm{V}_n(\beta)} = \sum_{i=1}^n \eta_i \tag{19}$$

is a sum of martingale differences, and that $(\eta_i, \mathcal{F}_i)_{i=1,\ldots,n}$ satisfies

$$\sum_{i=1}^n |\eta_i|^\beta \ = \ \sum_{i=1}^n \frac{|\xi_i|^\beta}{V_n(\beta)^\beta} \ = \ 1.$$

For any $x > 0$, define the stopping time $T$:

$$T(x) = \min\Big\{k \in [1, n] : \sum_{i=1}^k \eta_i \geq x\Big\},$$

with the convention that $\min \emptyset = 0$. Then it follows that

$$\mathbf{1}_{\left\{ \max_{1\leq k\leq n} S_k/V_n(\beta) \geq x\right\}} = \sum_{k=1}^n \mathbf{1}_{\{T(x)=k\}}.$$

For any nonnegative number $\lambda$, define the martingale $M(\lambda) = (M_k(\lambda), \mathcal{F}_k)_{k=0,\ldots,n}$, where

$$M_k(\lambda) = \prod_{i=1}^k \frac{\exp\{\lambda\eta_i\}}{\mathbf{E}\left[\exp\{\lambda\eta_i\}\,|\mathcal{F}_{i-1}\right]}, \qquad M_0(\lambda) = 1.$$

Since $T$ is a stopping time, then $M_{T\wedge k}(\lambda)$, $\lambda > 0$, is also a martingale. Define the conjugate probability measure $\mathbf{P}_\lambda$ on $(\Omega, \mathcal{F})$:

$$d\mathbf{P}_\lambda = M_{T\wedge n}(\lambda) d\mathbf{P}. \tag{20}$$

Denote by $\mathbf{E}_\lambda$ the expectation with respect to $\mathbf{P}_\lambda$. Using the change of probability measure (20), we have for all $x > 0$,

$$\begin{aligned}
\mathbf{P}\Big(\max_{1\leq k\leq n} \frac{S_k}{V_n(\beta)} \geq x\Big) \ &= \ \mathbf{E}_\lambda\Big[M_{T\wedge n}(\lambda)^{-1}\mathbf{1}_{\left\{ \max_{1\leq k\leq n} S_k/V_n(\beta) \geq x\right\}}\Big] \\
&= \ \sum_{k=1}^n \mathbf{E}_\lambda\Big[\exp\Big\{ -\lambda\sum_{i=1}^k \eta_i + \Psi_k(\lambda)\Big\}\mathbf{1}_{\{T(x)=k\}}\Big],
\end{aligned} \tag{21}$$

where

$$\Psi_k(\lambda) = \sum_{i=1}^{k} \log \mathbf{E}\left[\exp\{\lambda\eta_i\}\,\Big|\mathcal{F}_{i-1}\right].$$

Since $(\eta_i, \mathcal{F}_i)_{i=1,\dots,n}$ is conditionally symmetric, one has

$$\mathbf{E}\left[\exp\{\lambda\eta_i\}\,|\mathcal{F}_{i-1}\right] = \mathbf{E}\left[\exp\{-\lambda\eta_i\}\,|\mathcal{F}_{i-1}\right],$$

and thus it holds

$$\mathbf{E}\left[\exp\{\lambda\eta_i\}\,|\mathcal{F}_{i-1}\right] = \mathbf{E}\left[\cosh(\lambda\eta_i)|\mathcal{F}_{i-1}\right]. \tag{22}$$

Since

$$\cosh(x) = \sum_{k=0}^{\infty} \frac{1}{(2k)!} x^{2k}$$

is an even function, then $\cosh(\lambda\eta_i)$ is $\mathcal{F}_{i-1}-$measurable. Thus (22) implies that

$$\mathbf{E}[\exp\{\lambda\eta_i\}\,|\mathcal{F}_{i-1}] = \cosh(\lambda\eta_i).$$

Notice that the function $g(x) = \log(\cosh(x))$ is even and convex in $x \in \mathbf{R}$ and increasing in $x \in [0,\infty)$. Since $|\sum_{i=1}^n \eta_i| \le n^{1-1/\beta}(\sum_{i=1}^n |\eta_i|^\beta)^{1/\beta} = n^{1-1/\beta}$, it holds

$$\Psi_k(\lambda) \le \Psi_n(\lambda) = \sum_{i=1}^{n} g\Big(\lambda\eta_i\Big) \le ng\Big(\frac{1}{n}\sum_{i=1}^{n}\lambda\eta_i\Big) \le ng\Big(\frac{\lambda}{n^{1/\beta}}\Big).$$

By the fact $\sum_{i=1}^k \eta_i \ge x$ on the set $\{T(x) = k\}$, inequality (21) implies that for all $x > 0$,

$$\begin{aligned}
\mathbf{P}\Big(\max_{1\le k\le n}\frac{S_k}{V_n(\beta)} \ge x\Big) &\le \sum_{k=1}^{n} \mathbf{E}_\lambda\left[\exp\left\{-\lambda x + ng\Big(\frac{\lambda}{n^{1/\beta}}\Big)\right\}\mathbf{1}_{\{T=k\}}\right] \\
&\le \exp\left\{-\lambda x + ng\Big(\frac{\lambda}{n^{1/\beta}}\Big)\right\}\mathbf{E}_\lambda\left[\sum_{k=1}^{n}\mathbf{1}_{\{T=k\}}\right] \\
&\le \exp\left\{-\lambda x + ng\Big(\frac{\lambda}{n^{1/\beta}}\Big)\right\}. \qquad (23)
\end{aligned}$$

The last inequality attains its minimum at

$$\lambda = \lambda(x) = \frac{n^{1/\beta}}{2}\log\left(\frac{n^{(\beta-1)/\beta}+x}{n^{(\beta-1)/\beta}-x}\right), \qquad x \in (0, n^{(\beta-1)/\beta}).$$

Substituting $\lambda = \lambda(x)$ in (23), we obtain the desired inequality (5) for all $x \in (0, n^{(\beta-1)/\beta})$. When $x = n^{(\beta-1)/\beta}$, we have

$$\mathbf{P}\Big(\max_{1\le k\le n}\frac{S_k}{V_n(\beta)} \ge n^{(\beta-1)/\beta}\Big) = \lim_{x\nearrow n^{(\beta-1)/\beta}} \mathbf{P}\Big(\max_{1\le k\le n}\frac{S_k}{V_n(\beta)} \ge x\Big) \le \lim_{x\nearrow n^{(\beta-1)/\beta}} B_n(\beta, x) = 2^{-n}.$$

When $x > n^{(\beta-1)/\beta}$, the desired inequality follows from (7).

Notice that the function $h(x) = g(\sqrt{x})$ is convex and increasing in $x \in [0, \infty)$. Therefor $g(\sqrt{x})/x$ is increasing in $x$, and $g(\sqrt{\lambda^2/n})/(\lambda^2/n)$ is decreasing in $n$. Thus

$$B_n(2, x) = \inf_{\lambda \geq 0} \exp\left\{-\lambda x + ng\Big(\frac{\lambda}{n^{1/2}}\Big)\right\}$$

is increasing in $n$. Since $ng(\frac{\lambda}{n^{1/2}}) \to \lambda^2/2, n \to \infty$, we obtain

$$\lim_{n\to\infty} B_n(2, x) = \sup_n B_n(2, x) = \inf_{\lambda \geq 0} \exp\left\{-\lambda x + \frac{\lambda^2}{2}\right\} = \exp\left\{-\frac{x^2}{2}\right\}.$$

This completes the proof of Theorem 2.1. □

*Proof of Corollary 2.1.* Since $\cosh(x) \leq \exp\{x^2/2\}$, we have

$$ng\Big(\frac{\lambda}{n^{1/\beta}}\Big) \;\leq\; \frac{\lambda^2}{2} n^{1-\frac{2}{\beta}} \tag{24}$$

for all $\lambda > 0$. Thus, from (23), for all $x > 0$,

$$\begin{aligned} \mathbf{P}\Big(\max_{1\leq k\leq n} \frac{S_k}{V_n(\beta)} \geq x\Big) &\leq \inf_{\lambda>0} \exp\left\{-\lambda x + ng\Big(\frac{\lambda}{n^{1/\beta}}\Big)\right\} \\ &\leq \inf_{\lambda>0} \exp\left\{-\lambda x + \frac{\lambda^2}{2} n^{1-\frac{2}{\beta}}\right\} \\ &= \exp\left\{-\frac{x^2}{2}\, n^{\frac{2}{\beta}-1}\right\}, \end{aligned} \tag{25}$$

which gives the desired inequality (10). □

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