



Frequency-domain criterion on the stabilizability for infinite-dimensional linear control systems[☆]



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ABSTRACT

A quantitative frequency-domain condition related to the exponential stabilizability for infinite-dimensional linear control systems is presented. It is proven that this condition is necessary and sufficient for the stabilizability of special systems, while it is a necessary condition for the stabilizability in general. Applications are provided.

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R É S U M É

Une condition quantitative dans le domaine fréquentiel liée à la stabilisabilité exponentielle des systèmes de contrôle linéaires en dimension infinie est présentée. Nous prouvons que cette condition est nécessaire et suffisante pour la stabilisabilité de quelques systèmes, tandis qu'elle constitue une condition nécessaire pour la stabilisabilité en général. Plusieurs applications sont fournies.

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1. Introduction

Stabilization for linear control systems is one of the most important directions of control theory. How to determine whether a linear control system is stabilizable is one of the largest concerns in this direction. Over the past half-century, researchers have obtained many useful results on this issue (see, for instance, [1–15] and the references therein). These works have laid a solid foundation for the study of the stabilization of linear control systems. For finite-dimensional linear control systems, there is a well-known frequency-domain

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criterion to determine the stabilizability, which is the Hautus test condition (see [16]). Unfortunately, this criterion may not be valid in infinite-dimensional settings. For infinite-dimensional linear control systems, researchers have been trying to obtain the corresponding frequency-domain criteria for stabilization. This paper intends to provide a frequency-domain condition that is a necessary and sufficient condition for the stabilizability for special linear control systems, while it is a necessary condition in general. We begin by introducing the frequently used notation in this paper.

1.1. Notation

Let $\mathbb{N}^+ := \mathbb{N} \setminus \{0\}$, $\mathbb{R}^+ := (0, +\infty)$ and $\mathbb{R}^- := (-\infty, 0)$. If $\gamma \in \mathbb{R}$, we write $\mathbb{C}_\gamma^+ := \{z \in \mathbb{C} : \operatorname{Re} z > \gamma\}$ and $\mathbb{C}_\gamma^- := \{z \in \mathbb{C} : \operatorname{Re} z < \gamma\}$. Let i be the unit imaginary number. If S is a subset of \mathbb{C} , we denote its closure by \overline{S} . If $T > 0$, we let $[T] := \max\{n \in \mathbb{N} : n \leq T\}$. If X is a Banach space, we denote its norm and dual space by $\|\cdot\|_X$ and X^* , respectively. If X is a Hilbert space, we use $\langle \cdot, \cdot \rangle_X$ to denote its inner product. For Banach spaces X_1 and X_2 , $\mathcal{L}(X_1; X_2)$ denotes the space of all bounded linear operators from X_1 to X_2 . We write $\mathcal{L}(X_1) := \mathcal{L}(X_1; X_1)$ if $X_1 = X_2$. Given an unbounded (or bounded) linear operator L from X_1 to X_2 , its domain, kernel, adjoint operator, resolvent set, and spectrum are $D(L) := \{f \in X_1 : Lf \in X_2\}$, $\operatorname{Ker}(L) := \{f \in D(L) : Lf = 0\}$, L^* , $\rho(L)$ and $\sigma(L)$, respectively. Given two sets Λ_1 and Λ_2 in X , we let $\operatorname{Span}\{\Lambda_1, \Lambda_2\}$ be the space spanned by the elements of Λ_1 and Λ_2 . We use $C(\cdots)$ or $D(\cdots)$ to denote constants that depend on what is enclosed in the brackets.

1.2. Control problem

Let H and U be two separable and complex Hilbert spaces. We consider the control system $[A, B]$, i.e.,

$$y'(t) = Ay(t) + Bu(t), \quad t \in \mathbb{R}^+, \quad (1.1)$$

(where $u \in L^2(\mathbb{R}^+; U)$) under the following assumptions:

- (A₁) Operator A with its domain $D(A)$ generates a C_0 -semigroup $S(\cdot)$ on H .
- (A₂) Operator B belongs to $\mathcal{L}(U; H_{-1})$, where H_{-1} is the completion of H with respect to the norm $\|f\|_{-1} := \|(\rho_0 I - A)^{-1} f\|_H$ ($f \in H$), and $\rho_0 \in \rho(A) \cap \mathbb{R}^+$ is arbitrarily fixed.
- (A₃) For each $T > 0$, there is $C(T) > 0$ such that

$$\int_0^T \|B^* S^*(t) \varphi\|_U^2 dt \leq C(T) \|\varphi\|_H^2 \quad \text{for } \varphi \in D(A^*).$$

Remark 1. There are several remarks on the above assumptions as follows:

- (i) In this paper, we write H_1 for the space $D(A^*)$ with the norm: $\|(\rho_0 I - A^*)\varphi\|_H$, $\varphi \in D(A^*)$. (The space H_{-1} is the dual space of H_1 with respect to the pivot space H , see [17, Section 2.9, Chapter 2]); \tilde{A} is the unique extension of A in $\mathcal{L}(H; H_{-1})$, which is provided in the following manner (see [17, Proposition 2.10.3, Chapter 2]):

$$\langle \tilde{A}\varphi, \psi \rangle_{H_{-1}, H_1} = \langle \varphi, A^*\psi \rangle_H \quad \text{for } \varphi \in H, \quad \psi \in H_1. \quad (1.2)$$

We let $\tilde{S}(\cdot) := (\rho_0 I - \tilde{A})S(\cdot)(\rho_0 I - \tilde{A})^{-1}$, which is the C_0 -semigroup on H_{-1} generated by \tilde{A} (with its domain H) and an extension of $S(\cdot)$ (see [17, Proposition 2.10.4, Chapter 2]).

Moreover, by assumption (A₂), we have $B^* \in \mathcal{L}(H_1; U)$ and $B^*(\rho_0 I - A^*)^{-1} \in \mathcal{L}(H; U)$.

- (ii) By assumption (A_3) , we have that for any $u \in L^2(\mathbb{R}^+; U)$ and $y_0 \in H$, system (1.1), which corresponds to this u and with the initial condition $y(0) = y_0$, has a unique solution in $C(\mathbb{R}^+; H)$, and this solution can be expressed by $y(t; y_0, u) = \tilde{S}(t)y_0 + \int_0^t \tilde{S}(t-s)Bu(s)ds$, $t \in \mathbb{R}^+$. (See [17, Propositions 4.2.2, 4.2.5, Chapter 4].)

We are going to define the exponential/rapid stabilizability in H for system $[A, B]$, and ‘in H ’ will be omitted in what follows.

Definition 1. System $[A, B]$ is said to be exponentially stabilizable (or ‘stabilizable’ for short) if there is a constant $\alpha > 0$, a C_0 -semigroup $\mathcal{S}_\dagger(\cdot)$ on H (with its generator $\mathcal{A}_\dagger : D(\mathcal{A}_\dagger) \subset H \rightarrow H$), and an operator $K_\dagger \in \mathcal{L}(D(\mathcal{A}_\dagger); U)$ such that

- (a) there is a constant $C_1 > 0$ such that $\|\mathcal{S}_\dagger(t)\|_{\mathcal{L}(H)} \leq C_1 e^{-\alpha t}$ for all $t \in \mathbb{R}^+$;
- (b) for each $x \in D(\mathcal{A}_\dagger)$, $\mathcal{A}_\dagger x = \tilde{A}x + BK_\dagger x$, with \tilde{A} provided by (1.2);
- (c) there is a constant $C_2 > 0$ such that $\|K_\dagger \mathcal{S}_\dagger(\cdot)x\|_{L^2(\mathbb{R}^+; U)} \leq C_2 \|x\|_H$ for each $x \in D(\mathcal{A}_\dagger)$.

K_\dagger and α are called the feedback law and a stabilizable decay rate (or ‘a decay rate’ for short), respectively. If α , $\mathcal{S}_\dagger(\cdot)$, and K_\dagger exist, system $[A, B]$ is also said to be stabilizable with decay rate α .

Definition 2. System $[A, B]$ is rapidly (or completely) stabilizable if for each $\alpha > 0$, $[A, B]$ is stabilizable with the decay rate α .

Remark 2. Several notes on Definition 1 are provided as follows:

- (i) Definition 1 is originally from [18], where the authors proved that the solvability of the LQ problem: $V(y_0) = \inf_{u \in L^2(\mathbb{R}^+; U)} \int_0^{+\infty} [\|y(t; u, y_0)\|_H^2 + \|u(t)\|_U^2] dt$ (i.e., $V(y_0) < +\infty$ for all $y_0 \in H$) implies the stabilizability of system $[A, B]$ in the sense of Definition 1. The reverse was proven in [8, Proposition 3.9]. Hence, the solvability of the above LQ problem is equivalent to the stabilizability of system $[A, B]$ in the sense of Definition 1.
- (ii) If $B \in \mathcal{L}(U; H)$, the stabilizability of system $[A, B]$ is defined as follows: there is a $K \in \mathcal{L}(H; U)$ such that $e^{(A+BK)t}$ is exponentially stable. Using the weak observability inequality in [8, 14], one can easily show that Definition 1 is an extension of the above definition if $B \in \mathcal{L}(U; H)$.
- (iii) If $[A, B]$ is stabilizable in the sense of Definition 1, the feedback law can be constructed by the usual LQ theory.

1.3. Motivation and novelty

Motivation. The stabilizability of system $[A, B]$ is equivalent to the weak observability inequality. This equivalence can be summarized as the following lemma:

Lemma 1.

- (i) The following statements are equivalent:
 - (a) System $[A, B]$ is stabilizable.
 - (b) There are constants $\alpha > 0$ and $C(\alpha) > 0$ such that

$$\|S^*(t)\varphi\|_H^2 \leq C(\alpha) \left(\int_0^t \|B^*S^*(s)\varphi\|_U^2 ds + e^{-\alpha t} \|\varphi\|_H^2 \right) \quad \text{for } t \in \mathbb{R}^+, \varphi \in H_1. \quad (1.3)$$

(c) There are constants $T > 0$, $\delta \in (0, 1)$ and $C > 0$ such that

$$\|S^*(T)\varphi\|_H^2 \leq C \int_0^T \|B^*S^*(s)\varphi\|_U^2 ds + \delta \|\varphi\|_H^2 \text{ for } \varphi \in H_1. \quad (1.4)$$

(ii) System $[A, B]$ is rapidly stabilizable if and only if for each $\alpha > 0$, there is $C(\alpha) > 0$ that satisfies (1.3).

Remark 3. The equivalence of (a) and (c) above was originally obtained in [14, Theorem 1] for the case that B is bounded, and then extended to the case where B is admissible. The equivalence between (b) and (c) above was proved in [11, Proposition 3]. The conclusion (ii) of Lemma 1 was obtained in [8, Theorem 1.1, Theorem 3.4].

Inequality (1.3) (or (1.4)) can be considered as a time-domain criterion for the stabilizability of system $[A, B]$. Naturally, we would expect frequency-domain criteria for system $[A, B]$. For finite-dimensional settings, the following criterion on the stabilizability is well known:

Lemma 2. ([16, Theorems 3, 4]) Let $H := \mathbb{C}^n$, $U := \mathbb{C}^m$, $A \in \mathbb{C}^{n \times n}$ and $B \in \mathbb{C}^{n \times m}$ ($n, m \in \mathbb{N}^+$). Then, system $[A, B]$ is stabilizable if and only if the pair (A^*, B^*) satisfies the condition

$$(HSF): \text{Ker}(\lambda I - A^*, B^*)^\top = \{0\} \text{ for } \lambda \in \sigma(A^*) \cap \overline{\mathbb{C}_0^+}.$$

The time-domain criterion and frequency-domain criterion have their own merits. Unfortunately, for general infinite-dimensional linear control systems, the equivalence in Lemma 2 may not be true. Counterexamples can be founded in [19] (see also [20, 3.4 in Section 3, Chapter 4]). Thus, it is natural to ask for frequency-domain criteria/conditions on the stabilization for system $[A, B]$ in our setting.

Novelty. We provide a quantitative frequency-domain condition that can be considered as an extension of condition (HSF) in Lemma 2. We prove that this condition is a criterion on the stabilization for system $[A, B]$ under additional conditions beyond (A_1) – (A_3) , while it is a necessary condition for the stabilization in general. Our method of proving these results uses the weak observability inequality in Lemma 1. Combining time domain with frequency domain concepts appears to be novel for infinite dimensional stabilization problems.

1.4. Main results

To state our main results, we need the following definition:

Definition 3. (i) The pair (A^*, B^*) satisfies $(\text{HESI})_\beta$ for $\beta > 0$ if there exists $C(\beta) > 0$ such that

$$\|\varphi\|_H^2 \leq \frac{C(\beta)}{(\text{Re } \lambda + \beta)^2} (\|(\lambda I - A^*)\varphi\|_H^2 + \|B^*\varphi\|_U^2) \text{ for } \lambda \in \mathbb{C}_{-\beta}^+, \varphi \in H_1, \quad (1.5)$$

or equivalently, there exists $C(\beta) > 0$ such that

$$\|\varphi\|_H^2 \leq C(\beta) (\|(\lambda I - A^*)\varphi\|_H^2 + \|B^*\varphi\|_U^2) \text{ for } \lambda \in \mathbb{C}_{-\beta}^+, \varphi \in H_1. \quad (1.6)$$

(ii) The pair (A^*, B^*) satisfies (HESI) if it satisfies $(\text{HESI})_\beta$ for each $\beta > 0$.

Remark 4. Several notes on Definition 3 are as follows:

- (i) ‘(HESI)’ is the abbreviation of ‘**H**autus test condition for **E**xponential **S**tabilizability of **I**nfinite-dimensional systems’. The subscript β is used to characterize the boundary of the frequency-domain (i.e., the boundary of $\mathbb{C}_{-\beta}^+$) that appears in (1.5) (or (1.6)).
- (ii) The constants (including β) in $(\text{HESI})_\beta$, (1.5) and (1.6) are allowed to be different. The proof of the equivalence between (1.5) and (1.6) will be given in Appendix of the paper (see Proposition 4). It is worth mentioning that (1.5) is sharper than (1.6) to characterize the optimal decay rate of system $[A, B]$. For more details, we refer readers to Remark 13.
- (iii) $(\text{HESI})_\beta$ (for some $\beta > 0$) is a type of quantitative frequency-domain condition. The connection to (HSF) (in Lemma 2) is as follows: if (A^*, B^*) satisfies $(\text{HESI})_\beta$ for some $\beta > 0$, then $\text{Ker}(\lambda I - A^*, B^*)^\top = \{0\}$ for each $\lambda \in \mathbb{C}_{-\beta}^+$.
- (iv) $(\text{HESI})_\beta$ (for some $\beta > 0$) can be considered as an extension of the classical frequency-domain criterion for the exponential stability of the C_0 -semigroup $S(\cdot)$. Indeed, if $B^* = 0$ and the pair $(A^*, 0)$ satisfies $(\text{HESI})_\beta$ for some $\beta > 0$, then it follows from (1.5) that $\mathbb{C}_{-\beta}^+ \subset \rho(A^*)$ and $\sup_{\lambda \in \mathbb{C}_{-\alpha}^+} \|(\lambda I - A^*)^{-1}\|_{\mathcal{L}(H)} < +\infty$ for each $\alpha \in (0, \beta)$. Thus, by [20, Theorem 1.11, Chapter V] (see also [19, 21]), we see that $S^*(\cdot)$ is exponentially stable, and so is $S(\cdot)$.

The first two main theorems show that under additional conditions beyond (A_1) – (A_3) , $(\text{HESI})_\beta$ for some $\beta > 0$ is a necessary and sufficient condition on the stabilizability of system $[A, B]$.

Theorem 1. *Suppose that assumptions (A_1) – (A_3) hold. Further assume that A is a normal operator, and for each $\gamma > 0$, $\sigma(A) \cap \mathbb{C}_{-\gamma}^+$ is bounded. Then, the following conclusions are true:*

- (i) *System $[A, B]$ is stabilizable if and only if the pair (A^*, B^*) satisfies $(\text{HESI})_\beta$ for some $\beta > 0$.*
- (ii) *System $[A, B]$ is rapidly stabilizable if and only if the pair (A^*, B^*) satisfies (HESI) .*

Remark 5. Two notes on the assumptions in Theorem 1 are as follows:

- (i) If A is a normal operator, A^* is also normal and has a unique spectral measure (see [22, Chapter 6]). The corresponding spectral measure can be used to define an orthogonal projection on H which plays a key role in the proof of Theorem 1. The application of normal operators in partial differential equations covers a considerable wide. It includes self-adjoint operators and linear partial differential operators on the entire space with (real/complex-valued) constant coefficients among others (see [23, Theorem 13.24, Chapter 13]).
- (ii) If the semigroup $S(\cdot)$ is uniformly continuous/analytical/differentiable/compact, then $\sigma(A) \cap \mathbb{C}_{-\gamma}^+$ is bounded for each $\gamma > 0$ (see [24, Sections 2.3–2.6, Chapter 2]).

Theorem 2. *Suppose that the assumptions (A_1) – (A_3) hold. Assume that one of the following assumptions holds:*

- (a) *The semigroup $S^*(\cdot)$ is uniformly bounded;*
- (b) *For each $\alpha > 0$, there are two closed subspaces $Q_1 := Q_1(\alpha)$ and $Q_2 := Q_2(\alpha)$ of H (depending on α) such that (b₁) $H = Q_1 \oplus Q_2$; (b₂) Q_1 and Q_2 are invariant subspaces of $S^*(\cdot)$; (b₃) $A^*|_{Q_1}$, the restriction of A^* on Q_1 is bounded and satisfies that $\sigma(A^*|_{Q_1}) \subset \mathbb{C}_{-\alpha}^+$; (b₄) $S^*(\cdot)|_{Q_2}$, the restriction of $S^*(\cdot)$ on Q_2 is exponentially stable.*

Then, system $[A, B]$ is stabilizable if and only if the pair (A^, B^*) satisfies $(\text{HESI})_\beta$ for some $\beta > 0$.*

Remark 6. Several notes on Theorem 2 are as follows:

- (i) If A is skew-adjoint (i.e., $A = -A^*$), then the semigroup $S^*(\cdot)$ is uniformly bounded.
- (ii) If the semigroup $S(\cdot)$ is compact, then for each $\alpha > 0$, we can find two closed subspaces Q_1 and Q_2 of H such that Q_1 is finite-dimensional; conditions (b_1) – (b_4) in (b) of Theorem 2 hold. (See [20, Section 3, Chapter IV, Section 3, Chapter V].)
- (iii) In assumption (b) of Theorem 2, it is not required that Q_1 has a finite dimension.
- (iv) By the spectral theorem (e.g., [22, Theorem 6.47, Chapter 6]), one can check that if A satisfies the assumptions of Theorem 1, then it also satisfies assumption (b) in Theorem 2. Thus, (i) in Theorem 1 can be regraded as an application of Theorem 2. But, there are some difficulties in applying Theorem 2 to characterize the rapid stabilizability. Indeed, when A satisfies the assumptions of Theorem 1, $A + \beta I$ also satisfies these assumptions for each $\beta > 0$. This property plays a crucial role in the proof of (ii) of Theorem 1. It is not known to us if assumption (b) in Theorem 2 can guarantee this property for the general cases.

The last main result of this paper shows that $(\text{HESI})_\beta$ (for some $\beta > 0$)/ (HESI) is a necessary condition on the exponential/rapid stabilizability for system $[A, B]$, with (A_1) – (A_3) holding.

Theorem 3. *Under assumptions (A_1) – (A_3) , the following conclusions are true:*

- (i) *If system $[A, B]$ is stabilizable, then the pair (A^*, B^*) satisfies $(\text{HESI})_\beta$ for some $\beta > 0$.*
- (ii) *If system $[A, B]$ is rapidly stabilizable, then the pair (A^*, B^*) satisfies (HESI) .*

1.5. Related works

- H. O. Fattorini in [6] established a frequency-domain condition similar to Lemma 2 for the special infinite-dimensional setting, where $B \in \mathcal{L}(U; H)$; system $[A, B]$ can be decomposed into two decoupled subsystems: one is in a finite-dimensional subspace and controllable; the other is in an infinite-dimensional subspace and exponentially stable. For more studies in this direction, we refer the readers to [2, 25, 12] and the references therein. The setting in [6] is covered by our setting, where (A_1) – (A_3) and (b) in Theorem 2 hold (see (ii) in Remark 6). Thus, Theorem 2 can be considered as an extension of the related result in [6].
- K. Liu proved in [9] that if A is skew-adjoint and $B \in \mathcal{L}(U; H)$, then system $[A, B]$ is stabilizable if and only if $i\mathbb{R} \in \rho(A - BB^*)$ and $\sup_{\omega \in \mathbb{R}} \| (i\omega I - A + BB^*)^{-1} \|_{\mathcal{L}(H)} < +\infty$. Later, Q. Zhou and M. Yamamoto in [26] obtained that if A is skew-adjoint and $B \in \mathcal{L}(U; H)$, then system $[A, B]$ is stabilizable if and only if there was $C > 0$ such that

$$\|\varphi\|_H^2 \leq C(\|(i\omega I - A^*)\varphi\|_H^2 + \|B^*\varphi\|_U^2) \quad \text{for } \omega \in \mathbb{R}, \quad \varphi \in H_1.$$

These are two frequency-domain criteria on the stabilizability for the case $A = -A^*$ and $B \in \mathcal{L}(U; H)$. However, the condition that $A = -A^*$ implies condition (a) in Theorem 2. Thus, Theorem 2 can also be considered as an extension of the above frequency-domain criteria.

- R. Rebarber and H. Zwart in [13] introduced the concept of the open-loop stabilizability for system $[A, B]$ and provided necessary conditions in the frequency-domain for the open-loop stabilizability in infinite dimensional settings. Such stabilizability is defined as follows: *If there is $\sigma > 0$ such that for each $y_0 \in H$, there is a control $u \in \mathcal{D}'(\mathbb{R}^+; U) (= (C_0^\infty(\mathbb{R}^+; U))')$ such that the solution (in the sense of distribution) $y(\cdot; u, y_0)$ to system (1.1) (with the initial condition $y(0) = y_0$ and the control u) satisfies $e^{\sigma \cdot} y(\cdot; u, y_0) \in L^2(\mathbb{R}^+; H)$, then system $[A, B]$ is called open-loop stabilizable.* Clearly, the open-loop stabilizability is weaker than the closed-loop stabilizability. (Several examples that are open-loop stabilizable but not closed-loop stabilizable were provided in [13].) For more studies on the open-

loop stabilizability in infinite dimensional settings, we refer the readers to [27,15,28–30]. Our condition $(\text{HESI})_\beta$ is partially inspired by [13] and related works.

1.6. Plan of this paper

This paper is organized as follows: In Section 2, we present some criteria on the stabilizability for system $[A, B]$. In Section 3, we prove our main theorems. In Section 4, we provide selected applications. Section 5 is Appendix.

2. Other criteria on the stabilization

This section provides criteria on stabilizability for system $[A, B]$ from the perspective of integral transformation. Although they are not easily verifiable, they play important roles in the proofs of our main theorems.

To present them, we must introduce the following function spaces: Let X be a separable, complex Hilbert space and $\alpha \geq 0$. For each open and connected subset $S \subset \mathbb{C}$, we let $\mathcal{H}(S; X)$ be the set of all X -valued holomorphic functions on S . We define the following Hardy space:

$$\mathcal{H}^2(\mathbb{C}_{-\alpha}^+; X) := \{f \in \mathcal{H}(\mathbb{C}_{-\alpha}^+; X) : \mathcal{M}_\alpha(f) < +\infty\} \quad \text{with } \mathcal{M}_\alpha(f) := \sup_{\omega_1 > -\alpha} \int_{\mathbb{R}} \|f(\omega_1 + i\omega_2)\|_X^2 d\omega_2,$$

and the following weighted L^2 -space:

$$L_\alpha^2(\mathbb{R}^+; X) := \{h : \mathbb{R} \rightarrow X : e^{\alpha \cdot} h(\cdot) \in L^2(\mathbb{R}; X); \quad h(\cdot) = 0 \text{ on } \mathbb{R}^-\},$$

with the inner product:

$$\langle g, h \rangle_{L_\alpha^2(\mathbb{R}^+; X)} := \int_{\mathbb{R}^+} e^{2\alpha t} \langle g(t), h(t) \rangle_X dt.$$

One can easily check that $L_\alpha^2(\mathbb{R}^+; X)$ is a Hilbert space and continuously embedded into $L^2(\mathbb{R}^+; X)$. Throughout this paper, we extend each $f \in L^2(\mathbb{R}^+; X)$ over \mathbb{R} by setting it to be zero over \mathbb{R}^- (we denote this extension in same way). We have $L_0^2(\mathbb{R}^+; X) = L^2(\mathbb{R}^+; X)$.

Theorem 4. Suppose that (A_1) – (A_3) hold. Then, the following statements are equivalent:

- (i) System $[A, B]$ is stabilizable.
- (ii) There is $\alpha > 0$ such that for each $\beta \in [0, \alpha)$ and $y_0 \in H$, there is $(\xi(\cdot; y_0), \eta(\cdot; y_0)) \in \mathcal{H}^2(\mathbb{C}_{-\beta}^+; H) \times \mathcal{H}^2(\mathbb{C}_{-\beta}^+; U)$ such that

$$\langle \xi(\lambda; y_0), (\bar{\lambda}I - A^*)\varphi \rangle_H + \langle \eta(\lambda; y_0), B^*\varphi \rangle_U = \langle y_0, \varphi \rangle_H \quad \text{for } \lambda \in \mathbb{C}_{-\beta}^+, \quad \varphi \in H_1; \quad (2.1)$$

$$\|\xi(\lambda; y_0)\|_H \leq \frac{C(\beta)}{\operatorname{Re} \lambda + \beta} \|y_0\|_H, \quad \|\eta(\lambda; y_0)\|_U \leq \frac{D(\beta)}{\sqrt{\operatorname{Re} \lambda + \beta}} \|y_0\|_H \quad \text{for } \lambda \in \mathbb{C}_{-\beta}^+, \quad (2.2)$$

where $C(\beta) > 0$ and $D(\beta) > 0$ are two constants independent of y_0 .

- (iii) There is $\beta \geq 0$ such that for each $y_0 \in H$, there is $(\xi(\cdot; y_0), \eta(\cdot; y_0)) \in \mathcal{H}^2(\mathbb{C}_{-\beta}^+; H) \times \mathcal{H}^2(\mathbb{C}_{-\beta}^+; U)$ satisfying (2.1).

To prove Theorem 4, we need the following lemmas: The first lemma contains several results quoted from [25, Section A.6.3, Chapter A]:

Lemma 3. *Let $\alpha \geq 0$. Then, the following statements are true:*

- (i) $f \in \mathcal{H}^2(\mathbb{C}_{-\alpha}^+; X)$ if and only if there is a unique $h \in L_\alpha^2(\mathbb{R}^+; X)$ such that $f(\lambda) = \int_{\mathbb{R}^+} e^{-t\lambda} h(t) dt$ ($\lambda \in \mathbb{C}_{-\alpha}^+$), i.e., $\mathcal{H}^2(\mathbb{C}_{-\alpha}^+; X)$ and $L_\alpha^2(\mathbb{R}^+; X)$ are linear isomorphic. Moreover, $\frac{1}{2\pi} \mathcal{M}_\alpha(f) = \int_{\mathbb{R}^+} e^{2\alpha t} \|h(t)\|_X^2 dt$.
- (ii) For each $f \in \mathcal{H}^2(\mathbb{C}_{-\alpha}^+; X)$, there is a unique $f_\alpha^* \in L^2(\mathbb{R}; X)$ such that $\lim_{\omega_1 \rightarrow -\alpha} \|f(\omega_1 + i\cdot) - f_\alpha^*\|_{L^2(\mathbb{R}; X)} = 0$. Moreover, $\|f_\alpha^*\|_{L^2(\mathbb{R}; X)}^2 = \mathcal{M}_\alpha(f)$.
- (iii) $\mathcal{H}^2(\mathbb{C}_{-\alpha}^+; X)$, with inner product $\langle f, g \rangle_{\mathcal{H}^2(\mathbb{C}_{-\alpha}^+; X)} := \langle f_\alpha^*, g_\alpha^* \rangle_{L^2(\mathbb{R}; X)}$ ($f, g \in \mathcal{H}^2(\mathbb{C}_{-\alpha}^+; X)$), is a Hilbert space.

The second lemma is as follows, which is clear if $B \in \mathcal{L}(U; H)$. However, for our framework, we do not find accurate literature that provides its proof. Thus, we provide it for the completeness of the paper.

Lemma 4. *Suppose that (A_1) – (A_3) hold. If system $[A, B]$ is stabilizable with decay rate $\alpha > 0$, then for each $\beta \in (0, \alpha)$, system $[A + \beta I, B]$ is stabilizable.*

Proof. We arbitrarily fix $\beta \in (0, \alpha)$. Since system $[A, B]$ is stabilizable with decay rate $\alpha > 0$, we can use the same method as that used in *Step 1* of the proof of [8, Theorem 3.4] to find a positive constant $C(\alpha)$ such that

$$\|S^*(t)\varphi\|_H^2 \leq C(\alpha) \left(\int_0^t \|B^* S^*(s)\varphi\|_U^2 ds + e^{-2\alpha t} \|\varphi\|_H^2 \right) \text{ for } t \in \mathbb{R}^+, \varphi \in H_1.$$

This yields that for each $t \in \mathbb{R}^+$,

$$\|S_\beta^*(t)\varphi\|_H^2 \leq C(\alpha) e^{2\beta t} \int_0^t \|B^* S_\beta^*(s)\varphi\|_H^2 ds + C(\alpha) e^{-2(\alpha-\beta)t} \|\varphi\|_H^2 \text{ for } \varphi \in H_1, \quad (2.3)$$

where $S_\beta(\cdot)$ is the C_0 -semigroup on H generated by $A_\beta := A + \beta I$ with its domain $D(A_\beta) = D(A)$. Let $T > 0$ satisfy $\delta_T := C(\alpha) e^{-2(\alpha-\beta)T} < 1$. Then, it follows from (2.3) that

$$\|S_\beta^*(T)\varphi\|_H^2 \leq C(\alpha) e^{2\beta T} \int_0^T \|B^* S_\beta^*(s)\varphi\|_H^2 ds + \delta_T \|\varphi\|_H^2 \text{ for } \varphi \in H_1.$$

With Lemma 1, the above shows that system $[A_\beta, B]$ is stabilizable. This completes the proof. \square

Proof of Theorem 4. We organize the proof into the following steps.

Step 1. We prove (i) \Rightarrow (ii).

Suppose that system $[A, B]$ is stabilizable with decay rate $\alpha > 0$. We arbitrarily fix $\beta \in [0, \alpha)$. According to Lemma 4, system $[A_\beta, B]$ (where $A_\beta := A + \beta I$) is stabilizable, i.e., there is $\gamma > 0$, a C_0 -semigroup $\mathcal{S}_{\beta, \gamma}(\cdot)$ on H with the generator $\mathcal{A}_{\beta, \gamma} : D(\mathcal{A}_{\beta, \gamma}) \subset H \rightarrow H$, and a $K_{\beta, \gamma} \in \mathcal{L}(D(\mathcal{A}_{\beta, \gamma}); U)$ such that

- (a1) there is $C_{\beta, \gamma, 1} > 0$ such that $\|\mathcal{S}_{\beta, \gamma}(t)\|_{\mathcal{L}(H)} \leq C_{\beta, \gamma, 1} e^{-\gamma t}$ for all $t \in \mathbb{R}^+$;
- (b1) for each $x \in D(\mathcal{A}_{\beta, \gamma})$, $\mathcal{A}_{\beta, \gamma} x = \tilde{A}_\beta x + B K_{\beta, \gamma} x$, where $\tilde{A}_\beta := \tilde{A} + \beta I$, with \tilde{A} provided by (1.2);

(c1) there is $C_{\beta,\gamma,2} > 0$ such that $\|K_{\beta,\gamma}\mathcal{S}_{\beta,\gamma}(\cdot)x\|_{L^2(\mathbb{R}^+;U)} \leq C_{\beta,\gamma,2}\|x\|_H$ for each $x \in D(\mathcal{A}_{\beta,\gamma})$.

From these properties we deduce the following facts:

(O1) Based on (b1) and (c1), for each $y_0 \in D(\mathcal{A}_{\beta,\gamma})$,

$$u_{K_{\beta,\gamma}}(\cdot; y_0) := K_{\beta,\gamma}\mathcal{S}_{\beta,\gamma}(\cdot)y_0 = K_{\beta,\gamma}y_{K_{\beta,\gamma}}(\cdot; y_0) \in L_0^2(\mathbb{R}^+; U). \quad (2.4)$$

and

$$y_{K_{\beta,\gamma}}(t; y_0) := \mathcal{S}_{\beta,\gamma}(t)y_0 = \tilde{S}_\beta(t)y_0 + \int_0^t \tilde{S}_\beta(t-s)Bu_{K_{\beta,\gamma}}(s; y_0)ds \text{ for all } t \in \mathbb{R}^+,$$

where $\tilde{S}_\beta(\cdot) := e^{\beta\cdot}\tilde{S}(\cdot)$ is the C_0 -semigroup generated by \tilde{A}_β on H_{-1} (see (i) in Remark 1).

(O2) We arbitrarily fix $y_0 \in D(\mathcal{A}_{\beta,\gamma})$. By (a1) and (2.4), we obtain that if $t \in \mathbb{R}^+$,

$$\begin{aligned} \|u_{K_{\beta,\gamma}}(t; y_0)\|_U &= \|K_{\beta,\gamma}y_{K_{\beta,\gamma}}(t; y_0)\|_U = \|K_{\beta,\gamma}\mathcal{S}_{\beta,\gamma}(t)y_0\|_U \\ &\leq \|K_{\beta,\gamma}(\rho_1 I - \mathcal{A}_{\beta,\gamma})^{-1}\|_{\mathcal{L}(H;U)} \|\mathcal{S}_{\beta,\gamma}(t)(\rho_1 I - \mathcal{A}_{\beta,\gamma})y_0\|_H \\ &\leq C_{\beta,\gamma,1}e^{-\gamma t} \|K_{\beta,\gamma}(\rho_1 I - \mathcal{A}_{\beta,\gamma})^{-1}\|_{\mathcal{L}(H;U)} \|(\rho_1 I - \mathcal{A}_{\beta,\gamma})y_0\|_H, \end{aligned} \quad (2.5)$$

where $\rho_1 \in \rho(\mathcal{A}_{\beta,\gamma}) \cap \mathbb{R}^+$, and we use that $K_{\beta,\gamma} \in \mathcal{L}(D(\mathcal{A}_{\beta,\gamma}); U)$.

(O3) We arbitrarily fix $y_0 \in D(\mathcal{A}_{\beta,\gamma})$ and define

$$\xi(\lambda; y_0) := \int_{\mathbb{R}^+} e^{-\lambda t} y_{K_{\beta,\gamma}}(t; y_0) dt; \quad \eta(\lambda; y_0) := - \int_{\mathbb{R}^+} e^{-\lambda t} u_{K_{\beta,\gamma}}(t; y_0) dt, \quad \lambda \in \mathbb{C}_0^+. \quad (2.6)$$

Considering the above two functions, we have the following conclusions: First, by (2.6) and (i) of Lemma 3, we see that $\xi(\cdot; y_0) \in \mathcal{H}^2(\mathbb{C}_0^+; H)$ and $\eta(\cdot; y_0) \in \mathcal{H}^2(\mathbb{C}_0^+; U)$. Second, based on (2.4)-(2.6), (a1) and (c1), we find

$$\|\xi(\lambda; y_0)\|_H \leq \frac{C_{\beta,\gamma,1}}{\operatorname{Re} \lambda} \|y_0\|_H \text{ and } \|\eta(\lambda; y_0)\|_U \leq \frac{C_{\beta,\gamma,2}}{\sqrt{\operatorname{Re} \lambda}} \|y_0\|_H \text{ for } \lambda \in \mathbb{C}_0^+. \quad (2.7)$$

Third, based on (a1), (c1), and Lemma 3, we obtain

$$\begin{aligned} \|\xi(\cdot; y_0)\|_{\mathcal{H}^2(\mathbb{C}_0^+; H)}^2 &= 2\pi \int_{\mathbb{R}^+} \|y_{K_{\beta,\gamma}}(t; y_0)\|_H^2 dt \leq \gamma^{-1} \pi C_{\beta,\gamma,1}^2 \|y_0\|_H^2; \\ \|\eta(\cdot; y_0)\|_{\mathcal{H}^2(\mathbb{C}_0^+; U)}^2 &= 2\pi \int_{\mathbb{R}^+} \|u_{K_{\beta,\gamma}}(t; y_0)\|_U^2 dt \leq 2\pi C_{\beta,\gamma,2}^2 \|y_0\|_H^2. \end{aligned} \quad (2.8)$$

Next, we show Claim One: For each $y_0 \in H$, there is $(\xi(\cdot; y_0), \eta(\cdot; y_0)) \in \mathcal{H}^2(\mathbb{C}_0^+; H) \times \mathcal{H}^2(\mathbb{C}_0^+; U)$ that satisfies (2.7) and

$$\langle \xi(\lambda; y_0), (\bar{\lambda}I - A_\beta^*)\varphi \rangle_H + \langle \eta(\lambda; y_0), B^*\varphi \rangle_U = \langle y_0, \varphi \rangle_H \text{ for } \lambda \in \mathbb{C}_0^+, \varphi \in H_1. \quad (2.9)$$

The proof of Claim One will be organized using two cases.

Case 1. We consider that $y_0 \in D(\mathcal{A}_{\beta,\gamma})$.

First, based on (O3), we have (2.7) for this case. We now show (2.9) for this case. For this purpose, we arbitrarily fix $y_0 \in D(\mathcal{A}_{\beta,\gamma})$ and $\varphi \in H_1 = D(A_\beta^*)$. From (O1) and the main theorem in [31] (see also [24, Theorem on Page 259]), we obtain

$$\begin{cases} \frac{d}{dt} \langle y_{K_{\beta,\gamma}}(t; y_0), \varphi \rangle_H = \langle y_{K_{\beta,\gamma}}(t; y_0), A_\beta^* \varphi \rangle_H + \langle u_{K_{\beta,\gamma}}(t; y_0), B^* \varphi \rangle_U, & t \in \mathbb{R}^+, \\ y_{K_{\beta,\gamma}}(0) = y_0. \end{cases} \quad (2.10)$$

The combination of this result, (a1), and (2.5) yield

$$e^{-\lambda \cdot} \frac{d}{dt} \langle y_{K_{\beta,\gamma}}(\cdot; y_0), \varphi \rangle_H \in L^1(\mathbb{R}^+; \mathbb{C}) \text{ for each } \lambda \in \mathbb{C}_0^+. \quad (2.11)$$

Now, (a1), (2.11), and (2.6) lead to

$$\int_{\mathbb{R}^+} e^{-\lambda t} \frac{d}{dt} \langle y_{K_{\beta,\gamma}}(t; y_0), \varphi \rangle_H dt = \langle \lambda \xi(\lambda; y_0), \varphi \rangle_H - \langle y_0, \varphi \rangle_H = \langle \xi(\lambda; y_0), \bar{\lambda} \varphi \rangle_H - \langle y_0, \varphi \rangle_H, \quad \lambda \in \mathbb{C}_0^+.$$

From the above and (2.10), one can directly obtain that $(\xi(\cdot; y_0), \eta(\cdot; y_0))$ (which is defined by (2.6)) satisfies (2.9).

Case 2. We consider that $y_0 \in H$.

According to the density of $D(\mathcal{A}_{\beta,\gamma})$ in H , there is a sequence $\{y_0^n\}_{n \in \mathbb{N}^+} \subset D(\mathcal{A}_{\beta,\gamma})$ such that $y_0^n \rightarrow y_0$ in H as $n \rightarrow +\infty$. Thus, $\{y_0^n\}_{n \in \mathbb{N}^+}$ is a Cauchy sequence in H . Moreover, by (2.6) and (2.8), we have

$$\|\xi(\cdot; y_0^n) - \xi(\cdot; y_0^m)\|_{\mathcal{H}^2(\mathbb{C}_0^+; H)} = \|\xi(\cdot; y_0^n - y_0^m)\|_{\mathcal{H}^2(\mathbb{C}_0^+; H)} \leq \sqrt{\gamma^{-1} \pi} C_{\beta,\gamma,1} \|y_0^n - y_0^m\|_H \quad \forall n, m \in \mathbb{N}^+.$$

Hence, $\{\xi(\cdot; y_0^n)\}_{n \in \mathbb{N}^+}$ is a Cauchy sequence in $\mathcal{H}^2(\mathbb{C}_0^+; H)$. Then, according to (iii) in Lemma 3, there is $\xi^*(\cdot) \in \mathcal{H}^2(\mathbb{C}_0^+; H)$ such that

$$\xi(\cdot; y_0^n) \rightarrow \xi^*(\cdot) \text{ in } \mathcal{H}^2(\mathbb{C}_0^+; H) \text{ as } n \rightarrow +\infty. \quad (2.12)$$

By (i) in Lemma 3, we can find $h^*(\cdot)$ in $L^2(\mathbb{R}^+; H)$ such that $\xi^*(\cdot) = \int_{\mathbb{R}^+} e^{-\cdot t} h^*(t) ds$. The combination of this result, (2.6), (2.12) and (i) in Lemma 3 yield that if we write $h^n(t) := y_{K_{\beta,\gamma}}(t; y_0^n)$, $t \in \mathbb{R}^+$ ($n \in \mathbb{N}^+$), then we have

$$\int_{\mathbb{R}^+} \|h^n(t) - h^*(t)\|_H^2 dt = \frac{1}{2\pi} \|\xi(\cdot; y_0^n) - \xi^*(\cdot)\|_{\mathcal{H}^2(\mathbb{C}_0^+; H)}^2 \rightarrow 0 \text{ as } n \rightarrow +\infty.$$

Thus, for each $\lambda \in \mathbb{C}_0^+$,

$$\begin{aligned} \|\xi(\lambda; y_0^n) - \xi^*(\lambda)\|_H^2 &\leq \left(\int_{\mathbb{R}^+} e^{-\operatorname{Re} \lambda t} \|h^n(t) - h^*(t)\|_H dt \right)^2 \\ &\leq (2\operatorname{Re} \lambda)^{-1} \|h^n - h^*\|_{L^2(\mathbb{R}^+; H)}^2 \rightarrow 0 \text{ as } n \rightarrow +\infty. \end{aligned} \quad (2.13)$$

Similarly, we can show that there is $\eta^*(\cdot) \in \mathcal{H}^2(\mathbb{C}_0^+; U)$ such that for each $\lambda \in \mathbb{C}_0^+$,

$$\|\eta(\lambda; y_0^n) - \eta^*(\lambda)\|_U \rightarrow 0 \text{ as } n \rightarrow +\infty. \quad (2.14)$$

Therefore, by (2.9), (2.13) and (2.14), we obtain that for $\varphi \in H_1$ and $\lambda \in \mathbb{C}_0^+$,

$$\begin{aligned} & \langle \xi^*(\lambda), (\bar{\lambda}I - A^*)\varphi \rangle_H + \langle \eta^*(\lambda), B^*\varphi \rangle_U \\ &= \lim_{n \rightarrow +\infty} \langle \xi(\lambda; y_0^n), (\bar{\lambda}I - A^*)\varphi \rangle_H + \langle \eta(\lambda; y_0^n), B^*\varphi \rangle_U = \lim_{n \rightarrow +\infty} \langle y_0^n, \varphi \rangle_H = \langle y_0, \varphi \rangle_H, \end{aligned}$$

which leads to (2.9). Meanwhile, based on (2.13) and (2.14), we can directly observe that (2.7) holds for all $y_0 \in H$. Hence, Claim One has been proven.

Finally, we arbitrarily fix $y_0 \in H$. Letting $\lambda = \beta + \mu$ with $\mu \in \mathbb{C}_{-\beta}^+$, formulas (2.9) and (2.7) lead to

$$\langle \xi(\mu + \beta; y_0), (\bar{\mu}I - A^*)\varphi \rangle_H + \langle \eta(\mu + \beta; y_0), B^*\varphi \rangle_U = \langle y_0, \varphi \rangle_H \quad \text{for } \mu \in \mathbb{C}_{-\beta}^+, \varphi \in H_1; \quad (2.15)$$

$$\|\xi(\mu + \beta; y_0)\|_H \leq \frac{C_{\beta, \gamma, 1}}{\operatorname{Re} \mu + \beta} \|y_0\|_H \quad \text{and} \quad \|\eta(\mu + \beta; y_0)\|_U \leq \frac{C_{\beta, \gamma, 2}}{\sqrt{\operatorname{Re} \mu + \beta}} \|y_0\|_H \quad \text{for } \mu \in \mathbb{C}_{-\beta}^+. \quad (2.16)$$

One can directly check that $(\xi(\cdot + \beta; y_0), \eta(\cdot + \beta; y_0)) \in \mathcal{H}^2(\mathbb{C}_{-\beta}^+; H) \times \mathcal{H}^2(\mathbb{C}_{-\beta}^+; U)$ (since $(\xi(\cdot; y_0), \eta(\cdot; y_0)) \in \mathcal{H}^2(\mathbb{C}_0^+; H) \times \mathcal{H}^2(\mathbb{C}_0^+; U)$). Thus, (2.15) and (2.16) imply that (2.1) and (2.2) hold with $C(\beta) := C_{\beta, \gamma, 1}$ and $D(\beta) := C_{\beta, \gamma, 2}$, respectively. Hence, conclusion (ii) is true.

Step 2. The proof of (ii) \Rightarrow (iii) is trivial.

Step 3. We prove (iii) \Rightarrow (i).

Suppose that (iii) holds, i.e., there is $\beta \geq 0$ such that for each $y_0 \in H$, there is $(\xi(\cdot; y_0), \eta(\cdot; y_0)) \in \mathcal{H}^2(\mathbb{C}_{-\beta}^+; H) \times \mathcal{H}^2(\mathbb{C}_{-\beta}^+; U)$ satisfying (2.1). We arbitrarily fix $y_0 \in H$. Then, based on (i) in Lemma 3, there is a unique $(y(\cdot), u(\cdot)) \in L^2_{\beta}(\mathbb{R}^+; H) \times L^2_{\beta}(\mathbb{R}^+; U)$ such that

$$\xi(\lambda; y_0) = \int_{\mathbb{R}^+} e^{-\lambda t} y(t) dt; \quad \eta(\lambda; y_0) = - \int_{\mathbb{R}^+} e^{-\lambda t} u(t) dt, \quad \lambda \in \mathbb{C}_{\beta}^+. \quad (2.17)$$

Let $\omega := \max \{1, \lim_{t \rightarrow +\infty} t^{-1} \ln \|S(t)\|_{\mathcal{L}(H)}\}$. Then, $\xi(\cdot; y_0)$ and $\eta(\cdot; y_0)$ are well defined over \mathbb{C}_{ω}^+ , and moreover, by [20, Proposition 2.2, Chapter IV], we have $\mathbb{C}_{\omega}^+ \subset \rho(A) (= \rho(\tilde{A}))$. We arbitrarily fix $\lambda \in \mathbb{C}_{\omega}^+$. Then, we have that $(\bar{\lambda}I - A^*)^{-1} \in \mathcal{L}(H; H_1)$, and it follows from the proof of Theorem 3.1 in [24, Section 1.3, Chapter 1] that

$$(\bar{\lambda}I - A^*)^{-1}\varphi = \int_{\mathbb{R}^+} e^{-\bar{\lambda}t} S^*(t)\varphi dt \quad \text{for each } \varphi \in H. \quad (2.18)$$

We arbitrarily fix $\psi \in H_1$. There are two facts. First, replacing φ by $(\bar{\lambda}I - A^*)^{-1}\psi$ in (2.1) ($\lambda \in \mathbb{C}_{\omega}^+$) leads to

$$\langle \xi(\lambda; y_0), \psi \rangle_H + \langle \eta(\lambda; y_0), B^*(\bar{\lambda}I - A^*)^{-1}\psi \rangle_U = \langle y_0, (\bar{\lambda}I - A^*)^{-1}\psi \rangle_H. \quad (2.19)$$

Second, with assumption (A_2) , we have $B^* \in \mathcal{L}(H_1; U)$ and $B^*(\lambda I - A^*)^{-1} \in \mathcal{L}(H; U)$, thus,

$$\langle \eta(\lambda; y_0), B^*(\bar{\lambda}I - A^*)^{-1}\psi \rangle_U = \langle B\eta(\lambda; y_0), (\bar{\lambda}I - A^*)^{-1}\psi \rangle_{H_{-1}, H_1}. \quad (2.20)$$

Now, we claim that

$$e^{-\operatorname{Re} \lambda \cdot} \int_0^{\cdot} \tilde{S}(\cdot - s) B u(s) ds \in L^1(\mathbb{R}^+; H_{-1}) \cap L^2(\mathbb{R}^+; H_{-1}). \quad (2.21)$$

For this purpose, we first recall that ρ_0 is provided in assumption (A_2) , so we have $(\rho_0 I - \tilde{A})^{-1} B \in \mathcal{L}(U; H)$. Moreover, by [17, Proposition 2.10.3, Chapter 2], we have that $\rho_0 I - \tilde{A}$ is a unitary linear map from H to H_{-1} . Hence, for each $t \in \mathbb{R}^+$,

$$\begin{aligned}
e^{-\operatorname{Re}\lambda t} \left\| \int_0^t \tilde{S}(t-s)Bu(s)ds \right\|_{H_{-1}} &= e^{-\operatorname{Re}\lambda t} \left\| \int_0^t S(t-s)(\rho_0 I - \tilde{A})^{-1}Bu(s)ds \right\|_H \\
&\leq C(\omega) \|(\rho_0 I - \tilde{A})^{-1}B\|_{\mathcal{L}(U;H)} e^{-\operatorname{Re}\lambda t} \int_0^t e^{\omega(t-s)} \|u(s)\|_U ds \\
&\leq C(\omega) \|(\rho_0 I - \tilde{A})^{-1}B\|_{\mathcal{L}(U;H)} e^{-\operatorname{Re}\lambda t} \left(\int_0^t e^{2\omega s} ds \right)^{\frac{1}{2}} \left(\int_0^t \|u(s)\|_U^2 ds \right)^{\frac{1}{2}} \\
&\leq C(\omega) \|(\rho_0 I - \tilde{A})^{-1}B\|_{\mathcal{L}(U;H)} (2\omega)^{-\frac{1}{2}} \|u\|_{L^2_\beta(\mathbb{R}^+;U)} e^{-(\operatorname{Re}\lambda - \omega)t}.
\end{aligned}$$

Since $\operatorname{Re}\lambda > \omega$, the above leads to (2.21).

Based on (i) in Remark 1, (2.17), (2.18) and (2.21), using the Fubini theorem and $u = 0$ in $(-\infty, 0)$, we obtain

$$\begin{aligned}
&\langle B\eta(\lambda; y_0), (\bar{\lambda}I - A^*)^{-1}\psi \rangle_{H_{-1}, H_1} \\
&= - \left\langle B \int_{\mathbb{R}^+} e^{-\lambda t} u(t) dt, \int_{\mathbb{R}^+} e^{-\bar{\lambda}t} S^*(t)\psi dt \right\rangle_{H_{-1}, H_1} \\
&= - \left\langle (\rho_0 I - \tilde{A})^{-1}B \int_{\mathbb{R}^+} e^{-\lambda t} u(t) dt, (\rho_0 I - A^*) \int_{\mathbb{R}^+} e^{-\bar{\lambda}t} S^*(t)\psi dt \right\rangle_H \\
&= - \int_{\mathbb{R}^+} \int_t^{+\infty} \langle e^{-\lambda(\sigma-t)} (\rho_0 I - \tilde{A})^{-1}Bu(\sigma-t), e^{-\bar{\lambda}t} S^*(t)(\rho_0 I - A^*)\psi \rangle_H d\sigma dt \\
&= - \int_{\mathbb{R}^+} e^{-\lambda\sigma} \int_0^\sigma \langle (\rho_0 I - \tilde{A})^{-1}Bu(\sigma-t), S^*(t)(\rho_0 I - A^*)\psi \rangle_H dt d\sigma \\
&= - \int_{\mathbb{R}^+} e^{-\lambda\sigma} \left\langle \int_0^\sigma \tilde{S}(\sigma-t)Bu(t)dt, \psi \right\rangle_{H_{-1}, H_1} d\sigma.
\end{aligned}$$

The combination of this result, (2.19) and (2.20) imply that

$$\int_{\mathbb{R}} e^{-\lambda t} \langle F(t), \psi \rangle_{H_{-1}, H_1} dt = 0 \quad \text{for } \lambda \in \mathbb{C}_\omega^+, \quad \psi \in H_1, \quad (2.22)$$

where $F(t) := y(t) - \tilde{S}(t)y_0 - \int_0^t \tilde{S}(t-s)Bu(s)ds$ for $t \geq 0$, while $F(t) := 0$ for $t < 0$. With (2.21), we can apply the inverse Fourier transform to (2.22) with respect to $\operatorname{Im}\lambda$ to conclude that

$$y(t) = \tilde{S}(t)y_0 + \int_0^t \tilde{S}(t-s)Bu(s)ds \quad \text{a.e. } t \in \mathbb{R}^+,$$

which leads to $y(\cdot) = y(\cdot; y_0, u)$ a.e. in \mathbb{R}^+ . With $(y(\cdot), u(\cdot)) \in L^2_\beta(\mathbb{R}^+; H) \times L^2_\beta(\mathbb{R}^+; U) \subset L^2(\mathbb{R}^+; H) \times L^2(\mathbb{R}^+; U)$, we obtain

$$\mathcal{U}_{ad}(y_0) := \{u(\cdot) \in L^2(\mathbb{R}^+; U) : y(\cdot; y_0, u) \in L^2(\mathbb{R}^+; H)\} \neq \emptyset. \quad (2.23)$$

Since y_0 was arbitrarily taken from H , (2.23) and [8, Proposition 3.9] imply that system $[A, B]$ is exponentially stabilizable. Hence, we have completed the proof of Theorem 4. \square

3. Proofs of the main theorems

Before presenting the proofs of the main theorems, we need the following lemmas:

Lemma 5. *Suppose that $A \in \mathcal{L}(H)$ and $B \in \mathcal{L}(U; H)$. Then, the following statements are equivalent:*

- (i) *System $[A, B]$ is exactly controllable at some $T > 0$;*
- (ii) *There are constants $T > 0$ and $C(T) > 0$ such that*

$$\|\varphi\|_H^2 \leq C(T) \int_0^T \|B^* e^{A^* t} \varphi\|_U^2 dt \quad \text{for } \varphi \in H;$$

- (iii) *There is $n \in \mathbb{N}$ such that $\text{Span}\{BU, ABU, \dots, A^n BU\} = H$;*
- (iv) *For each $\lambda \in \mathbb{C}$, there is $C(\lambda) > 0$ such that $\|\varphi\|_H^2 \leq C(\lambda)(\|(\lambda I - A^*)\varphi\|_H^2 + \|B^*\varphi\|_U^2)$ for all $\varphi \in H$.*

Proof. It is well-known that (i) \Leftrightarrow (ii) (e.g., [17, Theorem 11.2.1, Chapter 11]). Using Baire category theorem, one can directly verify that (i) \Rightarrow (iii), while the proof of (iii) \Rightarrow (i) can be found in [32, Theorem 2.3]. The proof of (iii) \Leftrightarrow (iv) can be found in [33] (see the main theorem and the remark on it there). This completes the proof. \square

Lemma 6. *System $[A, B]$ is rapidly stabilizable if and only if for each $\alpha > 0$, system $[A_\alpha, B]$ is stabilizable, where $A_\alpha := A + \alpha I$.*

If $B \in \mathcal{L}(U; H)$, then Lemma 6 is well known. However, for our framework, we do not find an accurate literature with its proof. Thus, we prove it here.

Proof of Lemma 6. First, let $S_\alpha(\cdot)$ be the C_0 -semigroup generated by A_α . Then, we have $S_\alpha(t) = e^{\alpha t} S(t)$, $t \geq 0$.

Now, we suppose that system $[A, B]$ is rapidly stabilizable. Then, according to [8, Theorem 3.4], for each $\alpha > 0$, there is $C(\alpha) > 0$ such that

$$\|S^*(t)\varphi\|_H^2 \leq C(\alpha) \left(\int_0^t \|B^* S^*(s)\varphi\|_U^2 ds + e^{-(2\alpha+1)t} \|\varphi\|_H^2 \right) \quad \text{for } \varphi \in H_1, t \in \mathbb{R}^+.$$

Hence, for $\alpha > 0$,

$$\|S_\alpha^*(t)\varphi\|_H^2 \leq C(\alpha) e^{2\alpha t} \int_0^t \|B^* S_\alpha^*(s)\varphi\|_U^2 ds + C(\alpha) e^{-t} \|\varphi\|_H^2 \quad \text{for } \varphi \in H_1, t \in \mathbb{R}^+. \quad (3.1)$$

For each $\alpha > 0$, we let $T > 0$ satisfy $\delta := D(\alpha) e^{-T} < 1$. Then, based on (3.1), we have

$$\|S_\alpha^*(T)\varphi\|_H^2 \leq C_1(\alpha) e^{2\alpha T} \int_0^T \|B^* S_\alpha^*(s)\varphi\|_U^2 ds + \delta \|\varphi\|_H^2 \quad \text{for } \varphi \in H_1.$$

The combination of this and (i) of Lemma 1 yield that system $[A_\alpha, B]$ is stabilizable.

Conversely, we suppose that, for each $\alpha > 0$, system $[A_\alpha, B]$ is stabilizable. Then, according to (i) of Lemma 1, for each $\alpha > 0$, there is $C(\alpha) > 0$ such that

$$\|S_\alpha^*(t)\varphi\|_H^2 \leq C(\alpha) \left(\int_0^t \|B^* S_\alpha^*(s)\varphi\|_U^2 ds + \|\varphi\|_H^2 \right) \quad \text{for } \varphi \in H_1, t \in \mathbb{R}^+,$$

which implies that if $\alpha > 0$,

$$\|S^*(t)\varphi\|_H^2 \leq C(\alpha) \left(\int_0^t \|B^* S^*(s)\varphi\|_U^2 ds + e^{-\alpha t} \|\varphi\|_H^2 \right) \quad \text{for } \varphi \in H_1, t \in \mathbb{R}^+.$$

With (ii) of Lemma 1, the above leads to the rapid stabilizability of system $[A, B]$. This completes the proof of Lemma 6. \square

We start with proving Theorem 3.

Proof of Theorem 3. We first prove (i). Suppose that system $[A, B]$ is stabilizable. According to Theorem 4, there are constants $\beta_0 > 0$, $C(\beta_0) \geq 1$ and $D(\beta_0) > 0$ such that for each $y_0 \in H$, there is $(\xi(\cdot; y_0), \eta(\cdot; y_0)) \in \mathcal{H}^2(\mathbb{C}_{-\beta_0}^+; H) \times \mathcal{H}^2(\mathbb{C}_{-\beta_0}^+; U)$ that satisfies (2.1) and (2.2). Then, based on (2.1) and (2.2), for $\lambda \in \mathbb{C}_{-\beta_0}^+$ and $y_0 \in H$,

$$|\langle y_0, \varphi \rangle_H| \leq \left(\frac{C(\beta_0)}{\operatorname{Re} \lambda + \beta_0} \|(\bar{\lambda}I - A^*)\varphi\|_H + \frac{D(\beta_0)}{\sqrt{\operatorname{Re} \lambda + \beta_0}} \|B^*\varphi\|_U \right) \|y_0\|_H \quad \text{for } \varphi \in H_1.$$

Thus, for each $\lambda \in \mathbb{C}_{-\beta_0}^+$,

$$\|\varphi\|_H \leq \frac{C(\beta_0)}{\operatorname{Re} \lambda + \beta_0} \|(\bar{\lambda}I - A^*)\varphi\|_H + \frac{D(\beta_0)}{\sqrt{\operatorname{Re} \lambda + \beta_0}} \|B^*\varphi\|_U \quad \text{for } \varphi \in H_1. \quad (3.2)$$

Suppose that $\|S^*(t)\| \leq C(\omega)e^{\omega t}$ for each $t \in \mathbb{R}^+$ for some constants $\omega > 0$ and $C(\omega) > 0$. Taking $\beta \in (0, \beta_0)$. By [24, Theorem 5.3 and Remark 5.4, Section 1.5, Chapter 1], we have that, for each $\lambda \in \mathbb{C}_{\max\{\omega, 2|\beta - \omega| - \beta\}}^+$,

$$\begin{aligned} \|\varphi\|_H &\leq \frac{C(\omega)}{\operatorname{Re} \lambda - \omega} \|(\lambda I - A^*)\varphi\|_H \leq \frac{C(\omega)}{(\operatorname{Re} \lambda + \beta) - |\beta - \omega|} \|(\lambda I - A^*)\varphi\|_H \\ &= \frac{C(\omega)}{\frac{1}{2}(\operatorname{Re} \lambda + \beta) + (\frac{1}{2}(\operatorname{Re} \lambda + \beta) - |\beta - \omega|)} \|(\lambda I - A^*)\varphi\|_H \\ &\leq \frac{2C(\omega)}{\operatorname{Re} \lambda + \beta} \|(\lambda I - A^*)\varphi\|_H \quad \text{for } \varphi \in H_1. \end{aligned} \quad (3.3)$$

If $\lambda \in \mathbb{C}_{-\beta}^+ \setminus \mathbb{C}_{\max\{\omega, 2|\beta - \omega| - \beta\}}^+$, then

$$\frac{D(\beta)}{\sqrt{\operatorname{Re} \lambda + \beta_0}} \leq \frac{D(\beta)(\max\{\omega, 2|\beta - \omega| - \beta\} + \beta)}{\sqrt{\beta_0 - \beta}} \frac{1}{\operatorname{Re} \lambda + \beta}.$$

The combination of this relation, (3.3), (3.2) and Definition 3 imply that the pair (A^*, B^*) satisfies $(\text{HESI})_\beta$.

Next, we prove (ii). Suppose that system $[A, B]$ is rapidly stabilizable. Lemma 6 implies that, for an arbitrarily fixed $\alpha > 0$, system $[A + \alpha I, B]$ is stabilizable. Then, according to the conclusion (i) of Theorem 3, there is $C(\alpha) > 0$ such that if $\lambda \in \mathbb{C}_0^+$,

$$\|\varphi\|_H^2 \leq \frac{C(\alpha)}{(\operatorname{Re} \lambda)^2} (\|(\lambda I - (A^* + \alpha I))\varphi\|_H^2 + \|B^* \varphi\|_U^2) \quad \text{for } \varphi \in H_1.$$

Hence, if $\lambda \in \mathbb{C}_{-\alpha}^+$,

$$\|\varphi\|_H^2 \leq \frac{C(\alpha)}{(\operatorname{Re} \lambda + \alpha)^2} (\|(\lambda I - A^*)\varphi\|_H^2 + \|B^* \varphi\|_U^2) \quad \text{for } \varphi \in H_1.$$

Since $\alpha > 0$ was arbitrarily taken, the above shows that the pair (A^*, B^*) satisfies (HESI). This completes the proof of Theorem 3. \square

We now prove Theorem 1.

The proof of Theorem 1. First, since A^* is normal, we can write E^{A^*} for the unique spectral measure corresponding to A^* , which is provided by the spectral theorem (e.g., [22, Theorem 6.47, Chapter 6]). We divide the proof into two steps.

Step 1. We prove conclusion (i).

By Theorem 3, we have the necessity. The remainder is to show the sufficiency. We suppose that the pair (A^*, B^*) satisfies $(\text{HESI})_\beta$ for some $\beta > 0$. Then, there is $\beta > 0$ and $C(\beta) > 0$ satisfying (1.5). Without loss of generality, we can assume that $\mathbb{C}_{-\varepsilon}^+ \cap \sigma(A^*) \neq \emptyset$ for each $\varepsilon > 0$. (Otherwise, there is $\varepsilon^* > 0$ such that $\mathbb{C}_{-\varepsilon^*}^+ \cap \sigma(A^*) = \emptyset$. Then, it follows from [20, Corollary 3.4, Section 3, Chapter IV and Lemma 1.9, Section 1, Chapter V] that $S^*(\cdot)$ and $S(\cdot)$ are exponentially stable. Thus, by taking the feedback law as 0, we obtain the sufficiency.) We take $\omega > 0$ such that $\|S^*(t)\|_{\mathcal{L}(H)} \leq C(\omega)e^{\omega t}$ for each $t \in \mathbb{R}^+$. The remainder of the proof in this step is organized into two sub-steps.

Sub-step 1.1. We prove that for each $\beta^ \in (0, \beta)$, there are $T := T(\beta^*) > 0$ and $C(T, \beta^*) > 0$ such that*

$$\|S^*(T)E^{A^*}(\mathbb{C}_{-\beta^*}^+)\varphi\|_H^2 \leq C(T, \beta^*) \int_0^T \|B^*E^{A^*}(\mathbb{C}_{-\beta^*}^+)S^*(t)E^{A^*}(\mathbb{C}_{-\beta^*}^+)\varphi\|_U^2 dt \quad \text{for } \varphi \in H_1. \quad (3.4)$$

We arbitrarily fix $\beta^* \in (0, \beta)$. Since $\sigma(A^*) \cap \mathbb{C}_{-\gamma}^+$ is bounded (by our assumption), it follows from the spectral theorem that

- (a) $A^*E^{A^*}(\mathbb{C}_{-\beta^*}^+) = E^{A^*}(\mathbb{C}_{-\beta^*}^+)A^*$ is a bounded operator on H ;
- (b) $E^{A^*}(\mathbb{C}_{-\beta^*}^+)H$ is an invariant subspace of $A^*E^{A^*}(\mathbb{C}_{-\beta^*}^+)$.

Thus, we have $E^{A^*}(\mathbb{C}_{-\beta^*}^+)H = D(A^*E^{A^*}(\mathbb{C}_{-\beta^*}^+))$ and $A^*E^{A^*}(\mathbb{C}_{-\beta^*}^+) \in \mathcal{L}(E^{A^*}(\mathbb{C}_{-\beta^*}^+)H)$. Consequently, $E^{A^*}(\mathbb{C}_{-\beta^*}^+)S^*(\cdot) (= S^*(\cdot)E^{A^*}(\mathbb{C}_{-\beta^*}^+))$ is the C_0 -semigroup on H , which is generated by $A^*E^{A^*}(\mathbb{C}_{-\beta^*}^+)$. Based on (b), we know that $E^{A^*}(\mathbb{C}_{-\beta^*}^+)H$ is an invariant subspace of $S^*(t)$ for each $t \in \mathbb{R}^+$, so $E^{A^*}(\mathbb{C}_{-\beta^*}^+)S^*(\cdot)$ is a C_0 -semigroup on $E^{A^*}(\mathbb{C}_{-\beta^*}^+)H$. Meanwhile, based on (a) and assumption (A_2) (see also (i) in Remark 1), $B^*E^{A^*}(\mathbb{C}_{-\beta^*}^+) \in \mathcal{L}(E^{A^*}(\mathbb{C}_{-\beta^*}^+)H; U)$. Indeed, we have

$$\|B^*E^{A^*}(\mathbb{C}_{-\beta^*}^+)\varphi\|_U \leq \|(B^*(\rho_0 I - A^*)^{-1})\|_{\mathcal{L}(H; U)} \|(\rho_0 I - A^*)E^{A^*}(\mathbb{C}_{-\beta^*}^+)\varphi\|_{\mathcal{L}(H)} \|\varphi\|_H$$

for all $\varphi \in E^{A^*}(\mathbb{C}_{-\beta^*}^+)H$.

Now, we claim that there is $C(\beta, \beta^*) > 0$ such that if $\lambda \in \mathbb{C}$,

$$\|\varphi\|_H^2 \leq C(\beta, \beta^*) (\|(\lambda I - A^*E^{A^*}(\mathbb{C}_{-\beta^*}^+))\varphi\|_H^2 + \|B^*E^{A^*}(\mathbb{C}_{-\beta^*}^+)\varphi\|_U^2) \quad \text{for } \varphi \in E^{A^*}(\mathbb{C}_{-\beta^*}^+)H. \quad (3.5)$$

Indeed, (1.5) implies that for $\lambda \in \overline{\mathbb{C}_{-\frac{\beta^*+\beta}{2}}^+} (\subset \mathbb{C}_{-\beta}^+)$ and $\varphi \in E^{A^*}(\mathbb{C}_{-\beta^*}^+)H$,

$$\|\varphi\|_H^2 \leq \frac{4C(\beta)}{(\beta - \beta^*)^2} (\|(\lambda I - A^* E^{A^*}(\mathbb{C}_{-\beta^*}^+))\varphi\|_H^2 + \|B^* E^{A^*}(\mathbb{C}_{-\beta^*}^+)\varphi\|_U^2). \quad (3.6)$$

However, according to the spectral theorem,

$$\|(\lambda I - A^* E^{A^*}(\mathbb{C}_{-\beta^*}^+))^{-1}\varphi\|_H^2 \leq \frac{4}{(\beta - \beta^*)^2} \|\varphi\|_H^2 \quad \text{for } \lambda \in \mathbb{C}_{-\frac{\beta^* + \beta}{2}}^-, \varphi \in E^{A^*}(\mathbb{C}_{-\beta^*}^+)H,$$

which implies that

$$\|\varphi\|_H^2 \leq \frac{4}{(\beta - \beta^*)^2} \|(\lambda I - A^* E^{A^*}(\mathbb{C}_{-\beta^*}^+))\varphi\|_H^2 \quad \text{for } \lambda \in \mathbb{C}_{-\frac{\beta^* + \beta}{2}}^-, \varphi \in E^{A^*}(\mathbb{C}_{-\beta^*}^+)H.$$

This fact and (3.6) lead to (3.5) with $C(\beta, \beta^*) := \frac{4(1+C(\beta))}{(\beta - \beta^*)^2}$. Since

$$(A^* E^{A^*}(\mathbb{C}_{-\beta^*}^+), B^* E^{A^*}(\mathbb{C}_{-\beta^*}^+)) \in \mathcal{L}(E^{A^*}(\mathbb{C}_{-\beta^*}^+)H) \times \mathcal{L}(E^{A^*}(\mathbb{C}_{-\beta^*}^+)H; U),$$

(3.5) and Lemma 5 imply that there is $T := T(\beta^*) > 0$ and $C(T, \beta^*) > 0$ that satisfies (3.4). (Here, we use that $\|S^*(T)E^{A^*}(\mathbb{C}_{-\beta^*}^+)\varphi\|_H \leq C(\omega)e^{\omega T}\|E^{A^*}(\mathbb{C}_{-\beta^*}^+)\varphi\|_H$ for each $\varphi \in H_1$.) Thus, we have completed *Sub-step 1.1*.

Sub-step 1.2. We prove that the system $[A, B]$ is stabilizable.

We arbitrarily fix $\beta^* \in (0, \beta)$. The facts are as follows: First, according to *Substep 1.1*, there are constants $T_0 > 0$ and $C(T_0, \beta^*) > 0$ such that (3.4) (where $T = T_0$ and $C(T, \beta^*) = C(T_0, \beta^*)$) holds. Second, one can easily check that $A^* E^{A^*}(\overline{\mathbb{C}_{-\beta^*}^-})$ (with its domain $E^{A^*}(\overline{\mathbb{C}_{-\beta^*}^-})H_1$) generates the C_0 -semigroup $S^*(\cdot)E^{A^*}(\overline{\mathbb{C}_{-\beta^*}^-})$ on $E^{A^*}(\overline{\mathbb{C}_{-\beta^*}^-})H$. Third, the spectral theorem implies that $\sigma(A^* E^{A^*}(\overline{\mathbb{C}_{-\beta^*}^-})) \subset \overline{\mathbb{C}_{-\beta^*}^-}$ and thus

$$\sup\{\operatorname{Re}\lambda : \lambda \in \sigma(A^* E^{A^*}(\overline{\mathbb{C}_{-\beta^*}^-}))\} \leq -\beta^*. \quad (3.7)$$

Since A^* is a normal operator and $E^{A^*}(\overline{\mathbb{C}_{-\beta^*}^-})$ is an orthogonal projection, we have that $A^* E^{A^*}(\overline{\mathbb{C}_{-\beta^*}^-})$ is normal on $E^{A^*}(\overline{\mathbb{C}_{-\beta^*}^-})H$. Then, by (3.7) and [20, Corollary 3.4, Section 3, Chapter IV and Lemma 1.9, Section 1, Chapter V], we obtain that for each $\eta \in (0, \beta^*)$, there is $C(\eta) > 0$ such that

$$\|S^*(t)E^{A^*}(\overline{\mathbb{C}_{-\beta^*}^-})\|_{\mathcal{L}(H)} \leq C(\eta)e^{-\eta t} \quad \text{for } t \in \mathbb{R}^+. \quad (3.8)$$

Now, we claim that for each $T \geq 2T_0$, there is $C(T_0, \omega, \beta^*, \eta) > 0$ such that

$$\|S^*(T)\varphi\|_H^2 \leq C(T_0, \omega, \beta^*, \eta) \left(\int_0^T \|B^* S^*(t)\varphi\|_U^2 dt + e^{-2\eta T} \|\varphi\|_H^2 \right) \quad \text{for } \varphi \in H_1. \quad (3.9)$$

For this purpose, we arbitrarily fix $T \geq 2T_0$. Let $N := [T/T_0]$, then $N \geq 2$ and $NT_0 \leq T < (N+1)T_0$. Based on (3.4) (with $T = T_0$ and $C(T, \beta^*) = C(T_0, \beta^*)$) and (3.8), we have that, for each $\varphi \in H_1$,

$$\begin{aligned} \|S^*(T)\varphi\|_H^2 &= \|S^*(T - NT_0)S^*(NT_0)\varphi\|_H^2 \leq (C(\omega))^2 e^{2\omega T_0} \|S^*(NT_0)\varphi\|_H^2 \\ &= (C(\omega))^2 e^{2\omega T_0} \left(\|S^*(T_0)E^{A^*}(\mathbb{C}_{-\beta^*}^+)S^*((N-1)T_0)\varphi\|_H^2 + \|S^*(NT_0)E^{A^*}(\overline{\mathbb{C}_{-\beta^*}^-})\varphi\|_H^2 \right) \\ &\leq (C(\omega))^2 e^{2\omega T_0} \left(C(T_0, \beta^*) \int_0^{T_0} \|B^* S^*((N-1)T_0 + t)E^{A^*}(\mathbb{C}_{-\beta^*}^+)\varphi\|_U^2 dt + \|S^*(NT_0)E^{A^*}(\overline{\mathbb{C}_{-\beta^*}^-})\varphi\|_H^2 \right) \\ &\leq I_1 + I_2, \end{aligned} \quad (3.10)$$

where

$$I_1 := (C(\omega))^2 e^{2\omega T_0} \left(2C(T_0, \beta^*) \int_{(N-1)T_0}^{NT_0} \|B^* S^*(t) \varphi\|_U^2 dt + (C(\eta))^2 e^{-2\eta NT_0} \|\varphi\|_H^2 \right),$$

and

$$I_2 := 2(C(\omega))^2 C(T_0, \beta^*) e^{2\omega T_0} \int_0^{T_0} \|B^* S^*((N-1)T_0 + t) E^{A^*} (\overline{\mathbb{C}_{-\beta^*}^-}) \varphi\|_H^2 dt.$$

Based on assumption (A_3) and (3.8), one can directly check that

$$I_1 \leq 2(C(\omega))^2 C(T_0, \beta^*) e^{2\omega T_0} \int_0^T \|B^* S^*(t) \varphi\|_U^2 dt + (C(\omega) C(\eta))^2 e^{2(\omega+\eta)T_0} e^{-2\eta T} \|\varphi\|_H^2; \quad (3.11)$$

$$\begin{aligned} I_2 &= 2(C(\omega))^2 C(T_0, \beta^*) e^{2\omega T_0} \int_0^{T_0} \|B^* S^*(t) S^*((N-1)T_0) E^{A^*} (\overline{\mathbb{C}_{-\beta^*}^-}) \varphi\|_H^2 dt \\ &\leq 2(C(\omega))^2 C(T_0, \beta^*) C(T_0) e^{2\omega T_0} \|S^*((N-1)T_0) E^{A^*} (\overline{\mathbb{C}_{-\beta^*}^-}) \varphi\|_H^2 \\ &\leq 2(C(\omega))^2 C(T_0, \beta^*) C(T_0) (C(\eta))^2 e^{2\omega T_0} e^{-2\eta(N-1)T_0} \|E^{A^*} (\overline{\mathbb{C}_{-\beta^*}^-}) \varphi\|_H^2 \\ &\leq 2(C(\omega) C(\eta))^2 C(T_0, \beta^*) C(T_0) e^{2(\omega+2\eta)T_0} e^{-2\eta T} \|\varphi\|_H^2. \end{aligned}$$

The above, (3.10), and (3.11) lead to (3.9) with

$$C(T_0, \omega, \beta^*, \eta) := (C(\omega))^2 e^{2\omega T_0} \max\{2C(T_0, \beta^*), (C(\eta))^2 e^{2\eta T_0} (1 + 2C(T_0, \beta^*) C(T_0) e^{2\eta T_0})\}.$$

Using (3.9), we can find $\hat{T} > 0$ such that

$$\|S^*(\hat{T}) \varphi\|_H^2 \leq C(T_0, \omega, \beta^*, \eta) \int_0^{\hat{T}} \|B^* S^*(t) \varphi\|_U^2 dt + \frac{1}{2} \|\varphi\|_H^2 \quad \text{for } \varphi \in H_1.$$

The combination of this result and Lemma 1 yield that system $[A, B]$ is stabilizable. Thus, we have completed *Sub-step 1.2* and *Step 1*.

Step 2. We prove conclusion (ii).

The necessity is proven in Theorem 3. Thus, we must only prove the sufficiency. Suppose that the pair (A^*, B^*) satisfies (HESI). Then, for each $\beta > 0$, there is $C(\beta) > 0$ such that

$$\|\varphi\|_H^2 \leq \frac{C(\beta)}{(\operatorname{Re} \lambda + 2\beta)^2} (\|(\lambda I - A^*) \varphi\|_H^2 + \|B \varphi\|_U^2) \quad \text{for } \lambda \in \mathbb{C}_{-2\beta}^+, \varphi \in H_1,$$

which yields

$$\|\varphi\|_H^2 \leq \frac{C(\beta)}{(\operatorname{Re} \lambda + \beta)^2} (\|(\lambda I - (A^* + \beta I)) \varphi\|_H^2 + \|B \varphi\|_U^2) \quad \text{for } \lambda \in \mathbb{C}_{-\beta}^+, \varphi \in H_1.$$

The combination of this result and conclusion (i) of Theorem 1 imply that $[A + \beta I, B]$ is stabilizable. (Since A is normal, $A + \beta I$ is also normal for any $\beta \in \mathbb{R}$.) Since $\beta > 0$ can be arbitrarily chosen, Lemma 6 implies that $[A, B]$ is rapidly stabilizable.

Thus, we complete the proof of Theorem 1. \square

Finally, we prove Theorem 2.

Proof of Theorem 2. By Theorem 3, we only need to show the sufficiency. For this purpose, we suppose that there are constants $\beta > 0$ and $C(\beta) > 0$ that satisfy (1.5). We divide the remainder of the proof into two steps.

Step 1. We prove the stabilizability for system $[A, B]$ for case (a).

First, (a) in Theorem 2 implies that there is $C_A > 0$ such that $\|S^*(t)\|_{\mathcal{L}(H)} \leq C_A$ for each $t \geq 0$.

Next, we arbitrarily fix $\tau > 0$. Let

$$\Theta_\tau(t) := \begin{cases} \sin\left(\frac{\pi t}{\tau}\right), & \text{if } t \in [0, \tau], \\ 0, & \text{if } t \in \mathbb{R} \setminus [0, \tau], \end{cases} \quad (3.12)$$

which satisfies

$$\Theta_\tau(\cdot) \in H^1(\mathbb{R}) \quad (3.13)$$

and

$$\Theta'_\tau(t) = \begin{cases} \frac{\pi}{\tau} \cos\left(\frac{\pi t}{\tau}\right), & \text{if } t \in [0, \tau], \\ 0, & \text{if } t \in \mathbb{R} \setminus [0, \tau]. \end{cases} \quad (3.14)$$

Now, we arbitrarily fix $\varphi \in H_1$ and define

$$w(t) := \Theta_\tau(t)z(t), \quad t \in \mathbb{R}, \quad \text{where } z(t) := \begin{cases} S^*(t)\varphi, & \text{if } t \geq 0, \\ 0, & \text{if } t < 0. \end{cases} \quad (3.15)$$

By assumption, we have $\|z(t)\|_H \leq C_A \|\varphi\|_H$ for each $t \in \mathbb{R}$. The combination of this result, (3.15) and (3.13) imply that $w \in H^1(\mathbb{R}; H)$ and

$$w'(t) = \begin{cases} A^*w(t) + \Theta'_\tau(t)z(t) & \text{if } t \geq 0, \\ 0, & \text{if } t < 0. \end{cases} \quad (3.16)$$

Thus, we can apply the Fourier transform to (3.16) to obtain

$$(\mathrm{i}\varsigma I - A^*)\mathcal{F}[w](\varsigma) = \mathcal{F}[g](\varsigma) \quad \text{a.e. } \varsigma \in \mathbb{R}, \quad (3.17)$$

where $g(\cdot) := \Theta'_\tau(\cdot)z(\cdot)$, and $\mathcal{F}[f]$ denotes the Fourier transform of $f \in L^2(\mathbb{R}; H)$. Integrating (1.5) (where φ and λ are replaced by $\mathcal{F}[w](\varsigma)$ and $\mathrm{i}\varsigma$) with respect to ς over \mathbb{R} , using (3.17), we obtain

$$\int_{\mathbb{R}} \|\mathcal{F}[w](\varsigma)\|_H^2 d\varsigma \leq \beta^{-2} C(\beta) \left(\int_{\mathbb{R}} \|\mathcal{F}[g](\varsigma)\|_H^2 d\varsigma + \int_{\mathbb{R}} \|B^* \mathcal{F}[w](\varsigma)\|_U^2 d\varsigma \right). \quad (3.18)$$

It is clear that

$$\mathcal{F}[g](\cdot) \in L^2(\mathbb{R}; H). \quad (3.19)$$

Thus, the first integral on the right-hand side of (3.18) is finite. We now claim

$$B^* \mathcal{F}[w](\cdot) \in L^2(\mathbb{R}; U). \quad (3.20)$$

Two facts ensuring (3.20) are as follows: First, we have

$$\mathcal{F}[B^* w](\cdot) \in L^2(\mathbb{R}; U). \quad (3.21)$$

Indeed, it follows from (3.13) and assumption (A_2) (see (i) in Remark 1) that

$$\begin{aligned} \|B^* w(\cdot)\|_U &= \|\Theta_\tau(\cdot) B^* (\rho_0 I - A^*)^{-1} S^*(\cdot) (\rho_0 I - A^*) \varphi\|_U \\ &\leq C_A \|\Theta_\tau(\cdot)\| \|B^* (\rho_0 I - A^*)^{-1}\|_{\mathcal{L}(H; U)} \|(\rho_0 I - A^*) \varphi\|_H \in L^2(\mathbb{R}^+; \mathbb{R}), \end{aligned}$$

while it follows from (3.12) and (3.15) that $\|B^* w(t)\|_U = 0$ when $t \in \overline{\mathbb{R}^-}$. These relations lead to (3.21). Second, one can check that for a.e. $\varsigma \in \mathbb{R}$,

$$\begin{aligned} \mathcal{F}[B^* w](\varsigma) &= \int_{\mathbb{R}} e^{-i\varsigma t} B^* w(t) dt = \int_{\mathbb{R}^+} e^{-i\varsigma t} B^* (\rho_0 I - A^*)^{-1} \Theta_\tau(t) S^*(t) (\rho_0 I - A^*) \varphi dt \\ &= B^* (\rho_0 I - A^*)^{-1} \int_{\mathbb{R}^+} e^{-i\varsigma t} \Theta_\tau(t) S^*(t) (\rho_0 I - A^*) \varphi dt = B^* \int_{\mathbb{R}^+} e^{-i\varsigma t} \Theta_\tau(t) S^*(t) \varphi dt \\ &= B^* \int_{\mathbb{R}} e^{-i\varsigma t} w(t) dt = B^* \mathcal{F}[w](\varsigma), \end{aligned} \quad (3.22)$$

where we used the fact $w(\cdot) \in L^2(\mathbb{R}; H)$. Clearly, (3.20) follows from (3.21) and (3.22).

Now, based on (3.19), (3.20), (3.18), (3.12), (3.13) and Plancherel's theorem, we obtain

$$\int_0^\tau \|\Theta_\tau(t) z(t)\|_H^2 dt \leq \beta^{-2} C(\beta) \left(\int_0^\tau \|B^* \Theta_\tau(t) z(t)\|_U^2 dt + \int_0^\tau \|\Theta'_\tau(t) z(t)\|_H^2 dt \right). \quad (3.23)$$

We will use (3.23) to obtain the weak observability, which leads to the stabilizability of $[A, B]$. Indeed, since $\|S^*(t)\|_{\mathcal{L}(H)} \leq C_A$ for all $t \geq 0$, it follows from (3.12) that

$$\begin{aligned} \frac{1}{2} \tau C_A^{-2} \|S^*(\tau) \varphi\|_H^2 &= C_A^{-2} \int_{\frac{\tau}{4}}^{\frac{3\tau}{4}} \|S^*(\tau) \varphi\|_H^2 ds \leq \int_{\frac{\tau}{4}}^{\frac{3\tau}{4}} \|S^*(s) \varphi\|_H^2 ds; \\ \int_0^\tau \|\Theta_\tau(t) z(t)\|_H^2 dt &= \int_0^\tau \sin^2\left(\frac{\pi t}{\tau}\right) \|S^*(t) \varphi\|_H^2 dt \geq \int_{\frac{\tau}{4}}^{\frac{3\tau}{4}} \sin^2\left(\frac{\pi t}{\tau}\right) \|S^*(t) \varphi\|_H^2 dt \geq \frac{1}{2} \int_{\frac{\tau}{4}}^{\frac{3\tau}{4}} \|S^*(t) \varphi\|_H^2 dt. \end{aligned}$$

These relations yield that

$$\frac{1}{4} \tau C_A^{-2} \|S^*(\tau) \varphi\|_H^2 \leq \int_0^\tau \|\Theta_\tau(t) z(t)\|_H^2 dt. \quad (3.24)$$

Using (3.12) and (3.14), we further have

$$\begin{aligned} \int_0^\tau \|B^* \Theta_\tau(t) z(t)\|_U^2 dt &\leq \int_0^\tau \|B^* S^*(t) \varphi\|_U^2 dt; \\ \int_0^\tau \|\Theta'_\tau(t) z(t)\|_H^2 dt &\leq \tau^{-2} \pi^2 \int_0^\tau \|S^*(t) \varphi\|_H^2 dt \leq \tau^{-1} (\pi C_A)^2 \|\varphi\|_H^2. \end{aligned}$$

With (3.23) and (3.24), these relations imply

$$\|S^*(\tau) \varphi\|_H^2 \leq 4\tau^{-1} \beta^{-2} C(\beta) C_A^2 \int_0^\tau \|B^* S^*(t) \varphi\|_U^2 dt + 4\tau^{-2} \beta^{-2} C(\beta) \pi^2 C_A^4 \|\varphi\|_H^2. \quad (3.25)$$

Taking $\tilde{\tau} > 0$ such that $4\tilde{\tau}^{-2} \beta^{-2} C(\beta) \pi^2 C_A^4 \leq \frac{1}{2}$ in (3.25) leads to

$$\|S^*(\tilde{\tau}) \varphi\|_H^2 \leq 4\tilde{\tau}^{-1} \beta^{-2} C(\beta) C_A^2 \int_0^{\tilde{\tau}} \|B^* S^*(t) \varphi\|_U^2 dt + \frac{1}{2} \|\varphi\|_H^2.$$

The combination of this result and conclusion (i) in Lemma 1 imply that system $[A, B]$ is stabilizable.

Step 2. We prove the stabilizability for system $[A, B]$ for case (b).

According to assumption (b) (in Theorem 2), there are two closed subspaces $Q_1 := Q_1(\beta)$ and $Q_2 := Q_2(\beta)$ of H satisfying (b_1) – (b_4) (where α is replaced by β). Based on (b_1) , we can define $P : H \rightarrow Q_1$ in the following manner: $Pf = f_1$, for each $f \in H$, where $f = f_1 + f_2$ with $f_j \in Q_j$ ($j = 1, 2$). Based on assumption (b_2) , one can directly check the following:

$$PS^*(t) = S^*(t)P, \quad t \geq 0; \quad PH_1 \subset H_1; \quad A^*P = PA^* \text{ on } H_1. \quad (3.26)$$

We write $S_1^*(\cdot) := PS^*(\cdot)$ and $S_2^*(\cdot) := (I - P)S^*(\cdot)$; $A_1^* = A^*|_{Q_1}$ and $A_2^* = A^*|_{Q_2}$. Based on (3.26), one can easily check that $S_j^*(\cdot)$ is the C_0 -semigroup on Q_j , generated by A_j^* , $j = 1, 2$.

Two facts are as follows: First, by (b_3) , we have that $A_1^* \in \mathcal{L}(Q_1)$, which implies $D(A_1^*) = Q_1$. The combination of this result and assumption (A_2) (see also (i) in Remark 1) yield that

$$B^*P|_{Q_1} = (B^*(\rho_0 I - A^*)^{-1})(\rho_0 I - A_1^*) \in \mathcal{L}(Q_1; U). \quad (3.27)$$

Second, by (b_3) , we have $\sigma(A_1^*) \subset \mathbb{C}_{-\beta}^+$ which implies

$$\overline{\mathbb{C}_{-\beta}^-} \subset \rho(A_1^*). \quad (3.28)$$

Based on (3.28) and (1.5), for each $\lambda \in \mathbb{C}$, there is $C(\lambda) > 0$ such that

$$\|\varphi\|_H^2 \leq C(\lambda) (\|(\lambda I - A_1^*)\varphi\|_H^2 + \|B^*P\varphi\|_U^2) \quad \text{for } \varphi \in Q_1. \quad (3.29)$$

(Indeed, if $\lambda \in \mathbb{C}_{-\beta}^+$, we can use (1.5) to find $C(\lambda)$ above, while when $\lambda \notin \mathbb{C}_{-\beta}^+$, we can choose $C(\lambda) = \|(\lambda I - A_1^*)^{-1}\|_{\mathcal{L}(H)}^2$ because of (3.28).)

Now, based on (3.29), (3.27) and boundedness of A_1^* , we can apply Lemma 5 to find $T > 0$ and $C(T) > 0$ such that

$$\|S_1^*(T)\varphi\|_H^2 \leq C(T) \int_0^T \|B^*PS_1^*(t)\varphi\|_U^2 dt \quad \text{for } \varphi \in Q_1.$$

The combination of this result and (b_2) yield

$$\begin{aligned} \|PS^*(T)\varphi\|_H^2 &= \|S_1^*(T)P\varphi\|_H^2 \\ &\leq C(T) \int_0^T \|B^*PS_1^*(t)P\varphi\|_U^2 dt = C(T) \int_0^T \|B^*PS^*(t)\varphi\|_U^2 dt \quad \text{for } \varphi \in H_1. \end{aligned} \quad (3.30)$$

Based on (b_4) and (3.30), using a similar method as in *Sub-step 1.2* of the proof of Theorem 1, there are $\hat{T} > 0$ and $C(\hat{T}) > 0$ such that

$$\|S^*(\hat{T})\varphi\|_H^2 \leq \hat{C}(\hat{T}) \int_0^{\hat{T}} \|B^*S^*(t)\|_U^2 dt + \frac{1}{2}\|\varphi\|_H^2 \quad \text{for } \varphi \in H_1,$$

The combination of this result and (i) of Lemma 1 lead to the stabilizability of system $[A, B]$.

Hence, we have completed the proof of Theorem 2. \square

4. Applications

This section provides several applications of our main theorems to specific control PDEs. We start with introducing the concept of ‘thick sets’: *We say a measurable subset $E \subset \mathbb{R}^N$ (with $N \in \mathbb{N}^+$) to be thick, if there is $\varepsilon > 0$ and $L > 0$ such that*

$$|E \cap Q_L(x)| \geq \varepsilon L^N \quad \text{for each } x \in \mathbb{R}^N,$$

where $Q_L(x)$ denotes the closed cube in \mathbb{R}^N , centered at x and of side length L , and $|E \cap Q_L(x)|$ denotes the Lebesgue measure of $E \cap Q_L(x)$. Then, we quote the following lemma, which is related to the thick sets and will be used later:

Lemma 7. ([34, Theorem 1]) *If ω is a thick set, then for each $R > 0$, there exists $C(R, \omega) > 0$ such that, for each $f \in L^2(\mathbb{R}^N)$ with $\text{supp}(\mathcal{F}[f]) \subset [-R, R]^N$, the following estimate holds*

$$\|f\|_{L^2(\mathbb{R}^N)} \leq C(R, \omega) \|\chi_\omega f\|_{L^2(\mathbb{R}^N)}.$$

4.1. Ginzburg–Landau equation in \mathbb{R}^N

Let $a \in \mathbb{R}^+$ and $b \in \mathbb{R}$. Let $\omega \subset \mathbb{R}^N$ (with $N \in \mathbb{N}^+$) be a measurable set with its characteristic function χ_ω . We consider the controlled Ginzburg–Landau equation in \mathbb{R}^N :

$$\begin{cases} y_t = (a + ib)\Delta y + \chi_\omega u & \text{in } \mathbb{R}^+ \times \mathbb{R}^N, \\ y(0, \cdot) = y_0(\cdot) \in L^2(\mathbb{R}^N), \end{cases} \quad (4.1)$$

where $u \in L^2(\mathbb{R}^+; L^2(\mathbb{R}^N))$. Equation (4.1) can be put into our framework by setting: $H = U = L^2(\mathbb{R}^N)$; $A := (a + ib)\Delta$, with its domain $H^2(\mathbb{R}^N)$; $B := \chi_\omega$. One can directly check that (A_1) – (A_3) are true. One can also check that A is normal (using Fourier transform) and generates an analytic semigroup. The latter and (ii) of Remark 5 yield that $\sigma(A) \cap \mathbb{C}_{-\gamma}^+$ is bounded for each $\gamma > 0$. Therefore, Theorem 1 can be applied. It provides the following results:

Theorem 5. *If ω is a thick set, then equation (4.1) is rapidly stabilizable.*

Proof. According to (ii) in Theorem 1 and Definition 3, we only need to show the following: For each $\beta > 0$, there is $C(\beta) > 0$ such that

$$\|\varphi\|_{L^2(\mathbb{R}^N)}^2 \leq C(\beta)(\|(\lambda I - (a - ib)\Delta)\varphi\|_{L^2(\mathbb{R}^N)}^2 + \|\chi_\omega \varphi\|_{L^2(\mathbb{R}^N)}^2) \quad \text{for } \lambda \in \mathbb{C}_{-\beta}^+, \varphi \in H^2(\mathbb{R}^N). \quad (4.2)$$

For this purpose, we arbitrarily fix $\varphi \in H^2(\mathbb{R}^N)$, $\beta > 0$ and $\lambda \in \mathbb{C}_{-\beta}^+$. Plancherel's theorem implies that

$$\begin{aligned} \|(\lambda I - (a - ib)\Delta)\varphi\|_{L^2(\mathbb{R}^N)}^2 &= \|(\lambda + (a - ib)|\xi|^2)\mathcal{F}[\varphi]\|_{L^2(\mathbb{R}^N)}^2 \\ &= \int_{\mathbb{R}^N} (|\operatorname{Re}\lambda + a|\xi|^2|^2 + |\operatorname{Im}\lambda - b|\xi|^2|^2) |\mathcal{F}[\varphi](\xi)|^2 d\xi \\ &\geq \beta^2 \int_{\mathbb{R}^N} \chi_{|\xi| \geq \sqrt{2a-1}\beta} |\mathcal{F}[\varphi](\xi)|^2 d\xi. \end{aligned}$$

The combination of this result, Lemma 7, and the Plancherel theorem yield that there is $C_1(\beta, a, \omega) > 0$ such that

$$\begin{aligned} \|\varphi\|_{L^2(\mathbb{R}^N)}^2 &\leq \beta^{-2} \|(\lambda I - (a - ib)\Delta)\varphi\|_{L^2(\mathbb{R}^N)}^2 + \int_{\mathbb{R}^N} \chi_{|\xi| \leq \sqrt{2a-1}\beta} |\mathcal{F}[\varphi](\xi)|^2 d\xi \\ &\leq \beta^{-2} \|(\lambda I - (a - ib)\Delta)\varphi\|_{L^2(\mathbb{R}^N)}^2 + C_1(\beta, a, \omega) \int_{\mathbb{R}^N} \chi_\omega(x) |\mathcal{F}^{-1}[\chi_{|\xi| \leq \sqrt{2a-1}\beta} \mathcal{F}[\varphi]](x)|^2 dx \\ &\leq \beta^{-2} \|(\lambda I - (a - ib)\Delta)\varphi\|_{L^2(\mathbb{R}^N)}^2 + 2C_1(\beta, a, \omega) \int_{\mathbb{R}^N} \chi_\omega(x) |\varphi(x)|^2 dx \\ &\quad + 2C_1(\beta, a, \omega) \int_{\mathbb{R}^N} |\mathcal{F}^{-1}[\chi_{|\xi| \geq \sqrt{2a-1}\beta} \mathcal{F}[\varphi]](x)|^2 dx \\ &\leq \beta^{-2} (2C_1(\beta, a, \omega) + 1) \|(\lambda I - (a - ib)\Delta)\varphi\|_{L^2(\mathbb{R}^N)}^2 + 2C_1(\beta, a, \omega) \|\chi_\omega \varphi\|_{L^2(\mathbb{R}^N)}^2, \end{aligned}$$

which leads to (4.2) and completes the proof. \square

Remark 7. Theorem 5 tells us that, if ω is a thick set, then, for each $\alpha > 0$, there is $K := K(\alpha) \in \mathcal{L}(L^2(\mathbb{R}^N))$ such that the closed-loop system: $y_t = (a + ib)\Delta y + \chi_\omega K^\dagger y$ in $\mathbb{R}^+ \times \mathbb{R}^N$ is exponentially stable with decay rate α . In this case, the corresponding feedback law can be constructed by the classical LQR theory on $[0, +\infty)$.

4.2. Fractional heat equation in \mathbb{R}^N

Let $s \in (0, 1)$ and $N \in \mathbb{N}^+$. Let $\omega \subset \mathbb{R}^N$ be a measurable set with its characteristic function χ_ω . We consider the following controlled fractional heat equation:

$$\begin{cases} \partial_t y + (-\Delta)^{\frac{s}{2}} y = \chi_\omega u & \text{in } \mathbb{R}^+ \times \mathbb{R}^N, \\ y(0, \cdot) = y_0(\cdot) \in L^2(\mathbb{R}^N), \end{cases} \quad (4.3)$$

where $u \in L^2(\mathbb{R}^+; \mathbb{R}^N)$ and $(-\Delta)^{\frac{s}{2}}$ is defined by

$$(-\Delta)^{\frac{s}{2}} f := \mathcal{F}^{-1} [|\xi|^s \mathcal{F}[f]], \quad f \in C_c^\infty(\mathbb{R}^N). \quad (4.4)$$

Equation (4.3) can be put into our framework by setting: $H = U = L^2(\mathbb{R}^N)$; $A := -(-\Delta)^{\frac{s}{2}}$ with domain $D(A) := H^s(\mathbb{R}^N)$; $B := \chi_\omega(\in \mathcal{L}(L^2(\mathbb{R}^N)))$. One can easily check that (A_1) – (A_3) hold, $A^* = A$, and A generates an analytic semigroup. The latter and (ii) of Remark 5 yield that $\sigma(A) \cap \mathbb{C}_{-\gamma}^+$ is bounded for each $\gamma > 0$. Therefore, Theorem 1 can be applied. Moreover, the spectral measure E^{A^*} (corresponding to A^*) is provided as follows: for each Borel set $\Omega \subset \mathbb{C}$,

$$[E^{A^*}(\Omega)]f := \mathcal{F}^{-1} [\chi_{-\{|\xi|^s \in (\Omega \cap \overline{\mathbb{R}^-})\}} \mathcal{F}[f]] \quad \text{for } f \in L^2(\mathbb{R}^N). \quad (4.5)$$

Regarding equation (4.3), we have the following: *Fact 1:* If ω is thick, then equation (4.3) is not null controllable in general (see [35]). *Fact 2:* Equation (4.3) is rapidly stabilizable if and only if ω is thick (see [8, Theorem 4.5]), where it was proven by the weak observability inequality. Next, we utilize (ii) of Theorem 1 to provide a proof for the sufficiency by the frequency-domain inequality.

Proposition 1. *If ω is a thick set, then for each $\beta > 0$, there is $C(\beta) > 0$ such that*

$$\|\varphi\|_{L^2(\mathbb{R}^N)}^2 \leq \frac{C(\beta)}{(\operatorname{Re} \lambda + \beta)^2} \left(\|(\lambda I - A^*)\varphi\|_{L^2(\mathbb{R}^N)}^2 + \|B^*\varphi\|_{L^2(\mathbb{R}^N)}^2 \right) \quad \text{for } \lambda \in \mathbb{C}_{-\beta}^+, \varphi \in H^s(\mathbb{R}^N). \quad (4.6)$$

Proof. We arbitrarily fix $\beta > 0$. Let $k := k(\beta) = [\beta] + 2$. We set $E_k := E^{A^*}(\{z \in \mathbb{C} : \operatorname{Re} z \in [-k, 0]\})$, where E^{A^*} is provided in (4.5). Then, [7, Lemma 3.1] implies that

$$\|E_k \varphi\|_{L^2(\mathbb{R}^N)}^2 \leq e^{C_0 k^{\frac{1}{s}}} \|B^* E_k \varphi\|_{L^2(\mathbb{R}^N)}^2 \quad \text{for } \varphi \in L^2(\mathbb{R}^N), \quad (4.7)$$

where $C_0 > 0$ is a constant, which is independent of k . Let $A_k^* := A^* + (k - \frac{1}{2})I$. Now we claim

$$\|(I - E_k)\varphi\|_{L^2(\mathbb{R}^N)}^2 \leq \frac{1}{(\operatorname{Re} \lambda + \frac{1}{2})^2} \|(\lambda I - A_k^*)(I - E_k)\varphi\|_{L^2(\mathbb{R}^N)}^2 \quad \text{for } \lambda \in \mathbb{C}_{-\frac{1}{2}}^+, \varphi \in H^s(\mathbb{R}^N). \quad (4.8)$$

For this purpose, we arbitrarily fix $\lambda \in \mathbb{C}_{-\frac{1}{2}}^+$, $\varphi \in H^s(\mathbb{R}^N)$. Since $A = A^*$, it follows from (4.4) and (4.5) that

$$(\lambda I - A_k^*)(I - E_k)\varphi = \mathcal{F}^{-1} \left[\chi_{\{|\xi|^s > k\}} \left(\lambda - k + \frac{1}{2} + |\xi|^s \right) \mathcal{F}[\varphi] \right].$$

The combination of this result and the Plancherel theorem yield

$$\begin{aligned} \|(\lambda I - A_k^*)(I - E_k)\varphi\|_{L^2(\mathbb{R}^N)}^2 &= \int_{\mathbb{R}^N} \left| \chi_{\{|\xi|^s > k\}} \left(\lambda - k + \frac{1}{2} + |\xi|^s \right) \mathcal{F}[\varphi](\xi) \right|^2 d\xi \\ &\geq \left| \lambda + \frac{1}{2} \right|^2 \int_{\mathbb{R}^N} |\chi_{\{|\xi|^s > k\}} \mathcal{F}[\varphi](\xi)|^2 d\xi \geq \left(\operatorname{Re} \lambda + \frac{1}{2} \right)^2 \|(I - E_k)\varphi\|_{L^2(\mathbb{R}^N)}^2, \end{aligned}$$

which leads to (4.8).

Next, since $A^* E_k = E_k A^*$, (4.7) and (4.8) imply that for $\lambda \in \mathbb{C}_{-\frac{1}{4}}^+$ and $\varphi \in H^s(\mathbb{R}^N)$,

$$\begin{aligned} \|\varphi\|_{L^2(\mathbb{R}^N)}^2 &\leq \|(I - E_k)\varphi\|_{L^2(\mathbb{R}^N)}^2 + \|E_k \varphi\|_{L^2(\mathbb{R}^N)}^2 \leq \|(I - E_k)\varphi\|_{L^2(\mathbb{R}^N)}^2 + e^{C_0 k^{\frac{1}{s}}} \|B^* E_k \varphi\|_{L^2(\mathbb{R}^N)}^2 \\ &\leq \left(1 + 2e^{C_0 k^{\frac{1}{s}}} \right) \|(I - E_k)\varphi\|_{L^2(\mathbb{R}^N)}^2 + 2e^{C_0 k^{\frac{1}{s}}} \|B^* \varphi\|_{L^2(\mathbb{R}^N)}^2 \\ &\leq 16 \left(1 + 2e^{C_0 k^{\frac{1}{s}}} \right) \|(\lambda I - A_k^*)\varphi\|_{L^2(\mathbb{R}^N)}^2 + 2e^{C_0 k^{\frac{1}{s}}} \|B^* \varphi\|_{L^2(\mathbb{R}^N)}^2. \end{aligned}$$

The combination of this result and Definition 3 (see also Proposition 4 in Appendix) imply that there are $\gamma > 0$ and $C(\gamma, k) > 0$ such that

$$\|\varphi\|_{L^2(\mathbb{R}^N)}^2 \leq \frac{C(\gamma, k)}{(\operatorname{Re} \eta + \gamma)^2} \left(\|(\eta I - A_k^*)\varphi\|_{L^2(\mathbb{R}^N)}^2 + \|B^*\varphi\|_{L^2(\mathbb{R}^N)}^2 \right) \quad \text{for } \eta \in \mathbb{C}_{-\gamma}^+, \varphi \in H^s(\mathbb{R}^N). \quad (4.9)$$

Since $\beta + 1 \leq k$, letting $\eta = \lambda + k - \frac{1}{2}$ in (4.9) leads to (4.6) with $C(\beta) := C(\gamma, k)$. This completes the proof. \square

Remark 8. Proposition 1, together with Theorem 1, yields that, for each $\alpha > 0$, there is a $K := K(\alpha) \in \mathcal{L}(L^2(\mathbb{R}^N))$ such that the closed-loop system $y_t + (-\Delta)^{\frac{\alpha}{2}} y = \chi_\omega K y$ in $\mathbb{R}^+ \times \mathbb{R}^N$ is exponentially stable with the decay rate α . Again, the corresponding feedback law can be constructed by LQR theory on $[0, +\infty)$.

4.3. One-dimensional heat equation with point-wise controls

Let $c > \pi^2$, $x_0 \in (0, 1)$, and $\delta(\cdot)$ be the Dirac function at $x = 0 \in \mathbb{R}$. We consider the following heat equation with point-wise controls:

$$\begin{cases} y_t = (\partial_x^2 + c)y + \delta(\cdot - x_0)u & \text{in } \mathbb{R}^+ \times (0, 1), \\ y(\cdot, 0) = y(\cdot, 1) = 0 & \text{in } \mathbb{R}^+, \\ y(0, \cdot) = y_0(\cdot) \in L^2(0, 1), \end{cases} \quad (4.10)$$

where $u \in L^2(\mathbb{R}^+)$. Equation (4.10) can be put into our framework by setting: $H := L^2(0, 1)$; $U := \mathbb{R}$; $A := \partial_x^2 + c$ (with its domain $D(A) := H_1 = H_0^1(0, 1) \cap H^2(0, 1)$); $B := \delta(\cdot - x_0)$ (which is considered as a bounded operator from \mathbb{R} to in H_{-1} , see (i) in Remark 1 for the definition of H_{-1}). One can easily check that assumptions (A_1) – (A_3) are true, A is self-adjoint, and A generates an analytic semigroup (see [8, Example 4.3]). The latter and (ii) of Remark 5 yield that $\sigma(A) \cap \mathbb{C}_{-\gamma}^+$ is bounded for each $\gamma > 0$. Therefore, Theorem 1 can be applied. Moreover, since A generates a compact semigroup, assumption (b) in Theorem 2 holds (see (ii) in Remark 6). Thus, Theorem 2 can also be applied.

Regarding equation (4.10), we have the following: *Fact 1:* For some irrational number $x_0 \in (0, 1)$, equation (4.10) is not null controllable (see [8, Example 4.3]). *Fact 2:* Equation (4.10) is rapidly stabilizable if and only if $x_0 \in (0, 1)$ is irrational (see [8, Theorem 4.9]). Thus, a natural question is *what happens about equation (4.10) when x_0 is rational?* We will give the answer for this question by using Theorem 1 or Theorem 2.

Theorem 6. *System (4.10) is stabilizable in $L^2(0, 1)$ if and only if $x_0 \notin \{k/n \in (0, 1) : k \in \mathbb{N}^+, n = 1, 2, \dots, [\sqrt{c}/\pi]\}$.*

Proof. First of all, we give two facts. *Fact One:* The eigenvalues and the corresponding normalized eigenfunctions of A^* are as follows: $\lambda_n := -(n\pi)^2 + c$; $e_n(x) := \sqrt{2} \sin(n\pi x)$, $x \in (0, 1)$, $n \in \mathbb{N}^+$. *Fact Two:* $B^*\varphi = \varphi(x_0)$ for $\varphi \in H_1$ which is bounded from H_1 to \mathbb{R} .

Now we prove the necessity. By contradiction, we suppose that there is $x_0 = k/n \in (0, 1)$ (with $n \in \{1, 2, \dots, [\sqrt{c}/\pi]\}$, $k \in \mathbb{N}^+$) such that system (4.10) is stabilizable. Then, we have $e_n(x_0) = \sqrt{2} \sin(n\pi x_0) = 0$, which implies that $B^*e_n = 0$. Since $1 \leq n \leq [\sqrt{c}/\pi]$, one can directly check that

$$\lambda_n = -(n\pi)^2 + c \geq -([\sqrt{c}/\pi]\pi)^2 + c \geq 0, \quad \text{and thus } \lambda_n \in \mathbb{C}_0^+.$$

For any $\beta > 0$, the right-hand side of (1.5) (where $\lambda = \lambda_n$ and $\varphi = e_n$) is 0, while the left-hand side of (1.5) (where $\lambda = \lambda_n$ and $\varphi = e_n$) is 1. So (1.5) is not true for any $\beta > 0$. Thus, it follows from (i) of Theorem 1

(or Theorem 2) that system (4.10) is not stabilizable, which causes a contradiction and completes the proof of the necessity.

We next prove the sufficiency. We will show that if $x_0 \notin \{k/n \in (0, 1) : k \in \mathbb{N}^+, n = 1, 2, \dots, [\sqrt{c}/\pi]\}$, then there is $\beta > 0$ such that the pair (A^*, B^*) satisfies $(\text{HESI})_\beta$. We let $n^* := [\sqrt{c}/\pi]$. Since $x_0 \notin \{k/n \in (0, 1) : k \in \mathbb{N}^+, n = 1, 2, \dots, n^*\}$, we have

$$e_n(x_0) \neq 0 \text{ for each } n \in \{1, 2, \dots, n^*\}. \quad (4.11)$$

We define the following projection operator: $P_{n^*}\varphi := \sum_{n=1}^{n^*} \langle \varphi, e_n \rangle_H e_n$ ($\varphi \in H$). Let $\beta := -\frac{1}{2}\lambda_{n^*+1}$. Since $n^* = [\sqrt{c}/\pi]$, one can directly check that $\beta > 0$. Now, we claim that there is $C(\beta) > 0$ such that

$$\|\varphi\|_H^2 \leq C(\beta)(\|(\lambda I - A^*)\varphi\|_H^2 + \|B^*\varphi\|_U^2) \text{ for } \lambda \in \mathbb{C}_{-\beta}^+, \varphi \in H_1. \quad (4.12)$$

Indeed, based on (4.11), one can easily check that $\text{Ker}((\lambda I - A^*P_{n^*}, B^*P_{n^*})^\top|_{P_{n^*}H}) = \{0\}$ for $\lambda \in \mathbb{C}$. Since $P_{n^*}H$ and U are finite-dimensional, this, along with Kalman's controllability condition, yields that system $[P_{n^*}A, P_{n^*}B]$ is controllable (consequently, is rapidly stabilizable) on $P_{n^*}H$. Thus, by (ii) in Theorem 3, there is $C_0(\beta) > 0$ such that

$$\|P_{n^*}\varphi\|_H^2 \leq C_0(\beta)(\|(\lambda I - A^*)P_{n^*}\varphi\|_H^2 + \|B^*P_{n^*}\varphi\|_U^2) \text{ for } \lambda \in \mathbb{C}_{-\beta}^+, \varphi \in H_1. \quad (4.13)$$

Meanwhile, by the definition of β , one can directly check that, for $\lambda \in \mathbb{C}_{-\beta}^+$,

$$\|(\lambda I - A^*)\varphi\|_H^2 \geq \|(\lambda I - A^*)(I - P_{n^*})\varphi\|_H^2 \geq \beta^2\|(I - P_{n^*})\varphi\|_H^2 \text{ for } \varphi \in H_1. \quad (4.14)$$

Moreover, by the Hölder inequality, there exists $C_1(\beta) > 0$ such that, for $\lambda \in \mathbb{C}_{-\beta}^+$ and $\varphi \in H_1$,

$$\begin{aligned} \|B^*P_{n^*}\varphi\|_U^2 &\leq 2\|B^*\varphi\|_U^2 + 2\|B^*(I - P_{n^*})\varphi\|_U^2 \leq 2\|B^*\varphi\|_U^2 + 4\left|\sum_{n=n^*+1}^{+\infty} a_n \sin(n\pi x_0)\right|^2 \\ &\leq 2\|B^*\varphi\|_U^2 + 4\left(\sum_{n=n^*+1}^{+\infty} |a_n(\lambda - \lambda_n)|^2\right)\left(\sum_{n=n^*+1}^{+\infty} |\beta + \lambda_n|^{-2}\right) \\ &\leq 2\|B^*\varphi\|_U^2 + C_1(\beta)\|(\lambda I - A^*)\varphi\|_H^2. \end{aligned}$$

With (4.13) and (4.14), the above yields that for any $\lambda \in \mathbb{C}_{-\beta}^+$ and $\varphi \in H_1$,

$$\|\varphi\|_H^2 \leq \|P_{n^*}\varphi\|_H^2 + \|(I - P_{n^*})\varphi\|_H^2 \leq (1 + C_0(\beta))(1 + \beta^{-2} + C_1(\beta))\|(\lambda I - A^*)\varphi\|_H^2 + 2C_0(\beta)\|B^*\varphi\|_U^2,$$

which leads to (4.12). So (A^*, B^*) satisfies $(\text{HESI})_\beta$. Then, according to (i) in Theorem 1 (or Theorem 2), system (4.10) is stabilizable. Thus, the sufficiency has been proven.

Hence, we have completed the proof of Theorem 6. \square

Remark 9. From Theorem 6, we can claim that, when $x_0 \notin \{k/n \in (0, 1) : k \in \mathbb{N}^+, n = 1, 2, \dots, [\sqrt{c}/\pi]\}$, there is $K \in \mathcal{L}(L^2(0, 1))$ such that the closed-loop system: $y_t = (\partial_x^2 + c)y + \delta(\cdot - x_0)Ky$ in $\mathbb{R}^+ \times (0, 1)$ is exponentially stable. The feedback law K can be constructed by two manners: The first one is LQR theory on the infinite time horizon with unbounded controllers (see [18]); The second one is the Fattorini's strategy (see [2, 6]).

4.4. Heat equation with time delay in \mathbb{R}^N

Let $\tau > 0$, $N \in \mathbb{N}$. Let $\omega \subset \mathbb{R}^N$ be a subset of positive measure. Let χ_ω be the characteristic function of ω . We consider the following controlled heat equation with time delay in \mathbb{R}^N :

$$\begin{cases} y_t(t, x) = (\Delta - I)y(t, x) + y(t - \tau, x) + \chi_\omega u(t, x), & (t, x) \in \mathbb{R}^+ \times \mathbb{R}^N, \\ y(0, x) = y_0(x), & x \in \mathbb{R}^N, \\ y(t, x) = f(t, x), & (t, x) \in (-\tau, 0) \times \mathbb{R}^N, \end{cases} \quad (4.15)$$

where I is the identity operator on $L^2(\mathbb{R}^N)$, $y_0 \in L^2(\mathbb{R}^N)$, $f \in L^2(-\tau, 0; L^2(\mathbb{R}^N))$ and $u \in L^2(\mathbb{R}^+; L^2(\mathbb{R}^N))$. It is well known that (4.15) has a unique solution in $C([0, +\infty); L^2(\mathbb{R}^N))$ (see [36, Theorem 2.1]). We denote this solution by $y(\cdot, \cdot; y_0, f, u)$ if it is viewed as a real-valued function of (t, x) , while by $y(\cdot; y_0, f, u)$ if it is viewed as an $L^2(\mathbb{R}^N)$ -valued function of t .

In this subsection, we will apply Theorem 2 to show the stabilizability of (4.15) in the sense of the following definition:

Definition 4. System (4.15) is said to be stabilizable if there is $\alpha > 0$, $C > 0$ and a feedback law $K \in \mathcal{L}(L^2(\mathbb{R}^N) \times L^2(-\tau, 0; L^2(\mathbb{R}^N)); L^2(\mathbb{R}^N))$ such that for any $y_0 \in L^2(\mathbb{R}^N)$ and $f \in L^2(-\tau, 0; L^2(\mathbb{R}^N))$, the solution $y_K(\cdot; y_0, f)$ of the closed-loop system:

$$\begin{cases} y_t(t, x) = (\Delta - I)y(t, x) + y(t - \tau, x) + \chi_\omega [K(y(t), y(t + \cdot))^\top](x), & (t, x) \in \mathbb{R}^+ \times \mathbb{R}^N, \\ y(0, x) = y_0(x), & x \in \mathbb{R}^N, \\ y(t, x) = f(t, x), & (t, x) \in (-\tau, 0) \times \mathbb{R}^N, \end{cases} \quad (4.16)$$

satisfies that

$$\|y_K(t; y_0, f)\|_{L^2(\mathbb{R}^N)} \leq C e^{-\alpha t} (\|y_0\|_{L^2(\mathbb{R}^N)} + \|f\|_{L^2(-\tau, 0; L^2(\mathbb{R}^N))}) \quad \text{for } t \in \mathbb{R}^+. \quad (4.17)$$

For this purpose, we first embed system (4.15) into our framework. Let $\mathcal{U} := L^2(\mathbb{R}^N)$ and $\mathcal{H} := L^2(\mathbb{R}^N) \times L^2(0, 1; L^2(\mathbb{R}^N))$ with the inner product:

$$\langle (\varphi_1, \varphi_2)^\top, (f_1, f_2)^\top \rangle_{\mathcal{H}} := \langle \varphi_1, f_1 \rangle_{L^2(\mathbb{R}^N)} + \tau \int_0^1 \langle \varphi_2(\cdot, \rho), f_2(\cdot, \rho) \rangle_{L^2(\mathbb{R}^N)} d\rho$$

for $(\varphi_1, \varphi_2)^\top, (f_1, f_2)^\top \in L^2(\mathbb{R}^N) \times L^2(0, 1; L^2(\mathbb{R}^N))$. We define an unbounded operator on \mathcal{H} by

$$\begin{cases} \mathcal{A}\varphi := \begin{pmatrix} \mathcal{A}_1\varphi \\ \mathcal{A}_2\varphi \end{pmatrix} \quad \text{for } \varphi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} \in D(\mathcal{A}), \\ \text{with } \begin{pmatrix} [\mathcal{A}_1\varphi](x) \\ [\mathcal{A}_2\varphi](x, \rho) \end{pmatrix} = \begin{pmatrix} (\Delta - I)\varphi_1(x) + \varphi_2(x, 1) \\ -\tau^{-1}\partial_\rho\varphi_2(x, \rho) \end{pmatrix}, \quad (x, \rho) \in \mathbb{R}^N \times (0, 1), \\ D(\mathcal{A}) := \{(f_1, f_2)^\top \in \mathcal{H} : f_1 \in H^2(\mathbb{R}^N), f_2 \in H^1(0, 1; L^2(\mathbb{R}^N)), f_1(x) = f_2(x, 0) \text{ a.e. } x \in \mathbb{R}^N\}. \end{cases} \quad (4.18)$$

Let $\mathcal{B} := (\chi_\omega, 0)^\top$. Then one can easily check that $D(\mathcal{A})$ is dense in \mathcal{H} and $\mathcal{B} \in \mathcal{L}(\mathcal{U}; \mathcal{H})$. Moreover, we have the following result:

Lemma 8. *The operator \mathcal{A} generates a C_0 -semigroup of contractions on \mathcal{H} .*

Proof. We first prove that \mathcal{A} is dissipative, i.e.,

$$\operatorname{Re}\langle \mathcal{A}\varphi, \varphi \rangle_{\mathcal{H}} \leq 0 \quad \text{for } \varphi \in D(\mathcal{A}). \quad (4.19)$$

Indeed, it follows from (4.18) that for $\varphi = (\varphi_1, \varphi_2)^\top \in D(\mathcal{A})$,

$$\begin{aligned} \langle \mathcal{A}\varphi, \varphi \rangle_{\mathcal{H}} &= \left\langle \begin{pmatrix} \mathcal{A}_1\varphi \\ \mathcal{A}_2\varphi \end{pmatrix}, \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} \right\rangle_{\mathcal{H}} \\ &= \langle (\Delta - I)\varphi_1, \varphi_1 \rangle_{L^2(\mathbb{R}^N)} + \langle \varphi_2(\cdot, 1), \varphi_1(\cdot) \rangle_{L^2(\mathbb{R}^N)} - \langle \partial_\rho \varphi_2, \varphi_2 \rangle_{L^2(0,1;L^2(\mathbb{R}^N))} \\ &\leq -\frac{1}{2}\|\varphi_1\|_{L^2(\mathbb{R}^N)}^2 + \langle \varphi_2(\cdot, 1), \varphi_1(\cdot) \rangle_{L^2(\mathbb{R}^N)} - \frac{1}{2}\|\varphi_2(\cdot, 1)\|_{L^2(\mathbb{R}^N)}^2 \leq 0, \end{aligned}$$

which leads to (4.19).

Next, we show that for any $\lambda > 0$ and $f \in \mathcal{H}$, there is a unique $\varphi = (\varphi_1, \varphi_2)^\top \in D(\mathcal{A})$ such that

$$(\lambda I - \mathcal{A})\varphi = f. \quad (4.20)$$

To this end, we arbitrarily fix $\lambda > 0$ and $f = (f_1, f_2)^\top \in \mathcal{H}$. Then by (4.18), we see that (4.20) is equivalent to the system:

$$\begin{cases} (\lambda I - (\Delta - I))\varphi_1(x) - \varphi_2(x, 1) = f_1(x), & x \in \mathbb{R}^N, \\ \lambda \varphi_2(x, \rho) + \tau^{-1} \partial_\rho \varphi_2(x, \rho) = f_2(x, \rho), & (x, \rho) \in \mathbb{R}^N \times (0, 1), \\ \varphi_2(x, 0) = \varphi_1(x), & x \in \mathbb{R}^N. \end{cases} \quad (4.21)$$

Meanwhile, after some simple computations, one can easily see that the (4.21) is equivalent to the following system:

$$\begin{cases} \varphi_2(x, \rho) = e^{-\tau\lambda\rho}\varphi_1(x) + \tau \int_0^\rho e^{-\tau\lambda(\rho-s)} f_2(x, s) ds, & (x, \rho) \in \mathbb{R}^N \times (0, 1), \\ ((\lambda + 1 - e^{-\tau\lambda})I - \Delta)\varphi_1(x) = f_1(x) + \tau \int_0^1 e^{-\tau\lambda(1-s)} f_2(x, s) ds, & x \in \mathbb{R}^N. \end{cases} \quad (4.22)$$

However, since $\lambda + 1 - e^{-\tau\lambda} > 0$, we can use the Fourier transform to obtain that $(\lambda + 1 - e^{-\tau\lambda})I - \Delta$ is invertible from $H^2(\mathbb{R}^N)$ to $L^2(\mathbb{R}^N)$. Therefore, the second equation in (4.22) has a unique solution φ_1 in $H^2(\mathbb{R}^N)$. This, along with the first equation in (4.22), implies that (4.20) has a unique solution in $D(\mathcal{A})$.

Finally, by (4.19) and (4.20), we can apply the Lumer-Phillips theorem (see [24, Theorem 4.3, Chapter 1]) to conclude that \mathcal{A} generates a C_0 -semigroup of contractions on \mathcal{H} . This completes the proof of Lemma 8. \square

Now we consider the following system in \mathcal{H} :

$$\begin{cases} Y_t(t) = \mathcal{A}Y(t) + \mathcal{B}u(t) & t \in \mathbb{R}^+, \\ Y(0) = Y_0, \end{cases} \quad (4.23)$$

where $u \in L^2(\mathbb{R}^+; \mathcal{U})$ and $Y_0 \in \mathcal{H}$.

Remark 10. By Lemma 8, we have the following two facts: Fact 1. For each $Y_0 \in \mathcal{H}$ and each $u \in L^2(\mathbb{R}^+; \mathcal{U})$, system (4.23) has a unique solution $Y(\cdot; Y_0, u)$ in $C([0, +\infty); \mathcal{H})$; Fact 2. The pair $(\mathcal{A}, \mathcal{B})$ satisfies assumptions (A_1) – (A_3) (with $(H, U) = (\mathcal{H}, \mathcal{U})$).

We now give relationships between (4.15) and (4.23), which show that (4.15) can be embedded into our framework (4.23).

Proposition 2. (i) The relationship of solutions to (4.15) and (4.23) is as follows:

- (a) If $y(\cdot; y_0, f, u)$, with $y_0 \in L^2(\mathbb{R}^N)$, $f \in L^2(-\tau, 0; L^2(\mathbb{R}^N))$ and $u \in L^2(\mathbb{R}^+; L^2(\mathbb{R}^N))$, is the solution of (4.15), then $Y(\cdot) := (Y_1(\cdot), Y_2(\cdot))^\top$ is the solution of (4.23) with $Y_0 := (y_0, f(-\cdot\tau))^\top$, where $[Y_1(t)](x) := y(t, x; y_0, f, u)$ and $[Y_2(t)](x, \rho) := y(t - \rho\tau, x; y_0, f, u)$ for $(t, x, \rho) \in \mathbb{R}^+ \times \mathbb{R}^N \times (0, 1)$.
- (b) If $Y(\cdot; Y_0, u) = (Y_1(\cdot; Y_0, u), Y_2(\cdot; Y_0, u))^\top$, with $Y_0 := (h, g)^\top \in \mathcal{H}$ and $u \in L^2(\mathbb{R}^+; \mathcal{U})$, is the solution of (4.23), then the function $y(\cdot, \cdot)$ defined by

$$y(t, x) := \begin{cases} [Y_1(t; Y_0, u)](x), & (t, x) \in \mathbb{R}^+ \times \mathbb{R}^N, \\ g(x, -t\tau^{-1}), & (t, x) \in (-\tau, 0) \times \mathbb{R}^N, \end{cases} \quad (4.24)$$

is the solution of (4.15) with $y_0 = h$ and $f(t, x) = g(x, -t\tau^{-1})$ for $(t, x) \in (-\tau, 0) \times \mathbb{R}^N$. Moreover,

$$[Y_2(t; Y_0, u)](x, \rho) = y(t - \rho\tau, x) \text{ for } (t, x, \rho) \in \mathbb{R}^+ \times \mathbb{R}^N \times (0, 1). \quad (4.25)$$

(ii) The relationship of the stabilizability of (4.15) and (4.23) is as follows: System (4.15) is stabilizable in the sense of Definition 4 if and only if system (4.23) is stabilizable, i.e., there is a $K \in \mathcal{L}(\mathcal{H}; \mathcal{U})$ such that $e^{(\mathcal{A} + \mathcal{B}K)t}$ is exponentially stable.

Proof. We first prove (a) of (i). For this purpose, we arbitrarily fix $y_0 \in L^2(\mathbb{R}^N)$, $f \in L^2(-\tau, 0; L^2(\mathbb{R}^N))$ and $u \in L^2(\mathbb{R}^+; L^2(\mathbb{R}^N))$. Let

$$z(t, x, \rho; y_0, f, u) := y(t - \rho\tau, x; y_0, f, u) \text{ for } (t, x, \rho) \in \mathbb{R}^+ \times \mathbb{R}^N \times (0, 1).$$

Then, by (4.15), $(y(\cdot, \cdot; y_0, f, u), z(\cdot, \cdot, \cdot; y_0, f, u))$ satisfies the following equation:

$$\begin{cases} y_t(t, x) = (\Delta - I)y(t, x) + z(t, x, 1) + \chi_\omega u(t, x), & (t, x) \in \mathbb{R}^+ \times \mathbb{R}^N, \\ z_t(t, x, \rho) = -\tau^{-1} \partial_\rho z(t, x, \rho), & (t, x, \rho) \in \mathbb{R}^+ \times \mathbb{R}^N \times (0, 1), \\ z(t, x, 0) = y(t, x), & (t, x) \in \mathbb{R}^+ \times \mathbb{R}^N, \\ y(0, x) = y_0(x), & x \in \mathbb{R}^N, \\ z(0, x, \rho) = f(-\rho\tau, x), & (x, \rho) \in \mathbb{R}^N \times (0, 1). \end{cases} \quad (4.26)$$

Let

$$[Y_1(t)](x) := y(t, x; y_0, f, u); [Y_2(t)](x, \rho) := z(t, x, \rho; y_0, f, u) \text{ for } (t, x, \rho) \in \mathbb{R}^+ \times \mathbb{R}^N \times (0, 1).$$

It is clear that

$$[Y_1(0)](x) := y_0(x); [Y_2(0)](x, \rho) := f(-\rho\tau, x) \text{ for } x \in \mathbb{R}^N, \rho \in (0, 1).$$

Then, by (4.18) and (4.26), we see that $Y(\cdot) := (Y_1(\cdot), Y_2(\cdot))^\top$ is the solution to (4.23), with $Y_0 := (y_0, f(-\cdot\tau))^\top$, which leads to (a) of (i).

We next prove (b) of (i). To this end, we arbitrarily fix $Y_0 = (h, g)^\top \in \mathcal{H}$ and $u \in \mathcal{U}$. Let $Y(\cdot; Y_0, u) = (Y_1(\cdot; Y_0, u), Y_2(\cdot; Y_0, u))^\top$ be the solution of (4.23), with $Y_0 = (h, g)^\top \in \mathcal{H}$. Then we have

$$[Y_1(0; Y_0, u)](x) = h(x); [Y_2(0; Y_0, u)](x, \rho) = g(x, \rho) \text{ for } x \in \mathbb{R}^N, \rho \in (0, 1).$$

Let $y(\cdot, \cdot)$ be defined by (4.24). Then it is clear that $y(t, x) := [Y_1(t; Y_0, u)](x)$ in $(t, x) \in \mathbb{R}^+ \times \mathbb{R}^N$. Let $z(t, x, \rho) := [Y_2(t; Y_0, u)](x, \rho)$ for $(t, x, \rho) \in \mathbb{R}^+ \times \mathbb{R}^N \times (0, 1)$. By (4.18), we have that the pair $(y(\cdot, \cdot), z(\cdot, \cdot, \cdot))$

satisfies that equation (4.26) with $y_0 = h$ and $f(t, x) = g(x, -t\tau^{-1})$ for $(t, x) \in (-\tau, 0) \times \mathbb{R}^N$. By the second, third and fifth equations in (4.26), we can directly check that

$$z(t, x, \rho) = \begin{cases} y(t - \rho\tau, x) & \text{if } t \geq \rho\tau, \\ g(x, -(t - \rho\tau)\tau^{-1}) & \text{if } t < \rho\tau, \end{cases} \quad (t, x, \rho) \in \mathbb{R}^+ \times \mathbb{R}^N \times (0, 1). \quad (4.27)$$

This, along with the first equation in (4.26) and (4.24), implies

$$y_t(t, x) = (\Delta - I)y(t, x) + y(t - \tau, x) + \chi_\omega u(t, x), \quad (t, x) \in \mathbb{R}^+ \times \mathbb{R}^N.$$

Since $y(0) = Y_1(0; Y_0, u) = h$, the above shows that the function $y(\cdot, \cdot)$, defined by (4.24), is the solution of (4.15), with $y_0(x) = h(x)$ and $f(t, x) = g(x, -t\tau^{-1})$ for $(t, x) \in (-\tau, 0) \times \mathbb{R}^N$. Meanwhile, (4.25) follows from (4.24), (4.27) and the definition of $z(\cdot, \cdot, \cdot)$ at once. These lead to (b) of (i).

We finally prove (ii). First, we suppose that system (4.23) is stabilizable, i.e., there is $\alpha > 0$, $C > 0$, and $K \in \mathcal{L}(\mathcal{H}; \mathcal{U})$ such that for any $Y_0 \in \mathcal{H}$, the solution $Y^K(\cdot; Y_0)$ of the following equation:

$$\begin{cases} Y_t(t) = \mathcal{A}Y(t) + \mathcal{B}KY(t), & t \in \mathbb{R}^+, \\ Y(0) = Y_0, \end{cases}$$

satisfies

$$\|Y^K(t; Y_0)\|_{\mathcal{H}} \leq Ce^{-\alpha t} \|Y_0\|_{\mathcal{H}} \quad \text{for any } t \geq 0. \quad (4.28)$$

We arbitrarily fix $(y_0, f)^\top \in L^2(\mathbb{R}^N) \times L^2(-\tau, 0; L^2(\mathbb{R}^N))$. Then we define

$$h(x) := y_0(x); \quad g(x, \rho) := f(-\rho\tau, x) \quad \text{for } (x, \rho) \in \mathbb{R}^N \times (0, 1). \quad (4.29)$$

Let $Y_0 := (h, g)^\top$. We write

$$Y^K(t; Y_0) := (Y_1^K(t), Y_2^K(t))^\top; \quad u^K(t) := KY^K(t; Y_0) \quad \text{for } t \in \mathbb{R}^+. \quad (4.30)$$

Since $K \in \mathcal{L}(\mathcal{H}; \mathcal{U})$, it follows from (4.30) and (4.28) that $u^K \in L^2(\mathbb{R}^+; \mathcal{U})$. Then, according to (b) in (i) of Proposition 2, the function, defined by

$$y^K(t, x) := \begin{cases} [Y_1^K(t)](x) & (t, x) \in \mathbb{R}^+ \times \mathbb{R}^N, \\ g(x, -t\tau^{-1}) & (t, x) \in (-\tau, 0) \times \mathbb{R}^N, \end{cases} \quad (4.31)$$

satisfies

$$\begin{aligned} y_t^K(t, x) &= (\Delta - I)y^K(t, x) + y^K(t - \tau, x) + \chi_\omega u^K(t, x), \quad (t, x) \in \mathbb{R}^+ \times \mathbb{R}^N; \\ y^K(0, x) &= h(x), \quad x \in \mathbb{R}^N; \\ y^K(t, x) &= g(x, -t\tau^{-1}), \quad (t, x) \in (-\tau, 0) \times \mathbb{R}^N; \\ [Y_2^K(t)](x, \rho) &= y^K(t - \rho\tau, x) \quad \text{for } (t, x, \rho) \in \mathbb{R}^+ \times \mathbb{R}^N \times (0, 1). \end{aligned} \quad (4.32)$$

These, along with (4.31), (4.28) and (4.29), yield

$$\begin{aligned}
\|y^K(t)\|_{L^2(\mathbb{R}^N)}^2 &\leq \|Y^K(t; Y_0)\|_{\mathcal{H}}^2 \leq C^2 e^{-2\alpha t} \left(\|h\|_{L^2(\mathbb{R}^N)}^2 + \int_0^1 \|g(\cdot, \rho)\|_{L^2(\mathbb{R}^N)}^2 d\rho \right) \\
&\leq C^2 e^{-2\alpha t} \left(\|h\|_{L^2(\mathbb{R}^N)}^2 + \int_{-\tau}^0 \|f(s, \cdot)\|_{L^2(\mathbb{R}^N)}^2 ds \right) \quad \text{for } t \in \mathbb{R}^+.
\end{aligned} \tag{4.33}$$

Next, we will rewrite u^K in the feedback form required by Definition 4. To this end, we define an operator Λ_τ in the manner: for each $\hat{f} \in L^2(0, 1; L^2(\mathbb{R}^N))$, we let

$$\Lambda_\tau[\hat{f}](t, x) := \hat{f}(x, -t\tau^{-1}), \quad (t, x) \in (-\tau, 0) \times \mathbb{R}^N.$$

One can directly check that $\Lambda_\tau : L^2(0, 1; L^2(\mathbb{R}^N)) \rightarrow L^2(-\tau, 0; L^2(\mathbb{R}^N))$ is isomorphic. Then, by (4.30) and (4.32), we find that

$$u^K(t) = K \begin{pmatrix} I & 0 \\ 0 & \Lambda_\tau^{-1} \end{pmatrix} \begin{pmatrix} y^K(t) \\ y^K(t + \cdot) \end{pmatrix}.$$

The above, along with (4.33) and Definition 4, shows that system (4.15) is stabilizable in the sense of Definition 4 with the feedback law $K \begin{pmatrix} I & 0 \\ 0 & \Lambda^{-1} \end{pmatrix}$.

Conversely, we suppose that system (4.15) is stabilizable in the sense of Definition 4, i.e., there is $\alpha > 0$, $C > 0$ and a feedback law $K \in \mathcal{L}(L^2(\mathbb{R}^N) \times L^2(-\tau, 0; L^2(\mathbb{R}^N)); L^2(\mathbb{R}^N))$ such that for any $y_0 \in L^2(\mathbb{R}^N)$ and $f \in L^2(-\tau, 0; L^2(\mathbb{R}^N))$, the solution $y_K(\cdot; y_0, f)$ of the closed-loop system (4.16) satisfies (4.17). We arbitrarily fix $Y_0 = (h, g)^\top \in \mathcal{H}$. Let

$$y_0(x) = h(x); \quad f(t, x) = g(x, -t\tau^{-1}), \quad (x, t) \in \mathbb{R}^N \times (-\tau, 0).$$

Then $(y_0, f) \in L^2(\mathbb{R}^N) \times L^2(-\tau, 0; L^2(\mathbb{R}^N))$. Let

$$u_K(t) := K(y_K(t; y_0, f), y(t + \cdot; y_0, f))^\top, \quad t \geq 0. \tag{4.34}$$

By (4.34), one can easily check that

$$y_K(t, x; y_0, f) = y(t, x; y_0, f, u_K), \quad t \geq 0, x \in \mathbb{R}^N. \tag{4.35}$$

Given $t \geq 0$, $x \in \mathbb{R}^N$ and $\rho \in (0, 1)$, we let

$$[Y_1^K(t)](x) := y(t, x; y_0, f, u_K); \quad [Y_2^K(t)](x, \rho) := y(t - \rho\tau, x; y_0, f, u_K). \tag{4.36}$$

By (4.36), (4.35) and (a) in (i) of Proposition 2, we see that $Y^K(\cdot) := (Y_1^K(\cdot), Y_2^K(\cdot))^\top$ is the solution of (4.23), with $Y_0 := (y_0, f(-\cdot\tau))^\top$ and $u = u^K$. This, together with (4.34), (4.36) and (4.17), yields

$$\|Y^K(t)\|_{\mathcal{H}}^2 = \|y_K(t)\|_{L^2(\mathbb{R}^N)}^2 + \int_0^1 \|y_K(t - \rho\tau, \cdot; y_0, f)\|_{L^2(\mathbb{R}^N)}^2 d\rho \leq C^2(1 + e^{2\tau})e^{-2\alpha t} \|Y_0\|_{\mathcal{H}}^2 \quad \text{for } t \in \mathbb{R}^+.$$

With (4.34), the above leads to

$$\mathcal{U}_{ad}(Y_0) := \{u \in L^2(\mathbb{R}^+; \mathcal{U}) : Y(\cdot; Y_0, u) \in L^2(\mathbb{R}^+; \mathcal{H})\} \neq \emptyset, \tag{4.37}$$

where $Y(\cdot; Y_0, u)$ is the solution of (4.23), with the initial value Y_0 and the control u . By (4.37), we can apply [8, Proposition 3.9] to see that system (4.23) is stabilizable.

Thus, we complete the proof of Proposition 2. \square

Remark 11. From Proposition 2 and its proof, if system (4.15) is stabilizable in the sense of Definition 4, then a feedback law can be constructed by LQR theory on the infinite time horizon.

Next, we will show that system (4.15), with $u = 0$, is not exponentially stable in $L^2(\mathbb{R}^N)$. This shows the importance of studying the stabilizability of system (4.15).

Proposition 3. *System (4.15), with $u = 0$, is not exponentially stable.*

Proof. First of all, according to (i) in Proposition 2, system (4.15) (with $u = 0$) is exponentially stable if and only if system (4.23) (with $u = 0$) is exponentially stable.

Now, we let $\lambda_j := j^{-1}$ ($j \in \mathbb{N}^+$). Then let $f^j := (f_1^j, 0) \in \mathcal{H}$, with $\|f^j\|_{\mathcal{H}} = 1$ and $\text{supp}(\mathcal{F}[f_1^j]) \subset \{\xi \in \mathbb{R}^N \mid |\xi| \leq \sqrt{\lambda_j}\}$. It is obvious that $\lambda_j + 1 - e^{-\tau\lambda_j} > \lambda_j > 0$ for all $j \in \mathbb{N}^+$. Thus, equation (4.21) (equivalently, (4.22)), with $\lambda = \lambda_j$ and $f = f^j$, has a unique solution $\varphi^j := (\varphi_1^j, \varphi_2^j)$ in $D(\mathcal{A})$. Applying the Fourier transform to the second equation in (4.22) (with $\lambda = \lambda_j$), using the fact that $f_2 = 0$ and the Plancherel theorem, we obtain that as $j \rightarrow +\infty$,

$$\begin{aligned} \|\varphi^j\|_{\mathcal{H}}^2 &\geq \|\varphi_1^j\|_{L^2(\mathbb{R}^N)}^2 = \int_{\mathbb{R}^N} (\lambda_j + 1 - e^{-\tau\lambda_j} + |\xi|^2)^{-2} |\mathcal{F}[f_1^j](\xi)|^2 d\xi \\ &\geq (2\lambda_j + 1 - e^{-\tau\lambda_j})^{-2} \int_{\mathbb{R}^N} |\mathcal{F}[f_1^j](\xi)|^2 d\xi = (2j^{-1} + (1 - e^{-\tau j^{-1}}))^{-1} \rightarrow +\infty. \end{aligned}$$

This yields that $\|(\lambda_j I - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} \rightarrow +\infty$ as $j \rightarrow +\infty$. Thus, we have

$$\sup_{\lambda \in \mathbb{C}, \text{Re } \lambda > 0} \|(\lambda I - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} = +\infty.$$

The above, along with [20, Theorem 1.11, Chapter V], yields that (4.23) (with $u = 0$) is not exponentially stable. This completes the proof. \square

The main result of this subsection concerns with the stabilizability of system (4.15), which is proved by making use of Theorem 2.

Theorem 7. *If ω is a thick set, then system (4.15) is stabilizable in the sense of Definition 4.*

Proof. According to [37, Theorem 1.1 and Remark (b₁)], the controlled heat equation $y_t = \Delta y + \chi_\omega u$ (in $\mathbb{R}^+ \times \mathbb{R}^N$), with ω a thick set, is null controllable, and consequently is rapidly stabilizable. Thus, by (ii) in Theorem 3, we can conclude that for each $\gamma > 0$, there is $C_0(\gamma) > 0$ such that

$$\|\varphi\|_{L^2(\mathbb{R}^n)}^2 \leq C_0(\gamma) (\|(\lambda I - \Delta)\varphi\|_{L^2(\mathbb{R}^N)}^2 + \|\chi_\omega \varphi\|_{L^2(\mathbb{R}^N)}^2) \quad \text{for } \lambda \in \mathbb{C}_{-\gamma}^+, \quad \varphi \in H^2(\mathbb{R}^N). \quad (4.38)$$

Meanwhile, one can easily check that $\mathcal{B}^* = (\chi_\omega, 0)$ and

$$\left\{ \begin{array}{l} \mathcal{A}^* \varphi := \begin{pmatrix} \mathcal{A}_1^\# \varphi \\ \mathcal{A}_2^\# \varphi \end{pmatrix} \text{ for } \varphi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} \in D(\mathcal{A}^*), \\ \text{with } \begin{pmatrix} [\mathcal{A}_1^\# \varphi](x) \\ [\mathcal{A}_2^\# \varphi](x, \rho) \end{pmatrix} = \begin{pmatrix} (\Delta - I)\varphi_1(x) + \varphi_2(x, 0) \\ \tau^{-1} \partial_\rho \varphi_2(x, \rho) \end{pmatrix} \quad (x, \rho) \in \mathbb{R}^N \times (0, 1), \\ D(\mathcal{A}^*) := \{(f_1, f_2)^\top \in \mathcal{H} : f_1 \in H^2(\mathbb{R}^N), f_2 \in H^1(0, 1; L^2(\mathbb{R}^N)), f_1(x) = f_2(x, 1) \text{ a.e. } x \in \mathbb{R}^N\}. \end{array} \right. \quad (4.39)$$

We arbitrarily fix $\gamma_0 \in \mathbb{R}^+$. We claim that the pair $(\mathcal{A}^*, \mathcal{B}^*)$ satisfies $(\text{HESI})_{\gamma_0}$, i.e., there is $C(\gamma_0) > 0$ such that

$$\|\varphi\|_{\mathcal{H}}^2 \leq C(\gamma_0)(\|(\lambda I - \mathcal{A}^*)\varphi\|_{\mathcal{H}}^2 + D(\gamma_0)\|\mathcal{B}^*\varphi\|_{\mathcal{U}}^2) \text{ for } \lambda \in \mathbb{C}_{-\gamma_0}^+, \varphi \in D(\mathcal{A}^*). \quad (4.40)$$

For this purpose, we arbitrarily fix $\lambda \in \mathbb{C}_{-\gamma_0}^+$ and $\varphi = (\varphi_1, \varphi_2)^\top \in D(\mathcal{A}^*)$. Let

$$f = (f_1, f_2)^\top := (\lambda I - \mathcal{A}^*)\varphi. \quad (4.41)$$

By (4.39), we have

$$\left\{ \begin{array}{ll} ((\lambda + 1)I - \Delta)\varphi_1(x) - \varphi_2(x, 0) = f_1(x), & x \in \mathbb{R}^N, \\ (\lambda I - \tau^{-1} \partial_\rho)\varphi_2(x, \rho) = f_2(x, \rho), & (x, \rho) \in \mathbb{R}^N \times (0, 1), \\ \varphi_1(x) = \varphi_2(x, 1), & x \in \mathbb{R}^N. \end{array} \right. \quad (4.42)$$

From the second and third equations in (4.42), we deduce that for each $\rho \in (0, 1)$,

$$\varphi_2(x, \rho) = e^{-\tau\lambda(1-\rho)}\varphi_1(x) + \tau \int_\rho^1 e^{-\tau\lambda(s-\rho)} f_2(x, s) ds, \quad x \in \mathbb{R}^N. \quad (4.43)$$

The combination of (4.43) and the first equation in (4.42) leads to

$$((\lambda + 1 - e^{-\tau\lambda})I - \Delta)\varphi_1(x) = f_1(x) + \tau \int_0^1 e^{-\tau\lambda s} f_2(x, s) ds, \quad x \in \mathbb{R}^N. \quad (4.44)$$

Since

$$\operatorname{Re}(\lambda + 1 - e^{-\tau\lambda}) \geq \operatorname{Re}\lambda + 1 - e^{-\tau\operatorname{Re}\lambda} \geq -(\gamma_0 + e^{\tau\gamma_0} - 1) \text{ for each } \lambda \in \mathbb{C}_{-\gamma_0}^+$$

and

$$\gamma_0 + e^{\tau\gamma_0} - 1 > 0,$$

the combination of (4.38) (with $\gamma := \gamma_0 + e^{\tau\gamma_0} - 1$) and (4.44) yields that there is $C_1(\gamma_0) > 0$ such that

$$\begin{aligned} \|\varphi_1(\cdot)\|_{L^2(\mathbb{R}^N)}^2 &\leq C_1(\gamma_0) \left(\left\| f_1(\cdot) + \tau \int_0^1 e^{-\tau\lambda s} f_2(\cdot, s) ds \right\|_{L^2(\mathbb{R}^N)}^2 + \|\chi_\omega \varphi_1(\cdot)\|_{L^2(\mathbb{R}^N)}^2 \right) \\ &\leq 2C_1(\gamma_0)(1 + \tau e^{2\tau\gamma_0})\|f\|_{\mathcal{H}}^2 + C_1(\gamma_0)\|\mathcal{B}^*\varphi\|_{\mathcal{U}}^2. \end{aligned} \quad (4.45)$$

Meanwhile, it follows from (4.43) that

$$\begin{aligned} \|\varphi_2(\cdot, \cdot)\|_{L^2(0,1;L^2(\mathbb{R}^N))}^2 &\leq 2e^{2\tau\gamma_0} \|\varphi_1(\cdot)\|_{L^2(\mathbb{R}^N)}^2 + 2\tau^2 e^{2\tau\gamma_0} \int_0^1 \|f_2(\cdot, s)\|_{L^2(\mathbb{R}^N)}^2 ds \\ &\leq [4C_1(\gamma_0)e^{2\tau\gamma_0}(1 + \tau e^{2\tau\gamma_0}) + 2\tau e^{2\tau\gamma_0}] \|f\|_{\mathcal{H}}^2 \\ &\quad + 2C_1(\gamma_0)e^{2\tau\gamma_0} \|\mathcal{B}^* \varphi\|_{\mathcal{U}}^2. \end{aligned} \quad (4.46)$$

Now, by (4.41), (4.45) and (4.46), we obtain (4.40), with

$$C(\gamma_0) := 4C_1(\gamma_0)(1 + \tau e^{2\tau\gamma_0})(1 + \tau e^{2\tau\gamma_0}) + 2\tau^2 e^{2\tau\gamma_0}, \quad D(\gamma_0) := C_1(\gamma_0)(1 + 2e^{2\tau\gamma_0}).$$

Finally, according to Lemma 8, the C_0 -semigroup generated by \mathcal{A}^* is uniformly bounded, which leads to the condition (a) in Theorem 2. Thus, we can apply Theorem 2 and (4.40) to conclude that system (4.23) is stabilizable. Then by (ii) of Proposition 2, we find that system (4.15) is stabilizable in the sense of Definition 4. This completes the proof. \square

Remark 12. It appears to us that Theorem 7 is new. It serves well to illustrate the strength of our results.

5. Appendix

Proposition 4. Suppose that (A_1) – (A_3) hold. Then the inequalities (1.5) and (1.6) are equivalent.

Proof. We first show (1.5) \Rightarrow (1.6). Suppose that (1.5) holds. Let $\beta_1 \in (0, \beta)$. Then, it follows from (1.5) that for $\lambda \in \mathbb{C}_{-\beta_1}^+$,

$$\|\varphi\|_H^2 \leq \frac{C(\beta)}{(\beta - \beta_1)^2} (\|(\lambda I - A^*)\varphi\|_H^2 + \|B^*\varphi\|_U^2) \quad \text{for } \varphi \in H_1,$$

which leads to (1.6) with a different $C(\beta) > 0$.

Next, we show (1.6) \Rightarrow (1.5). Suppose that (1.6) is true. First, there are two constants $\omega > 0$ and $C(\omega) > 0$ such that $\|S^*(t)\| \leq C(\omega)e^{\omega t}$ for all $t \in \mathbb{R}^+$, which implies that $\mathbb{C}_\omega^+ \subset \rho(A^*)$ and that for each $\lambda \in \mathbb{C}_\omega^+$, $\|(\lambda I - A^*)^{-1}\|_{\mathcal{L}(H)} \leq C(\omega)(\operatorname{Re} \lambda - \omega)^{-1}$ (see [24, Theorem 5.3 and Remark 5.4, Section 1.5, Chapter 1]). These facts, together with the same argument in (3.3), imply that for each $\lambda \in \mathbb{C}_{\max\{\omega, 2|\beta - \omega| - \beta\}}^+$,

$$\|\varphi\|_H \leq \frac{2C(\omega)}{\operatorname{Re} \lambda + \beta} \|(\lambda I - A^*)\varphi\|_H \quad \text{for } \varphi \in H_1. \quad (5.1)$$

Meanwhile, it follows from (1.6) that for $\lambda \in \mathbb{C}_{-\beta}^+ \setminus \mathbb{C}_{\max\{\omega, 2|\beta - \omega| - \beta\}}^+$, we have

$$\|\varphi\|_H^2 \leq \frac{(\beta + \max\{\omega, 2|\beta - \omega| - \beta\})^2}{(\operatorname{Re} \lambda + \beta)^2} C(\beta) (\|(\lambda I - A^*)\varphi\|_H^2 + \|B^*\varphi\|_U^2) \quad \text{for } \varphi \in H_1,$$

where we recall that the quotient in front of $C(\beta)$ is larger than or equal to 1. The combination of this result and (5.1) yield

$$\|\varphi\|_H^2 \leq \frac{C(\beta, \omega)}{(\operatorname{Re} \lambda + \beta)^2} (\|(\lambda I - A^*)\varphi\|_H^2 + \|B^*\varphi\|_U^2) \quad \text{for } \lambda \in \mathbb{C}_{-\beta}^+, \varphi \in H_1,$$

where $C(\beta, \omega) := (\beta + \max\{\omega, 2|\beta - \omega| - \beta\})^2 C(\beta) + 4(C(\omega))^2$. This implies that (1.5) holds. This completes the proof. \square

Remark 13. The following example shows that (1.5) is more sharp than (1.6) to describe the optimal decay rate of system $[A, B]$: Suppose that $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times m}$ ($n, m \in \mathbb{N}^+$), i.e., system $[A, B]$ is a finite-dimensional system in \mathbb{R}^n . Further, we assume that $[A, B]$ is stabilizable, but not controllable. It follows that there is an invertible matrix $P \in \mathbb{R}^{n \times n}$ such that

$$PAP^{-1} = \begin{pmatrix} A_1 & A_2 \\ 0 & A_3 \end{pmatrix}, \quad PB = \begin{pmatrix} B_1 \\ 0 \end{pmatrix},$$

where $[A_1, B_1]$ is controllable and $\sigma(A_3)$ is non-empty and in \mathbb{C}_0^- , i.e., $\sigma^* := \max\{\operatorname{Re} \lambda : \lambda \in \sigma(A_3)\} < 0$. We define the optimal decay rate of system $[A, B]$ as follows:

$$\sigma^\sharp := \inf\{\alpha \in \mathbb{R} : \exists F \in \mathbb{R}^{m \times n} \text{ s.t. } A + BF \text{ is exponentially stable with the decay rate } \alpha\}.$$

One can directly check that $\sigma^\sharp = -\sigma^*$. If there is $F^\sharp \in \mathbb{R}^{m \times n}$ such that $A + BF^\sharp$ is exponentially stable with decay rate α^\sharp , then we say the optimal decay rate of $[A, B]$ can be reached, otherwise, we say that it can not be reached. We take $\lambda^* \in \sigma(A_3) \subset \sigma(A)$ such that $\operatorname{Re} \lambda^* = \sigma^*$ and suppose $\operatorname{Re} \lambda < \sigma^*$ for any $\lambda \in \sigma(A_3) \setminus \{\lambda^*\}$. If the geometric multiplicity of λ^* equals to its algebraic multiplicity, then, by the classical argument, we can directly check that the optimal decay rate of $[A, B]$ can be reached. Moreover, by the Laplace transform, we can conclude that (1.5) holds for $\beta = \sigma^\sharp$, but (1.6) holds only for $\beta < \sigma^\sharp$. If the geometric multiplicity of λ^* is strictly less than its algebraic multiplicity, then, we can directly check that the optimal decay rate of $[A, B]$ can not be reached. In this case, we can show that (1.5) and (1.6) hold only for $\beta < \sigma^\sharp$. In summary, (1.5) holds for $\beta = \alpha^\sharp$ in some cases, while (1.6) holds only for $\beta < \alpha^\sharp$. Therefore, we say that (1.5) is more sharp than (1.6).

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