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Research paper

Regularity for infinitely degenerate inhomogenous elliptic equations, Part I: The Moser Method

Lyudmila Korobenko [a], Cristian Rios [b],*, Eric Sawyer [c], Ruipeng Shen [d]

[a] Mathematics and Statistics Department, Reed College, 3203 Southeast Woodstock Boulevard, Portland, 97202-8199, OR, USA
[b] Department of Mathematics and Statistics, University of Calgary, 2500 University Dr. NW, Calgary, T2N 1N4, AB, Canada
[c] Department of Mathematics and Statistics, McMaster University, 1280 Main Street West, Hamilton, L8S 4L8, ON, Canada
[d] Center for Applied Mathematics, Tianjin University, Tianjin, 300072, Tianjin, PR China

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ABSTRACT

We show that if $\mathbb{R}^n$ is equipped with a certain non-doubling metric and an Orlicz-Sobolev inequality holds for a special family of Young functions Φ, then weak solutions to quasilinear infinitely degenerate elliptic equations of the form $-\mathrm{div}\mathcal{A}(x,u)\nabla u = \phi_0 - \mathrm{div}_A\vec{\phi}_1$ are locally bounded. This is obtained by the implementation of a Moser iteration method, what constitutes the first instance of such technique applied to infinite degenerate equations. The results presented here partially extend previously known estimates for solutions of similar equations in which the right hand side does not have a drift term. We also obtain bounds for small negative powers of nonnegative solutions, which will be applied in a subsequent paper to prove continuity of solutions. We also provide examples of geometries in which our abstract theorem is applicable. We consider the family of functions $f_{k,\sigma}(x) = |x|^{\left(\ln^{(k)}\frac{1}{|x|}\right)^{\sigma}}$, $k \in \mathbb{N}$, $\sigma > 0$, $-\infty < x < \infty$, infinitely degenerate at the origin, and show that all weak solutions to $-\mathrm{div}A(x,y,u)\nabla u = \phi(x,y) - \mathrm{div}_A\vec{\phi}_1(x,y)$, $A(x,y,z) \sim \begin{bmatrix} 1 & 0 \\ 0 & f_{k,\sigma}(x)^2 \end{bmatrix}$, with rough data $A, \phi_0, \vec{\phi}_1$, are locally bounded when $k=1$ and $0<\sigma<1$.

1. Introduction and main results

We consider divergence form quasilinear degenerate elliptic equations of the form

$$\mathcal{L}u \equiv -\nabla^{\mathrm{tr}}\mathcal{A}(x,u(x))\nabla u = \phi_0 - \mathrm{div}_A\vec{\phi}_1, \qquad x \in \Omega \tag{1}$$

in a bounded domain $\Omega \subset \mathbb{R}^n$. The matrix $\mathcal{A}(x,z) \in \mathfrak{A}(A,\Lambda,\lambda)$ uniformly in $z \in \mathbb{R}$, where $\mathfrak{A}(A,\Lambda,\lambda)$ denotes the class of *nonnegative symmetric* matrices $\tilde{A}(x)$ satisfying

$$0 \le \lambda\, \xi^{\mathrm{tr}}A(x)\xi \le \xi^{\mathrm{tr}}\tilde{A}(x)\xi \le \Lambda\, \xi^{\mathrm{tr}}A(x)\xi\ , \tag{2}$$

for a.e. $x \in \Omega$, $\xi \in \mathbb{R}^n$, and some fixed $0 < \lambda \le \Lambda < \infty$; i.e. $\mathcal{A}(x,\cdot)$ is assumed to be equivalent to a degenerate elliptic matrix $A(x)$ in the sense of quadratic forms. We further assume that the reference matrix A satisfies that $\sqrt{A}$ is a bounded Lipschitz continuous $n \times n$ real-valued nonnegative definite matrix in Ω, and define the A-gradient and the A-divergence operators by

$$\nabla_A = \sqrt{A(x)}\nabla\ , \qquad \mathrm{div}_A = \mathrm{div}\left(\sqrt{A(x)}\cdot\right), \tag{3}$$

* Corresponding author.
E-mail address: crios@ucalgary.ca (C. Rios).

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To obtain local bounds for weak solutions u of the second order quasilinear equation (1) it suffices to consider the linear operator

$$L_{\tilde{A}}u = -\mathrm{div}\tilde{A}\nabla u = -\mathrm{div}_{\tilde{A}}\nabla_{\tilde{A}}u = \phi_0 - \mathrm{div}_A\vec{\phi}_1, \qquad x \in \Omega \tag{4}$$

where the matrix $\tilde{A} \in \mathfrak{A}(A, \Lambda, \lambda)$, i.e. it satisfies the equivalences (2).

We first work in an abstract setting which requires the existence of an underlying metric d satisfying some geometric compatibility with the differential structure induced by A, including the validity of a certain Orlicz-Sobolev inequality (Definition 1.4) for compactly supported Lipschitz functions on d-metric balls.

The Moser iteration developed here is the first instance of such technique being implemented for infinite degenerate equations, and as such it has an interest on its own. This iteration scheme requires the composition of Orlicz norms, which has been so far an insurmountable technical obstacle. We overcome this problem by considering a specially designed family of Young functions $\Phi_m(t) \sim t\exp\left((\ln t)^{\frac{m-1}{m}}\right)$ as $t \to \infty$, $m > 1$, which are well-behaved under successive compositions. These "exp-log" Young functions[1] were introduce in the study of *bump conditions* in weighted norm inequalities by Cruz-Uribe and Fiorenza [1], and were later shown by Cruz-Uribe et al. [2] to have properties more akin to Lebesgue norms[2] than to the Orlicz norms induced by the so-called log-bump functions $A(t) = t^p \log(e+t)^q$. Note that the Young functions Φ_m are *much larger* than the log-bump functions $A_{1,N}$ considered in [9], so in the end our results hold in a more restrictive family of geometries. Nevertheless, we do extend in these geometries the boundedness results in [9] to operators in which the right hand side also has a drift term.

Another reason to implement the Moser iteration is that it yields L^2-L^∞ estimates for *small negative powers* u^α of nonnegative solutions u, which combined with similar estimates for small positive powers of nonnegative solutions can render a Harnack-type inequality which, in turn, can be used to obtain continuity of solutions. We have been unable to obtain these same estimates with the De Giorgi approach used in [9]. On the other hand, our present methods require convexity of the power functions t^α, limiting our results to exponents $\alpha < 0$ or $\alpha \geq 1$. In a subsequent work we will establish estimates for small positive powers of nonnegative solutions via the De Giorgi method, and we will combine these results to prove continuity of solutions .

The abstract results are of interest in themselves because of their greater generality, but they prove their true relevance in actual geometric settings where they can yield new boundedness theorems. We provide in this paper an application of our abstract theory to a *two-dimensional* quasilinear operator comparable to a diagonal linear operator with degeneracy controlled by a function f that only depends on one of the variables. The current implementation of the Moser method requires a rather restrictive assumption on the type of the degeneracy that is allowed, and does not handle as large a range of degeneracies as is covered by the De Giorgi iteration in [9], or by the trace method in [10]. However, it does guarantee boundedness of solutions to degenerate quasilinear equations as in [9,10] while including the case of non-zero right hand side. In this application, the structural assumptions on A will ensure that A is elliptic away from the hyperplane $x_1 = 0$, and that the Carnot-Carathéodory metric d_A induced by A is topologically equivalent to the Euclidean metric $d_{\mathbb{E}}$, although these will not be equivalent metrics since the d_A-balls are *not doubling* when centered on that hyperplane. We prove that the assumptions necessary for the abstract theory, including an Orlicz-Sobolev embedding, all hold, thereby obtaining boundedness of weak solutions to $-\mathrm{div}\mathcal{A}(x,u)\nabla u = \phi_0 - \mathrm{div}_A\vec{\phi}_1$ for these operators in the plane (Theorem 1.1). The right hand side pair $\left(\phi_0, \vec{\phi}_1\right)$ is required to be admissible as given in Definition 1.5 below, which basically requires the $\left(\phi_0, \vec{\phi}_1\right)$ to belong to the dual of the homogeneous degenerate Sobolev space $W^{1,1}_{A,0}$ (see Section 1.2 for the definition of these spaces).

We now present the two-dimensional geometric application, the boundedness Theorem 1.1. For this result we will specifically consider the geometry of balls induced by diagonal matrices

$$A(x) = \begin{bmatrix} 1 & 0 \\ 0 & f(x)^2 \end{bmatrix} \tag{5}$$

where $f = f_{k,\sigma} = e^{-F_{k,\sigma}}$ with

$$F_{k,\sigma}(r) = \left(\ln\frac{1}{r}\right)\left(\ln^{(k)}\frac{1}{r}\right)^\sigma, \qquad r > 0,\ k \in \mathbb{N}, \text{ and } \sigma > 0.$$

That is, $f_{k,\sigma}(r) = e^{-F_{k,\sigma}(r)} = r^{\left(\ln^{(k)}\frac{1}{r}\right)^\sigma}$ vanishes to infinite order at $r = 0$, and $f_{k,\sigma}$ vanishes faster than $f_{k',\sigma'}$ if either $k < k'$ or if $k = k'$ and $\sigma > \sigma'$. These geometries are particular examples of the general geometries F considered in our abstract theory defined by the structural conditions 5.1 in Section 5 below. In [9] we consider $F_\sigma = F_{0,\sigma} = r^{-\sigma}$ $(k = 0)$ with $0 < \sigma < 1$, so $f_{1,\sigma}(r) \gg f_\sigma = e^{-\frac{1}{r^\sigma}}$ near $r = 0$. The boundedness results obtained here, albeit having a drift term on the right hand side and being able to treat small negative powers of supersolutions, do not include the case $k = 0$, $0 < \sigma < 1$, as in [9]; this is due to the current technical limitations for implementing a Moser iteration in the infinite degenerate setting.

Theorem 1.1 (*Geometric Local Boundedness*). *Let* $\{(0,0)\} \subset \Omega \subset \mathbb{R}^2$ *and* $\mathcal{A}(x,z)$ *be a nonnegative semidefinite matrix in* $\Omega \times \mathbb{R}$ *that satisfies the degenerate elliptic condition* (2) *where* $A(x)$ *is given by* (5) *with* $f = f_{k,\sigma}$. *Then every weak subsolution of* (1):

$$\mathcal{L}u \equiv -\nabla^{\mathrm{tr}}\mathcal{A}(x,u(x))\nabla u = \phi_0 - \mathrm{div}_A\vec{\phi}_1$$

is locally bounded above in $\Omega \subset \mathbb{R}^2$ *provided that:*

[1] We want to thank the referee for pointing out the earliest introduction of these "exp-log" bump functions Φ_m, together with their applications to weighted norm inequalities and the analysis of some their properties.

[2] In particular, the Orlicz norms determined by Φ_m are one-sided comparable to the non-homogenous quasi-norms given in Definition 1.4 below. See also Lemma 3.2.

1. *the right hand side pair* $\left(\phi_0, \vec{\phi}_1\right)$ *satisfies* $\phi_0 \in L^{\Phi^*}_{\text{loc}}(\Omega)$, *where* Φ^* *is the adjoint Young function to* Φ_m, *for some* $m > 2$, *and* $\left|\vec{\phi}_1\right| \in L^{\infty}_{\text{loc}}(\Omega)$,
2. *at least one of the following two conditions hold:*
   (a) $k = 1$ *and* $0 < \sigma < 1$,
   (b) $k \geq 2$ *and* $\sigma > 0$.

1.1. Relation to other results in the literature

Apart from the two papers by the authors [9,10] , mentioned earlier, there have been very few related results obtained by other authors, since this current paper first appeared on the arXiv in 2015. The two most recent and relevant ones are Cruz-Uribe and Rodney [3] and Di Fazio et al. [5]. In [3] the authors obtain boundedness of weak solutions to a certain class of degenerate elliptic Dirichlet problems using an adaptation of the De Giorgi technique developed in [9]. The results there are of abstract type where one assumes a weighted Sobolev inequality, and these results are similar, but incomparable, to our abstract results. However, they obtain a quantitative bound for a much larger class of inhomogeneous data. On the other hand, there are no geometric theorems there, which would require verification of complicated hypotheses, such as a Sobolev inequality. In this paper, as in the original version in the arXiv in 2015 [8], the use of the Moser iteration is crucial, this despite the comment made in ([3], page 5) to the effect that "We were unable to adapt Moser iteration to work in the context of Orlicz norms, and it remains an open question whether such an approach is possible in this setting".

More recently in [5] the authors consider quasilinear degenerate equations of this nature, and they use Moser iteration to obtain abstract results on Harnack inequalities and Hölder continuity of solutions. Similar to Cruz-Uribe and Rodney [3], the authors use an axiomatic approach, where the relevant (weighted) Sobolev and Poincaré inequalities, as well as the doubling property of the weights on the metric balls, are assumed to hold *a priori*. Since there are no geometric theorems established in [5], their results are also incomparable to those in our paper.

From the point of view of abstract results, the current paper also makes a new significant contribution. In both Cruz-Uribe and Rodney [3] and Di Fazio et al. [5] the authors use (q, p) Sobolev inequalities with $q > p$ and do not perform Moser or De Giorgi iterations using a weaker Orlicz-Sobolev inequality employed in this paper. Due to the inhomogeneous nature of the Orlicz norm, adapting these techniques to this new setting was a highly technical nontrivial task which required new ideas. This allows to establish regularity of solutions in the case when the metric balls are non-doubling with respect to Lebesgue measure, that is, the metric space is *not of homogenous type*; see [7].

1.2. The abstract setting

We work in an open, bounded domain $\Omega \subset \mathbb{R}^n$ and as described above we consider nonnegative symmetric real valued matrices A in Ω such that $\sqrt{A(x)}$ is uniformly bounded and uniformly Lipschitz in Ω. The degenerate Sobolev space $W^{1,2}_A(\Omega)$ associated to A has norm

$$\|v\|_{W^{1,2}_A} \equiv \sqrt{\int_\Omega |v|^2 + \left((\nabla v)^{\text{tr}} A \nabla v\right)} = \sqrt{\int_\Omega \left(|v|^2 + |\nabla_A v|^2\right)}.$$

Since $\sqrt{A}$ is Lipschitz then $\text{div}\sqrt{A(x)} \in (L^\infty(\Omega))^n$, hence the space $W^{1,2}_A(\Omega)$ is a Hilbert space (see [13], Theorem 2) contained in $L^2(\Omega)$, with inner product given by the bilinear form

$$a_1(u, v) = \int_\Omega \nabla_A v \cdot \nabla_A w \, dx + \int_\Omega vw \, dx, \qquad v, w \in W^{1,2}_A(\Omega)$$

where $\nabla_A v = \sqrt{A}\nabla v$. The associated homogeneous subspace $W^{1,2}_{A,0}(\Omega)$ is defined as the closure in $W^{1,2}_A(\Omega)$ of Lipschitz functions with compact support, $\text{Lip}_c(\Omega)$. If a global (1-1)-Sobolev inequality holds in Ω, i.e.

$$\int_\Omega |g| \, dx \leq C_\Omega \int_\Omega \left|\nabla_A g\right| \, dx \qquad \text{for some } C_\Omega > 0 \text{ and all } g \in \text{Lip}_c(\Omega), \tag{6}$$

it follows that the Hilbert space structure in $W^{1,2}_{A,0}(\Omega)$ has the equivalent inner product

$$a(u, v) = \int_\Omega A(x) \nabla v \cdot \nabla w \, dx = \int_\Omega \nabla_A v \cdot \nabla_A w \, dx, \qquad v, w \in W^{1,2}_{A,0}(\Omega).$$

In this case we have that $\|v\|_{W^{1,2}_A(\Omega)} \approx \left\|\nabla_A v\right\|_{L^2(\Omega)}$ for all $v \in W^{1,2}_{A,0}(\Omega)$. In ([9], Section 8.2) we show that inequality (6) holds for a wide variety of infinitely degenerate geometries.

Note that $\nabla_A : W^{1,2}_A(\Omega) \to \left(L^2(\Omega)\right)^n$ and $\text{div}_A : \left(L^2(\Omega)\right)^n \to \left(W^{1,2}_{A,0}(\Omega)\right)^*$ are bounded linear operators, where $\left(W^{1,2}_{A,0}(\Omega)\right)^*$ is the dual space of $W^{1,2}_{A,0}(\Omega)$. The derivatives in $W^{1,2}_A(\Omega)$ are understood in the weak sense, i.e., $\vec{f} = \nabla_A u$ in Ω if and only if $\vec{f} \in \left(L_1(\Omega)_{\text{loc}}\right)^n$ and for all $\vec{v} \in \left(\text{Lip}_c(\Omega)\right)^n$

$$\int_\Omega f \cdot \vec{v} \, dx = \int_\Omega u \text{div}_A \vec{v} \, dx,$$

note that the right hand side is integrable since $\mathrm{div}_A\vec{v} \in L^\infty(\Omega)$ and $u \in L^2(\Omega)$. When $u \in W_A^{1,2}(\Omega)$ and $\tilde{A} \in \mathfrak{A}(A, \Lambda, \lambda)$ we define the equivalent $\tilde{A}$ -gradient and $\mathrm{div}_{\tilde{A}}$ operators associated to by setting $\nabla_{\tilde{A}} v = \sqrt{\tilde{A}}\nabla v$ and $\langle \mathrm{div}_{\tilde{A}}\vec{w}, v\rangle = -\int \vec{w} \cdot \nabla_{\tilde{A}} v$ for all $v \in \mathrm{Lip}_c(\Omega)$. From (2) it is clear that $|\nabla_{\tilde{A}} v(x)| \approx |\nabla_A v(x)|$ for a.e. $x \in \Omega$. Each $u \in W_A^{1,2}(\Omega)$ then defines the bilinear form

$$\tilde{a}(v, w) = \int_\Omega \tilde{A}(x)\nabla v \cdot \nabla w = \int_\Omega \nabla_{\tilde{A}} v \cdot \nabla_{\tilde{A}} w \, dx, \qquad v, w \in W_A^{1,2}(\Omega).$$

The assumptions (2) imply that $\tilde{a} \approx a$ as bilinear forms, which are bounded on $W_A^{1,2}(\Omega)$, that is $|\tilde{a}(v,w)| \lesssim |a(v,w)| \lesssim \|v\|_{W_A^{1,2}} \|w\|_{W_A^{1,2}}$. In the presence of a (1-1)-Sobolev inequality (6) we moreover have that a and $\tilde{a}$ are coercive on $W_{A,0}^{1,2}(\Omega)$, i.e. $\tilde{a}(v,v) \gtrsim a(v,v) \gtrsim \|v\|^2_{W_A^{1,2}(\Omega)}$.

Definition 1.2 (*Weak Solutions*)**.** Let Ω be a bounded domain in $\mathbb{R}^n$. Assume that $\phi_0, |\vec{\phi}_1| \in L^2_{\mathrm{loc}}(\Omega)$. We say that $u \in W_A^{1,2}(\Omega)$ is a *weak solution* to $L_{\tilde{A}} u = -\mathrm{div}\tilde{A}\nabla u = \phi_0 - \mathrm{div}_A\vec{\phi}_1$ provided

$$\int_\Omega \tilde{A}(x)\nabla u \cdot \nabla w \, dx = \int_\Omega \phi_0 w + \vec{\phi}_1 \cdot \nabla_A w \, dx \tag{7}$$

for all $w \in \mathrm{Lip}_c(\Omega)$. Eq. (7) may be written as $\tilde{a}(u,w) = F(w)$ where F is the operator defined by the right hand side of (7), which is a bounded linear operator on $W_{A,0}^{1,2}(\Omega)$. With this notation we similarly define the notion of subsolution (supersolution) by saying that $u \in W_A^{1,2}(\Omega)$ is a (*weak*) *subsolution* (*supersolution*) to $L_{\tilde{A}} u = \phi_0 - \mathrm{div}_A\vec{\phi}_1$, and write $L_{\tilde{A}} u \leq \phi_0 - \mathrm{div}_A\vec{\phi}_1$ ($L_{\tilde{A}} u \geq \phi_0 - \mathrm{div}_A\vec{\phi}_1$), if and only if

$$\tilde{a}(u,w) \leq F(w) \quad (\tilde{a}(u,w) \geq F(w)) \quad \textit{for all} \text{ nonnegative } w \in \mathrm{Lip}_c(\Omega).$$

Finally, we say that $u \in W_A^{1,2}(\Omega)$ is a weak solution (subsolution, supersolution) to $\mathcal{L}u = -\mathrm{div}\mathcal{A}(x,u)\nabla u = \phi_0 - \mathrm{div}_A\vec{\phi}_1$ provided u is a weak solution (subsolution, supersolution) to $L_{\tilde{A}} u = \phi_0 - \mathrm{div}_A\vec{\phi}_1$ for $\tilde{A}(x) = \mathcal{A}(x, u(x))$.

Note that our structural condition (2) implies that the integral on the left above is absolutely convergent, and our assumption that $\phi_0, |\vec{\phi}_1| \in L^2_{\mathrm{loc}}(\Omega)$ implies that the integrals on the right above are absolutely convergent. In Definition 1.5 below we weaken the assumptions on the right hand side pair $(\phi_0, \vec{\phi}_1)$.

In this abstract setting we work with the differential structure defined through the matrix A, inducing the Sobolev spaces $W_A^{1,2}(\Omega)$. We further assume the existence of a metric $d : \mathbb{R}^n \times \mathbb{R}^n \to [0,\infty)$ satisfying certain geometric compatibility with this differential structure, namely conditions (i), (ii), and (iii) in Theorem 1.6. We now describe each assumption in more detail.

Definition 1.3 (*Standard Sequence of Accumulating Lipschitz Functions*)**.** Let Ω be a bounded domain in $\mathbb{R}^n$ and let $d : \mathbb{R}^n \times \mathbb{R}^n \to [0,\infty)$ be a metric. Fix $r > 0$, $\nu \in (0,1)$, and $x \in \Omega$. We define an (A,d)- *standard* sequence of Lipschitz cutoff functions $\{\psi_j\}_{j=1}^\infty$ at (x,r), along with sets $B(x, r_j) \supset \mathrm{supp}\psi_j$, to be a sequence satisfying $\psi_j = 1$ on $B(x, r_{j+1})$, $r_1 = r$, $r_\infty \equiv \lim_{j\to\infty} r_j = \nu r$, $r_j - r_{j+1} = \frac{c}{j^2}(1-\nu)r$ for a uniquely determined constant c, and $\|\nabla_A \psi_j\|_\infty \lesssim \frac{j^2}{(1-\nu)r}$ with ∇_A as in (3) (see e.g. [12]).

A sufficient condition for the existence of these cutoffs would be the existence of a constant $C_d > 0$ such that whenever $0 < r < R < \infty$ and $B(x,R) \subset \Omega$, then there exists a Lipschitz function $\psi = \psi_{x,r,R} \in \mathrm{Lip}_c(B_R)$ such that $0 \leq \psi \leq 1$, $\psi \equiv 1$ in B_r and $\|\nabla_A \psi\|_\infty \leq \frac{C_d}{R-r}$. This is indeed the case $d = d_A$ is the Carnot-Carathéodory metric induced by a continuous matrix A, and this metric is topologically equivalent to the Euclidean metric (see Lemma 5.3).

We will need to assume the following single scale (Φ, A, φ)-Orlicz-Sobolev bump inequality:

Definition 1.4 (*Orlicz-Sobolev Inequality*)**.** Let Ω be a bounded domain in $\mathbb{R}^n$, the (Φ, A)-Orlicz-Sobolev bump inequality for Ω is

$$\Phi^{-1}\left(\int_\Omega \Phi(w)\, dx\right) \leq C \|\nabla_A w\|_{L^1(\Omega)}, \qquad w \in \mathrm{Lip}_c(\Omega), \tag{8}$$

where dx is Lebesgue measure in $\mathbb{R}^n$ and C depends on n, A, Φ, and Ω but not on w.

Fix $x \in \Omega$ and $r > 0$ such that $B(x,r) \subset \Omega$, the (Φ, A, φ)-Orlicz-Sobolev bump inequality at (x,r) is:

$$\Phi^{(-1)}\left(\int_{B(x,\rho)} \Phi(w)\, d\mu_{x,\rho}\right) \leq \varphi(\rho) \|\nabla_A w\|_{L^1(\mu_{x,\rho})}, \qquad 0 < \rho \leq r, \tag{9}$$

for all $w \in \mathrm{Lip}_c(B(x,\rho))$, where $d\mu_{x,\rho}(y) = \frac{1}{|B(x,\rho)|}\mathbf{1}_{B(x,\rho)}(y)\, dy$, and the function $\varphi(r)$, dubbed the superradius, is continuous, nondecreasing, and it satisfies $\varphi(0) = 0$, $\varphi(\rho) \geq \rho$ for all $0 \leq \rho \leq r$.

Finally, we say that the *single scale*[3] (Φ, A, φ) -Orlicz-Sobolev bump inequality holds at (x,r) if (9) holds for $\rho = r$ (and not necessarily for $0 < \rho < r$).

[3] As opposed to the multi-scale Sobolev bump inequalities assumed for continuity, that require $0 < \rho < r_0$.

The particular family of Orlicz bump functions Φ_m required above that is crucial for our theorem is the family

$$\Phi_m(t) = e^{\left((\ln t)^{\frac{1}{m}}+1\right)^m}, \qquad t > E_m = e^{2^m},\ m > 1, \tag{10}$$

which is then extended in (34) below to be *linear* on the interval $[0, E_m]$, continuous and submultiplicative on $[0,\infty)$; we discuss this in more detail in Section 5.1.

Finally, we describe the notion of admissible right hand side pair.

Definition 1.5 (*Admissible Right Hand Sides*). Let Ω be a bounded domain in $\mathbb{R}^n$ and let $\phi_0 : \Omega \to \mathbb{R}$, $\vec{\phi}_1 : \Omega \to \mathbb{R}^n$ be locally integrable. We call $\left(\phi_0, \vec{\phi}_1\right)$ a right hand side pair (although we may just refer them as just a "pair"). Fix $x \in \Omega$ and $\rho > 0$, we say that the right hand side pair $\left(\phi_0, \vec{\phi}_1\right)$ is *A-admissible* at (x, ρ) if

$$\left\|\left(\phi, \vec{\phi}_1\right)\right\|_{\mathcal{X}(B(x,\rho))} \equiv \sup_{v\in\mathcal{W}_1}\left|\int_{B(x,\rho)} v\phi_0\, dy\right| + \sup_{v\in\mathcal{W}_1}\left|\int_{B(x,\rho)} \nabla_A v\cdot\vec{\phi}_1\, dy\right| < \infty. \tag{11}$$

where $\mathcal{W}_1 = \left\{v \in \left(W^{1,1}_{A,0}\right)(B(x,\rho)) : \int_{B(x,\rho)} |\nabla_A v|\, dy = 1\right\}$. Similarly, we say the pair $\left(\phi_0, \vec{\phi}_1\right)$ is *A-admissible* for Ω if (11) holds with Ω replacing $B(x,\rho)$.

For convenience we also introduce the concept of *strongly A-admissible* pair. We say that $\left(\phi_0, \vec{\phi}_1\right)$ is strongly A-admissible at (x, ρ) if

$$\left\|\left(\phi, \vec{\phi}_1\right)\right\|_{\mathcal{X}^*(B(x,\rho))} \equiv \sup_{v\in\mathcal{W}_1}\int_{B(x,\rho)} |v\phi_0|\, dy + \sup_{v\in\mathcal{W}_1}\int_{B(x,\rho)} \left|\nabla_A v\cdot\vec{\phi}_1\right| dy < \infty.$$

It is clear that if $\left(\phi_0, \vec{\phi}_1\right)$ is strongly A -admissible at (x,ρ) then it is A-admissible at (x,ρ).

In the above definition an A-admissible right hand side pair at (x, r) defines a bounded linear operator $T_{\left(\phi,\vec{\phi}_1\right)}$ on the space $W^{1,1}_{A,0}(B(x,r))$ by setting

$$T_{\left(\phi,\vec{\phi}_1\right)}(v) = \int_{B(x,\rho)} v\phi_0\, dy + \int_{B(x,\rho)} \nabla_A v\cdot\vec{\phi}_1\, dy.$$

Recall that a measurable function u in Ω is *locally bounded above* at x if u can be modified on a set of measure zero so that the modified function $\tilde{u}$ is bounded above in some neighborhood of x.

Theorem 1.6 (*Abstract Local Boundedness*). *Let Ω be a bounded domain in $\mathbb{R}^n$. Suppose that $\mathcal{A}(x,z)$ is a nonnegative semidefinite matrix in $\Omega\times\mathbb{R}$ that satisfies the degenerate elliptic condition (2). Let $d(x,y)$ be a symmetric metric in Ω, and suppose that $B(x,r) = \{y \in \Omega : d(x,y) < r\}$ with $x \in \Omega$ are the corresponding metric balls. Fix $x \in \Omega$. Then every weak subsolution (supersolution) of (1) is locally bounded above (locally bounded below) at x provided there is $r_0 > 0$ such that:*

i. *the right hand side pair $\left(\phi_0, \vec{\phi}_1\right)$ is A-admissible at (x, r_0),*
ii. *the single scale (Φ, A, φ) -Orlicz-Sobolev bump inequality (9) holds at (x, r_0) with $\Phi = \Phi_m$ as in (10) for some $m > 2$,*
iii. *there exists an (A, d)-standard accumulating sequence of Lipschitz cutoff functions at (x, r_0).*
Similarly, under the above three conditions every weak supersolution of (1) is locally bounded below at x, and every weak solution of (1) is locally bounded at x.
In particular, every weak solution (supersolution) of (1) is locally bounded at x.

Proof. This local boundedness result is an immediate consequence of Theorem 4.1 for $\beta = 1$, proven in Section 4.1. Indeed, setting $\tilde{A}(x) = \mathcal{A}(x, u(x))$ because of the equivalences (2) we have that $\tilde{A}$ satisfies (2). By hypothesis, the (Φ, A, φ)-Orlicz-Sobolev bump inequality (9) holds at (x, r_0) with $\Phi = \Phi_m$ for some $m > 2$ and an (A, d)-*standard* accumulating sequence of Lipschitz cutoff functions at (x, r_0).

Thus, if u is a weak subsolution of (1), then it is a weak subsolution of $L_{\tilde{A}}u = -\mathrm{div}\tilde{A}\nabla u = \phi_0 - \mathrm{div}_A\vec{\phi}_1$, and all the hypotheses of Theorem 4.1 are satisfied, therefore u is locally bounded above ($u^+ \in L^\infty_{\mathrm{loc}}(\Omega)$). In fact, Theorem 4.1 provides precise estimates: for $v_0 \le v < 1$, with $v_0 = 1 - \frac{\delta_0(r)}{r}$, where $\delta_x(r)$ is the doubling increment of $B(x,r)$, defined by (12), we have that there exists a constant $C = C(\varphi, m, \lambda, \Lambda, r, v)$ such that

$$\|u^+ + \phi^*\|_{L^\infty(B(x,vr))} \le C\ \|u^+ + \phi^*\|_{L^2(B(x,r),d\mu_r)} < \infty$$

where . The last inequality follows form the fact that since $u \in W^{1,2}_A(B(x,r))$, then

$$\|u^+ + \phi^*\|_{L^2(B(x,r),d\mu_r)} = \left(\frac{1}{|B(x,r)|}\int_{B(x,r)} \left(u^+ + \phi^*\right)^2 dx\right)^{\frac{1}{2}} < \infty.$$

Similarly, if u is a weak supersolution of (1) we conclude that

$$\|u^- + \phi^*\|_{L^\infty(B(x,vr))} \le C\ \|u^- + \phi^*\|_{L^2(B(x,r),d\mu_r)} < \infty. \quad \square$$

Remark 1.7. The hypotheses required for local boundedness of weak solutions to $L_{\tilde{A}}u = \phi_0 - \mathrm{div}_A \vec{\phi}_1$ at a single *fixed* point x in Ω are quite weak; namely we only need the existence of cutoff functions for $B(x, r_0)$ for some $r_0 > 0$, that the inhomogeneous couple $(\phi_0, \vec{\phi}_1)$ is *A-admissible* at **just one** point (x, r_0), and the *single scale* condition relating the geometry to the equation at **the one** point (x, r_0).

Remark 1.8. We could take the metric d to be the Carnot-Carathéodory metric associated with A, but the present formulation allows for additional flexibility in the choice of balls used for Moser iteration.

The specific relation between the metric and the Orlicz-Sobolev embedding will be given in terms of the concept of *doubling increment* of a ball and its connection with the superradius φ. The bounds in Theorems 4.1 and 4.2 are the embedding norms of $L^\infty\left(B_{r-\delta_x(r)}\right)$ into $L^2(B_r)$.

Definition 1.9. Let Ω be a bounded domain in $\mathbb{R}^n$. Let $\delta_x(r)$ be defined implicitly by

$$\left|B\left(x, r - \delta_x(r)\right)\right| = \frac{1}{2}|B(x,r)|, \tag{12}$$

We refer to $\delta_x(r)$ as the *doubling increment* of the ball $B(x,r)$.

2. Caccioppoli inequalities for weak subsolutions and supersolutions

In this section we establish various Caccioppoli inequalities for subsolutions and supersolutions of (4) (see Definition 1.2). In order to prove a Caccioppoli inequality, we assume that the inhomogeneous pair $(\phi_0, \vec{\phi}_1)$ in (7) is admissible for A in the whole domain Ω in sense of Definition 1.5.

What is usually called a Caccioppoli inequality is a reverse Sobolev inequality which is valid only for functions satisfying an equation of the form $L_{\tilde{A}}u \geq \phi_0 - \mathrm{div}_A \vec{\phi}_1$ or $L_{\tilde{A}}u \leq \phi_0 - \mathrm{div}_A \vec{\phi}_1$. The Moser iteration is based on these type of inequalities obtained from the equation when the test function is an appropriate function of the solution. If $u \in W_A^{1,2}(\Omega)$, and h is a $C^{0,1}$ or $C^{1,1}$ function on $[0,\infty)$, then $h(u)$ formally satisfies the equation

$$L_{\tilde{A}}(h(u)) = -\mathrm{div}\tilde{A}\nabla(h(u)) = -\mathrm{div}\tilde{A}h'(u)\nabla u = h'(u)Lu - h''(u)\left|\nabla_{\tilde{A}}u\right|^2.$$

Indeed, if $w \in W_{A,0}^{1,2}(\Omega)$ and u is a positive subsolution or supersolution of (4) in Ω, we have

$$\begin{aligned}\int \nabla_{\tilde{A}}w \cdot \nabla_{\tilde{A}}h(u) &= \int h'(u)\nabla_{\tilde{A}}w \cdot \nabla_{\tilde{A}}u \\ &= \int \nabla_{\tilde{A}}\left(h'(u)w\right)\cdot\nabla_{\tilde{A}}u - \int wh''(u)\nabla_{\tilde{A}}u\cdot\nabla_{\tilde{A}}u \\ &\leq \int wh'(u)\phi_0 + \int \nabla_A\left(wh'(u)\right)\cdot\vec{\phi}_1 \\ &\quad - \int wh''(u)\left|\nabla_{\tilde{A}}u\right|^2\end{aligned}$$

provided that $wh'(u) \in W_{A,0}^{1,2}(\Omega)$ and that it is nonnegative if u is a subsolution, and nonpositive if u is a supersolution. Note that $wh'(u) \in W_{A,0}^{1,2}(\Omega)$ if in addition we have that h' is bounded.

We will establish two Caccioppoli inequalities. Lemma 2.1 holds for *convex increasing* functions h applied to $u^\pm$; this estimate is utilized to implement a Moser iteration scheme to obtain boundedness of solutions without restrictions on their sign. The other result, Lemma 2.3, applies to convex functions of nonnegative subsolutions or supersolutions, and the function h will satisfy suitable structural properties which will allow us to obtain (through a Moser iteration) inner ball inequalities for negative powers u^β of the solution,.

Lemma 2.1. *Assume that $u \in W_A^{1,2}(B)$ is a weak subsolution to $L_{\tilde{A}}u = \phi_0 - \mathrm{div}_A\vec{\phi}_1$ in $B = B(x,r)$, where $(\phi_0, \vec{\phi}_1)$ is an admissible pair and $\tilde{A} \in \mathfrak{A}(A, \Lambda, \lambda)$ (i.e. it satisfies the equivalences (2) for some $0 < \lambda \leq \Lambda < \infty$). Let $h(t) \geq 0$ be a Lipschitz convex function which satisfies $0 < h'_-(t) \leq C_h \frac{h(t)}{t}$, for $t > 0$ and it is piecewise twice continuously differentiable except possibly at finitely many points, where $C_h \geq 1$ is a constant. Then the following reverse Sobolev inequality holds for any $\psi \in \mathrm{Lip_c}(B)$:*

$$\int_B \psi^2\left|\nabla_A\left[h\left(u^+ + \phi^*\right)\right]\right|^2 dx \leq C_{\lambda,\Lambda}C_h^2\int_B h\left(u^+ + \phi^*\right)^2\left(\left|\nabla_A\psi\right|^2 + \psi^2\right), \tag{13}$$

where $\phi^* = \phi^*(x,r) = \left\|\left(\phi, \vec{\phi}_1\right)\right\|_{X(B(x,r))}$ as given in Definition 1.5. Moreover, if $u \in W_A^{1,2}(B)$ is a weak supersolution to $L_{\tilde{A}}u = \phi_0 - \mathrm{div}_A\vec{\phi}_1$ in B, then (13) holds with u^+ replaced by u^-.

Proof. From the hypothesis, if $t > 0$ is a discontinuity point of h', then h' has simple jump discontinuity there, and both the left and right derivatives are defined with $h'_+(t) - h'_-(t) > 0$. Following the proof of Theorem 8.15 in [6], for $N \gg \phi^* \geq 0$ larger than the last point of discontinuity of h', we define $H \in C^{0,1}([\phi^*, \infty))$ by

$$H(t) = \begin{cases} h(t) - h(\phi^*) & t \in [\phi^*, N] \\ h(N) - h(\phi^*) + h'_-(N)(t-N) & t > N \end{cases},$$

and let $\omega(t) = \int_{\phi^*}^t \left(H'(s)\right)^2 ds$ for $t \geq \phi^*$, i.e.

$$\omega(t) = \begin{cases} \int_{\phi^*}^t \left(h'(s)\right)^2 ds & t \in [\phi^*, N] \\ \int_{\phi^*}^N \left(h'(s)\right)^2 ds + \left(h'_-(N)\right)^2 (t-N) & t > N \end{cases}.$$

Then ω is continuous and piecewise differentiable for all $t \geq 0$, with $\omega'(t)$ having at most finitely many simple jump discontinuities. Since h is convex we have that $H'(t)$ is increasing, and therefore

$$\omega(t) = \int_{\phi^*}^t \left(H'(s)\right)^2 ds \leq H'(t) \int_{\phi^*}^t H'(s)\, ds = H'(t) H(t). \tag{14}$$

Note also that, since h is convex, $H(t) \leq h(t)$ for all $t \geq 0$. Now, since both h and h' are locally bounded on $[0, \infty)$, it follows the function $w(x) = \omega\left(u^+(x) + \phi^*\right) \in W_A^{1,2}(\Omega)$ whenever $u \in W_A^{1,2}(\Omega)$, moreover, $\operatorname{supp} w = \operatorname{supp} u^+$ and $\nabla_A w = \left(H'\left(u^+(x) + \phi^*\right)\right)^2 \nabla_A u^+$.

If u is a subsolution to $L_{\tilde{A}} u = \phi_0 - \operatorname{div}_A \vec{\phi}_1$ in $B(0, r)$ and $\psi \in \operatorname{Lip}_c(B(0, r))$, then we have that $\psi^2 w \in W_{A,0}^{1,2}(\Omega)$ and we have

$$\int \nabla_{\tilde{A}} u \cdot \nabla_{\tilde{A}} \left(\psi^2 w\right) \leq \int \psi^2 w \phi_0 + \int \nabla_A \left(\psi^2 w\right) \cdot \vec{\phi}_1.$$

Write $v(x) = H\left(u^+(x) + \phi^*\right)$, and $v'(x) = H'\left(u^+(x) + \phi^*\right)$ then the left hand side equals

$$\begin{aligned} \int \nabla_{\tilde{A}} u \cdot \nabla_{\tilde{A}} \left(\psi^2 w\right) &= \int \psi^2 \nabla_{\tilde{A}} u \cdot \nabla_{\tilde{A}} w + 2 \int \psi w \nabla_{\tilde{A}} u \cdot \nabla_{\tilde{A}} \psi \\ &= \int \psi^2 \left(v'\right)^2 \nabla_{\tilde{A}} u^+ \cdot \nabla_{\tilde{A}} u^+ + 2 \int \psi w \nabla_{\tilde{A}} u^+ \cdot \nabla_A \psi \\ &= \int \psi^2 \left|\nabla_{\tilde{A}} v\right|^2 + 2 \int \psi w \nabla_{\tilde{A}} u^+ \cdot \nabla_A \psi, \end{aligned}$$

where we used that $\operatorname{supp} w = \operatorname{supp} u^+$; we obtain

$$\int \psi^2 \left|\nabla_{\tilde{A}} v\right|^2 \leq -2 \int \psi w \nabla_{\tilde{A}} u^+ \cdot \nabla_{\tilde{A}} \psi + \int \psi^2 w \phi_0 + \int \nabla_A \left(\psi^2 w\right) \cdot \vec{\phi}_1.$$

From (14) we have

$$w(x) = \omega\left(u^+(x) + \phi^*\right) \leq H'\left(u^+(x) + \phi^*\right) H\left(u^+(x) + \phi^*\right) = v'(x) v(x),$$

so we can estimate the first term on the right hand side by

$$\begin{aligned} 2 \int \psi w \left|\nabla_A u^+\right| \left|\nabla_A \psi\right| &\leq 2 \int \psi v v' \left|\nabla_{\tilde{A}} u^+\right| \left|\nabla_{\tilde{A}} \psi\right| = 2 \int \psi v \left|\nabla_{\tilde{A}} v\right| \left|\nabla_{\tilde{A}} \psi\right| \\ &\leq \frac{1}{2} \int \psi^2 \left|\nabla_{\tilde{A}} v\right|^2 + 2 \int \left|\nabla_{\tilde{A}} \psi\right|^2 v^2, \end{aligned}$$

Substituting above and absorbing into the left, we obtain

$$\int \psi^2 \left|\nabla_{\tilde{A}} v\right|^2 \leq 4 \int \left|\nabla_{\tilde{A}} \psi\right|^2 v^2 + 2 \int \psi^2 w \phi_0 + 2 \int \nabla_A \left(\psi^2 w\right) \cdot \vec{\phi}_1. \tag{15}$$

Now, since $\left(\phi_0, \vec{\phi}_1\right)$ is admissible, we have that

$$\begin{aligned} \left|\int \psi^2 w \phi_0\right| + \left|\int \nabla_A \left(\psi^2 w\right) \cdot \vec{\phi}_1\right| &\leq \phi^* \int \left|\nabla_A \left(\psi^2 w\right)\right| \\ &\leq 2\phi^* \int \psi \left|\nabla_A \psi\right| w + \phi^* \int \psi^2 \left|\nabla_A w\right|. \end{aligned} \tag{16}$$

We assume now that $\phi^* > 0$, if this is not the case, then we substitute ϕ^* by a small constant $c > 0$ and let $c \to 0$ at the end of the proof. By the inequality $h'(t) \leq C_h \frac{h(t)}{t}$ and the definition of H we have that

$$H'(t) = \begin{cases} h'(t) & t \in [\phi^*, N] \\ h'(N) & t > N \end{cases} \leq C_h \begin{cases} \frac{h(t)}{t} & t \in [\phi^*, N] \\ \frac{h(N)}{N} & t > N \end{cases} \leq C_h \frac{h(t)}{\phi^*}. \tag{17}$$

Then by (17) we have that $v'(x) \leq C_h \frac{h\left(u^+(x) + \phi^*\right)}{\phi^*}$, and writing $\tilde{v}(x) = h\left(u^+(x) + \phi^*\right)$, the first term on the right of (16) is bounded by

$$2\phi^* \int \psi \left|\nabla_A \psi\right| w \leq 2\phi^* \int \psi \left|\nabla_A \psi\right| v v' \leq 2C_h \int \psi \left|\nabla_A \psi\right| v \tilde{v}$$

$$\leq C_h \int \left(\psi^2 \tilde{v}^2 + \left|\nabla_A \psi\right|^2 v^2\right).$$

Similarly, the second term on the right of (16) is bounded by

$$\begin{aligned}\phi^* \int \psi^2 \left|\nabla_A w\right| &= \phi^* \int \psi^2 \left(v'\right)^2 \left|\nabla_A u^+\right| = \phi^* \int \psi^2 v' \left|\nabla_A v\right| \\ &\leq C_h \int \psi^2 \tilde{v} \left|\nabla_A v\right| \leq \frac{\lambda}{4} \int \psi^2 \left|\nabla_A v\right|^2 + \frac{C_h^2}{\lambda} \int \psi^2 \tilde{v}^2.\end{aligned}$$

where $\lambda > 0$ is as in (2) and we also used (17). Plugging these estimates into (16) and substituting into (15) yields

$$\begin{aligned}\int \psi^2 \left|\nabla_{\tilde{A}} v\right|^2 &\leq 4 \int \left|\nabla_{\tilde{A}} \psi\right|^2 v^2 + 2C_h \int \left(\psi^2 \tilde{v}^2 + \left|\nabla_A \psi\right|^2 v^2\right) \\ &+ \frac{\lambda}{2} \int \psi^2 \left|\nabla_A v\right|^2 + \frac{2C_h^2}{\lambda} \int \psi^2 \tilde{v}^2.\end{aligned}$$

Using the structural assumptions (2) yields

$$\begin{aligned}\lambda \int \psi^2 \left|\nabla_A v\right|^2 &\leq 4\Lambda \int \left|\nabla_A \psi\right|^2 v^2 + 2C_h \int \left(\psi^2 \tilde{v}^2 + \left|\nabla_A \psi\right|^2 v^2\right) \\ &+ \frac{\lambda}{2} \int \psi^2 \left|\nabla_A v\right|^2 + \frac{2C_h^2}{\lambda} \int \psi^2 \tilde{v}^2,\end{aligned}$$

absorbing in to the left we obtain

$$\int \psi^2 \left|\nabla_A v\right|^2 \leq 16C_h^2 \left(\frac{\Lambda}{\lambda} + \frac{1}{\lambda^2}\right) \int \left(\psi^2 + \left|\nabla_A \psi\right|^2\right) \tilde{v}^2,$$

where we used the inequality $v(x) = H\left(u^+(x) + \phi^*\right) \leq h\left(u^+(x) + \phi^*\right) = \tilde{v}(x)$. This is

$$\int \psi^2 \left|\nabla_A \left[H\left(u^+ + \phi^*\right)\right]\right|^2 dx \leq C_{\lambda,\Lambda} C_h^2 \int \left(\psi^2 + \left|\nabla_A \psi\right|^2\right) \left(h\left(u^+(x) + \phi^*\right)\right)^2,$$

the lemma the follows in this case by letting $N \to \infty$.

When u is a weak supersolution to $L_{\tilde{A}} u = \phi_0 - \mathrm{div}_A \vec{\phi}_1$ in B, then $-u$ is a weak subsolution to $L(-u) = -\phi_0 - \mathrm{div}_A \left(-\vec{\phi}_1\right)$ with the same admissible norm ϕ^*, and $(-u)^+ = u^-$, so (13) holds in this case with u^+ replaced by u^-. □

Remark 2.2. Taking $h(t) \equiv t$ in Lemma 2.1 we have that

$$\int_{B(0,r)} \psi^2 \left|\nabla_A u^+\right|^2 dx \leq C_{\lambda,\Lambda} \int_{B(0,r)} \left(u^+ + \phi^*\right)^2 \left(\left|\nabla_A \psi\right|^2 + \psi^2\right)\, dx$$

when u is a subsolution to $L_{\tilde{A}} u = \phi_0 - \mathrm{div}_A \vec{\phi}_1$, and the same estimate holds for u^- ($|u|$) when u is a supersolution (solution).

The following variation of Caccioppoli requires stronger hypotheses on the function h, however h is allowed to be *decreasing* when applied to supersolutions. In particular, h needs to be $C^{1,1}$ since the second derivative of h explicitly appears within the integrals in the calculations. When h is $C^{1,1}$ the second derivative may be discontinuous (piece-wise discontinuous in our applications) but discontinuities will only be jump discontinuities, which do not affect the integrals.

Lemma 2.3. *Assume that $u \in W_A^{1,2}(\Omega)$ is a nonnegative weak subsolution or supersolution to $L_{\tilde{A}} u = \phi_0 - \mathrm{div}_A \vec{\phi}_1$ in $B(0,r)$, where $\left(\phi_0, \vec{\phi}_1\right)$ is an A-admissible pair with norm ϕ^* and $\tilde{A}$ satisfies the equivalences (2) for some $0 < \lambda \leq \Lambda < \infty$. Let $h(t) \geq 0$ be a convex monotonic C^1 and piecewise twice continuously differentiable function on $(0,\infty)$ that satisfies the following conditions except possibly at finitely many points when $t \in (0,\infty)$:*

I. *$Y(t) = h(t)h''(t) + \left(h'(t)\right)^2$ satisfies $c_1 \left(h'(t)\right)^2 \leq Y(t) \leq C_1 \left(h'(t)\right)^2$ at every point of continuity of h'', where $0 < c_1 \leq 1 \leq C_1 < \infty$ are constant;*
II. *The derivative $h'(t)$ satisfies the inequality $0 < \left|h'(t)\right| \leq C_2 \frac{h(t)}{t}$, where $C_2 \geq 1$ is a constant;*
Furthermore, we assume that
III. *if u is a weak subsolution then $h' \geq 0$, and if u is a weak supersolution then $h' \leq 0$.*
Then the following reverse Sobolev inequality holds for any $\psi \in \mathrm{Lip}_c(B(0,r))$:

$$\int_{B(x,r)} \psi^2 \left|\nabla_A \left[h\left(u + \phi^*\right)\right]\right|^2 dx \leq C_{\lambda,\Lambda} \frac{C_1^2 C_2^2}{c_1^2} \int_{B(x,r)} h\left(u + \phi^*\right)^2 \left(\left|\nabla_A \psi\right|^2 + \psi^2\right). \tag{18}$$

Proof. We will prove the lemma with an extra assumption that $h'(t)$ is bounded and $h(u + \phi^*) \in L^2(B(0,r))$. These assumptions can be dropped by the following limiting argument. Using standard truncations as in [12]. If h is increasing we define for $N \gg 1$,

$$h_N(t) \equiv \begin{cases} h(t) & \text{if} \quad 0 \leq t \leq N \\ h(N) + h'(N)(t - N) & \text{if} \quad t \geq N \end{cases}.$$

while if h is decreasing we let

$$h_N(t) \equiv \begin{cases} h\left(\frac{1}{N}\right) + h'\left(\frac{1}{N}\right)\left(t - \frac{1}{N}\right) & \text{if} \quad 0 \le t \le \frac{1}{N} \\ h(t) & \text{if} \quad t \ge \frac{1}{N} \end{cases}$$

We note that either function h_N still satisfy conditions (I)–(III) in the lemma with the same constants C_1 and C_2, if we can obtain a reverse Sobolev inequality similar to (18) for h_N, then the dominated converge theorem applies to establish (18) in general. Moreover, note that since h_N is linear for large t when h is increasing and for small t when h is decreasing, then $h(u+\phi^*) \in L^2(B(0,r)) \iff u \in L^2(B(0,r))$. Hence, if $\phi^* > 0$ from (II) it follows that also $\left(h'(u+\phi^*)\right)^2$ and $\Upsilon(u+\phi^*) \in L^1(B(0,r))$. If $\phi^* = 0$ we replace it by a small positive $\varepsilon > 0$ and then let $\varepsilon \to 0$ at the end. Thus, in what follows we will assume that $h'(t)$ and $h''(t)$ are bounded on the range of $u+\phi^*$, and that all integrals below are finite.

Assume that h is C^1, convex, and piecewise twice differentiable in $(0,\infty)$ with bounded first and second derivatives. By these assumptions it follows that h is twice differentiable everywhere except a finitely many points where h'' has finite jump discontinuities.

Let $\psi \in \mathrm{Lip}_c(B(0,r))$, $v(x) = h(u(x)+\phi^*)$ and write $v'(x) = h'(u(x)+\phi^*)$, $v''(x) = h''(u(x)+\phi^*)$. Then we have that $w(x) = \psi^2(x)v(x)v'(x)$ is in the space $W^{1,2}_{A,0}(B(0,r))$. Now, by assumption (III) we have that $w \ge 0$ when u is a subsolution, and $w \le 0$ when u is a supersolution, then we have

$$\int \nabla_{\tilde{A}} u \cdot \nabla_{\tilde{A}} w \le \int w\phi_0 + \int \nabla_A w \cdot \vec{\phi}_1 \tag{19}$$

Since $\nabla_{\tilde{A}} v = v' \nabla_{\tilde{A}} u$, and $\left(v'\right)^2 + vv'' = \Upsilon(u+\phi^*)$, the left side of (19) equals

$$\begin{aligned} \int \nabla_{\tilde{A}} u \cdot \nabla_{\tilde{A}} w &= \int \nabla_{\tilde{A}} u \cdot v' \nabla_{\tilde{A}}\left(\psi^2 v\right) + \int \psi^2 v v'' \nabla_{\tilde{A}} u \cdot \nabla_{\tilde{A}} u \\ &= \int \nabla_{\tilde{A}} v \cdot \nabla_{\tilde{A}}\left(\psi^2 v\right) + \int \psi^2 v v'' \left|\nabla_{\tilde{A}} u\right|^2 \\ &= 2\int \psi v \nabla_{\tilde{A}} v \cdot \nabla_{\tilde{A}} \psi + \int \psi^2 \Upsilon\left(u+\phi^*\right)\left|\nabla_{\tilde{A}} u\right|^2. \end{aligned}$$

Combining this and (19), we obtain

$$\int \psi^2 \Upsilon\left(u+\phi^*\right)\left|\nabla_{\tilde{A}} u\right|^2 \le -2\int \psi v \nabla_{\tilde{A}} v \cdot \nabla_{\tilde{A}} \psi + \int w\phi_0 + \int \nabla_A w \cdot \vec{\phi}_1. \tag{20}$$

By property (I) and the equivalences (2) we obtain:

$$c_1\lambda \int \psi^2 \left|\nabla_A v\right|^2 \le 2\Lambda \int \psi v \left|\nabla_A v\right| \left|\nabla_A \psi\right| + \int w\phi_0 + \int \nabla_A w \cdot \vec{\phi}_1$$

By Schwartz inequality we can estimate the first term on the right hand side by

$$2\Lambda \int \psi v \left|\nabla_A v\right| \left|\nabla_A \psi\right| \le \frac{c_1\lambda}{2} \int \psi^2 \left|\nabla_A v\right|^2 + \frac{4\Lambda^2}{c_1\lambda} \int v^2 \left|\nabla_A \psi\right|^2.$$

Substituting above and absorbing into the left, we obtain

$$\frac{c_1\lambda}{2} \int \psi^2 \left|\nabla_A v\right|^2 \le \frac{4\Lambda^2}{c_1\lambda} \int v^2 \left|\nabla_A \psi\right|^2 + \int w\phi_0 + \int \nabla_A w \cdot \vec{\phi}_1. \tag{21}$$

Since $\left(\phi_0, \vec{\phi}_1\right)$ is admissible, we have that

$$\begin{aligned} \left|\int \psi^2 v v' \phi_0\right| + \left|\int \nabla_A\left(\psi^2 v v'\right) \cdot \vec{\phi}_1\right| &\le \phi^* \int \left|\nabla_A\left(\psi^2 v v'\right)\right| \\ &\le 2\phi^* \int \psi \left|\nabla_A \psi\right| v \left|v'\right| \\ &\quad + \phi^* \int \psi^2 \Upsilon\left(u+\phi^*\right)\left|\nabla_A u\right| \end{aligned}$$

By property (II) we have that $\left|v'\right| = \left|h'(u+\phi^*)\right| \le C_2 \frac{h(u+\phi^*)}{u+\phi^*} = C_2 \frac{v}{u+\phi^*}$; applying this to the first term on the right, and properties (I)–(II) to the second, we obtain

$$\begin{aligned} &\left|\int w\phi_0\right| + \left|\int \nabla_A w \cdot \vec{\phi}_1\right| \\ &\le 2C_2\phi^* \int \psi \left|\nabla_A \psi\right| \frac{v^2}{u+\phi^*} + C_1\phi^* \int \psi^2 \left(v'\right)^2 \left|\nabla_A u\right| \\ &\le 2C_2 \int \psi \left|\nabla_A \psi\right| v^2 + C_1 C_2 \phi^* \int \psi^2 \frac{v}{u+\phi^*} \left|\nabla_A v\right|. \\ &\le C_2 \int \left(\psi^2 + \left|\nabla_A \psi\right|^2\right) v^2 + C_1 C_2 \int \psi^2 v \left|\nabla_A v\right| \end{aligned}$$

$$\leq C_2 \int \left(\psi^2 + \left|\nabla_A \psi\right|^2\right) v^2 + \frac{2C_1^2 C_2^2}{c_1 \lambda} \int \psi^2 v^2 + \frac{c_1 \lambda}{4} \int \psi^2 \left|\nabla_A v\right|^2 . \tag{22}$$

Replacing this on the right of (21) and operating yields

$$\begin{aligned} \int \psi^2 \left|\nabla_A v\right|^2 &\leq \frac{16\Lambda^2}{c_1^2 \lambda^2} \int v^2 \left|\nabla_A \psi\right|^2 + 4 \frac{C_2}{c_1 \lambda} \int \left(\psi^2 + \left|\nabla_A \psi\right|^2\right) v^2 + \frac{8C_1^2 C_2^2}{c_1^2 \lambda^2} \int \psi^2 v^2 \\ &\leq 16 \frac{C_1^2 C_2^2}{c_1^2} \left(\frac{\Lambda^2}{\lambda^2} + \frac{1}{\lambda^2} + \frac{1}{\lambda}\right) \int \left(\psi^2 + \left|\nabla_A \psi\right|^2\right) v^2. \quad \square \end{aligned}$$

3. Preliminaries on Young functions

In this section introduce some basic concepts from Orlicz spaces and define the particular families of Young function that we will use in our applications. We also compute successive compositions of these functions and their inverses and obtain estimates for their derivatives.

3.1. The Orlicz norm and the Orlicz quasidistance

Suppose that μ is a σ-finite measure on a set X, and $\Phi : [0,\infty) \to [0,\infty)$ is a Young function, which for our purposes is an increasing, convex, piecewise differentiable (meaning there are at most finitely many points where the derivative of Φ may fail to exist, but right and left hand derivatives exist everywhere) function such that $\Phi(0) = 0$. The homogeneous Luxemburg norm associated to a Young function Φ is given by

$$\|f\|_{L^{\Phi}(X,d\mu)} = \inf \left\{ t > 0 : \int_X \Phi\left(\frac{|f|}{t}\right) d\mu \leq 1 \right\} \in [0,\infty], \tag{23}$$

where it is understood that $\inf\left(\emptyset\right) = \infty$. The completion of the space of μ-measurable functions in X with respect to this norm (see [11], page 20 for more details) is the Orlicz space $L^{\Phi}(X, d\mu)$ which is a Banach space by definition. The conjugate Young function Φ^* is defined through the relation $(\Phi^*)' = \left(\Phi'\right)^{-1}$ and it can be used to give an equivalent norm

$$\|f\|_{L_*^{\Phi}(\mu)} \equiv \sup \left\{ \int_X |fg| \, d\mu : \int_X \Phi^*(|g|) \, d\mu \leq 1 \right\}.$$

The conjugate function Φ^* is equivalently defined as

$$\Phi^*(s) = \sup_{t>0} (st - \Phi(t)), \qquad \text{for all } s > 0. \tag{24}$$

If Φ and Φ^* are conjugate Young functions, then we have the Orlicz-Hölder inequality

$$\int_X |fg| \, d\mu \leq 2 \, \|f\|_{L^{\Phi}(\mu)} \, \|g\|_{L^{\Phi^*}(\mu)} \tag{25}$$

for all $f \in L^{\Phi}(X, d\mu)$ and $g \in L^{\Phi}(X, d\mu)$ (see [11], (4)-page 58).

Given a Young function Φ and a measure μ we will define a non-homogeneous norm as follows. We let $L_*^{\Phi}(\mu)$ be the set of measurable functions $f : X \to \mathbb{R}$ such that the integral

$$\int_X \Phi(|f|) \, d\mu,$$

is finite, where as usual, functions that agree almost everywhere are identified. The set $L_*^{\Phi}(\mu)$ may not in general be closed under scalar multiplication, but if Φ is K -submultiplicative for some constant $K > 0$, i.e.

$$\Phi(st) \leq K\Phi(s)\Phi(t) \qquad \text{for all } s, t \geq 0$$

then clearly $\int_X \Phi(|Cf|) \, d\mu \leq K\Phi(C) \int_X \Phi(|f|) \, d\mu$ and $L_*^{\Phi}(\mu)$ is a vector space because if $f, g \in L_*^{\Phi}(\mu)$ then

$$\begin{aligned} \int_X \Phi(|f+g|) \, d\mu &= \int_0^\infty \Phi'(t) \mu\{|f+g| > t\} \, dt \\ &\leq \int_0^\infty \Phi'(t) \mu\left\{|f| > \frac{t}{2}\right\} dt + \int_0^\infty \Phi'(t) \mu\left\{|g| > \frac{t}{2}\right\} dt \\ &= \int_X \Phi(2|f|) \, d\mu + \int_X \Phi(2|g|) \, d\mu \\ &< K\Phi(2) \left\{ \int_X \Phi(|f|) \, d\mu + \int_X \Phi(|g|) \, d\mu \right\} < \infty. \end{aligned}$$

We claim that if Φ is an K-submultiplicative Young function then the function

$$\|f\|_{D^{\Phi}(\mu)} \equiv \Phi^{-1}\left(\int_X \Phi(|f|) \, d\mu\right) \tag{26}$$

is a *nonhomogeneous quasi-norm* in $L_*^\Phi(\mu)$, that is, $\|\cdot\|_{D^\Phi(\mu)} : L_*^\Phi(\mu) \to [0,\infty)$ satisfies

$$\|f\|_{D^\Phi(\mu)} = 0 \iff f \equiv 0$$
$$\|f+g\|_{D^\Phi(\mu)} \le C_\Phi \left(\|f\|_{D^\Phi(\mu)} + \|g\|_{D^\Phi(\mu)}\right).$$

Indeed, it is clear that $\|f\|_{D^\Phi(\mu)} \ge 0$ and $\|f\|_{D^\Phi(\mu)} = 0 \iff f = 0$, and that $\|f-g\|_{D^\Phi(\mu)} = \|g-f\|_{D^\Phi(\mu)}$. From the above computation we also have that

$$\begin{aligned}
\Phi\left(\|f+g\|_{D^\Phi(\mu)}\right) &= \int_X \Phi(|f+g|)\ d\mu \\
&\le K\Phi(2)\left\{\int_X \Phi(|f|)\ d\mu + \int_X \Phi(|g|)\ d\mu\right\} \\
&= K\Phi(2)\left\{\Phi\left(\|f\|_{D^\Phi(\mu)}\right) + \Phi\left(\|g\|_{D^\Phi(\mu)}\right)\right\} \\
&\le 2K\Phi(2)\ \Phi\left(\|f\|_{D^\Phi(\mu)} + \|g\|_{D^\Phi(\mu)}\right) \\
&\le \Phi\left(2K\Phi(2)\left\{\|f\|_{D^\Phi(\mu)} + \|g\|_{D^\Phi(\mu)}\right\}\right)
\end{aligned}$$

where we used that Φ is increasing and that $C\Phi(t) \le \Phi(Ct)$ since Φ is increasing convex with $\Phi(0) = 0$. Thus, we have

$$\|f+g\|_{D^\Phi(\mu)} \le C_\Phi\left(\|f\|_{D^\Phi(\mu)} + \|g\|_{D^\Phi(\mu)}\right) \qquad \text{for all } f,g \in L_*^\Phi(\mu).$$

The same proof provides an inequality for any general finite sum of functions $\sum_{j=1}^N f_j$:

$$\left\|\sum_{j=1}^N f_j\right\|_{D^\Phi(\mu)} \le C_{\Phi,N}\left(\sum_{j=1}^N \left\|f_j\right\|_{D^\Phi(\mu)}\right) \qquad \text{whenever } f_j \in L_*^\Phi(\mu),\ j = 1,\dots,N, \tag{27}$$

where $C_{\Phi,N} = NK\ \Phi(N)$.

The function $\|\cdot\|_{D^\Phi(\mu)}$ in general would not be a quasinorm because it may fail to be absolutely homogeneous, i.e., in general$\|Cf\|_{D^\Phi(\mu)} = |C|\,\|f\|_{D^\Phi(\mu)}$ may not hold. It is clear though that $d_\Phi(f,g) = \|f-g\|_{D^\Phi(\mu)}$ is a quasi-distance in $L_*^\Phi(\mu)$, i.e. the function $d_\Phi(\cdot,\cdot) : L_*^\Phi(\mu) \times L_*^\Phi(\mu) \to [0,\infty)$ is symmetric, $d_\Phi(f,g) = 0 \iff f \equiv g$, and satisfies a triangle inequality with a constant C_Φ that may be bigger than 1. We note that the same conclusion may be reached if Φ is K -supermultiplicative, i.e.

$$K\Phi(st) \ge \Phi(s)\Phi(t) \qquad \text{for all } s,t > 0.$$

Indeed, we have that for any $C > 0$ and $f \in L_*^\Phi(\mu)$

$$\int_X \Phi(|Cf|)\ d\mu = \frac{1}{\Phi\left(\frac{1}{C}\right)}\int_X \Phi\left(\frac{1}{C}\right)\Phi(|Cf|)\ d\mu \le \frac{K}{\Phi\left(\frac{1}{C}\right)}\int_X \Phi(|f|)\ d\mu < \infty,$$

and it similarly follows as above that $f+g \in L_*^\Phi(\mu)$ for all $f,g \in L_*^\Phi(\mu)$. We have shown the following:

Proposition 3.1. *If Φ is a K-submultiplicative or K -supermultiplicative Young function in $[0,\infty)$ for some $K > 0$ then the space*

$$L_*^\Phi(\mu) = \left\{f\ :\ \int_X \Phi(|f|)\ d\mu < \infty\right\}$$

is a vector space and the function $\|\cdot\|_{D^\Phi(\mu)} : L_*^\Phi(\mu) \to [0,\infty)$ defined in (26) is a nonhomogeneous quasi-norm in $L_*^\Phi(\mu)$.

In this paper we consider Young functions which satisfy the hypotheses of the above proposition, so our Moser iteration may be considered as an iteration scheme in quasi-metric spaces. The homogeneity of the norm $\|f\|_{L^\Phi(\mu)}$ is not that important, but rather it is the iteration of Orlicz expressions that is critical. The following lemma shows the relations between the Orlicz norm and the quasi-norm when the Young function is sub- or supermultiplicative.

Lemma 3.2. *If a Young function Φ is K-submultiplicative for some constant $K \ge 1$, then*

$$\Phi^{-1}\left(\int_{B(x,\rho)} \Phi(v)\,d\mu_{x,\rho}\right) \le K\,\|v\|_{L^\Phi(\mu_{x,\rho})}.$$

On the other hand, if Φ is a K-supermultiplicative Young function for some $K \ge 1$, then

$$\|v\|_{L^\Phi(\mu_{x,\rho})} \le K\Phi^{(-1)}\left(\int_{B(x,\rho)} \Phi(v)\,d\mu_{x,\rho}\right).$$

Proof. Recall that we have by definition

$$\|v\|_{L^\Phi(\mu_{x,\rho})} = \inf\left\{t > 0 : \int_{B(x,\rho)} \Phi\left(\frac{|v|}{t}\right)\ d\mu_{x,\rho} \le 1\right\}.$$

Let $\kappa = \|v\|_{D^\Phi(\mu)} = \Phi^{-1}\left(\int_{B(x,\rho)} \Phi(|v|)\ d\mu_{x,\rho}\right)$, by the submultiplicativity of Φ we have

$$\Phi(|v|) = \Phi\left(\frac{|v|}{\kappa}\cdot\kappa\right) \le K\Phi\left(\frac{|v|}{\kappa}\right)\Phi(\kappa) = K\Phi\left(\frac{|v|}{\kappa}\right)\int_{B(x,\rho)} \Phi(|v|)\ d\mu_{x,\rho}$$

$$\leq \Phi\left(K\frac{|v|}{\kappa}\right)\int_{B(x,\rho)}\Phi(|v|)\ d\mu_{x,\rho},$$

where we used that $C\Phi(t)\leq\Phi(Ct)$ for all $C\geq 1$. Integrating gives

$$\int_{B(x,\rho)}\Phi(|v|)\ d\mu_{x,\rho}\leq\int_{B(x,\rho)}\Phi\left(K\frac{|v|}{\kappa}\right)\ d\mu_{x,\rho}\cdot\int_{B(x,\rho)}\Phi(|v|)\ d\mu_{x,\rho}$$

so that $\int_{B(x,\rho)}\Phi\left(K\frac{|v|}{\kappa}\right)\ d\mu_{x,\rho}\geq 1$, which yields

$$\|v\|_{L^\Phi(\mu_{x,\rho})}\geq\frac{\kappa}{K}=\frac{1}{K}\Phi^{-1}\left(\int_{B(x,\rho)}\Phi(|v|)\ d\mu_{x,\rho}\right).$$

Now assume that Φ is a K-supermultiplicative Young function, i.e. $K\Phi(st)\geq\Phi(s)\Phi(t)$ for all $s,t\geq 0$. We have

$$\begin{aligned}K\int_{B(x,\rho)}\Phi(|v|)\ d\mu_{x,\rho}&=K\int_{B(x,\rho)}\Phi\left(\|v\|_{L^\Phi(\mu_{x,\rho})}\frac{|v|}{\|v\|_{L^\Phi(\mu_{x,\rho})}}\right)d\mu_{x,\rho}\\&\geq\Phi\left(\|v\|_{L^\Phi(\mu_{x,\rho})}\right)\int_{B(x,\rho)}\Phi\left(\frac{|v|}{\|v\|_{L^\Phi(\mu_{x,\rho})}}\right)d\mu_{x,\rho}.\end{aligned}$$

By Fatou's lemma we see that for any $\delta>0$

$$\begin{aligned}0&\leq\int_{B(x,\rho)}\Phi\left(\frac{|v|}{\|v\|_{L^\Phi(\mu_{x,\rho})}-\delta}\right)d\mu_{x,\rho}-\int_{B(x,\rho)}\limsup_{\varepsilon\to0^+}\Phi\left(\frac{|v|}{\|v\|_{L^\Phi(\mu_{x,\rho})}-\varepsilon}\right)d\mu_{x,\rho}\\&=\int_{B(x,\rho)}\liminf_{\varepsilon\to0^+}\left(\Phi\left(\frac{|v|}{\|v\|_{L^\Phi(\mu_{x,\rho})}-\delta}\right)-\Phi\left(\frac{|v|}{\|v\|_{L^\Phi(\mu_{x,\rho})}-\varepsilon}\right)\right)d\mu_{x,\rho}\\&\leq\liminf_{\varepsilon\to0^+}\int_{B(x,\rho)}\left(\Phi\left(\frac{|v|}{\|v\|_{L^\Phi(\mu_{x,\rho})}-\delta}\right)-\Phi\left(\frac{|v|}{\|v\|_{L^\Phi(\mu_{x,\rho})}-\varepsilon}\right)\right)d\mu_{x,\rho}\\&=\int_{B(x,\rho)}\Phi\left(\frac{|v|}{\|v\|_{L^\Phi(\mu_{x,\rho})}-\delta}\right)d\mu_{x,\rho}-\limsup_{\varepsilon\to0^+}\int_{B(x,\rho)}\Phi\left(\frac{|v|}{\|v\|_{L^\Phi(\mu_{x,\rho})}-\varepsilon}\right)d\mu_{x,\rho}.\end{aligned}$$

Hence

$$\begin{aligned}K\int_{B(x,\rho)}\Phi(|v|)\ d\mu_{x,\rho}&\geq\Phi\left(\|v\|_{L^\Phi(\mu_{x,\rho})}\right)\int_{B(x,\rho)}\limsup_{\varepsilon\to0^+}\Phi\left(\frac{|v|}{\|v\|_{L^\Phi(\mu_{x,\rho})}-\varepsilon}\right)d\mu_{x,\rho}\\&\geq\Phi\left(\|v\|_{L^\Phi(\mu_{x,\rho})}\right)\limsup_{\varepsilon\to0^+}\int_{B(x,\rho)}\Phi\left(\frac{|v|}{\|v\|_{L^\Phi(\mu_{x,\rho})}-\varepsilon}\right)d\mu_{x,\rho}\\&\geq\Phi\left(\|v\|_{L^\Phi(\mu_{x,\rho})}\right).\end{aligned}$$

where we applied the definition of $\|v\|_{L^\Phi(\mu_{x,\rho})}$. Then, since $\Phi^{-1}(cs)\leq c\Phi^{-1}(s)$ for all $0<c\leq 1$, we have that

$$\Phi^{-1}\left(\int_{B(x,\rho)}\Phi(|v|)\ d\mu_{x,\rho}\right)\geq\Phi^{-1}\left(\frac{1}{K}\Phi\left(\|v\|_{L^\Phi(\mu_{x,\rho})}\right)\right)\geq\frac{\|v\|_{L^\Phi(\mu_{x,\rho})}}{K}.\quad\square$$

3.2. *Orlicz norms and admissibility*

The next proposition gives sufficient conditions for strong admissibility.

Proposition 3.3. *Given a right hand side pair $\left(\phi_0,\vec{\phi}_1\right)$ defined in a bounded domain Ω. Suppose that $\vec{\phi}_1\in L^\infty(\Omega)$ and that there exists a submultiplicative bump function Φ and a constant C_Ω such that the global (Φ,A)-Orlicz-Sobolev bump inequality (8) holds, and such that $\phi_0\in L^{\Phi^*}(\Omega)$ where Φ^* is the conjugate Young function to Φ. Then $\left(\phi_0,\vec{\phi}_1\right)$ is strongly admissible in Ω as given in Definition 1.5 with norm*

$$\left\|\left(\phi,\vec{\phi}_1\right)\right\|_{\mathcal{X}^*(\Omega)}\leq 2C_\Omega\|\phi_0\|_{L^{\Phi^*}(\Omega)}+\left\|\vec{\phi}_1\right\|_{L^\infty(\Omega)}<\infty.$$

Proof. First, note that for any $v\in\mathrm{Lip}_c(\Omega)$

$$\int_\Omega\left|\nabla_A v\cdot\vec{\phi}_1\right|\ dx\leq\left\|\vec{\phi}_1\right\|_{L^\infty(\Omega)}\|\nabla_A v\|_{L^1(\Omega)},$$

so $\left\|\vec{\phi}_1\right\|_{\mathcal{X}^*(\Omega)}\leq\left\|\vec{\phi}_1\right\|_{L^\infty(\Omega)}$. Next, by the Orlicz-Hölder inequality (25), the global Orlicz-Sobolev inequality (8), and Lemma 3.2, for any $v\in\mathrm{Lip}_c(\Omega)$

$$\int_\Omega|v\phi_0|\ dx\ \leq\ 2\|\phi_0\|_{L^{\Phi^*}(\Omega)}\|v\|_{L^\Phi(\Omega)}$$

$$\leq 2C_{\Omega} \left\|\phi_0\right\|_{L^{\Phi^*}(\Omega)} \left\|\nabla_A v\right\|_{L^1(\Omega)}$$

this is $\left\|\phi_0\right\|_{\mathcal{X}^*(B(y,R_0))} \leq 2C_{\Omega} \left\|\phi_0\right\|_{L^{\Phi^*}(\Omega)}$. □

3.3. Submultiplicative extensions

In our application to Moser iteration the convex bump function $\Phi(t)$ is assumed to satisfy in addition:

- The function $\frac{\Phi(t)}{t}$ is positive, nondecreasing and tends to ∞ as $t \to \infty$;
- Φ is submultiplicative on an interval (E, ∞) for some $E > 1$:

$$\Phi(ab) \leq \Phi(a)\Phi(b), \qquad a, b > E. \tag{28}$$

Note that if we consider more generally the quasi-submultiplicative condition or K-submultiplicativity,

$$\Phi(ab) \leq K\Phi(a)\Phi(b), \qquad a, b > E, \tag{29}$$

for some constant K, then $\Phi(t)$ satisfies (29) if and only if $\Phi_K(t) \equiv K\Phi(t)$ satisfies (28). Thus we can always rescale a quasi-submultiplicative function to be submultiplicative.

Now let us consider the *linear extension* of Φ defined on $[E, \infty)$ to the entire positive real axis $(0, \infty)$ defined by

$$\Phi(t) = \frac{\Phi(E)}{E} t, \qquad 0 \leq t \leq E.$$

We claim that this extension of Φ is submultiplicative on $(0, \infty)$, i.e.

$$\Phi(ab) \leq \Phi(a)\Phi(b), \qquad a, b > 0.$$

In fact, the identity $\Phi(t)/t = \Phi(\max\{t, E\})/\max\{t, E\}$ and the monotonicity of $\Phi(t)/t$ imply

$$\frac{\Phi(ab)}{ab} \leq \frac{\Phi(\max\{a, E\}\max\{b, E\})}{\max\{a, E\}\max\{b, E\}} \leq \frac{\Phi(\max\{a, E\})}{\max\{a, E\}} \cdot \frac{\Phi(\max\{b, E\})}{\max\{b, E\}} = \frac{\Phi(a)}{a}\frac{\Phi(b)}{b}.$$

Conclusion 3.4. *If $\Phi : [E, \infty) \to \mathbb{R}^+$ is a submultiplicative piecewise differentiable convex function so that $\Phi(t)/t$ is nondecreasing, then we can extend Φ to a submultiplicative piecewise differentiable convex function on $[0, \infty)$ that vanishes at 0 if and only if*

$$\Phi'(E) \geq \frac{\Phi(E)}{E}. \tag{30}$$

3.4. An explicit family of Orlicz bumps

We now consider the *near* power bump case $\Phi(t) = \Phi_m(t) = e^{\left((\ln t)^{\frac{1}{m}}+1\right)^m}$ for $m > 1$. In the special case that $m > 1$ is an integer we can expand the mth power in

$$\ln \Phi(e^s) = \left(s^{\frac{1}{m}} + 1\right)^m = \sum_{k=0}^{m} \binom{m}{k} s^{\frac{k}{m}},$$

and using the inequality $1 \leq \left(\frac{s}{s+t}\right)^{\alpha} + \left(\frac{t}{s+t}\right)^{\alpha}$ for $s, t > 0$ and $0 \leq \alpha \leq 1$, we see that $\Theta_m(s) \equiv \ln \Phi_m(e^s)$ is subadditive on $(0, \infty)$, hence Φ_m is submultiplicative on $(1, \infty)$. In fact, it is not hard to see that for $m > 1$, $\Theta_m(s) = \left(s^{\frac{1}{m}} + 1\right)^m$ is subadditive on $(0, \infty)$, and so Φ_m is submultiplicative on $(1, \infty)$.

We will show that Φ is increasing and convex in $[E, \infty)$. For any $t > 1$ we have

$$\begin{aligned}\Phi'(t) &= \Phi(t)\, m\left((\ln t)^{\frac{1}{m}} + 1\right)^{m-1} \frac{1}{m}(\ln t)^{\frac{1}{m}-1}\frac{1}{t} \\ &= \frac{\Phi(t)}{t}\left(1 + \frac{1}{(\ln t)^{\frac{1}{m}}}\right)^{m-1} \equiv \frac{\Phi(t)}{t}\Omega(t),\end{aligned} \tag{31}$$

with $\Omega(t) = \Omega_m(t) = \left(1 + (\ln t)^{-\frac{1}{m}}\right)^{m-1} > 1$; and so for any $E > 1$ we have

$$\Phi'(E) > \frac{\Phi(E)}{E}. \tag{32}$$

Next, we compute

$$\begin{aligned}\Phi''(t) &= \frac{\Phi(t)}{t^2}\left((\Omega(t))^2 - \Omega(t) + t\Omega'(t)\right) \\ &= \frac{\Phi_m(t)}{t^2}\left((\Omega(t))^2 - \Omega(t) + t\Omega'(t)\right).\end{aligned}$$

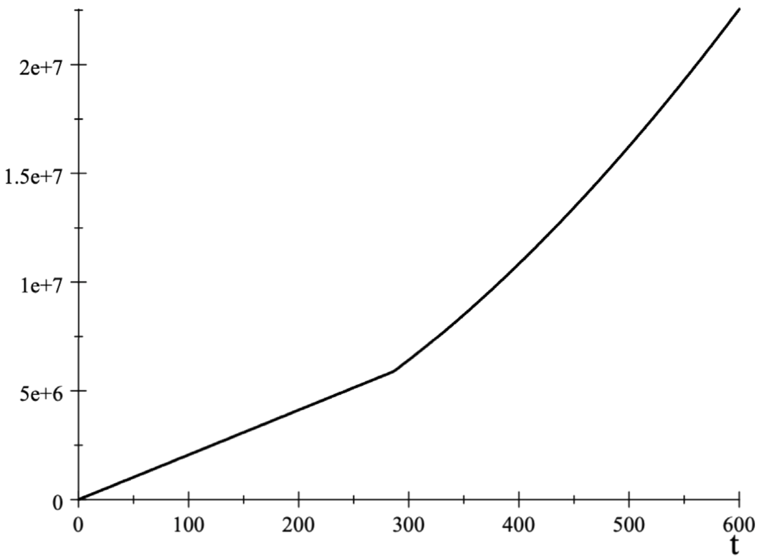

Fig. 1. The Young function $\Phi_{(5/2)}(t)$.

Since $\Omega'(t) = -\frac{m-1}{m}\frac{1}{t}\left(1+(\ln t)^{-\frac{1}{m}}\right)^{m-2}(\ln t)^{-\frac{1}{m}-1} = -\frac{m-1}{m}\frac{1}{t}\Omega^{\frac{m-2}{m-1}}(\ln t)^{-1-\frac{1}{m}}$, for $t>1$ we have

$$\begin{aligned}\Phi''(t) &= \frac{\Phi(t)}{t^2}\left((\Omega(t))^2 - \Omega(t) - \frac{m-1}{m}\Omega^{\frac{m-2}{m-1}}(\ln t)^{-\frac{1}{m}-1}\right)\\ &= \frac{\Phi_m(t)}{t^2}\Omega(t)\left(\Omega(t) - 1 - \frac{\frac{m-1}{m}}{\Omega^{\frac{1}{m-1}}(\ln t)^{1+\frac{1}{m}}}\right)\\ &= \frac{\Phi_m(t)}{t^2}\Omega(t)\Gamma(t),\end{aligned} \tag{33}$$

where

$$\Gamma(t) = \Gamma_m(t) = \Omega(t) - 1 - \frac{\frac{m-1}{m}}{\Omega^{\frac{1}{m-1}}(\ln t)^{1+\frac{1}{m}}}.$$

since $\Omega(t) - 1 = \left(1+(\ln t)^{-\frac{1}{m}}\right)^{m-1} - 1 \geq (m-1)(\ln t)^{-\frac{1}{m}}$, it follows that

$$\Gamma(t) \geq \frac{m-1}{(\ln t)^{\frac{1}{m}}}\left(1 - \frac{\frac{1}{m}}{\Omega^{\frac{1}{m-1}}\ln t}\right) > \frac{C_{m,E}}{(\ln t)^{\frac{1}{m}}} > 0$$

for all $t \geq e$ and $m > 1$. This shows that Φ is convex on $[e,\infty)$, and so by (32) and Conclusion 3.4 we can extend Φ to a positive increasing *submultiplicative* convex function on $[0,\infty)$. However, due to technical calculations below, it is convenient to take $E = E_m = e^{2^m}$, $F = F_m = e^{3^m}$, and so we will work from now on with the definition

$$\Phi(t) = \Phi_m(t) \equiv \begin{cases} e^{\left((\ln t)^{\frac{1}{m}}+1\right)^m} & \text{if} \quad t \geq E = e^{2^m} \\ \frac{F}{E}t & \text{if} \quad 0 \leq t \leq E = e^{2^m} \end{cases}, \tag{34}$$

where $m > 1$ will be explicitly mentioned or understood from the context (see Fig. 1).

The function Φ_m is clearly continuous and piecewise C^∞. In some of our applications we will require that the Young function should be C^1 and piece-wise smooth, so the second derivative only has at most jump discontinuities. For this reason we define a variation $\tilde{\Phi}_m$ of the Young function Φ_m which has the same growth as $t \to \infty$, and has the required smoothness. We define

$$\tilde{\Phi}_m(t) \equiv \begin{cases} \Phi_m(t) & \text{if} \quad t \geq E \\ \varrho_m(t) & \text{if} \quad \frac{BE^2}{F} \leq t \leq E \\ \frac{1}{B}\frac{F}{E}t & \text{if} \quad 0 \leq t \leq \frac{BE^2}{F} \end{cases} \tag{35}$$

where $B > 1$ and $\varrho_m(t)$ is an increasing convex function satisfying

$$\begin{array}{ll} \varrho_m(a) = E, & \varrho_m(E) = F \\ \varrho_m'(a) = \frac{1}{B}\frac{F}{E} & \varrho_m'(E) = \frac{F}{E}\left(\frac{3}{2}\right)^{m-1} \end{array} \quad \text{where } a = \frac{BE^2}{F}. \tag{36}$$

For example, we can take

$$\varrho_m(t) = E + \frac{1}{B}\frac{F}{E}(t-a) + \alpha(t-a)^{\beta}, \tag{37}$$

where α and β are determined by (36):

$$\beta = \frac{B}{B-1}\left(1-\frac{a}{E}\right)\left(\left(\frac{3}{2}\right)^{m-1} - \frac{1}{B}\right), \qquad \text{and} \qquad \alpha = \frac{B-1}{B}\frac{F}{(E-a)^{\beta}}.$$

Indeed, since $\beta > 1$ we have that both $\alpha(t-a)^{\beta}$ and its derivative vanish at $t = a$, and so

$$\varrho_m(a) = E \qquad \text{and} \qquad \varrho_m'(t) = \frac{1}{B}\frac{F}{E}.$$

Moreover,

$$\begin{aligned} \varrho_m(E) &= E + \frac{1}{B}\frac{F}{E}(E-a) + \frac{B-1}{B}\frac{F}{(E-a)^{\beta}}(E-a)^{\beta} \\ &= E + \frac{1}{B}\frac{F}{E}(E-a) + \frac{B-1}{B}F \\ &= F + E - \frac{1}{B}\frac{F}{E}a = F + E - \frac{1}{B}\frac{F}{E}\frac{BE^2}{F} = F, \end{aligned}$$

and

$$\begin{aligned} \varrho_m'(E) &= \frac{1}{B}\frac{F}{E} + \alpha\beta(E-a)^{\beta-1} = \frac{1}{B}\frac{F}{E} + \beta\frac{B-1}{B}\frac{F}{E-a} \\ &= \frac{1}{B}\frac{F}{E} + \left(1-\frac{a}{E}\right)\left(\left(\frac{3}{2}\right)^{m-1} - \frac{1}{B}\right)\frac{F}{E-a} \\ &= \frac{1}{B}\frac{F}{E} + \left(\left(\frac{3}{2}\right)^{m-1} - \frac{1}{B}\right)\frac{F}{E} = \frac{F}{E}\left(\frac{3}{2}\right)^{m-1}. \end{aligned}$$

This proves that ϱ_m defined by (37) satisfies (36). Note that since

$$\begin{aligned} \beta = \beta(m) &= \frac{B}{B-1}\left(1-\frac{a}{E}\right)\left(\left(\frac{3}{2}\right)^{m-1} - \frac{1}{B}\right) \\ &= \frac{B}{B-1}\left(1-\frac{B}{e^{3^m-2^m}}\right)\left(\left(\frac{3}{2}\right)^{m-1} - \frac{1}{B}\right), \end{aligned}$$

then β is increasing in m, and therefore for each fixed $m_0 > 1$ we have $\beta \geq \beta(m)$ for all $m \geq m_0$. Given a fixed $m_0 > 1$ we will choose B so that $\beta(m_0) = 2$, i.e. B is a root of the quadratic equation

$$\left(1-\frac{B}{e^{3^{m_0}-2^{m_0}}}\right)\left(\left(\frac{3}{2}\right)^{m_0-1} B - 1\right) = 2(B-1). \tag{38}$$

This choice of B guarantees that $0 \leq \varrho_m''(t) \leq M < \infty$ for t in $[a, E]$ for all $m \geq m_0$, and hence the function $\tilde{\Phi}_m$ is in $C^{1,1}(\mathbb{R})$. To see that $B > 1$ can be chosen we write Eq. (38) in the equivalent form

$$(\nu+1)(B-1)^2 - (\nu\mu - \mu - \nu - 2)(B-1) - \mu\nu = 0$$

where $\nu = \left(\frac{3}{2}\right)^{m_0-1} - 1$ and $\mu = e^{3^{m_0}-2^{m_0}} - 1$. The choice of root

$$B = 1 + \frac{(\nu\mu - \mu - \nu - 2) + \sqrt{(\nu\mu - \mu - \nu - 2)^2 + 4\mu\nu(\nu+1)}}{2(\nu+1)}$$

show that B is clearly bigger than 1. In our applications it will suffice to take $m_0 = 2$, for which we have

$$B = \frac{1}{3}\left(\sqrt{5e^5 + \frac{1}{4}e^{10} + 1} - \frac{e^5-2}{2}\right) \approx 1.949450754..$$

Moreover, we have $\varrho_m''(t) = \alpha\beta(\beta-1)(t-a)^{\beta-2} > 0$ for $a \leq t \leq E$, so ϱ_m is clearly convex (see Fig. 2).

In our application we will also require $\gamma_{\frac{1}{2}}(t) = \sqrt{\varrho_m(t^2)}$ to be convex. Since

$$\gamma_{\frac{1}{2}}'(t) = \left(\sqrt{\varrho_m(t^2)}\right)' = \frac{t\varrho_m'(t^2)}{\sqrt{\varrho_m(t^2)}} = t\frac{\frac{1}{B}\frac{F}{E} + \alpha\beta\left(t^2-a\right)^{\beta-1}}{\sqrt{E + \frac{1}{B}\frac{F}{E}\left(t^2-a\right) + \alpha\left(t^2-a\right)^{\beta}}}$$

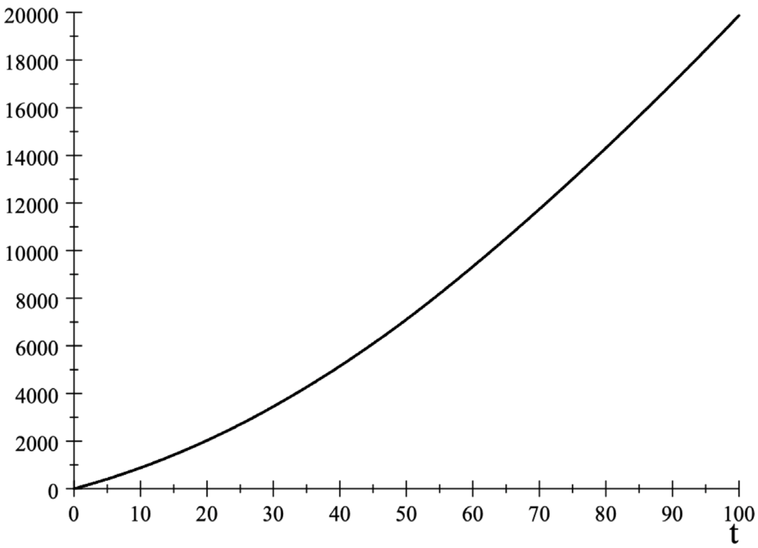

Fig. 2. The Young function $\tilde{\Phi}_2(t)$ with $\beta = 2$, $a \approx 0.7$ and $E \approx 55$.

for $a \le t^2 \le E$. With the change of variables $s = t^2 - a$ and since $\frac{1}{B}\frac{F}{E} = \frac{E}{a}$, this is

$$\gamma'_{\frac{1}{2}}\left(\sqrt{s+a}\right) = \sqrt{\frac{s+a}{E + \frac{E}{a}s + \alpha s^\beta}}\left(\frac{E}{a} + \alpha\beta s^{\beta-1}\right), \qquad 0 \le s \le E - a$$

which is increasing in s when $\beta \ge 2$. Indeed, the derivative of $\left(\gamma'_{\frac{1}{2}}\left(\sqrt{s+a}\right)\right)^2$ is

$$\begin{aligned}
&\frac{d}{ds}\left\{\frac{s+a}{E + \frac{E}{a}s + \alpha s^\beta}\left(\frac{E}{a} + \alpha\beta s^{\beta-1}\right)^2\right\} \\
&= \frac{\left(E + \frac{E}{a}s + \alpha s^\beta\right) - (s+a)\left(\frac{E}{a} + \alpha\beta s^{\beta-1}\right)}{\left(E + \frac{E}{a}s + \alpha s^\beta\right)^2} + \frac{(s+a)\left(\frac{E}{a} + \alpha\beta s^{\beta-1}\right) 2\alpha\beta(\beta-1)s^{\beta-2}}{E + \frac{E}{a}s + \alpha s^\beta} \\
&= \frac{-\alpha s^\beta(\beta-1) - a\alpha\beta s^{\beta-1}}{\left(E + \frac{E}{a}s + \alpha s^\beta\right)^2} + \frac{(s+a)\left(\frac{E}{a} + \alpha\beta s^{\beta-1}\right) 2\alpha\beta(\beta-1)s^{\beta-2}}{E + \frac{E}{a}s + \alpha s^\beta} \\
&= \frac{-\alpha s^\beta(\beta-1) - a\alpha\beta s^{\beta-1} + (s+a)\left(\frac{E}{a} + \alpha\beta s^{\beta-1}\right)\left(E + \frac{E}{a}s + \alpha s^\beta\right) 2\alpha\beta(\beta-1)s^{\beta-2}}{\left(E + \frac{E}{a}s + \alpha s^\beta\right)^2} \\
&\ge \frac{-\alpha s^\beta(\beta-1) - a\alpha\beta s^{\beta-1} + 2\left(\frac{E}{a}\right)^2 \alpha\beta(\beta-1)s^\beta + 2\frac{E^2}{a}\alpha\beta(\beta-1)s^{\beta-1}}{\left(E + \frac{E}{a}s + \alpha s^\beta\right)^2} \\
&= \frac{\left\{2\left(\frac{E}{a}\right)^2\beta - 1\right\}\alpha s^\beta(\beta-1) + \left\{\frac{E^2}{a^2}(2\beta-2) - 1\right\} a\alpha\beta s^{\beta-1}}{\left(E + \frac{E}{a}s + \alpha s^\beta\right)^2} \ge 0.
\end{aligned}$$

Therefore $\gamma'_{\frac{1}{2}}(t)$ is increasing in t, and so $\gamma_{\frac{1}{2}}(t) = \sqrt{\varrho_m\left(t^2\right)}$ is convex. We also note that the upper bound

$$\frac{\gamma_{\frac{1}{2}}(t)\,\gamma''_{\frac{1}{2}}(t)}{\left(\gamma'_{\frac{1}{2}}(t)\right)^2} \le C_m \qquad \text{for } a \le t^2 \le E, \tag{39}$$

readily follows from the definitions.

Notice that $\tilde{\Phi}_m(t) \equiv \Phi_m(t)$ for $t \geq E$, while $\frac{1}{C_m}\Phi_m(t) \leq \tilde{\Phi}_m(t) \leq \Phi_m(t)$ for all $t \geq 0$. It follows that if an Orlicz Sobolev inequality holds for Φ_m with superradius φ, then we have that the Orlicz Sobolev inequality holds for $\tilde{\Phi}_m$ with superradius $C_m\varphi$ for some constant C_m. Indeed, if $v \in \mathrm{Lip}_c(B(x,r))$ for a ball $B(x,r)$, then

$$\begin{aligned}\tilde{\Phi}^{(-1)}\left(\int_{B(x,r)} \tilde{\Phi}(|v|)\frac{dx}{|B(x,r)|}\right) &\leq \Phi^{(-1)}\left(C_m\int_{B(x,r)} \Phi(|v|)\frac{dx}{|B(x,r)|}\right)\\ &\leq \Phi^{(-1)}\left(\int_{B(x,r)} \Phi\left(C_m|v|\right)\frac{dx}{|B(x,r)|}\right)\\ &\leq C_m\varphi(x,r)\int_{B(x,r)} \left|\nabla_A v\right|\frac{dx}{|B(x,r)|}. \end{aligned} \tag{40}$$

Moreover, $\tilde{\Phi}_m$ is defined to be linear on $[0,a]$ with $\tilde{\Phi}(a) = E$ to facilitate computing successive compositions $\tilde{\Phi}_m^{(n)}(t)$; indeed, for t small these compositions are just linear, for $t \geq E$ these are $\tilde{\Phi}_m^{(n)}(t) = \Phi_m^{(n)}(t)$, and when $a \leq t \leq E$ then $\Phi(t) \geq E$, so the modified formula in the *middle* appears at most once in any composition. See Corollary 3.7 for details.

3.5. Iterates of increasing functions

In this subsection we consider the specific families of test functions h that arise in our proofs. To implement the Moser iteration scheme we are interested in estimates for the iterates $h_j(t) = h\circ h\circ\cdots\circ h$ (j times), in particular, to apply the previous Caccioppoli inequalities, we want to estimate the quotients $\frac{th_j'(t)}{h_j(t)}$ and $\frac{th_j''(t)}{h_j'(t)}$, as well as the function $Y_j(t) = \left(\frac{1}{2}h_j^2(t)\right)'' = h_j(t)h_j''(t) + \left(h_j'(t)\right)^2$.

One family of test functions we consider is

$$h_j(t) = \Gamma_m^{(j)}(t) \equiv \Gamma_m\circ\Gamma_m\circ\ldots\Gamma_m(t) \quad (j \text{ times}), \tag{41}$$

where the function $\Gamma_m(t) \equiv \sqrt{\Phi_m(t^2)}$ for $m > 1$. When $t > \sqrt{E_m} = e^{2^{m-1}}$, we have the explicit formula

$$\Gamma_m(t) \equiv \sqrt{\Phi_m\left(t^2\right)} = e^{\frac{1}{2}\left((2\ln t)^{\frac{1}{m}}+1\right)^m} > t.$$

Proposition 3.5. *Let $m > 1$, the function $h(t) = h_j(t) = \sqrt{\Phi_m^{(j)}\left(t^2\right)}$ defined in (41) for each $j \geq 1$ satisfies*

$$h'(t)^2 \leq Y(t) \leq 2h'(t)^2 \qquad \textit{and} \qquad 1 \leq \frac{th'(t)}{h(t)} \leq C_m j^{m-1},$$

where $Y(t) = \left(\frac{1}{2}h^2(t)\right)'' = h(t)h''(t) + \left(h'(t)\right)^2$. Moreover, we have that $h''(t) \geq 0$ for all $t > 0$.

Proof. From the definition (34) of Φ_m, we have

$$h_1(t) = \begin{cases}\gamma_0(t) & 0 \leq t < e^{2^{m-1}}\\ \gamma_1(t) & e^{2^{m-1}} \leq t\end{cases}.$$

where $\gamma_0(t) = \tau t$ with $\tau = \exp\left(\frac{1}{2}\left(3^m - 2^m\right)\right)$, and $\gamma_1(t) = e^{\frac{1}{2}\left((2\ln t)^{\frac{1}{m}}+1\right)^m}$. Then, defining the intervals $I_0 = \left(0, \tau^{-(j-1)}e^{2^{m-1}}\right)$, $I_k = \left[\tau^{-(j-k)}e^{2^{m-1}}, \tau^{-(j-k-1)}e^{2^{m-1}}\right)$ for $k = 1,\ldots,j-1$, and $I_j = \left[e^{2^{m-1}}, \infty\right)$, we have the expression

$$h(t) = h_j(t) = \begin{cases}\gamma_0^{(j)}(t) & t \in I_0;\\ \gamma_1^{(k)}\left(\gamma_0^{(j-k)}(t)\right) & t \in I_k\ k = 1,\ldots,j-1;\,.\\ \gamma_1^{(j)}(t)\,. & t \in I_j.\end{cases} \tag{42}$$

Since $\gamma_0^{(j)}(t) = \tau^j t$ it is clear that for all $j \geq 1$ and for $t \in I_0$

$$Y_j(t) \equiv \gamma_0^{(j)}(t)\left(\gamma_0^{(j)}(t)\right)'' + \left(\left(\gamma_0^{(j)}(t)\right)'\right)^2 = \left(\left(\gamma_0^{(j)}(t)\right)'\right)^2 \qquad \text{and} \qquad \frac{\left(\gamma_0^{(j)}(t)\right)'t}{\gamma_0^{(j)}(t)} = 1 \tag{43}$$

Now, for $\ell \geq 1$, and $t \geq e^{2^{m-1}}$

$$\left(\gamma_1^{(\ell)}(t)\right)' = \left(e^{\frac{1}{2}\left((2\ln t)^{\frac{1}{m}}+\ell\right)^m}\right)' = \frac{\gamma_1^{(\ell)}(t)}{t}\Omega_\ell^*(t) \tag{44}$$

with $\Omega_\ell^*(t) = \left(1 + \frac{\ell}{(2\ln t)^{\frac{1}{m}}}\right)^{m-1}$, and

$$\left(\gamma_1^{(\ell)}(t)\right)'' = \frac{\gamma_1^{(\ell)}(t)}{t^2}\left(\left(\Omega_\ell^*(t)\right)^2 - \Omega_\ell^*(t) + t\left(\Omega_\ell^*(t)\right)'\right). \tag{45}$$

So from (43) and (44) (since $\gamma_0^{(j-k)}(t) \geq e^{2^{m-1}}$ whenever $t \in I_k$, $k = 1, \ldots, j$), for $t \in I_j$ we obtain

$$1 \leq \frac{t\left(\gamma_1^{(j)}(t)\right)'}{\gamma_1^{(j)}(t)} = \Omega_j^*(t) \leq \left(1 + \frac{j}{2}\right)^{m-1}, \tag{46}$$

and

$$\begin{aligned}\frac{\Upsilon_j(t)}{\left(\left(\gamma_1^{(j)}(t)\right)'\right)^2} &\equiv \frac{\gamma_1^{(j)}(t)\left(\gamma_1^{(j)}(t)\right)'' + \left(\left(\gamma_1^{(j)}(t)\right)'\right)^2}{\left(\left(\gamma_1^{(j)}(t)\right)'\right)^2} \\ &= \frac{\left(\Omega_j^*(t)\right)^2 - \Omega_j^*(t) + t\left(\Omega_j^*(t)\right)'}{\left(\Omega_j^*(t)\right)^2} + 1. \end{aligned} \tag{47}$$

Since $\left(\Omega_j^*(t)\right)' = -\frac{\frac{m-1}{m}j}{2^{\frac{1}{m}}(\ln t)^{\frac{1}{m}+1}t}\left(1 + \frac{j}{(2\ln t)^{\frac{1}{m}}}\right)^{m-2}$, using the estimate

$$(1+x)^{m-1} - 1 \geq (1+x)^{-(2-m)_+}(m-1)x \geq (1+x)^{-1}(m-1)x$$

when $m > 1$, $x \geq 0$, we have that with $x = \frac{j}{(2\ln t)^{\frac{1}{m}}}$

$$\begin{aligned}\Omega_j^*(t) - 1 + t\frac{\left(\Omega_j^*(t)\right)'}{\Omega_j^*(t)} &= \Omega_j^*(t) - 1 - \frac{\frac{m-1}{m}j}{2^{\frac{1}{m}}(\ln t)^{\frac{1}{m}+1}\left(1 + \frac{j}{(2\ln t)^{\frac{1}{m}}}\right)} \\ &\geq \frac{1}{\left(1 + \frac{j}{(2\ln t)^{\frac{1}{m}}}\right)}\left(\frac{(m-1)j}{(2\ln t)^{\frac{1}{m}}} - \frac{\frac{m-1}{m}j}{2^{\frac{1}{m}}(\ln t)^{\frac{1}{m}+1}}\right) \\ &= \frac{\frac{(m-1)j}{(2\ln t)^{\frac{1}{m}}}}{\left(1 + \frac{j}{(2\ln t)^{\frac{1}{m}}}\right)}\left(1 - \frac{\frac{1}{m}}{\ln t}\right) \geq 0, \qquad \text{since } t \geq e^{2^{m-1}}.\end{aligned}$$

It follows that $\left(\gamma_1^{(j)}(t)\right)'' \geq 0$, and thus

$$0 \leq \frac{\gamma_1^{(j)}(t)\left(\gamma_1^{(j)}(t)\right)''}{\left(\left(\gamma_1^{(j)}(t)\right)'\right)^2} = \frac{\left(\Omega_j^*(t)\right)^2 - \Omega_j^*(t) + t\left(\Omega_j^*(t)\right)'}{\left(\Omega_j^*(t)\right)^2} \leq 1 \tag{48}$$

since $\left(\Omega_j^*\right)' \leq 0$. Substituting in (47) yields

$$1 \leq \frac{\Upsilon_j(t)}{\left(\left(\gamma_1^{(j)}(t)\right)'\right)^2} \leq 2. \tag{49}$$

Then from the expression (42) for h_j and (43)–(46)–(49) it follows that $h_j(t)$ satisfies the estimates claimed in the proposition both for $t \in I_0$ and $t \in I_j$. Now, note that when $t \in I_k$, $k = 1, \ldots, j-1$, we have $\gamma_0^{(j-k)}(t) \in I_j$, and also

$$\begin{aligned} h_j'(t) &= \left(\gamma_1^{(k)}\right)'\left(\gamma_0^{(j-k)}(t)\right) \cdot \left(\gamma_0^{(j-k)}\right)'(t) \\ h_j''(t) &= \left(\gamma_1^{(k)}\right)''\left(\gamma_0^{(j-k)}(t)\right) \cdot \left(\left(\gamma_0^{(j-k)}\right)'(t)\right)^2 + \left(\gamma_1^{(k)}\right)'\left(\gamma_0^{(j-k)}(t)\right) \cdot \left(\gamma_0^{(j-k)}\right)''(t) \\ &= \left(\gamma_1^{(k)}\right)''\left(\gamma_0^{(j-k)}(t)\right) \cdot \left(\left(\gamma_0^{(j-k)}\right)'(t)\right)^2 \end{aligned}$$

since $\left(\gamma_0^{(j-k)}\right)'' \equiv 0$. Then, in these intervals, by (43) we have

$$\frac{th_j'(t)}{h_j(t)} = \frac{t\left(\gamma_1^{(k)}\right)'\left(\gamma_0^{(j-k)}(t)\right) \cdot \left(\gamma_0^{(j-k)}\right)'(t)}{\gamma_1^{(k)}\left(\gamma_0^{(j-k)}(t)\right)}$$

$$= \frac{\gamma_0^{(j-k)}(t)\left(\gamma_1^{(k)}\right)'\left(\gamma_0^{(j-k)}(t)\right)}{\gamma_1^{(k)}\left(\gamma_0^{(j-k)}(t)\right)} \frac{t\left(\gamma_0^{(j-k)}\right)'(t)}{\gamma_0^{(j-k)}(t)}$$

$$= \frac{\gamma_0^{(j-k)}(t)\left(\gamma_1^{(k)}\right)'\left(\gamma_0^{(j-k)}(t)\right)}{\gamma_1^{(k)}\left(\gamma_0^{(j-k)}(t)\right)}$$

so $1 \le \frac{th_j'(t)}{h_j(t)} \le \left(1+\frac{j}{2}\right)^{m-1} \le C_m j^{m-1}$ by (46) for $t \in I_k$, $k = 0,\dots,j$; this finishes the proof of the second set of inequalities in the lemma. Also, for $t \in I_k$, $k = 1,\dots,j-1$

$$0 \le \frac{h_j(t)\,h_j''(t)}{\left(h_j'(t)\right)^2} = \frac{\gamma_1^{(k)}\left(\gamma_0^{(j-k)}\right)\cdot\left(\gamma_1^{(k)}\right)''\left(\gamma_0^{(j-k)}(t)\right)\cdot\left(\left(\gamma_0^{(j-k)}\right)'(t)\right)^2}{\left(\left(\gamma_1^{(k)}\right)'\left(\gamma_0^{(j-k)}(t)\right)\cdot\left(\gamma_0^{(j-k)}\right)'(t)\right)^2}$$

$$= \frac{\gamma_1^{(k)}\left(\gamma_0^{(j-k)}\right)\cdot\left(\gamma_1^{(k)}\right)''\left(\gamma_0^{(j-k)}\right)}{\left(\left(\gamma_1^{(k)}\right)'\left(\gamma_0^{(j-k)}\right)\right)^2} \le 1$$

by (48). Hence we also have $1 \le \frac{Y_j(t)}{\left(h_j'(t)\right)^2} \le 2$ for $t \in I_k$, what finishes the proof of the first pair of inequalities. □

Remark 3.6. Note that the identity, $h(t) = t$, trivially satisfies the conclusions in the previous proposition.

The following is a corollary of the proof of Proposition 3.5, which extends its conclusions for the Lipschitz Young function Φ_m to the $C^{1,1}$ Young functions $\tilde{\Phi}_m$.

Corollary 3.7. *Let $m > 1$, then for any integer $j \ge 1$ the function $\tilde{h}(t) = \tilde{h}_j(t) = \sqrt{\tilde{\Phi}_m^{(j)}(t^2)}$ with $\tilde{\Phi}_m(t)$ defined in (35) satisfies*

$$\tilde{h}'(t)^2 \le \tilde{Y}(t) \le C_m\, \tilde{h}'(t)^2 \qquad \text{and} \qquad 1 \le \frac{t\tilde{h}'(t)}{\tilde{h}(t)} \le C_m j^{m-1},$$

where $\tilde{Y}(t) = \left(\frac{1}{2}\tilde{h}^2(t)\right)'' = \tilde{h}(t)\tilde{h}''(t) + \left(\tilde{h}'(t)\right)^2$. Moreover, we have that $\tilde{h}''(t) \ge 0$ for all $t > 0$.

Proof. The proof is the same as for Proposition 3.5, with the appropriate modification of the explicit formula for the compositions. Indeed, for $t \in \left[0, \left(\frac{2E^2}{F}\right)^{\frac{1}{2}}\right] \equiv [0,\tilde{a}]$ we have that $\tilde{h}_1(t) = \sqrt{\tilde{\Phi}_m(t^2)} = \tilde{\tau}t$, with $\tilde{\tau} = \sqrt{\frac{1}{2}\frac{F}{E}}$. Using the definition (35) of $\tilde{\Phi}_m$, we write

$$\tilde{h}_1(t) = \sqrt{\tilde{\Phi}_m(t^2)} \equiv \begin{cases} \sqrt{\frac{1}{2}\frac{F}{E}}t & \text{if} \quad 0 \le t \le \left(\frac{2E^2}{F}\right)^{\frac{1}{2}} \\ \sqrt{\varrho_m(t^2)} & \text{if} \quad \left(\frac{2E^2}{F}\right)^{\frac{1}{2}} \le t \le E^{\frac{1}{2}} \\ \sqrt{\Phi_m(t^2)} & \text{if} \quad t \ge E^{\frac{1}{2}} \end{cases}$$

$$\equiv \begin{cases} \gamma_0(t) & \text{if} \quad 0 \le t \le \left(\frac{2E^2}{F}\right)^{\frac{1}{2}} \\ \gamma_{\frac{1}{2}}(t) & \text{if} \quad \left(\frac{2E^2}{F}\right)^{\frac{1}{2}} \le t \le E^{\frac{1}{2}} \\ \gamma_1(t) & \text{if} \quad t \ge E^{\frac{1}{2}} \end{cases}.$$

Then, defining the intervals $\tilde{I}_0 = \left(0, \tilde{\tau}^{-(j-1)}\tilde{a}\right)$, $\tilde{I}_k = \left[\tilde{\tau}^{-(j-k)}\tilde{a}, \tilde{\tau}^{-(j-k-1)}\tilde{a}\right)$ for $k = 1,\dots,j-1$, $\tilde{I}_j = \left[\tilde{a}, E^{\frac{1}{2}}\right)$, and $\tilde{I}_{j+1} = \left[E^{\frac{1}{2}}, \infty\right)$, we have that

$$\tilde{h}_j(t) \equiv \begin{cases} \gamma_0^{(j)}(t) & \text{if} \quad t \in \tilde{I}_0 \\ \gamma_1^{(k-1)} \circ \gamma_{\frac{1}{2}}(t) \circ \gamma_0^{(j-k)}(t) & \text{if} \quad t \in \tilde{I}_k, \quad k = 1,\dots,j \\ \gamma_1^{(j)}(t) = h_j(t) = \sqrt{\Phi_m^{(j)}(t^2)} & \text{if} \quad t \in \tilde{I}_{j+1} \end{cases}$$

The proof when $t \in \tilde{I}_0$ or $t \in \tilde{I}_{j+1}$ is the same as before (note that now γ_0 replaces $\gamma_{\frac{1}{2}}$ in the previous proof), while if $t \in \tilde{I}_k$, $k = 1,\dots,j$,

$$\left(\tilde{h}_j(t)\right)' = \left(\gamma_1^{(k-1)}\right)'\left(\gamma_{\frac{1}{2}}(t)\left(\gamma_0^{(j-k)}\right)\right)\cdot\left(\gamma_{\frac{1}{2}}(t)\right)'\left(\gamma_0^{(j-k)}\right)\cdot\left(\gamma_0^{(j-k)}\right)'$$

$$\left(\tilde{h}_j(t)\right)'' = \left(\gamma_1^{(k-1)}\right)''\left(\gamma_{\frac{1}{2}}(t)\left(\gamma_0^{(j-k)}(t)\right)\right)\cdot\left(\left(\gamma_{\frac{1}{2}}(t)\right)'\left(\gamma_0^{(j-k)}\right)\cdot\left(\gamma_0^{(j-k)}\right)'\right)^2 + \left(\gamma_1^{(k-1)}\right)'\left(\gamma_{\frac{1}{2}}(t)\left(\gamma_0^{(j-k)}\right)\right)\cdot\left(\left(\gamma_{\frac{1}{2}}\right)''\left(\gamma_0^{(j-k)}\right)\right)\cdot\left(\left(\gamma_0^{(j-k)}\right)'\right)^2$$

where we used that $\gamma_0'' \equiv 0$. Since by the chain rule we have that for any smooth functions $a(t), b(t), c(t)$

$$\frac{a(b(c(t)))\cdot(a(b(c(t))))''}{\left((a(b(c(t))))'\right)^2} = \frac{a(b(c))\cdot a''(b(c))}{(a'(b(c)))^2} + \frac{a(b(c))}{a'(b(c))\cdot b(c)}\frac{b(c)\cdot b''(c)}{(b'(c)')^2} + \frac{a(b(c))}{a'(b(c))\cdot b(c)}\frac{b(c)}{b'(c)\cdot c}\frac{c\cdot c''}{(c')^2}$$

we have

$$\frac{\tilde{h}_j(t)\left(\tilde{h}_j(t)\right)''}{\left(\left(\tilde{h}_j(t)\right)'\right)^2} = \frac{\gamma_1^{(k-1)}\left(\gamma_{\frac{1}{2}}(t)\left(\gamma_0^{(j-k)}\right)\right)\cdot\left(\gamma_1^{(k-1)}\right)''\left(\gamma_{\frac{1}{2}}(t)\left(\gamma_0^{(j-k)}(t)\right)\right)}{\left(\left(\gamma_1^{(k-1)}\right)'\left(\gamma_{\frac{1}{2}}(t)\left(\gamma_0^{(j-k)}\right)\right)\right)^2} + \frac{\gamma_1^{(k-1)}\left(\gamma_{\frac{1}{2}}(t)\left(\gamma_0^{(j-k)}\right)\right)}{\left(\gamma_1^{(k-1)}\right)'\left(\gamma_{\frac{1}{2}}(t)\left(\gamma_0^{(j-k)}\right)\right)\gamma_{\frac{1}{2}}(t)\left(\gamma_0^{(j-k)}\right)}\frac{\gamma_{\frac{1}{2}}\left(\gamma_0^{(j-k)}\right)\left(\gamma_{\frac{1}{2}}\right)''\left(\gamma_0^{(j-k)}\right)}{\left(\left(\gamma_{\frac{1}{2}}(t)\right)'\left(\gamma_0^{(j-k)}\right)\right)^2}$$

Then, since from (46) we have $\gamma_1^{(k-1)}(s)/\left[s\left(\gamma_1^{(k-1)}(s)\right)\right] \le 1$, and from Proposition 3.5 we have

$$\frac{\gamma_1^{(k-1)}(s)\cdot\left(\gamma_1^{(k-1)}\right)''(s)}{\left(\left(\gamma_1^{(k-1)}\right)'(s)\right)^2} \le \frac{\Upsilon(s)}{h'(s)^2} \le 2,$$

it follows that

$$0 \le \frac{\tilde{h}_j(t)\left(\tilde{h}_j(t)\right)''}{\left(\left(\tilde{h}_j(t)\right)'\right)^2} \le 2 + \frac{\gamma_{\frac{1}{2}}\left(\gamma_0^{(j-k)}\right)\left(\gamma_{\frac{1}{2}}\right)''\left(\gamma_0^{(j-k)}\right)}{\left(\left(\gamma_{\frac{1}{2}}(t)\right)'\left(\gamma_0^{(j-k)}\right)\right)^2} \le C_m$$

where we used inequality (39) and the fact that $\left(\gamma_{\frac{1}{2}}\right)'' \ge 0$ (see the discussion just before (39)). Then $1 \le \frac{\tilde{\Upsilon}(t)}{\tilde{h}'(t)^2} = \frac{\tilde{h}_j(t)\left(\tilde{h}_j(t)\right)''}{\left(\left(\tilde{h}_j(t)\right)'\right)^2} + 1 \le C_m$. □

We now consider $h_\beta(t) \equiv \sqrt{\Phi^{(j)}\left(t^{2\beta}\right)} \equiv \Gamma_m^{(j)}(t^\beta)$. We will show that this h satisfies the hypotheses of Lemma 2.1 for $\beta < 0$ and $\beta \ge 1$.

Proposition 3.8. *The function $h_{j,\beta}(t) = h_\beta = h_j\left(t^\beta\right)$, $\beta < 0$ or $\beta \ge 1$, where $h(t) = h_j(t) = \sqrt{\Phi_m^{(j)}\left(t^2\right)}$ is defined in (41) for each $j \ge 1$, satisfies $h_\beta''(t) \ge 0$ and*

$$1 \le \frac{\Upsilon_{j,\beta}(t)}{\left(h_{j,\beta}'(t)\right)^2} \le 2 + \frac{|\beta-1|}{|\beta|} \quad \text{and} \quad |\beta| \le \frac{t\left|h_{j,\beta}'(t)\right|}{h_{j,\beta}(t)} \le C_m|\beta|j^{m-1},$$

where $\Upsilon_\beta(t) = \left(\frac{1}{2}h_\beta^2(t)\right)'' = h_\beta(t)h_\beta''(t) + \left(h_\beta'(t)\right)^2$. Moreover, when $\beta \ge 1$ we that have $h_{j,\beta}$ is increasing.

Moreover, if $\tilde{h}_{j,\beta}(t) = \tilde{h}_\beta = \tilde{h}_j\left(t^\beta\right)$ with $\tilde{h}_j(t) = \sqrt{\tilde{\Phi}_m^{(j)}\left(t^2\right)}$ as in Corollary 3.7, then for $\tilde{\Upsilon}_\beta(t) = \left(\frac{1}{2}\tilde{h}_\beta^2(t)\right)'' = \tilde{h}_\beta(t)\tilde{h}_\beta''(t) + \left(\tilde{h}_\beta'(t)\right)^2$

$$1 \le \frac{\tilde{\Upsilon}_{j,\beta}(t)}{\left(\tilde{h}_{j,\beta}'(t)\right)^2} \le C_m + \frac{|\beta-1|}{|\beta|} \quad \text{and} \quad |\beta| \le \frac{t\left|\tilde{h}_{j,\beta}'(t)\right|}{\tilde{h}_{j,\beta}(t)} \le C_m|\beta|j^{m-1},$$

Proof. Since for all $\beta \ne 0$

$$h_\beta'(t) = \beta t^{\beta-1}h'\left(t^\beta\right) \quad \text{and} \quad h_\beta''(t) = \beta(\beta-1)t^{\beta-2}h'\left(t^\beta\right) + \beta^2 t^{2\beta-2}h''\left(t^\beta\right). \tag{50}$$

From the first equality it is clear that h_β is increasing when $\beta > 0$, and therefore so it is $h_{j,\beta}$. The lower bound $h_\beta''(t) \ge 0$ follows from the second identity and the facts that h is an increasing convex function, and $\beta(\beta-1) \ge 0$ when $\beta < 0$ or $\beta \ge 1$. Now, by

Proposition 3.5 we have

$$|\beta| \leq |\beta| \frac{t^{\beta} h'\left(t^{\beta}\right)}{h\left(t^{\beta}\right)} = \frac{t\left|h_{\beta}'(t)\right|}{h_{\beta}(t)} \leq C_m |\beta| j^{m-1}. \tag{51}$$

Similarly,

$$\begin{aligned}\frac{\Upsilon_{\beta}(t)}{\left(h_{\beta}'(t)\right)^2} &= \frac{\left(\frac{1}{2}h_{\beta}(t)^2\right)''}{h_{\beta}'(t)^2} = \frac{h_{\beta}(t)h_{\beta}''(t) + h_{\beta}'(t)^2}{\left(h_{\beta}'(t)\right)^2} \\ &= \frac{h\left(t^{\beta}\right)\left(\beta(\beta-1)t^{\beta-2}h'\left(t^{\beta}\right) + \beta^2 t^{2\beta-2}h''\left(t^{\beta}\right)\right)}{\left(\beta t^{\beta-1}h'\left(t^{\beta}\right)\right)^2} + 1 \\ &= \frac{\beta-1}{\beta}\frac{h\left(t^{\beta}\right)}{t^{\beta}h'\left(t^{\beta}\right)} + \frac{h\left(t^{\beta}\right)h''\left(t^{\beta}\right)}{\left(h'\left(t^{\beta}\right)\right)^2} + 1 \\ &= \frac{\beta-1}{\beta}\frac{h\left(t^{\beta}\right)}{t^{\beta}h'\left(t^{\beta}\right)} + \frac{\Upsilon\left(t^{\beta}\right)}{\left(h'\left(t^{\beta}\right)\right)^2}. \end{aligned} \tag{52}$$

By Proposition 3.5 we have that $\frac{1}{C_m j^{m-1}} \leq \frac{h(t^{\beta})}{t^{\beta}h'(t^{\beta})} \leq 1$, and $1 \leq \frac{\Upsilon(t^{\beta})}{(h'(t^{\beta}))^2} \leq 2$, so if $\beta < 0$ or $\beta \geq 1$ we have

$$1 \leq \frac{\beta-1}{\beta}\frac{1}{C_m j^{m-1}} + 1 \leq \frac{\Upsilon_{\beta}(t)}{\left(h_{\beta}'(t)\right)^2} \leq \frac{\beta-1}{\beta} + 2,$$

where we used that $\frac{\beta-1}{\beta} \geq 0$. The proof for $\tilde{h}_{j,\beta}$ is identical, using instead the estimates from Corollary 3.7. □

3.6. The L^{∞} norm

The following proposition establishes sufficient conditions for the iterated integrals to converge to the supremum norm.

Proposition 3.9. *Suppose that Θ is a nonnegative strictly increasing function such that $\Theta(0) = 0$ and with the following property:*

$$\liminf_{j\to\infty} \frac{\Theta^{(j)}(M)}{\Theta^{(j)}\left(M_1\right)} = \infty \qquad \text{for all } M > M_1 > 0. \tag{53}$$

Let $D \Subset D_1$ be nonempty open bounded sets in $\mathbb{R}^n$, and let $\left\{D_j\right\}_{j=1}^{\infty}$ be a sequence of nested open bounded sets satisfying

$$D_1 \Supset D_2 \Supset \cdots \Supset D_j \Supset D_{j+1} \Supset \cdots \Supset D$$

and such that $\overline{D} = \bigcap_{j=1}^{\infty} D_j$, Let ω be a Borel measure in D_1, with $\omega\left(D_1\right) < \infty$, such that $\mathfrak{m} \ll \omega$ where $\mathfrak{m}$ denotes Lebesgue's measure. Then, if f is ω-measurable in D_1we have

$$\|f\|_{L^{\infty}(D)} \leq \liminf_{j\to\infty} \Theta^{(-j)}\left(\int_{D_j} \Theta^{(j)}(|f(x)|)\, d\omega\right) \qquad \text{and}$$

$$\lim_{j\to\infty} \|f\|_{L^{\infty}(D_j)} \geq \limsup_{j\to\infty} \Theta^{(-j)}\left(\int_{D_j} \Theta^{(j)}(|f(x)|)\, d\omega\right).$$

Proof. Since $\Theta(0) = 0$ and Θ is strictly increasing, it is invertible, and $\Theta^{(j)}$, $\Theta^{(-j)}$ are nonnegative and strictly increasing for all $j \geq 1$. From the hypothesis (53) on Θ we have that for all $\delta \in (0,1)$, the inequality

$$\delta\Theta^{(j)}(M) \geq \Theta^{(j)}\left(M_1\right) \tag{54}$$

holds for each sufficiently large $j > N\left(M, M_1, \delta\right)$. Note that if $\delta \geq 1$ the inequality trivially holds since Θ is increasing. We have that (54) implies the range of Θ is $[0,\infty)$, i.e. $\Theta([0,\infty)) = [0,\infty)$, so Θ^{-1} is also defined on $[0,\infty)$. Indeed, for any $A \gg 1$ there exists $N \in \mathbb{N}$ such that

$$\frac{\Theta^{(j)}(2)}{\Theta^{(j)}(1)} \geq \frac{A}{\Theta(1)} \qquad \text{if} \quad j > N,$$

note that $\Theta(1) > 0$ since Θ is strictly increasing and $\Theta(0) = 0$. It then follows that $\frac{\Theta^{(j)}(2)}{\Theta^{(j)}(t)} \geq \frac{A}{\Theta(1)}$ for all $0 < t \leq 1$, in particular for $0 < t_j < 1$ given by $\Theta^{(j)}\left(t_j\right) = \Theta(1)$. Thus, we see that for all $A \gg 1$ there exists N such that $j > N \implies \Theta^{(j)}(2) \geq A$, since A is arbitrary it follows that $\Theta^{(j)}(2) \to \infty$ as $j \to \infty$, so $\Theta([0,\infty)) = [0,\infty)$.

Since $\mathfrak{m} \ll \omega$, we have that $\omega\left(D_j\right) > 0$ and in general $\omega(U) > 0$ for all nonempty open sets U. The sequence $\|f\|_{L^\infty(D_j)}$ is decreasing and bounded below by $\|f\|_{L^\infty(D)}$, it follows that $F = \lim_{j\to\infty} \|f\|_{L^\infty(D_j)}$ exists, and $F \geq \|f\|_{L^\infty(D)}$. Now, for each fixed $k \geq 1$ and $j \geq k$ we have

$$\begin{aligned}\Theta^{(-j)}\left(\int_{D_j} \Theta^{(j)}(|f(x)|)\ d\omega\right) &\leq \Theta^{(-j)}\left(\int_{D_j} \Theta^{(j)}\left(\|f\|_{L^\infty(D_k)}\right)\ d\omega\right) \\ &\leq \Theta^{(-j)}\left(\omega\left(D_k\right)\Theta^{(j)}\left(\|f\|_{L^\infty(D_k)}\right)\right).\end{aligned} \tag{55}$$

For $\varepsilon > 0$ and $j \geq N_k = \max\left\{k, N\left(\|f\|_{L^\infty(D_k)} + \varepsilon, \|f\|_{L^\infty(D_k)}, \frac{1}{\omega(D_k)}\right)\right\}$, we have that $\omega\left(D_k\right)\Theta^{(j)}\left(\|f\|_{L^\infty(D_k)}\right) \leq \Theta^{(j)}\left(\|f\|_{L^\infty(D_k)} + \varepsilon\right)$, so

$$\Theta^{(-j)}\left(\int_{D_j} \Theta^{(j)}(|f(x)|)\ d\omega\right) \leq \Theta^{(-j)}\left(\Theta^{(j)}\left(\|f\|_{L^\infty(D_k)} + \varepsilon\right)\right) = \|f\|_{L^\infty(D_k)} + \varepsilon$$

for all $j \geq N_k$. Then

$$\limsup_{j\to\infty} \Theta^{(-j)}\left(\int_{D_j} \Theta^{(j)}(|f(x)|)\ d\omega\right) \leq \|f\|_{L^\infty(D_k)} + \varepsilon.$$

Since $k, \varepsilon > 0$ are arbitrary, this proves that $\limsup_{j\to\infty} \Theta^{(-j)}\left(\int_{D_j} \Theta^{(j)}(|f(x)|)\ d\omega\right) \leq F = \lim_{k\to\infty} \|f\|_{L^\infty(D_k)}$.

On the other hand, for $0 < 2\varepsilon < \|f\|_{L^\infty(D)}$ (assume f is not trivially zero in D), define $\Delta_\varepsilon = \left\{x \in D : |f(x)| \geq \|f\|_{L^\infty(D)} - \varepsilon\right\}$. Then we have that $0 < \omega\left(\Delta_\varepsilon\right) < \infty$ (here we used that $\omega\left(D_1\right) < \infty$) and

$$\int_{D_j} \Theta^{(j)}(|f(x)|)\ d\omega \geq \omega\left(\Delta_\varepsilon\right)\Theta^{(j)}\left(\|f\|_{L^\infty(D)} - \varepsilon\right).$$

Hence, from (54), for $j \geq N\left(\|f\|_{L^\infty(D)} - \varepsilon, \|f\|_{L^\infty(D)} - 2\varepsilon, \omega\left(\Delta_\varepsilon\right)\right)$ it follows that

$$\begin{aligned}\Theta^{(-j)}\left(\int_{D_j} \Theta^{(j)}(|f(x)|)\ d\omega\right) &\geq \Theta^{(-j)}\left(\omega\left(\Delta_\varepsilon\right)\Theta^{(j)}\left(\|f\|_{L^\infty(D)} - \varepsilon\right)\right) \\ &\geq \Theta^{(-j)}\left(\Theta^{(j)}\left(\|f\|_{L^\infty(D)} - 2\varepsilon\right)\right) = \|f\|_{L^\infty(D)} - 2\varepsilon.\end{aligned}$$

Letting $j \to \infty$ we obtain

$$\liminf_{j\to\infty} \Theta^{(-j)}\left(\int_{D_j} \Theta^{(j)}(|f(x)|)\ d\mu_j\right) \geq \|f\|_{L^\infty(D)} - 2\varepsilon,$$

and since $\varepsilon > 0$ is arbitrary we conclude that

$$\|f\|_{L^\infty(D)} \leq \liminf_{j\to\infty} \Theta^{(-j)}\left(\int_{D_j} \Theta^{(j)}(|f(x)|)\ d\mu_j\right).$$

This finishes the proof. ☐

Remark 3.10. Note that in the previous result we cannot in general guarantee that

$$\|f\|_{L^\infty(D)} = \lim_{j\to\infty} \Theta^{(-j)}\left(\int_{D_j} \Theta^{(j)}(|f(x)|)\ d\omega\right)$$

unless we have $\|f\|_{L^\infty(D)} = \lim_{j\to\infty} \|f\|_{L^\infty(D_j)}$. This will be the case if, for example, f is continuous.

Remark 3.11. Proposition 3.9 also holds with $d\omega$ replaced by $d\mu_j = \frac{d\omega}{\omega(D_j)}$ in each D_j, the proof is the same.

Remark 3.12. The Young functions $\Phi = \Phi_m$ defined on (34) satisfies the hypotheses of Proposition 3.9. Indeed, it is clear that Φ_m is nonnegative, strictly increasing, and vanishes at the origin. Given any $M > M_1 > 0$, there exists N_0 such that $\Phi^{(N_0)}\left(M_1\right) \geq E$, so for all $N \geq 1$ we have

$$\frac{\Phi^{N+N_0}(M)}{\Phi^{N+N_0}\left(M_1\right)} = \exp\left((a+N)^m - (b+N)^m\right),$$

where $a = \left(\ln \Phi^{N_0}(M)\right)^{\frac{1}{m}} > \left(\ln \Phi^{N_0}\left(M_1\right)\right)^{\frac{1}{m}} = b$. Since for $m > 1$, we have

$$\lim_{N\to\infty}\left[(a+N)^m - (b+N)^m\right] \geq \lim_{N\to\infty}(a-b)\cdot m(b+N)^{m-1} = \infty,$$

we see that the growth condition (53) holds for Φ. Note that in terms of the associated Orlicz quasidistance (26) we have that $\|f\|_{L^\infty(D)} \leq \lim_{j\to\infty} \|f\|_{D^{\Phi^{(j)}}(D,\mu)}$.

In [4], Cruz-Uribe and Rodney established a general result for Orlicz norms with Young functions $B_{pq}(t) = t^p\left(\log\left(e_0+t\right)\right)^q$, $1 \le p < \infty$, $q > 0$, $e_0 = e - 1$. They showed that if f is measurable in a general measure space $(X, \mathcal{M}, \mu)$ then $\lim_{q\to\infty} \|f\|_{B_{pq}} = \|f\|_\infty$, where $\|f\|_{B_{pq}}$ is the Orlicz norm of f in X. Even though the results seem of a similar type, Proposition 3.9 neither contains nor it is contained in the theorem in [4], since the integrals $\Theta^{(-j)}\left(\int_{D_j} \Theta^{(j)}(|f|)\ d\omega\right)$ are not in general the Orlicz norms associated with the Young functions $\Theta^{(j)}$, but rather the quadi-distances $\|\cdot\|_{D\Theta^{(j)}}$ defined in Section 3.1.

4. The Moser method - Abstract local boundedness

In this section we prove the abstract boundedness result under the presence of an Orlicz-Sobolev inequality (9) and a standard sequence of Lipschitz cutoff functions (Definition 1.3) for the Young functions Φ_m given in (10).

4.1. Boundedness of subsolutions and supersolutions

Recall that $\sqrt{A(x)}$ is a bounded Lipschitz continuous $n\times n$ real-valued nonnegative definite matrix in $\mathbb{R}^n$, and $\mathfrak{A}(A,\Lambda,\lambda)$ denotes the set of symmetric $n\times n$ matrices which are equivalent to A within constants $0<\lambda\le\Lambda<\infty$ as given in (2), i.e. $\lambda\ A \le \tilde{A} \le \Lambda\ A$ in the sense of quadratic forms. In what follows $B = B(x,r)$, $0<r$ denotes a d-metric ball where d is a fixed metric in $\mathbb{R}^n$.

Theorem 4.1. *Let $\Phi(t) = \Phi_m(t)$ be as in (34) with $m > 2$; suppose that there exists a superradius φ so that the (Φ_m, A, φ)-Sobolev bump inequality (9) holds in $B = B(x,r)$ for some $0 < r \le 1$, and that an (A,d)-standard sequence of Lipschitz cutoff functions, as given in Definition 1.3, exists.*

Let $\nu_0 = 1 - \frac{\delta_x(r)}{r}$, where $\delta_x(r)$ is the doubling increment of $B(x,r)$, defined by (12). Then for all $\nu \in [\nu_0, 1)$ and $\beta \in [1,\infty)$ there exists a constant $C(\varphi, m, \lambda, \Lambda, r, \nu, \beta)$ such that if $\tilde{A} \in \mathfrak{A}(A,\Lambda,\lambda)$, and u is a weak ***subsolution*** *to the equation $L_{\tilde{A}} u = -\mathrm{div}_{\tilde{A}} \nabla_{\tilde{A}} u = \phi_0 - \mathrm{div}_A \vec{\phi}_1$ in $B(x,r)$, with A-admissible right hand side $\left(\phi_0, \vec{\phi}_1\right)$ (see Definition 1.5), then*

$$\left\|\left(u^+ + \phi^*\right)^\beta\right\|_{L^\infty(B(x,\nu r))} \le C(\varphi, m, \lambda, \Lambda, r, \nu, \beta)\ \left\|\left(u^+ + \phi^*\right)^\beta\right\|_{L^2(B(x,r), d\mu_r)} \qquad \beta \ge 1, \tag{56}$$

where $\phi^* = \left\|\left(\phi, \vec{\phi}_1\right)\right\|_{\mathcal{X}(B(x,r))}$ and $.d\mu_r = \frac{dx}{|B(x,r)|}$. In fact, we can choose

$$C(\varphi, m, \lambda, \Lambda, r, \nu, \beta) = \exp\left(C_{m,\lambda,\Lambda}\left((\beta-1)^m + \left(\ln\frac{\varphi(r)}{(1-\nu)r}\right)^m\right)\right).$$

Furthermore, if u is a weak ***supersolution*** *to the equation $L_{\tilde{A}} u = \phi_0 - \mathrm{div}_A \vec{\phi}_1$ in $B(x,r)$, then (56) holds with u^+ replaced by u^-. In particular, if u is a* ***solution*** *to $L_{\tilde{A}} u = \phi_0 - \mathrm{div}_{\tilde{A}}\left(\vec{\phi}_1\right)$ in $B(x,r)$, then u is locally bounded in $B(x,r)$ and (56) holds for $|u|$ and all $\nu \in [\nu_0, 1)$.*

Proof. Let us start by considering the *standard* sequence of Lipschitz cutoff functions $\left\{\psi_j\right\}_{j=1}^\infty$ depending on r as given in Definition 1.3, along with the balls $B_j = B(x, r_j) \supset \mathrm{supp}\psi_j$, so that $r = r_1 > r_2 > \cdots > r_j > r_{j+1} > \cdots r_\infty \equiv \lim_{j\to\infty} r_j = \nu r$, and $\left\|\nabla_A \psi_j\right\|_\infty \le \frac{Cj^2}{(1-\nu)r}$ with ∇_A as in (3) and $1 - \frac{\delta_x(r)}{r} = \nu_0 \le \nu < 1$.

Note that a priori we do not know whether $|u| + \phi^* \in L^{2\beta}(B)$ when $\beta > 1$, however, the proof will proceed with the assumption that $|u| + \phi^* \in L^{2\beta}(B)$ for all β, and then, a posteriori, the case $\beta = 1$ implies that $u^\pm + \phi^* \in L^{2\beta}(B(x,\nu r))$ for all $0<\nu<1$, $\phi^* \ge 0$, $\beta \ge 1$. Let u be a subsolution or supersolution of $L_{\tilde{A}} u = \phi_0 - \mathrm{div}_A \vec{\phi}_1$ in $B(x,r)$. Then we have that if

$$\tilde{a} = \frac{e^{\frac{2^{m-1}}{\beta}}}{\|u^\pm + \phi^*\|_{L^{2\beta}(d\mu_r)}} \tag{57}$$

then $\tilde{u} = \tilde{a}u$ is a subsolution or supersolution (respectively) of $L_{\tilde{A}} \tilde{u} = \tilde{\phi}_0 - \mathrm{div}_A\left(\vec{\tilde{\phi}}_1\right)$ in $B(x,r)$ with $\tilde{\phi}_0 = \tilde{a}\phi_0$, $\vec{\tilde{\phi}}_1 = \tilde{a}\vec{\phi}_1$. Moreover, $\tilde{\phi}^* \equiv \left\|\left(\tilde{\phi}_0, \vec{\tilde{\phi}}_1\right)\right\|_{X(B)} = \frac{\phi^* e^{\frac{2^{m-1}}{\beta}}}{\|u^\pm + \phi^*\|_{L^{2\beta}(d\mu_r)}} \le e^{\frac{2^{m-1}}{\beta}}$, and

$$\left\|\left(\tilde{u}^\pm + \tilde{\phi}^*\right)^\beta\right\|_{L^2(d\mu_r)}^{\frac{1}{\beta}} = \|\tilde{u}^\pm + \tilde{\phi}^*\|_{L^{2\beta}(d\mu_r)} = \left\|\frac{u^\pm + \phi^*}{\|u^\pm + \phi^*\|_{L^{2\beta}(d\mu_r)}}\right\|_{L^{2\beta}(d\mu_r)} e^{\frac{2^{m-1}}{\beta}} = e^{\frac{2^{m-1}}{\beta}}. \tag{58}$$

For simplicity, in what follows we write $v = \tilde{u}^\pm + \tilde{\phi}^*$, explicitly,

$$v = \begin{cases} \tilde{u}^+ + \tilde{\phi}^* & \text{if } L_{\tilde{A}}\tilde{u} \le \tilde{\phi}_0 - \mathrm{div}_A\left(\vec{\tilde{\phi}}_1\right) \\ \tilde{u}^- + \tilde{\phi}^* & \text{if } L_{\tilde{A}}\tilde{u} \ge \tilde{\phi}_0 - \mathrm{div}_A\left(\vec{\tilde{\phi}}_1\right) \end{cases} \tag{59}$$

By Proposition 3.8 we have that $h(t) = h_{j,\beta}(t) = \sqrt{\Phi_m^{(j-1)}\left(t^{2\beta}\right)}$, $j \ge 1$, (where $\Phi^{(0)}(t) = t$, see Remark 3.6) satisfies the hypotheses of Lemma 2.1 with constant $C_{h_{j,\beta}} = C_m|\beta|j^{m-1}$, namely, $h'_{j,\beta}(t) = \left|h'_{j,\beta}(t)\right| \le C_m|\beta|j^{m-1}\frac{h_{j,\beta}(t)}{t}$. We apply Lemma 2.1 to $h(t) = h_{j,\beta}(t)$, with $\psi = \psi_j$, $d\mu_j \equiv \frac{dx}{|B_j|}$, to obtain

$$\int_{B_j} \psi_j^2 \left|\nabla_A \left[h_j(v)\right]\right|^2 d\mu_j \le C_{m,\lambda,A}^2 \beta^2 j^{2(m-1)} \int_{B_j} (h(v))^2 \left(|\nabla_A \psi|^2 + \psi^2\right) d\mu_j, \tag{60}$$

where we used the estimates in Proposition 3.8, namely, $\left|h'_{j,\beta}(t)\right| \le C_m |\beta| j^{m-1} \frac{h_{j,\beta}(t)}{t}$. It follows that

$$\begin{aligned}
\left\|\nabla_A[\psi_j h(v)]\right\|_{L^2(\mu_j)}^2 &\le 2\left\|\psi_j \nabla_A h(v)\right\|_{L^2(\mu_j)}^2 + 2\left\|\left|\nabla_A \psi_j\right| h(v)\right\|_{L^2(\mu_j)}^2 \\
&\le C_{m,\lambda,A}^2 \beta^2 j^{2(m-1)} \int_{B_j} h(v)^2 \left(|\nabla_A \psi_j|^2 + \psi_j^2\right) d\mu_j \\
&\quad + 2\left\|\left|\nabla_A \psi_j\right| h(v)\right\|_{L^2(\mu_j)}^2 \\
&\le C_{m,\lambda,A}^2 (\beta+1)^2 j^{2(m-1)} \left\|\nabla_A \psi_j\right\|_{L^\infty}^2 \|h(v)\|_{L^2(B_j,\mu_j)}^2 \\
&\le C_{m,\lambda,A}^2 \frac{(\beta+1)^2}{(1-\nu)^2 r^2} j^{2(m+1)} \|h(v)\|_{L^2(B_j,\mu_j)}^2,
\end{aligned} \tag{61}$$

where we used the inequalities $\left\|\psi_j\right\|_\infty \le 1 \le \left\|\nabla_A \psi_j\right\|_{L^\infty} \le C j^2/((1-\nu) r)$, and the fact $r_j \le r \le 1$.

Taking $w = \psi_j^2 h(v)^2$ in the Orlicz-Sobolev inequality (9), and since $\frac{|B(x,r_j)|}{|B(x,r_{j+1})|} \le 2$ by the choice of the sequence of radii, yields

$$\begin{aligned}
&\Phi^{(-1)}\left(\int_{B_{j+1}} \Phi(h(v)^2) d\mu_{j+1}\right) \le \Phi^{(-1)}\left(\int_{B_j} 2\Phi(\psi_j^2 h(v)^2) d\mu_j\right) \\
&\le \Phi^{(-1)}\left(\int_{B_j} \Phi(2\psi_j^2 h(v)^2) d\mu_j\right) \le C\varphi(r_j) \left\|\nabla_A\left(\left(\psi_j h(v)\right)^2\right)\right\|_{L^1(B_j,\mu_j)} \\
&\le 2C\varphi(r_j) \left\|\nabla_A\left(\psi_j h(v)\right)\right\|_{L^2(B_j,\mu_j)} \left\|\psi_j h(v)\right\|_{L^2(B_j,\mu_j)} \\
&\le C_{m,\lambda,A}(\beta+1) \frac{\varphi(r_j)}{(1-\nu) r} j^{m+1} \|h(v)\|_{L^2(B_j,\mu_j)}^2
\end{aligned}$$

where we applied (61). Recalling the definition of $h(v) = \sqrt{\Phi^{(j-1)}\left(t^{2\beta}\right)}$ with $\Phi = \Phi_m$, this is

$$\begin{aligned}
\int_{B_{j+1}} \Phi^{(j)}\left(v^{2\beta}\right) d\mu_{j+1} &= \int_{B_{j+1}} \Phi(h(v)^2) d\mu_{j+1} \\
&\le \Phi\left(C_{m,\lambda,A}(\beta+1) \frac{\varphi(r)}{(1-\nu) r} j^{m+1} \int_{B_j} \Phi^{(j-1)}\left(v^{2\beta}\right) d\mu_j\right).
\end{aligned}$$

Thus, setting

$$K = K_{\text{standard}}(\varphi, r) = C_{m,\lambda,A}(\beta+1) \frac{\varphi(r)}{(1-\nu) r} > 1, \tag{62}$$

we have that

$$\int_{B(x,r_{j+1})} \Phi^{(j)}\left(v^{2\beta}\right) d\mu_{j+1} \le \Phi\left(K j^{m+1} \int_{B(x,r_j)} \Phi^{(j-1)}\left(v^{2\beta}\right) d\mu_j\right). \tag{63}$$

Now define a sequence by

$$b_1 = \int_{B_1} |v|^{2\beta} d\mu_1, \qquad b_{j+1} = \Phi\left(K j^{m+1} b_j\right). \tag{64}$$

The inequality (63) and a basic induction shows that

$$\int_{B_j} \Phi^{(j-1)}\left(v^{2\beta}\right) d\mu_j \le b_j. \tag{65}$$

Now we apply Lemma 4.3 with $b_1 = \int_{B(x,r_1)} |v|^{2\beta} d\mu_{r_1}$, $b_{j+1} = \Phi\left(K j^\gamma b_j\right)$, and $\gamma = m+1$, then there exists a positive number $C^* = C^*\left(b_1, K, m\right)$ such that the inequality $\Phi^{(j)}(C^*) \ge b_{j+1}$ holds for each positive number j. Moreover, since from (58) we have that $b_1 = \|v\|_{L^{2\beta}(d\mu_r)}^{2\beta} = \left(e^{\frac{2^{m-1}}{\beta}}\right)^{2\beta} = e^{2^m}$, we can take

$$\exp\left(\left(C_m \ln K\right)^m\right) \le \exp\left(C_{m,\lambda,A}\left((\beta-1)^m + \left(\ln \frac{\varphi(r)}{(1-\nu) r}\right)^m\right)\right) \equiv C^*.$$

It follows that

$$\Phi^{-(j)}\left(\int_{B(x,r_{j+1})}\Phi^{(j)}\left(v^{2\beta}\right)\,d\mu_{j+1}\right)\le\Phi^{-(j)}\left(b_{j+1}\right)\le C^{*}.$$

On the other hand, by Proposition 3.9 (and Remark 3.12) we have that

$$\left\|v^{2\beta}\right\|_{L^{\infty}(B_{\infty})}\le\liminf_{j\to\infty}\Phi^{-(j+1)}\left(\int_{B_{j+1}}\Phi^{(j+1)}\left(v^{2}\right)\,d\mu_{j+1}\right),$$

hence

$$\begin{aligned}\left\|v^{2\beta}\right\|_{L^{\infty}(B(x,\nu r))}&=\left\|\left(\tilde{u}^{+}+\tilde{\phi}\right)^{2\beta}\right\|_{L^{\infty}(B(x,\nu r))}=\|\tilde{u}^{+}+\tilde{\phi}\|^{2\beta}_{L^{\infty}(B(x,\nu r))}\\&=e^{2^{m}}\left\|\frac{u+\phi^{*}}{\|u^{+}+\phi^{*}\|_{L^{2\beta}(d\mu_r)}}\right\|^{2\beta}_{L^{\infty}(B(x,\nu r))}\\&=e^{2^{m}}\frac{\left\|(u+\phi^{*})^{\beta}\right\|^{2}_{L^{\infty}(B(x,\nu r))}}{\left\|(u^{+}+\phi^{*})^{\beta}\right\|^{2}_{L^{2}(B(x,r),d\mu_r)}}\\&\le\exp\left(C_{m,\lambda,\Lambda}\left((\beta-1)^{m}+\left(\ln\frac{\varphi(r)}{(1-\nu)r}\right)^{m}\right)\right)\\&\equiv\left(C(\varphi,m,\lambda,\Lambda,r,\nu,\beta)\right)^{2}.\end{aligned}$$

Recalling now that we wrote u for $\tilde{u}$ defined in (57), and by the choice of v in (59), this yields

$$\left\|\left(u^{+}+\phi^{*}\right)^{\beta}\right\|_{L^{\infty}(B(x,\nu r))}\le C(\varphi,m,\lambda,\Lambda,r,\nu,\beta)\ \left\|u^{+}+\phi^{*}\right\|^{\beta}_{L^{2\beta}(B(x,r),d\mu_r)}$$

for all $\beta\ge 1$ when $L_{\tilde{A}}u\le\phi_0-\mathrm{div}_A\left(\vec{\phi}_1\right)$, while we obtain

$$\left\|\left(u^{-}+\phi^{*}\right)^{\beta}\right\|_{L^{\infty}(B(x,\nu r))}\le C(\varphi,m,\lambda,\Lambda,r,\nu,\beta)\ \left\|u^{-}+\phi^{*}\right\|^{\beta}_{L^{2\beta}(B(x,r),d\mu_r)}$$

for all $\beta\ge 1$ when $L_{\tilde{A}}u\ge\phi_0-\mathrm{div}_A\left(\vec{\phi}_1\right)$. □

In the previous theorem we obtain abstract local boundedness of weak solutions of $Lu=\phi_0-\mathrm{div}_A\vec{\phi}_1$, when the right hand side only had the first term this was obtained in [9]. In order to obtain continuity, we need L^{∞} bounds for powers of solutions u^{β} for β in a neighborhood of $\beta=0$. When $\beta<0$ this can be done with a slight modification of the previous argument via the application of a different Caccioppoli estimate (Lemma 2.3). Note that we only consider nonnegative weak supersolutions, as this suffices for our applications.

Theorem 4.2. *Under the hypotheses of Theorem 4.1, for all $\nu\in\left[\nu_0,1\right)$ and $\beta<0$ there exists a constant $C(\varphi,m,\lambda,\Lambda,r,\nu)$ such that if u is a **nonnegative** weak **supersolution** to the equation $L_{\tilde{A}}u=\phi_0-\mathrm{div}_A\vec{\phi}_1$ in $B(0,r)$, then*

$$\left\|\left(u+\phi^{*}\right)^{\beta}\right\|_{L^{\infty}(B(0,\nu r))}\le C(\varphi,m,\lambda,\Lambda,r,\nu,\beta)\ \left\|\left(u+\phi^{*}\right)^{\beta}\right\|_{L^{2}(d\mu_r)}\qquad\beta<0\tag{66}$$

In fact, we can choose

$$C(\varphi,m,\lambda,\Lambda,r,\nu)=\exp\left(C_{m,\lambda,\Lambda}\left((|\beta|+1)^{m}+\left(\ln\frac{\varphi(r)}{(1-\nu)r}\right)^{m}\right)\right).$$

Proof. We proceed as in the proof of Theorem 4.1, to consider a *standard* sequence of Lipschitz cutoff functions $\left\{\psi_j\right\}_{j=1}^{\infty}$ depending on r as given in Definition 1.3, along with the balls $B_j=B(0,r_j)\supset\mathrm{supp}\psi_j$, so that $r=r_1>\cdots>r_j\searrow r_{\infty}\equiv\lim_{j\to\infty}r_j=\nu r$, and $\left\|\nabla_A\psi_j\right\|_{\infty}\le\dfrac{Cj^2}{(1-\nu)r}$ with $1-\frac{\delta_0(r)}{r}=\nu_0\le\nu<1$.

Let u be a nonnegative supersolution of $L_{\tilde{A}}u=-\mathrm{div}_{\tilde{A}}\nabla_{\tilde{A}}u=\phi_0-\mathrm{div}_A\vec{\phi}_1$ in $B(0,r)$, then we have that $(u+\phi^{*})^{\beta}$ is locally bounded for all $\beta<0$, $\phi^{*}>0$. If $\phi^{*}=0$ we replace it by a small positive ε and let $\varepsilon\to 0$ at the end of the argument. As in the previous proof, we have that if

$$\tilde{a}=\frac{e^{\frac{2^{m-1}}{\beta}}}{\|u+\phi^{*}\|_{L^{2\beta}(d\mu_r)}}$$

then $\tilde{u}=\tilde{a}u$ is a supersolution of $L\tilde{u}=\tilde{\phi}_0-\mathrm{div}_A\left(\vec{\tilde{\phi}}_1\right)$ in $B(0,r)$ with

$$\tilde{\phi}_0=\tilde{a}\phi_0,\qquad\vec{\tilde{\phi}}_1=\tilde{a}\vec{\phi}_1,\qquad\tilde{\phi}^{*}\equiv\left\|\left(\tilde{\phi}_0,\vec{\tilde{\phi}}_1\right)\right\|_{X(B)}=\frac{\phi^{*}e^{\frac{2^{m-1}}{\beta}}}{\|u+\phi^{*}\|_{L^{2\beta}(d\mu_r)}}\le e^{\frac{2^{m-1}}{\beta}},$$

and

$$\left\|\left(\tilde{u}+\tilde{\phi}^{*}\right)^{\beta}\right\|_{L^{2}(d\mu_{r})}^{\frac{1}{\beta}}=\left\|\tilde{u}+\tilde{\phi}^{*}\right\|_{L^{2\beta}(d\mu_{r})}=\left\|\frac{u+\phi^{*}}{\|u+\phi^{*}\|_{L^{2\beta}(d\mu_{r})}}\right\|_{L^{2\beta}(d\mu_{r})}e^{\frac{2^{m-1}}{\beta}}=e^{\frac{2^{m-1}}{\beta}}.$$

By Proposition 3.8 we have that $h(t)=h_{j,\beta}(t)=\sqrt{\tilde{\Phi}_{m}^{(j-1)}\left(t^{2\beta}\right)}$, $j\geq 1$, (where $\tilde{\Phi}^{(0)}(t)=t$, see Remark 3.6) satisfies the hypotheses of Lemma 2.3 . Explicitly, for $Y(t)=Y_{j,\beta}(t)=h(t)h''(t)+h'(t)^{2}>0$, we have

$$1\leq\frac{Y_{j,\beta}(t)}{\left(h'_{j,\beta}(t)\right)^{2}}\leq C_{m}+\frac{|\beta-1|}{|\beta|}\qquad\text{and}\qquad\left|h'_{j,\beta}\right|\leq C_{m}|\beta|\,j^{m-1}\frac{h_{j,\beta}(t)}{t},$$

and $h'(t)<0$. Notice that here we are using the modified Young function $\tilde{\Phi}_{m}$ (35) so we may apply Lemma 2.3 with $c_{1}=1$, $C_{1}=C_{m}N^{m-1}+\frac{|\beta-1|}{|\beta|}$, $C_{2}=C_{m}|\beta|\,j^{m-1}$, and

$$\frac{C_{1}^{2}C_{2}^{2}}{c_{1}^{2}}=\left(C_{m}+\frac{|\beta-1|}{|\beta|}\right)^{2}\left(C_{m,\lambda,\Lambda}|\beta|\,j^{m-1}\right)^{2}\leq C_{m,\lambda,\Lambda}|\beta|^{2}j^{2(m-1)},$$

to obtain for $v=\tilde{u}+\tilde{\phi}^{*}$, $h=h_{j,\beta}$

$$\int_{B_{j}}\psi_{j}^{2}\left|\nabla_{A}h(v)\right|^{2}d\mu_{j}\leq C_{m,\lambda,\Lambda}|\beta|^{2}j^{2(m-1)}\int_{B_{j}}h(v)^{2}\left(\left|\nabla_{A}\psi_{j}\right|^{2}+\psi_{j}^{2}\right)d\mu_{j}.$$

This is a similar estimate to (60) in the previous proof of Theorem 4.1. Recall that since then (Φ_{m},φ)-Sobolev bump inequality (9) holds in B, then for some $C_{m}\geq 1$ we have that from (40) then $(\tilde{\Phi}_{m},C_{m}\varphi)$-Sobolev bump inequality holds in B. The proof proceeds now identically as before, to obtain (66) with the given constants. □

4.2. Proof of recurrence inequalities

Now we provide the proof of the recurrence estimate used in Section 4.1 to prove boundedness of solutions.

Lemma 4.3. *Let $m>2$, $K>1$ and $\gamma>0$. Consider the sequence defined by*

$$b_{1}\geq e^{2^{m}},\qquad b_{n+1}=\Phi(Kn^{\gamma}b_{n}).$$

Then there exists a positive number $C^{*}=C^{*}\left(m,b_{1},K,\gamma\right)$, such that the inequality $\Phi^{(n-1)}(C^{*})\geq b_{n}$ holds for each positive integer n. In fact, we can choose

$$C^{*}=\exp\left(\left(\ln b_{1}\right)^{\frac{1}{m}}+C_{m}(\gamma+\ln K)\right)^{m}$$

where C_{m} only depends on m. Now we prove the growth estimate which allowed the Moser iteration to yield the boundedness theorem.

Proof. Let $m>2$, $K>1$, $\gamma>0$, and

$$b_{1}=\int_{B(0,r_{1})}|u|^{2}\,d\mu_{r_{1}}\geq e^{2^{m}},\qquad b_{n+1}=\Phi(Kn^{\gamma}b_{n}).$$

We want to estimate $\Phi^{(-j)}\left(b_{j+1}\right)$. Let us define another sequence by

$$\beta_{1}=C^{*},\qquad\beta_{n+1}=\Phi(\beta_{n}),\qquad n\geq 0$$

Thus we are trying to find a number C^{*} such that $\beta_{n}=\Phi^{(n-1)}\left(\beta_{1}\right)\geq b_{n}$ holds for all $n\geq 0$. Next we define the two related sequences:

$$\alpha_{n}=\left(\ln\beta_{n}\right)^{1/m},\qquad\text{and}\qquad\beta_{n}=\left(\ln b_{n}\right)^{1/m}.$$

The sequence $\{\alpha_{n}\}$ satisfies $\alpha_{1}=(\ln C^{*})^{1/m}$ and

$$\alpha_{n+1}=\left(\ln\beta_{n+1}\right)^{1/m}=\left(\ln\Phi\left(\beta_{n}\right)\right)^{1/m}=(\ln\beta_{n})^{1/m}+1=\alpha_{n}+1$$

for all $n\geq 1$. As for the other sequence, it is clear that $\beta_{1}=\left(\ln b_{1}\right)^{1/m}>2$, but the recurrence relation for b_{n} is a bit more complicated, we have:

$$\begin{aligned}\beta_{n+1}&=\left(\ln b_{n+1}\right)^{1/m}=\left(\ln\Phi\left(Kn^{\gamma}b_{n}\right)\right)^{1/m}=\left(\ln\left(Kn^{\gamma}b_{n}\right)\right)^{1/m}+1\\&=\left(\beta_{n}^{m}+\ln(Kn^{\gamma})\right)^{1/m}+1.\end{aligned}$$

This is clear that $\beta_{n+1}>\beta_{n}+1$ thus we have a rough lower bound

$$\beta_{n+1}\geq n+b_{1}.\tag{67}$$

Since the function $g(x)=x^{1/m}$ is concave, we have

$$\beta_{n+1}=\left(\beta_{n}^{m}+\ln(Kn^{\gamma})\right)^{1/m}+1=\beta_{n}\left\{1+\frac{\ln(Kn^{\gamma})}{\beta_{n}^{m}}\right\}^{1/m}+1\leq\beta_{n}+\frac{\ln(Kn^{\gamma})}{m\cdot\beta_{n}^{m-1}}+1.$$

Thus

$$\beta_{n+1} \le b_1 + n + \frac{1}{m}\sum_{j=1}^{n}\frac{\ln\left(Kj^{\gamma}\right)}{\beta_j^{m-1}} \quad\Longrightarrow\quad \alpha_n - \beta_n \ge \alpha_1 - b_1 - \frac{1}{m}\sum_{j=1}^{n}\frac{\ln\left(Kj^{\gamma}\right)}{\beta_j^{m-1}}. \tag{68}$$

Because $m > 2$, by (67) we have

$$\sum_{j=1}^{n}\frac{\ln\left(Kj^{\gamma}\right)}{\beta_j^{m-1}} < \sum_{j=1}^{\infty}\frac{\ln\left(Kj^{\gamma}\right)}{\left(\beta_1 + j - 1\right)^{m-1}} \le \frac{C_m}{\beta_1^{m-2}}\left(\gamma + \ln K\right) \le C_m\left(\gamma + \ln K\right) < \infty,$$

where we used that $\beta_1 = \left(\ln b_1\right)^{\frac{1}{m}} \ge 2$. Therefore, choosing $\alpha_1 = \beta_1 + C_m\left(\gamma + \ln K\right)$, (68) guarantees $\alpha_n > \beta_n$ for all $n \ge 1$, and so

$$\Phi^{(n-1)}\left(C^*\right) = \Phi^{(n-1)}\left(a_1\right) > b_n,$$

where $C^* = C^*(b_1, K, \gamma)$ is

$$\begin{aligned} C^* &= \exp\left(\alpha_1^m\right) = \exp\left(\beta_1 + C_m\left(\gamma + \ln K\right)\right)^m \\ &= \exp\left(\left(\ln b_1\right)^{\frac{1}{m}} + C_m\left(\gamma + \ln K\right)\right)^m. \quad \square \end{aligned}$$

Remark 4.4. Lemma 4.3 fails for $m \le 2$ even with $\gamma = 0$ and $K > e$. Indeed, then from the calculations above we have

$$\begin{aligned} \beta_{n+1} &= \beta_n\left(1 + \frac{\ln\left(Kn^{\gamma}\right)}{\beta_n^m}\right)^{1/m} + 1 \\ &\ge \beta_n + \frac{\ln\left(Kn^{\gamma}\right)}{m\beta_n^{m-1}} + 1 \ge \beta_n + \frac{\ln K}{m\beta_n^{m-1}} + 1 \end{aligned}$$

which when iterated gives

$$\beta_{n+1} \ge \beta_1 + n + \sum_{j=1}^{n}\frac{\ln K}{m\beta_j^{m-1}} \ge \beta_1 + n + \frac{\ln K}{2}\sum_{j=1}^{n}\frac{1}{\beta_j}.$$

So if there is a positive constant A such that $\beta_{n+1} \le n + A$ for n large, then we would have

$$\beta_{n+1} \ge \beta_1 + n + \frac{\ln K}{2}c\ln n$$

for some positive constant c, which is a contradiction to our assumption. Thus $\beta_{n+1} \le \alpha_0 + n$ for all $n \ge 1$ is impossible. That is, we have

$$\Phi^{(-n)}\left(b_n\right) = e^{\left[\left(\ln b_n\right)^{\frac{1}{m}} - n\right]^m} = e^{\left[\beta_n - n\right]^m} \ge e^{\left[\beta_1 + \frac{\ln K}{2}c\ln n\right]^m} \nearrow \infty$$

as $n \to \infty$, so Lemma 4.3 does not hold.

5. The geometric setting

In order to obtain *geometric* applications, we will take the metric d in Theorem 1.6 to be the Carnot-Carathéodory metric associated with the vector field ∇_A for appropriate matrices A, and we will show that the hypotheses of our abstract theorems hold in this geometry. For this we need to introduce a family of infinitely degenerate geometries that are simple enough so that we can compute the balls explicitly, prove the required Orlicz-Sobolev bump inequality, and define an appropriate accumulating sequence of Lipschitz cutoff functions. We will work solely in the plane and consider linear operators of the form

$$Lu(x,y) \equiv \nabla^{\mathrm{tr}} A(x,y)\nabla u(x,y), \qquad (x,y) \in \Omega,$$

where $\Omega \subset \mathbb{R}^2$ is a planar domain, and where the 2×2 matrix is

$$A(x,y) = \begin{bmatrix} 1 & 0 \\ 0 & f(x)^2 \end{bmatrix},$$

where $f(x) = e^{-F(x)}$ is even and there is $R > 0$ such that F satisfies the following five structure conditions for some constants $C \ge 1$ and $\varepsilon > 0$:

Definition 5.1 (*Structural Conditions*).

1. $\lim_{x \to 0^+} F(x) = +\infty$;
2. $F'(x) < 0$ and $F''(x) > 0$ for all $x \in (0, R)$;
3. $\frac{1}{C}\left|F'(r)\right| \le \left|F'(x)\right| \le C\left|F'(r)\right|$ for $\frac{1}{2}r < x < 2r < R$;
4. $\frac{1}{-xF'(x)}$ is increasing in the interval $(0, R)$ and satisfies $\frac{1}{-xF'(x)} \le \frac{1}{\varepsilon}$ for $x \in (0, R)$;
5. $\frac{F''(x)}{-F'(x)} \approx \frac{1}{x}$ for $x \in (0, R)$.

Remark 5.2. We make no smoothness assumption on f other than the existence of the second derivative f'' on the open interval $(0, R)$. Note also that at one extreme, f can be of finite type, namely $f(x) = x^\alpha$ for any $\alpha > 0$, and at the other extreme, f can be of strongly degenerate type, namely $f(x) = e^{-\frac{1}{x^\alpha}}$ for any $\alpha > 0$. Assumption (1) rules out the elliptic case $f(0) > 0$.

Under the general structural conditions 5.1 we will find further sufficient conditions on F so that the (Φ, A, φ)-Orlicz-Sobolev bump inequality (9) holds for a particular Φ in this geometry, where the superradius φ will depend on F (see Proposition 5.6). In ([9], Section 8.2) we showed that these geometries support both the $(1,1)$-Poincaré and the $(1,1)$-Sobolev inequalities.

In particular, we consider specific functions F satisfying the structural conditions 5.1, namely, the geometries $F_{k,\sigma}$ defined by

$$F_{k,\sigma}(r) = \left(\ln \frac{1}{r}\right)\left(\ln^{(k)} \frac{1}{r}\right)^\sigma, \qquad k \in \mathbb{N}, \quad \sigma > 0.$$

Note that $f_{k,\sigma} = e^{-F_{k,\sigma}(r)} = e^{-\left(\ln \frac{1}{r}\right)\left(\ln^{(k)} \frac{1}{r}\right)^\sigma}$ vanishes to infinite order at $r = 0$, and that $f_{k,\sigma}$ vanishes to a faster order than $f_{k',\sigma'}$ if either $k < k'$ or if $k = k'$ and $\sigma > \sigma'$.

To see that in the geometries $F_{k,\sigma}$ there exists a standard sequence of Lipschitz cutoff functions in $B = B(x,r)$, as given in Definition 1.3, we will prove the following general lemma for the Carnot-Carathéodory metric induced by a continuous nonnegative semidefinite quadratic form.

Lemma 5.3. *Let $\xi^t A(x)\xi$ be a continuous nonnegative semidefinite quadratic form. Suppose that the subunit metric d associated to $A(x)$ is topologically equivalent to the Euclidean metric d_E in the sense that for all $B(x,r) \subset \Omega$ there exist Euclidean balls $B_E\left(x, r_E(x,r)\right)$ and $B_E\left(x, R_E(x,r)\right)$ such that*

$$B_E\left(x, r_E(x,r)\right) \subseteq B(x,r) \subseteq B_E\left(x, R_E(x,r)\right). \tag{69}$$

Then for each ball $B(x,R) \subset \Omega$ and $0 < r < R$ there exists a cutoff function $\phi_{r,R} \in \mathrm{Lip}(\Omega)$ satisfying

$$\begin{cases} \operatorname{supp}\left(\phi_{r,R}\right) & \subseteq B(x,R), \\ \left\{x : \phi_{r,R}(x) = 1\right\} & \supseteq B(x,r), \\ \|\nabla_A \phi_{r,R}\|_{L^\infty(B(x,R))} & \leq \dfrac{C_n}{R-r}. \end{cases} \tag{70}$$

Proof. For any $\varepsilon \geq 0$ let $A^\varepsilon(x,\xi) = \xi^t A(x)\xi + \varepsilon^2|\xi|^2$. It has been shown in ([12], Lemma 65) that under the hypothesis of Lemma 5.3 the subunit metric $d^\varepsilon(x,y)$ associated to A^ε satisfies

$$\left|\nabla_A d^\varepsilon(x,y)\right| \leq \sqrt{n}, \quad x, y \in \Omega$$

uniformly in $\varepsilon > 0$. Moreover, $d^\varepsilon(\cdot,y) \nearrow d(\cdot,y)$, the convergence is monotone and d is continuous (in the Euclidean distance), therefore, $d^\varepsilon(\cdot,y) \to d(\cdot,y)$ uniformly on compact subsets of Ω.

Define $g(t)$ to vanish for $t \geq R - \frac{R-r}{4}$, to equal 1 for $t \leq r$ and to be linear on the interval $[r, R - \frac{R-r}{4}]$. Let $\phi_{r,R}(x) = g(d^{\varepsilon^*}(x,y))$, with ε^* to be chosen later. Since $d^{\varepsilon^*}(x,y) \leq d(x,y)$ we have

$$\phi_{r,R}(x) = 1 \quad \text{when} \quad d(x,y) \leq r.$$

And since $\phi_{r,R}(x) = 0$ when $d^{\varepsilon^*}(x,y) \geq R - \frac{R-r}{4}$, by choosing ε^* small enough, we obtain that $\phi_{r,R}(x) = 0$ when $d(x,y) \geq R$. This shows that $\operatorname{supp}\left(\phi_{r,R}\right) \subseteq B(x,R)$ and $\left\{x : \phi_{r,R}(x) = 1\right\} \supseteq B(x,r)$. Next,

$$\left|\nabla_A \phi_{r,R}(x)\right| \leq \|g'\|_\infty \left|\nabla_A d^{\varepsilon^*}\right| \leq \frac{4}{3}\frac{1}{R-r}\sqrt{n} = \frac{C_n}{R-r}. \tag{71}$$

This completes the proof. □

Remark 5.4. Note that the condition that $A(x)$ is continuous cannot be easily omitted. In [14] the author constructs an example of a locally unbounded (therefore discontinuous) solution to a degenerate linear elliptic equation (see Theorem 1.3 and Conjecture 7). However, the matrix Q in that case is discontinuous and this requirement seems to be essential for the construction.

5.1. Geometric Orlicz-Sobolev inequality

In this section we use subrepresentation inequalities proved in [9] to prove the relevant Sobolev and Poincaré inequalities. More precisely, we will use ([9], Lemma 58), which says that for every Lipschitz function w there holds

$$\left|w(x) - \mathbb{E}_{x,r_1} w\right| \leq C \int_{\Gamma(x,r)} \left|\nabla_A w(y)\right| \frac{\hat{d}(x,y)}{|B(x,d(x,y))|} dy, \tag{72}$$

where

$$\hat{d}(x,y) \equiv \min\left\{ d(x,y), \frac{1}{\left|F'\left(x_1 + d(x,y)\right)\right|} \right\}. \tag{73}$$

Here $\Gamma(x,r)$ is a cusp-like region defined as

$$\Gamma(x,r)=\bigcup_{k=1}^{\infty}\mathrm{co}\left[E\left(x,r_k\right)\cup E\left(x,r_{k+1}\right)\right],$$

where the sets $E\left(x,r_k\right)$ are curvilinear trapezoidal sets on which the function f does not change much, and which satisfy

$$\left|E\left(x,r_k\right)\right|\approx\left|E\left(x,r_k\right)\bigcap B\left(x,r_k\right)\right|\approx\left|B\left(x,r_k\right)\right|\text{ for all }k\geq 1. \tag{74}$$

Finally, we use the following notation for averages

$$\mathbb{E}_{x,r_1}w\equiv\frac{1}{|E(x,r_1)|}\int\int_{E(x,r_1)}w.$$

In our setting of infinitely degenerate metrics in the plane, the metrics we consider are elliptic away from the x_2 axis, and are invariant under vertical translations. As a consequence, we need only consider Sobolev inequalities for the metric balls $B(0,r)$ centered at the origin. So from now on we consider $X=\mathbb{R}^2$ and the metric balls $B(0,r)$ associated to one of the geometries F considered in ([9], Part 2).

First we recall that the optimal form of the degenerate Orlicz-Sobolev *norm* inequality for balls is

$$\|w\|_{L^{\Theta}\left(\mu_{r_0}\right)}\leq Cr_0\left\|\nabla_A w\right\|_{L^{\Omega}\left(\mu_{r_0}\right)},$$

where $d\mu_{r_0}(x)=\frac{dx}{\left|B(0,r_0)\right|}$, the balls $B\left(0,r_0\right)$ are control balls for a metric A, and the Young function Θ is a 'bump up' of the Young function Ω. We will instead obtain the nonhomogeneous form of this inequality where $L^{\Omega}\left(\mu_{r_0}\right)=L^1\left(\mu_{r_0}\right)$ is the usual Lebesgue space, and the factor r_0 on the right hand side is replaced by a suitable superradius $\varphi\left(r_0\right)$, namely

$$\Phi^{(-1)}\left(\int_{B(0,r_0)}\Phi(w)\ d\mu_{r_0}\right)\leq C\varphi\left(r_0\right)\ \left\|\nabla_A w\right\|_{L^1\left(\mu_{r_0}\right)},\qquad w\in\mathrm{Lip}_c(X), \tag{75}$$

which we refer to as the (Φ,A,φ)*-Sobolev Orlicz bump inequality*. In fact, consider the positive operator $T_{B(0,r_0)}:L^1\left(\mu_{r_0}\right)\to L^{\Phi}\left(\mu_{r_0}\right)$ defined by

$$T_{B(0,r_0)}g(x)\equiv\int_{B(0,r_0)}K_{B(0,r_0)}(x,y)\,g(y)dy$$

with kernel $K_{B(0,r_0)}$ defined as

$$K_{B(0,r_0)}(x,y)=\frac{\widehat{d}(x,y)}{|B(x,d(x,y))|}\mathbf{1}_{\Gamma(x,r_0)}(y). \tag{76}$$

We will obtain the following stronger inequality,

$$\Phi^{(-1)}\left(\int_{B(0,r_0)}\Phi\left(T_{B(0,r_0)}g\right)d\mu_{r_0}\right)\leq C\varphi\left(r_0\right)\ \|g\|_{L^1\left(\mu_{r_0}\right)}, \tag{77}$$

which we refer to as the *strong* (Φ,A,φ) -Sobolev Orlicz bump inequality, and which is stronger by the subrepresentation inequality $w\lesssim T_{B(0,r_0)}\nabla_A w$ on $B\left(0,r_0\right)$. But this inequality cannot in general be reversed. When we wish to emphasize that we are working with (75), we will often call it the *standard* (Φ,A,φ)-Sobolev Orlicz bump inequality.

Recall the operator $T_{B(0,r_0)}:L^1\left(\mu_{r_0}\right)\to L^{\Phi}\left(\mu_{r_0}\right)$ defined by

$$T_{B(0,r_0)}g(x)\equiv\int_{B(0,r_0)}K_{B(0,r_0)}(x,y)\,g(y)dy$$

with kernel K defined as in (76). We begin by proving that the bound (77) holds if the following endpoint inequality holds:

$$\Phi^{-1}\left(\sup_{y\in B}\int_B\Phi\left(K(x,y)|B|\alpha\right)d\mu(x)\right)\leq C\alpha\varphi(r)\ . \tag{78}$$

for all $\alpha>0$. Indeed, if (78) holds, then with $g=\left|\nabla_A w\right|$ and $\alpha=\|g\|_{L^1}=\left\|\nabla_A w\right\|_{L^1}$, we have using first the subrepresentation inequality, and then Jensen's inequality applied to the convex function Φ,

$$\begin{aligned}\int_B\Phi(w)d\mu(x)&\lesssim\int_B\Phi\left(\int_B K(x,y)\ |B|\ \|g\|_{L^1(\mu)}\frac{g(y)\,d\mu(y)}{\|g\|_{L^1(\mu)}}\right)d\mu(x)\\&\leq\int_B\int_B\Phi\left(K(x,y)\ |B|\ \|g\|_{L^1(\mu)}\right)\frac{g(y)\,d\mu(y)}{\|g\|_{L^1(\mu)}}d\mu(x)\\&\leq\int_B\left\{\sup_{y\in B}\int_B\Phi\left(K(x,y)\ |B|\ \|g\|_{L^1(\mu)}\right)d\mu(x)\right\}\frac{g(y)\,d\mu(y)}{\|g\|_{L^1(\mu)}}\\&\leq\Phi\left(C\varphi(r)\ \|g\|_{L^1(\mu)}\right)\int_B\frac{g(y)\,d\mu(y)}{\|g\|_{L^1(\mu)}}=\Phi\left(C\varphi(r)\ \|g\|_{L^1(\mu)}\right),\end{aligned}$$

and so

$$\Phi^{-1}\left(\int_B \Phi(w)d\mu(x)\right) \lesssim C\varphi(r)\ \|\nabla_A w\|_{L^1(\mu)}.$$

The converse follows from Fatou's lemma, but we will not need this. Note that (78) is obtained from (77) by replacing $g(y)\,dy$ with the point mass $|B|\alpha\delta_x(y)$ so that $Tg(x) \to K(x,y)\ |B|\ \alpha$.

Remark 5.5. The inhomogeneous condition (78) is in general stronger than its homogeneous counterpart

$$\sup_{y\in B(0,r_0)} \left\| K_{B(0,r_0)}(\cdot,y)\left|B\left(0,r_0\right)\right|\right\|_{L^\Phi\left(\mu_{r_0}\right)} \le C\varphi\left(r_0\right)\ ,$$

but is equivalent to it when Φ is submultiplicative. We will not however use this observation.

Now we turn to the explicit near power bumps Φ in (34), which satisfy

$$\Phi(t) = \Phi_m(t) = e^{\left((\ln t)^{\frac{1}{m}}+1\right)^m}, \qquad t > e^{2^m},$$

for $m \in (1,\infty)$. Let $\psi(t) = \left(1+(\ln t)^{-\frac{1}{m}}\right)^m - 1$ for $t > E = e^{2^m}$ and write $\Phi(t) = t^{1+\psi(t)}$.

Proposition 5.6. *Let $0 < r_0 < 1$ and $C_m > 0$. Suppose that the geometry F satisfies the monotonicity property:*

$$\varphi(r) \equiv \frac{1}{|F'(r)|} e^{C_m\left(\frac{|F'(r)|^2}{F''(r)}+1\right)^{m-1}} \quad \text{is an increasing function of } r \in \left(0, r_0\right). \tag{79}$$

Then the (Φ,φ)-Sobolev inequality (77) holds with geometry F, with φ as in (79) and with Φ as in (34), $m > 1$.

For fixed $\Phi = \Phi_m$ with $m > 1$, we now consider the geometry of balls defined by

$$F_{k,\sigma}(r) = \left(\ln\frac{1}{r}\right)\left(\ln^{(k)}\frac{1}{r}\right)^\sigma;$$

$$f_{k,\sigma}(r) = e^{-F_{k,\sigma}(r)} = e^{-\left(\ln\frac{1}{r}\right)\left(\ln^{(k)}\frac{1}{r}\right)^\sigma},$$

where $k \in \mathbb{N}$ and $\sigma > 0$.

Corollary 5.7. *The strong (Φ,φ)-Sobolev inequality (77) with $\Phi = \Phi_m$ as in (34), $m > 1$, and geometry $F = F_{k,\sigma}$ holds if* ***(either)*** *$k \ge 2$ and $\sigma > 0$ and $\varphi(r_0)$ is given by*

$$\varphi(r_0) = r_0^{\,1-C_m\frac{\left(\ln^{(k)}\frac{1}{r_0}\right)^{\sigma(m-1)}}{\ln\frac{1}{r_0}}}, \qquad \text{for } 0 < r_0 \le \beta_{m,\sigma},$$

for positive constants C_m and $\beta_{m,\sigma}$ depending only on m and σ;
(or) *$k = 1$ and $\sigma < \frac{1}{m-1}$ and $\varphi(r_0)$ is given by*

$$\varphi(r_0) = r_0^{\,1-C_m\frac{1}{\left(\ln\frac{1}{r_0}\right)^{1-\sigma(m-1)}}}, \qquad \text{for } 0 < r_0 \le \beta_{m,\sigma},$$

for positive constants C_m and $\beta_{m,\sigma}$ depending only on m and σ.

Conversely, the standard (Φ,φ)-Sobolev inequality (75) with Φ as in (34), $m > 1$, fails if $k = 1$ and $\sigma > \frac{1}{m-1}$.

Proof of Proposition 5.6. It suffices to prove the endpoint inequality (78), namely

$$\Phi^{-1}\left(\sup_{y\in B}\int_B \Phi\left(K(x,y)|B|\alpha\right)d\mu(x)\right) \le C\alpha\varphi\left(r(B)\right)\ , \qquad \alpha > 0,$$

for the balls and kernel associated with our geometry F, the Orlicz bump Φ, and the function $\varphi(r)$ satisfying (79). Fix parameters $m > 1$ and $t_m > 1$. Following the proof of ([9], Proposition 80) we consider the specific function $\omega(r(B))$ given by

$$\omega(r(B)) = \frac{1}{t_m\left|F'\left(r(B)\right)\right|}.$$

Using the submultiplicativity of Φ we have

$$\begin{aligned}\int_B \Phi\left(K(x,y)|B|\alpha\right)d\mu(x) &= \int_B \Phi\left(\frac{K(x,y)|B|}{\omega(r(B))}\alpha\omega(r(B))\right)d\mu(x)\\ &\le \Phi\left(\alpha\omega(r(B))\right)\int_B \Phi\left(\frac{K(x,y)|B|}{\omega(r(B))}\right)d\mu(x)\end{aligned}$$

and we will now prove

$$\int_B \Phi\left(\frac{K(x,y)|B|}{\omega(r(B))}\right) d\mu(x) \leq C_m \varphi(r(B)) \left|F'(r(B))\right|, \tag{80}$$

for all small balls B of radius $r(B)$ centered at the origin. Altogether this will give us

$$\int_B \Phi(K(x,y)|B|\alpha)\, d\mu(x) \leq C_m \varphi(r(B)) \left|F'(r(B))\right| \Phi\left(\frac{\alpha}{t_m |F'(r(B))|}\right).$$

Now we note that $x\Phi(y) = xy\frac{\Phi(y)}{y} \leq xy\frac{\Phi(xy)}{xy} = \Phi(xy)$ for $x \geq 1$ since $\frac{\Phi(t)}{t}$ is monotone increasing. But from (79) we have $\varphi(r)\left|F'(r)\right| = e^{C_m\left(\frac{\left|F'(r)\right|^2}{F''(r)}+1\right)^{m-1}} \gg 1$ and so

$$\begin{aligned}\int_B \Phi(K(x,y)|B|\alpha)\, d\mu(x) &\leq \Phi\left(C_m \varphi(r(B)) \left|F'(r(B))\right| \alpha \frac{1}{t_m |F'(r(B))|}\right)\\ &= \Phi\left(\frac{C_m}{t_m}\alpha\varphi(r(B))\right),\end{aligned}$$

which is (78) with $C = \frac{C_m}{t_m}$. Thus it remains to prove (80).

So we now take $B = B(0,r_0)$ with $r_0 \ll 1$ so that $\omega(r(B)) = \omega(r_0)$. First, from [9] we have the estimates

$$\left|B(0,r_0)\right| \approx \frac{f(r_0)}{|F'(r_0)|^2},$$

and in $\Gamma(x,r)$

$$K(x,y) \approx \frac{1}{h_{y_1-x_1}} \approx \begin{cases} \frac{1}{rf(x_1)}, & 0 < r = y_1 - x_1 < \frac{1}{|F'(x_1)|} \\ \frac{|F'(x_1+r)|}{f(x_1+r)}, & 0 < r = y_1 - x_1 \geq \frac{1}{|F'(x_1)|} \end{cases}.$$

Next, write $\Phi(t)$ as

$$\Phi(t) = t^{1+\psi(t)}, \qquad \text{for } t > 0,$$

where for $t \geq E$,

$$\begin{aligned} t^{1+\psi(t)} &= \Phi(t) = e^{\left((\ln t)^{\frac{1}{m}}+1\right)^m} = t^{\left(1+(\ln t)^{-\frac{1}{m}}\right)^m}\\ &\Longrightarrow \psi(t) = \left(1+(\ln t)^{-\frac{1}{m}}\right)^m - 1 \approx \frac{m}{(\ln t)^{1/m}},\end{aligned}$$

and for $t < E$,

$$\begin{aligned} t^{1+\psi(t)} &= \Phi(t) = \frac{\Phi(E)}{E}t\\ &\Longrightarrow (1+\psi(t))\ln t = \ln\frac{\Phi(E)}{E} + \ln t\\ &\Longrightarrow \psi(t) = \frac{\ln\frac{\Phi(E)}{E}}{\ln t}.\end{aligned}$$

Now temporarily fix $y = (y_1, y_2) \in B_+(0,r_0) \equiv \left\{x \in B(0,r_0) : x_1 > 0\right\}$. We then have for $0 < a < b < r_0$ that

$$\begin{aligned}\mathcal{I}_{a,b}(y) &\equiv \int_{\{x\in B_+(0,r_0):a\leq y_1-x_1\leq b\}\cap\Gamma^*(y,r_0)} \Phi\left(K_{B(0,r_0)}(x,y)\frac{\left|B(0,r_0)\right|}{\omega(r_0)}\right)\frac{dx}{\left|B(0,r_0)\right|}\\ &= \int_{y_1-b}^{y_1-a}\left\{\int_{y_2-h_{y_1-x_1}}^{y_2+h_{y_1-x_1}} \Phi\left(\frac{1}{h_{y_1-x_1}}\left|B(0,r_0)\right|\frac{\left|B(0,r_0)\right|}{\omega(r_0)}\right)dx_2\right\}\frac{dx_1}{\left|B(0,r_0)\right|}\\ &= \int_{y_1-b}^{y_1-a} 2h_{y_1-x_1}\Phi\left(\frac{1}{h_{y_1-x_1}}\frac{\left|B(0,r_0)\right|}{\omega(r_0)}\right)\frac{dx_1}{\left|B(0,r_0)\right|}\\ &= \int_{y_1-b}^{y_1-a} 2h_{y_1-x_1}\left(\frac{1}{h_{y_1-x_1}}\frac{\left|B(0,r_0)\right|}{\omega(r_0)}\right)\left(\frac{1}{h_{y_1-x_1}}\frac{\left|B(0,r_0)\right|}{\omega(r_0)}\right)^{\psi\left(\frac{1}{h_{y_1-x_1}}\frac{\left|B(0,r_0)\right|}{\omega(r_0)}\right)}\frac{dx_1}{\left|B(0,r_0)\right|}\end{aligned}$$

which simplifies to

$$\mathcal{I}_{a,b}(y) = \frac{2}{\omega(r_0)} \int_{y_1-b}^{y_1-a} \left(\frac{1}{h_{y_1-x_1}} \frac{\left|B(0,r_0)\right|}{\omega(r_0)} \right)^{\psi\left(\frac{1}{h_{y_1-x_1}} \frac{\left|B(0,r_0)\right|}{\omega(r_0)} \right)} dx_1$$

$$= \frac{2}{\omega(r_0)} \int_{a}^{b} \left(\frac{1}{h_r} \frac{\left|B(0,r_0)\right|}{\omega(r_0)} \right)^{\psi\left(\frac{1}{h_r} \frac{\left|B(0,r_0)\right|}{\omega(r_0)} \right)} dr.$$

Thus we have

$$\int_{B_+(0,r_0)} \Phi\left(K_{B(0,r_0)}(x,y) \frac{\left|B(0,r_0)\right|}{\omega(r_0)} \right) \frac{dx}{\left|B(0,r_0)\right|}$$
$$= \mathcal{I}_{0,y_1}(x)$$
$$= \frac{2}{\omega(r_0)} \int_{0}^{y_1} \left(\frac{1}{h_r} \frac{\left|B(0,r_0)\right|}{\omega(r_0)} \right)^{\psi\left(\frac{1}{h_r} \frac{\left|B(0,r_0)\right|}{\omega(r_0)} \right)} dr \ .$$

To prove (80) it thus suffices to show

$$\mathcal{I}_{0,y_1} = \frac{1}{\omega(r_0)} \int_{0}^{y_1} \left(\frac{\left|B(0,r_0)\right|}{h_r\omega(r_0)} \right)^{\psi\left(\frac{\left|B(0,r_0)\right|}{h_r\omega(r_0)} \right)} dr \le C_m\ \varphi(r_0) \left|F'(r_0)\right| \ , \tag{81}$$

where C_0 is a sufficiently large positive constant.

To prove this we divide the interval $(0, y_1)$ of integration in r into three regions:

(1): the small region $\mathcal{S}$ where $\frac{|B(0,r_0)|}{h_r\omega(r_0)} \le E$,

(2): the big region $\mathcal{R}_1$ that is disjoint from $\mathcal{S}$ and where $r = y_1 - x_1 < \frac{1}{\left|F'(x_1)\right|}$ and

(3): the big region $\mathcal{R}_2$ that is disjoint from $\mathcal{S}$ and where $r = y_1 - x_1 \ge \frac{1}{\left|F'(x_1)\right|}$.

In the small region $\mathcal{S}$ we use that Φ is linear on $[0, E]$ to obtain that the integral in the right hand side of (81), when restricted to those $r \in (0, y_1)$ for which $\frac{|B(0,r_0)|}{h_r\omega(r_0)} \le E$, is equal to

$$\frac{1}{\omega(r_0)} \int_{0}^{y_1} \left(\frac{\left|B(0,r_0)\right|}{h_r\omega(r_0)} \right)^{\frac{\ln \frac{\Phi(E)}{E}}{\ln\left(\frac{\left|B(0,r_0)\right|}{h_r\omega(r_0)} \right)}} dr$$
$$= \frac{1}{\omega(r_0)} \int_{0}^{y_1} e^{\ln \frac{\Phi(E)}{E}} dr = \frac{1}{\omega(r_0)} \frac{\Phi(E)}{E} y_1$$
$$\le \frac{\Phi(E)}{E} t_m\ r_0 \left|F'(r_0)\right| \ ,$$

since $\omega(r_0) = \frac{1}{t_m\left|F'(r_0)\right|}$. We now turn to the first big region $\mathcal{R}_1$ where we have $h_{y_1-x_1} \approx rf(x_1) = rf(y_1 - r)$. The integral to be evaluated is

$$\frac{1}{\omega(r_0)} \int_{0}^{y_1} \left(\frac{|B(0,r_0)|}{h_r\omega(r_0)} \right)^{\psi\left(\frac{|B(0,r_0)|}{h_r\omega(r_0)} \right)} dr, \quad \text{where} \quad \frac{|B(0,r_0)|}{h_r\omega(r_0)} \approx \frac{|B(0,r_0)|}{rf(y_1-r)\omega(r_0)}$$

Now we note that since $x_1 < y_1$, we have $\frac{1}{\left|F'(x_1)\right|} \le \frac{1}{\left|F'(y_1)\right|}$, and thus in this region we have $x_1 < y_1 \le x_1 + \frac{1}{\left|F'(y_1)\right|}$, and it is sufficient to evaluate

$$\frac{1}{\omega(r_0)} \int_{0}^{\frac{1}{\left|F'(y_1)\right|}} \left(\frac{|B(0,r_0)|}{h_r\omega(r_0)} \right)^{\psi\left(\frac{|B(0,r_0)|}{h_r\omega(r_0)} \right)} dr.$$

From the inequalities for y_1 it also follows that $f(x_1) \approx f(y_1)$, so $h_{y_1-x_1} \approx rf(y_1)$. Write

$$\frac{|B(0,r_0)|}{h_r\omega(r_0)} \le C' \frac{|B(0,r_0)|}{rf(y_1)\omega(r_0)} \le C \frac{t_m f(r_0)}{rf(y_1)\left|F'(r_0)\right|},$$

and we will now evaluate the following integral

$$\frac{1}{\omega\left(r_0\right)}\int_0^{\frac{1}{|F'(y_1)|}}\left(\frac{A}{r}\right)^{\psi\left(\frac{A}{r}\right)}dr,\quad \text{where}\quad A=C\frac{t_m f(r_0)}{f(y_1)\left|F'\left(r_0\right)\right|}.$$

Making a change of variables

$$R=\frac{A}{r}=\frac{A\left(y_1\right)}{r},$$

we obtain

$$\frac{1}{\omega\left(r_0\right)}\int_0^{\frac{1}{|F'(y_1)|}}\left(\frac{A}{r}\right)^{\psi\left(\frac{A}{r}\right)}dr=\frac{1}{\omega\left(r_0\right)}A\int_{A\left|F'\left(y_1\right)\right|}^{\infty}R^{\psi(R)-2}dR.$$

Integrating by parts gives

$$\begin{aligned}\int_{A\left|F'\left(y_1\right)\right|}^{\infty}R^{\psi(R)-2}dR&=\int_{A\left|F'\left(y_1\right)\right|}^{\infty}R^{\psi(R)+1}\left(-\frac{1}{2R^2}\right)'dR\\&=-\frac{R^{\psi(R)+1}}{2R^2}\Big|_{A\left|F'\left(x_1\right)\right|}^{\infty}+\int_{A\left|F'\left(x_1\right)\right|}^{\infty}\left(R^{\psi(R)+1}\right)'\frac{1}{2R^2}dR\\&\leq\frac{\left(A|F'(y_1)|\right)^{\psi\left(A|F'(y_1)|\right)}}{2A|F'(y_1)|}+\int_{A\left|F'\left(y_1\right)\right|}^{\infty}\frac{1}{2}R^{\psi(R)-2}\left(1+C\frac{m-1}{(\ln R)^{\frac{1}{m}}}\right)dR\\&\leq\frac{\left(A|F'(y_1)|\right)^{\psi\left(A|F'(y_1)|\right)}}{2A|F'(y_1)|}+\frac{1+C\frac{m-1}{(\ln E)^{\frac{1}{m}}}}{2}\int_{A\left|F'\left(y_1\right)\right|}^{\infty}R^{\psi(R)-2}dR,\end{aligned}$$

where we used

$$\left|\psi'(R)\right|\leq C\frac{1}{R}\frac{1}{(\ln R)^{\frac{m+1}{m}}}.$$

Taking E large enough depending on m we can assure

$$\frac{1+C\frac{m-1}{(\ln E)^{\frac{1}{m}}}}{2}\leq\frac{3}{4},$$

which gives

$$\int_{A\left|F'\left(y_1\right)\right|}^{\infty}R^{\psi(R)-2}dR\lesssim\frac{\left(A|F'(y_1)|\right)^{\psi\left(A|F'(y_1)|\right)}}{A|F'(y_1)|},$$

and therefore

$$\begin{aligned}\mathcal{I}_{0,\frac{1}{\left|F'\left(y_1\right)\right|}}(x)&\lesssim\frac{1}{\omega\left(r_0\right)}A\int_{A\left|F'\left(y_1\right)\right|}^{\infty}R^{\psi(R)-2}dR\\&\lesssim\frac{1}{\omega\left(r_0\right)|F'(y_1)|}\left(A\left(y_1\right)|F'(y_1)|\right)^{\psi\left(A(y_1)|F'(y_1)|\right)}.\end{aligned}$$

We now look for the maximum of the function on the right hand side

$$\begin{aligned}\mathcal{F}(y_1)&\equiv\frac{1}{\omega\left(r_0\right)|F'(y_1)|}\left(A\left(y_1\right)|F'(y_1)|\right)^{\psi\left(A(y_1)|F'(y_1)|\right)}\\&=t_m\left|F'\left(r_0\right)\right|\frac{1}{\left|F'\left(y_1\right)\right|}\left(c(r_0)\frac{\left|F'\left(y_1\right)\right|}{f\left(y_1\right)}\right)^{\psi\left(c(r_0)\frac{\left|F'\left(y_1\right)\right|}{f\left(y_1\right)}\right)}\end{aligned}$$

where

$$c(r_0)=f\left(y_1\right)A\left(y_1\right)=\frac{Ct_m f(r_0)}{|F'(r_0)|}.$$

Using the definition of $\psi(t)$ and $B\left(y_1\right)\equiv\ln\left[c(r_0)\frac{\left|F'\left(y_1\right)\right|}{f\left(y_1\right)}\right]$, we can rewrite $\mathcal{F}(y_1)$ as

$$\mathcal{F}(y_1)=t_m\left|F'\left(r_0\right)\right|\frac{1}{\left|F'\left(y_1\right)\right|}\exp\left(\left(1+B\left(y_1\right)^{\frac{1}{m}}\right)^m-B\left(y_1\right)\right).\tag{82}$$

Let $y_1^* \in (0, r_0]$ be the point at which $\mathcal{F}$ takes its maximum. Differentiating $\mathcal{F}(y_1)$ with respect to y_1 and then setting the derivative equal to zero, we obtain that y_1^* satisfies the equation,

$$\frac{F''(y_1^*)}{|F'(y_1^*)|^2} = \left(\left(1 + B\left(y_1^*\right)^{-\frac{1}{m}}\right)^{m-1} - 1\right)\left(1 + \frac{F''(y_1^*)}{|F'(y_1^*)|^2}\right).$$

Simplifying gives the following implicit expression for y_1^* that maximizes $\mathcal{F}(y_1)$

$$B\left(y_1^*\right) = \ln\left[c(r_0)\frac{\left|F'\left(y_1^*\right)\right|}{f\left(y_1^*\right)}\right] = \left(\left(1 + \frac{F''(y_1^*)}{|F'(y_1^*)|^2 + F''(y_1^*)}\right)^{\frac{1}{m-1}} - 1\right)^{-m}.$$

To estimate $\mathcal{F}(y_1^*)$ in an effective way, we set $b\left(y_1^*\right) \equiv \frac{F''(y_1^*)}{|F'(y_1^*)|^2 + F''(y_1^*)}$ and begin with

$$\begin{aligned}
&\left(1 + B\left(y_1\right)^{\frac{1}{m}}\right)^m - B\left(y_1\right) = \left(1 + \left(\ln\left[c(r_0)\frac{\left|F'\left(y_1^*\right)\right|}{f\left(y_1^*\right)}\right]\right)^{\frac{1}{m}}\right)^m - \ln\left[c(r_0)\frac{\left|F'\left(y_1^*\right)\right|}{f\left(y_1^*\right)}\right] \\
&= \frac{\left(1 + \frac{F''(y_1^*)}{|F'(y_1^*)|^2 + F''(y_1^*)}\right)^{\frac{m}{m-1}} - 1}{\left(\left(1 + \frac{F''(y_1^*)}{|F'(y_1^*)|^2 + F''(y_1^*)}\right)^{\frac{1}{m-1}} - 1\right)^m} = \frac{\left(1 + b\left(y_1^*\right)\right)^{\frac{m}{m-1}} - 1}{\left(\left(1 + b\left(y_1^*\right)\right)^{\frac{1}{m-1}} - 1\right)^m} \\
&\le C_m\left(\frac{1}{b\left(y_1^*\right)}\right)^{m-1} = C_m\left(\frac{|F'(y_1^*)|^2 + F''(y_1^*)}{F''(y_1^*)}\right)^{m-1} = C_m\left(1 + \frac{|F'(y_1^*)|^2}{F''(y_1^*)}\right)^{m-1},
\end{aligned}$$

where in the last inequality we used (**1**) the fact that $b\left(y_1^*\right) = \frac{F''(y_1^*)}{|F'(y_1^*)|^2 + F''(y_1^*)} < 1$ provided $y_1^* \le r$, which we may assume since otherwise we are done, and (**2**) the inequality

$$\frac{(1+b)^{\frac{m}{m-1}} - 1}{\left((1+b)^{\frac{1}{m-1}} - 1\right)^m} \le \frac{1}{2} m\,(2m-1)\,(m-1)^{2m}\, b^{1-m}, \qquad 0 \le b < 1,$$

which follows easily from upper and lower estimates on the binomial series. Combining this with (82) we thus obtain the following upper bound

$$\mathcal{F}(y_1) \le t_m\left|F'\left(r_0\right)\right| \frac{1}{\left|F'\left(y_1^*\right)\right|} e^{C_m\left(1 + \frac{|F'(y_1^*)|^2}{F''(y_1^*)}\right)^{m-1}} = t_m\left|F'\left(r_0\right)\right|\ \varphi\left(y_1^*\right),$$

with φ as in (79). Using the monotonicity of φ_F we therefore obtain

$$\mathcal{I}_{0, \frac{1}{\left|F'(y_1)\right|}}(x) \lesssim \mathcal{F}(y_1) \le t_m\left|F'\left(r_0\right)\right|\ \varphi\left(r_0\right) = t_m\left|F'\left(r_0\right)\right|\varphi\left(r_0\right),$$

which is the estimate required in (81).

For the second big region $\mathcal{R}_2$ we have

$$\frac{1}{h_{y_1 - x_1}} \approx \frac{|F'(x_1 + r)|}{f(x_1 + r)} = \frac{|F'(y_1)|}{f(y_1)},$$

and the integral to be estimated becomes

$$\begin{aligned}
I_{\mathcal{R}_2} &\equiv \frac{1}{\omega\left(r_0\right)} \int_{x_1 \in \mathcal{R}_2} \left(c(r_0)\frac{\left|F'\left(y_1\right)\right|}{f\left(y_1\right)}\right)^{\psi\left(c(r_0)\frac{\left|F'(y_1)\right|}{f(y_1)}\right)} dx_1 \\
&\le \frac{y_1}{\omega\left(r_0\right)} \left(c(r_0)\frac{\left|F'\left(y_1\right)\right|}{f\left(y_1\right)}\right)^{\psi\left(c(r_0)\frac{\left|F'(y_1)\right|}{f(y_1)}\right)} \\
&= t_m|F'(r_0)|y_1\left(c(r_0)\frac{\left|F'\left(y_1\right)\right|}{f\left(y_1\right)}\right)^{\psi\left(c(r_0)\frac{\left|F'(y_1)\right|}{f(y_1)}\right)},
\end{aligned}$$

where

$$c(r_0)=\frac{t_m\, f(r_0)}{|F'(r_0)|}.$$

We now look for the maximum of the function

$$\mathcal{G}(y_1)\equiv t_m\left|F'\left(r_0\right)\right| y_1\left(c(r_0)\frac{\left|F'\left(y_1\right)\right|}{f\left(y_1\right)}\right)^{\psi\left(c(r_0)\frac{\left|F'(y_1)\right|}{f(y_1)}\right)},$$

and look for the maximum of $\mathcal{G}(y_1)$ on $\left(0,r_0\right]$. We claim that a bound for $\mathcal{G}$ can be obtained in a similar way and yields

$$\mathcal{G}(y_1)\le C_m\left|F'\left(r_0\right)\right|\varphi\left(r_0\right),$$

where $\varphi\left(r_0\right)$ satisfies (79) with a constant C_m slightly bigger than in the case of $\mathcal{F}$. Indeed, rewriting $\mathcal{G}(y_1)$ in a form similar to (82) we have

$$\begin{aligned}\mathcal{G}(y_1)&=t_m\left|F'\left(r_0\right)\right| y_1\exp\left(\left(1+\left(\ln\left[c(r_0)\frac{\left|F'\left(y_1\right)\right|}{f\left(y_1\right)}\right]\right)^{\frac{1}{m}}\right)^m-\ln\left[c(r_0)\frac{\left|F'\left(y_1\right)\right|}{f\left(y_1\right)}\right]\right)\\&=t_m\left|F'\left(r_0\right)\right| y_1\exp\left(\left(1+B\left(y_1\right)^{\frac{1}{m}}\right)^m-B\left(y_1\right)\right)\end{aligned}$$

Again, we differentiate and equate the derivative to zero to obtain the following implicit expression for y_1^* maximizing $\mathcal{G}(y_1)$:

$$1=\left(\left(1+\left(\ln\left[c(r_0)\frac{\left|F'\left(y_1^*\right)\right|}{f\left(y_1^*\right)}\right]\right)^{-\frac{1}{m}}\right)^{m-1}-1\right)y_1\left(|F'(y_1^*)|+\frac{F''(y_1^*)}{|F'(y_1^*)|}\right).$$

A calculation similar to the one for the function $\mathcal{F}$ gives

$$\begin{aligned}&\left(1+\left(\ln\left[c(r_0)\frac{\left|F'\left(y_1^*\right)\right|}{f\left(y_1^*\right)}\right]\right)^{\frac{1}{m}}\right)^m-\ln\left[c(r_0)\frac{\left|F'\left(y_1^*\right)\right|}{f\left(y_1^*\right)}\right]\\&=\frac{\left(1+\frac{\left|F'\left(y_1^*\right)\right|}{y_1^*|F'(y_1^*)|^2+y_1^*F''(y_1^*)}\right)^{\frac{m}{m-1}}-1}{\left(\left(1+\frac{\left|F'\left(y_1^*\right)\right|}{y_1^*|F'(y_1^*)|^2+y_1^*F''(y_1^*)}\right)^{\frac{1}{m-1}}-1\right)^m}\\&\le C_m\left(\frac{y_1^*|F'(y_1^*)|^2+y_1^*F''(y_1^*)}{\left|F'\left(y_1^*\right)\right|}\right)^{m-1}\le\tilde{C}_m\left(1+\frac{|F'(y_1^*)|^2}{F''(y_1^*)}\right)^{m-1},\end{aligned}$$

where we used $|F'(r)/F''(r)|\approx r$. From this and the monotonicity condition we obtain

$$I_{\mathcal{R}_2}\lesssim\mathcal{G}(y_1)\le C_m\left|F'\left(r_0\right)\right|\varphi\left(r_0\right),$$

which concludes the estimate for the region $\mathcal{R}_2$. □

Now we turn to the proof of Corollary 5.7.

Proof of Corollary 5.7. We must first check that the monotonicity property (79) holds for the indicated geometries $F_{k,\sigma}$, where

$$\begin{aligned}f\left(r\right)&=f_{k,\sigma}\left(r\right)\equiv\exp\left\{-\left(\ln\frac{1}{r}\right)\left(\ln^{(k)}\frac{1}{r}\right)^{\sigma}\right\};\\F\left(r\right)&=F_{k,\sigma}\left(r\right)\equiv\left(\ln\frac{1}{r}\right)\left(\ln^{(k)}\frac{1}{r}\right)^{\sigma}.\end{aligned}$$

Consider first the case $k=1$. Then $F\left(r\right)=F_{1,\sigma}\left(r\right)=\left(\ln\frac{1}{r}\right)^{1+\sigma}$ satisfies

$$F'\left(r\right)=-\left(1+\sigma\right)\frac{\left(\ln\frac{1}{r}\right)^{\sigma}}{r}\text{ and }F''\left(r\right)=-\left(1+\sigma\right)\left\{-\frac{\left(\ln\frac{1}{r}\right)^{\sigma}}{r^2}-\sigma\frac{\left(\ln\frac{1}{r}\right)^{\sigma-1}}{r^2}\right\},$$

which shows that

$$\varphi(r) = \frac{1}{1+\sigma}\exp\left\{-\ln\frac{1}{r} - \sigma\ln\ln\frac{1}{r} + C_m\left(\frac{(1+\sigma)^2\frac{\left(\ln\frac{1}{r}\right)^{2\sigma}}{r^2}}{(1+\sigma)\left\{\frac{\left(\ln\frac{1}{r}\right)^{\sigma}}{r^2} + \sigma\frac{\left(\ln\frac{1}{r}\right)^{\sigma-1}}{r^2}\right\}} + 1\right)^{m-1}\right\}$$

$$= \frac{1}{1+\sigma}\exp\left\{-\ln\frac{1}{r} - \sigma\ln\ln\frac{1}{r} + C_m(1+\sigma)^{m-1}\left(\frac{\left(\ln\frac{1}{r}\right)^{\sigma}}{\left\{1+\sigma\frac{1}{\ln\frac{1}{r}}\right\}} + \frac{1}{1+\sigma}\right)^{m-1}\right\},$$

is increasing in r provided both $\sigma(m-1) < 1$ and $0 \le r \le \alpha_{m,\sigma}$, where $\alpha_{m,\sigma}$ is a positive constant depending only on m and σ. Hence we have the upper bound

$$\varphi(r) \le \exp\left\{-\ln\frac{1}{r} + C_m\left(\ln\frac{1}{r}\right)^{\sigma(m-1)}\right\} = r^{1-C_m\frac{1}{\left(\ln\frac{1}{r}\right)^{1-\sigma(m-1)}}}, \qquad 0 \le r \le \beta_{m,\sigma},$$

where $\beta_{m,\sigma} > 0$ is chosen even smaller than $\alpha_{m,\sigma}$ if necessary.

Thus in the case $\Phi = \Phi_m$ with $m > 2$ and $F = F_\sigma$ with $0 < \sigma < \frac{1}{m-1}$, we see that the norm $\varphi(r_0)$ of the Sobolev embedding satisfies

$$\varphi(r_0) \le r_0^{1-C_m\frac{1}{\left(\ln\frac{1}{r_0}\right)^{1-\sigma(m-1)}}}, \qquad \text{for } 0 < r_0 \le \beta_{m,\sigma},$$

and hence that

$$\frac{\varphi(r_0)}{r_0} \le \left(\frac{1}{r_0}\right)^{\frac{C_m}{\left(\ln\frac{1}{r_0}\right)^{1-\sigma(m-1)}}} \qquad \text{for } 0 < r_0 \le \beta_{m,\sigma}.$$

Now consider the case $k \ge 2$. Our first task is to show that $F_{k,\sigma}$ satisfies the structure conditions in Definition 5.1. Only condition (5) is not obvious, so we now turn to that. We have $F(r) = F_{k,\sigma}(r) = \left(\ln\frac{1}{r}\right)\left(\ln^{(k)}\frac{1}{r}\right)^{\sigma}$ satisfies

$$F'(r) = -\frac{\left(\ln^{(k)}\frac{1}{r}\right)^{\sigma}}{r} - \left(\ln\frac{1}{r}\right)\frac{\sigma\left(\ln^{(k)}\frac{1}{r}\right)^{\sigma-1}}{\left(\ln^{(k-1)}\frac{1}{r}\right)\left(\ln^{(k-2)}\frac{1}{r}\right)\dots\left(\ln\frac{1}{r}\right)r}$$

$$= -\frac{\left(\ln^{(k)}\frac{1}{r}\right)^{\sigma}}{r} - \frac{\sigma\left(\ln^{(k)}\frac{1}{r}\right)^{\sigma-1}}{\left(\ln^{(k-1)}\frac{1}{r}\right)\left(\ln^{(k-2)}\frac{1}{r}\right)\dots\left(\ln^{(2)}\frac{1}{r}\right)r}$$

$$= -\frac{\left(\ln^{(k)}\frac{1}{r}\right)^{\sigma}}{r}\left\{1 + \frac{\sigma}{\left(\ln^{(k)}\frac{1}{r}\right)\left(\ln^{(k-1)}\frac{1}{r}\right)\left(\ln^{(k-2)}\frac{1}{r}\right)\dots\left(\ln^{(2)}\frac{1}{r}\right)}\right\}$$

$$= -\frac{F(r)}{r\ln\frac{1}{r}}\left\{1 + \frac{\sigma}{\left(\ln^{(k)}\frac{1}{r}\right)\left(\ln^{(k-1)}\frac{1}{r}\right)\dots\left(\ln^{(2)}\frac{1}{r}\right)}\right\}$$

$$\equiv -\frac{F(r)\Lambda_k(r)}{r\ln\frac{1}{r}},$$

and

$$F''(r) = -\frac{F'(r)\Lambda_k(r)}{r\ln\frac{1}{r}} - \frac{F(r)\Lambda_k'(r)}{r\ln\frac{1}{r}} - F(r)\Lambda_k(r)\frac{d}{dr}\left(\frac{1}{r\ln\frac{1}{r}}\right)$$

$$= -\frac{F'(r)\Lambda_k(r)}{r\ln\frac{1}{r}} - \frac{F(r)\Lambda_k'(r)}{r\ln\frac{1}{r}} + \frac{F(r)\Lambda_k(r)}{r^2\ln\frac{1}{r}}\left(1 - \frac{1}{\ln\frac{1}{r}}\right),$$

where

$$\Lambda_k'(r) = \frac{d}{dr}\left(\frac{\sigma}{\left(\ln^{(k)}\frac{1}{r}\right)\left(\ln^{(k-1)}\frac{1}{r}\right)\dots\left(\ln^{(2)}\frac{1}{r}\right)}\right)$$

$$= -\sigma \sum_{j=2}^{k} \frac{\left(\ln^{(j)} \frac{1}{r}\right)}{\left(\ln^{(k)} \frac{1}{r}\right) \dots \left(\ln^{(2)} \frac{1}{r}\right)} \frac{1}{\left(\ln^{(j-1)} \frac{1}{r}\right) \dots \left(\ln \frac{1}{r}\right) r}$$

$$= -\sigma \frac{1}{\left(\ln^{(k)} \frac{1}{r}\right) \dots \left(\ln^{(2)} \frac{1}{r}\right) r} \sum_{j=2}^{k} \frac{\ln^{(j)} \frac{1}{r}}{\left(\ln^{(j-1)} \frac{1}{r}\right) \dots \left(\ln \frac{1}{r}\right)}$$

$$= -\sigma \frac{1}{\left(\ln^{(k)} \frac{1}{r}\right) \dots \left(\ln^{(2)} \frac{1}{r}\right) r} \left(\frac{\ln^{(2)} \frac{1}{r}}{\ln \frac{1}{r}} + \sum_{j=3}^{k} \frac{\ln^{(j)} \frac{1}{r}}{\left(\ln^{(j-1)} \frac{1}{r}\right) \dots \left(\ln \frac{1}{r}\right)} \right)$$

$$= -\sigma \frac{1}{\left(\ln^{(k)} \frac{1}{r}\right) \dots \left(\ln^{(2)} \frac{1}{r}\right)\left(\ln \frac{1}{r}\right) r} \left(\ln^{(2)} \frac{1}{r} + \sum_{j=3}^{k} \frac{\ln^{(j)} \frac{1}{r}}{\left(\ln^{(j-1)} \frac{1}{r}\right) \dots \left(\ln^{(2)} \frac{1}{r}\right)} \right).$$

Now

$$\ln^{(2)} \frac{1}{r} + \sum_{j=3}^{k} \frac{\ln^{(j)} \frac{1}{r}}{\left(\ln^{(j-1)} \frac{1}{r}\right) \dots \left(\ln^{(2)} \frac{1}{r}\right)} \approx \ln^{(2)} \frac{1}{r},$$

and so

$$-\Lambda_k'(r) \approx \begin{cases} \frac{\sigma}{\left(\ln \frac{1}{r}\right) r} & \text{for} \quad k=2 \\ \frac{\sigma}{\left(\ln^{(k)} \frac{1}{r}\right) \dots \left(\ln^{(3)} \frac{1}{r}\right)\left(\ln \frac{1}{r}\right) r} & \text{for} \quad k \geq 3 \end{cases}.$$

We also have $\Lambda_k(r) \approx 1$, which then gives

$$-F'(r) \approx \frac{F(r)}{r \ln \frac{1}{r}},$$

and

$$F''(r) \approx \frac{F(r)}{r^2 \left(\ln \frac{1}{r}\right)^2} + \frac{\sigma F(r)}{\left(\ln^{(k)} \frac{1}{r}\right) \dots \left(\ln^{(3)} \frac{1}{r}\right)\left(\ln \frac{1}{r}\right)^2 r^2} + \frac{F(r)}{r^2 \ln \frac{1}{r}} \approx \frac{F(r)}{r^2 \ln \frac{1}{r}}.$$

From these two estimates we immediately obtain structure condition (5) of Definition 5.1.

We also have

$$\frac{\left|F'(r)\right|^2}{F''(r)} \approx \frac{F(r)^2}{\left(r \ln \frac{1}{r}\right)^2} \frac{r^2 \ln \frac{1}{r}}{F(r)} = \frac{F(r)}{\ln \frac{1}{r}} = \left(\ln^{(k)} \frac{1}{r}\right)^{\sigma}, \qquad 0 \leq r \leq \beta_{m,\sigma},$$

and then from the definition of $\varphi(r) \equiv \frac{1}{|F'(r)|} e^{C_m \left(\frac{|F'(r)|^2}{F''(r)} + 1\right)^{m-1}}$ in (79), we obtain

$$\varphi(r) = \frac{1}{|F'(r)|} e^{C_m \left(\frac{|F'(r)|^2}{F''(r)} + 1\right)^{m-1}} \approx r \frac{e^{C_m \left(\ln^{(k)} \frac{1}{r}\right)^{\sigma(m-1)}}}{\left(\ln^{(k)} \frac{1}{r}\right)^{\sigma}}$$

$$\lesssim r e^{C_m \left(\ln^{(k)} \frac{1}{r}\right)^{\sigma(m-1)}} \approx r^{1 - C_m \frac{\left(\ln^{(k)} \frac{1}{r}\right)^{\sigma(m-1)}}{\ln \frac{1}{r}}}, \qquad 0 \leq r \leq \beta_{m,\sigma}.$$

This completes the proof of the monotonicity property (79) and the estimates for $\varphi(r)$ for each of the two cases in Corollary 5.7.

Finally, we must show that the standard (Φ, φ) -Sobolev inequality (75) with Φ as in (34), $m > 1$, fails if $k = 1$ and $\sigma > \frac{1}{m-1}$, and for this it is convenient to use the identity $\left|\nabla_A v\right| = \left|\frac{\partial v}{\partial r}\right|$ for radial functions v, see ([9], Appendix C.). Take $f(r) = f_{1,\sigma}(r) = r^{\left(\ln \frac{1}{r}\right)^{\sigma}}$ and with $\eta(r) \equiv \begin{cases} 1 & \text{if} \quad 0 \leq r \leq \frac{r_0}{2} \\ 2\left(1 - \frac{r}{r_0}\right) & \text{if} \quad \frac{r_0}{2} \leq r \leq r_0 \end{cases}$, we define the radial function

$$w(x,y) = w(r) = e^{\left(\ln \frac{1}{r}\right)^{\sigma+1}} = \frac{\eta(r)}{f(r)}, \qquad 0 < r < r_0.$$

From $\left|\nabla_A r\right| = 1$, we obtain the equality $\left|\nabla_A w(x,y)\right| = \left|\nabla_A r\right| \left|w'(r)\right| = \left|w'(r)\right|$, and combining this with $\left|\nabla_A \eta(r)\right| \leq \frac{2}{r_0} \mathbf{1}_{\left[\frac{r_0}{2}, r_0\right]}$ and the estimate (7.8) from [9], we have

$$\int \int_{B(0,r_0)} \left|\nabla_A w(x,y)\right| dx dy \lesssim \int_0^{r_0} \left|w'(r)\right| \frac{f(r)}{|F'(r)|} dr + \frac{2}{r_0} \int_{\frac{r_0}{2}}^{r_0} \frac{1}{|F'(r)|} dr$$

$$\approx \int_0^{r_0} \frac{f'(r)}{f(r)^2} \frac{f(r)^2}{f'(r)} dr + \frac{2}{r_0} \int_{\frac{r_0}{2}}^{r_0} C r dr \approx r_0 .$$

On the other hand, $\Phi_m(t) \geq t^{1+\frac{m}{(\ln t)^{\frac{1}{m}}}}$ and $|F'(r)| = (\sigma+1)\left(\ln \frac{1}{r}\right)^{\sigma} \frac{1}{r}$, so we obtain

$$\begin{aligned}
&\int\int_{B(0,r_0)} \Phi_m(w(x,y))\, dx dy \\
&\gtrsim \int_0^{\frac{r_0}{2}} \Phi_m\left(\frac{1}{f(r)}\right) \frac{f(r)}{|F'(r)|} dr \geq \int_0^{\frac{r_0}{2}} \left(\frac{1}{f(r)}\right)^{1+\frac{m}{F(r)^{\frac{1}{m}}}} \frac{f(r)}{|F'(r)|} dr \\
&\approx \int_0^{\frac{r_0}{2}} f(r)^{-\frac{m}{\left(\ln \frac{1}{r}\right)^{\frac{\sigma}{m}}}} \frac{1}{\left(\ln \frac{1}{r}\right)^{\sigma} \frac{1}{r}} dr = \int_0^{\frac{r_0}{2}} e^{m\left(\ln \frac{1}{r}\right)^{(\sigma+1)\left(1-\frac{1}{m}\right)}} \frac{r dr}{\left(\ln \frac{1}{r}\right)^{\sigma}} = \infty
\end{aligned}$$

if $(\sigma+1)\left(1-\frac{1}{m}\right) > 1$, i.e. $\sigma > \frac{1}{m-1}$. This finishes the proof of Corollary 5.7. □

5.2. Proof of the geometric theorem

In this section we prove the geometric Theorem 1.1 as consequence of the abstract Theorem 1.6 and of the geometric Orlicz-Sobolev inequality established in Section 5.1.

Proof of Theorem 1.1. Theorem 1.1 is a consequence of the abstract Theorem 1.6 and the geometric results described in Section 5.1, once we show that under the hypotheses of Theorem 1.1 conditions (i), (ii), and (iii) of Theorem 1.6 are satisfied. It suffices to consider the case that u is a weak *subsolution* of (1) in Ω, with right hand side pair as in condition (1) of Theorem 1.1.

Since $\phi_0 \in L^{\Phi^*}(B(0,r))$, and $\vec{\phi}_1 \in L^{\infty}(B(0,r))$, then the pair $\left(\phi_0, \vec{\phi}_1\right)$ is strongly A-admissible at $(0,r)$ by Proposition 3.3, so condition (i) from Theorem 1.6 holds.

Since the matrix $A(x)$ in (5) is elliptic away from the line $x_1 = 0$ and it is independent of the second variable x_2, it suffices to prove the theorem for a ball $B\left(0,r_0\right) \Subset \Omega$. By Corollary 5.7 in Section 5.1, when $k=1$ and $0<\sigma<\frac{1}{m-1}$ or $k \geq 2$ and $\sigma > 0$, we have that there exists $0 < r_0 = r_0(m,\sigma)$ such that the single scale (Φ, A, φ)-Orlicz-Sobolev bump inequality (9) holds with $\Phi = \Phi_m$ at $(0,r)$ for some $m>2$ and superradius $\varphi(r)$ given by

$$\frac{\varphi(r)}{r} = \exp\left(C_m\left(\ln^{(k)} \frac{1}{r}\right)^{\sigma(m-1)}\right), \qquad \text{for } 0 < r \leq r_0 \leq 1. \tag{83}$$

Hence condition (ii) from Theorem 1.6 is satisfied because of condition (2) of Theorem 1.1.

Finally, given $B\left(x, r_0\right) \Subset \Omega$, the existence of an (A,d)-*standard* accumulating sequence of Lipschitz cutoff functions at $\left(x, r_0\right)$ follows directly from Lemma 5.3 above, so condition (iii) from Theorem 1.6 holds. Therefore, applying Theorem 1.6, u is *locally bounded above* in Ω. □

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