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Asymptotic behaviour of non-radiative solution to the wave equations

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Abstract

In this work we consider weakly non-radiative solutions to both linear and non-linear wave equations. We first characterize all weakly non-radiative free waves, without the radial assumption. Then in dimension 3 we show that the asymptotic behaviours of non-radiative solutions to a wide range of nonlinear wave equations are similar to those of non-radiative free waves. This generalizes the already known results about radial solutions to the non-radial case.
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1. Introduction

1.1. Background and main topics

Channel of energy The channel of energy method plays an important role in the study of asymptotic behaviour of solutions to non-linear wave equations in the past decade. This method mainly discusses the distribution of energy as time tends to infinity. More precisely, if u is a solution to either linear or non-linear wave equation defined for all time, then the following limits are considered for a given constant R.

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$$\lim_{t\to\pm\infty}\int_{|x|>R+|t|}|\nabla_{t,x}u(x,t)|^2\mathrm{d}x.$$

Here for convenience we use the notation $\nabla_{t,x}u=(u_t,\nabla u)$. This theory was first established for solutions to homogeneous linear wave equation, i.e. free waves, then applied to the study of non-linear wave equations. Please see, for instance, Côte-Kenig-Schlag [3], Duyckaerts-Kenig-Merle [4,8] and Kenig et al. [15,16] for linear theory; and Duyckaerts-Kenig-Merle [6,10] for the applications of the channel of energy on soliton resolution of the focusing, energy-critical wave equations.

Non-radiative solutions A crucial part of the channel of energy theory is to discuss the property of non-radiative solutions. Let u be a solution to the wave equation with a finite energy. We call it a non-radiative solution if and only if

$$\lim_{t\to\pm\infty}\int_{|x|>|t|}|\nabla_{t,x}u(x,t)|^2\mathrm{d}x=0.$$

We may also consider a more general case. Let $R\geq 0$ be a constant. We call a solution u to be R-weakly non-radiative if and only if

$$\lim_{t\to\pm\infty}\int_{|x|>R+|t|}|\nabla_{t,x}u(x,t)|^2\mathrm{d}x=0.$$

Let us first consider (weakly) non-radiative solutions to the homogeneous linear wave equation in $\mathbb{R}^d$. It has been proved that any non-radiative free wave must be zero, see Duyckaerts-Kenig-Merle [5,8]. All radial weakly non-radiative free waves have also been well understood. The following result was first proved for odd dimensions $d\geq 3$ by Kenig et al. [16] then generalized to the even dimensions $d\geq 2$ in Li-Shen-Wei [17].

Proposition 1.1 *(Radial weakly non-radiative solutions). Let $d\geq 2$ be an integer and $R>0$ be a constant. If initial data $(u_0,u_1)\in\dot{H}^1\times L^2$ are radial, then the corresponding solution to the homogeneous linear wave equation u is R-weakly non-radiative, i.e.*

$$\lim_{t\to\pm\infty}\int_{|x|>|t|+R}|\nabla_{t,x}u(x,t)|^2\mathrm{d}x=0,$$

if and only if the restriction of (u_0,u_1) in the exterior region $\{x\in\mathbb{R}^d:|x|>R\}$ is contained in

$$Span\left\{(r^{2k_1-d},0),(0,r^{2k_2-d}):1\leq k_1\leq\left\lfloor\frac{d+1}{4}\right\rfloor,1\leq k_2\leq\left\lfloor\frac{d-1}{4}\right\rfloor\right\}.$$

Here the notation $\lfloor q\rfloor$ represents the integer part of q. In particular, all radial R-weakly non-radiative solution in dimension 2 are supported in $\{(x,t):|x|\leq|t|+R\}$.

Goals of this work The aim of this paper is two-fold. The first goal of this paper is to characterize all (possibly non-radial) initial data so that the corresponding solutions to free wave equation are R-weakly non-radiative. For convenience we define

$$P(R) \doteq \left\{ (u_0, u_1) \in \dot{H}^1 \times L^2(\mathbb{R}^d) : \lim_{t \to \pm\infty} \int_{|x|>R+|t|} |\nabla_{t,x} \mathbf{S}_L(u_0, u_1)|^2 \mathrm{d}x = 0 \right\}.$$

Here $\mathbf{S}_L(u_0, u_1)$ is the corresponding solution of the free wave equation with given initial data (u_0, u_1). We will give a decomposition of every element $(u_0, u_1) \in P(R)$ in term of spherical harmonic functions, whose details are given in Section 2.

The second goal is to show that in the 3-dimensional case any weakly non-radiative solution to a wide range of non-linear wave equations share the same asymptotic behaviour as weakly non-radiative free waves, as given in Section 3. This kind of result has been proved in all odd dimensions $d \geq 3$ in the radial case of focusing wave equation by Duyckaerts-Kenig-Merle [8], as a first step to prove the soliton resolution of solutions. In this work we give the first result of this kind in the non-radial setting, as far as the authors know. Our argument depends on suitable decay estimates of weakly non-radiative free waves in the exterior region $\{(x,t) : |x| > |t| + R\}$, as given in (12)-(15). The decay estimates of this kind can be verified via a direct calculation for radial non-radiative solutions, whose initial data are given explicitly in Proposition 1.1. Although we expect that a similar decay estimate holds for all non-radial non-radiative solutions in all dimensions $d \geq 2$ as well, this has been proved only in dimension 3, and recently in odd dimensions $d \geq 5$, as far as the author knows. For more details of the decay estimates, please refer to our work [18,21]. For convenience we restrict our discussion of nonlinear non-radiative solutions to dimension 3 in this work, although a similar argument works for all odd dimensions $d \geq 5$ as well. Please note that the even dimensions are much more difficult, due to the presence of a Hilbert transform in the symmetry of the radiation profiles. More details can be found in Section 3.

Our ultimate goal is to characterize all non-radiative solutions to the nonlinear wave equation, especially for the classic focusing wave equation

$$\partial_t^2 u - \Delta u = |u|^{\frac{4}{d-2}} u.$$

This characterization is highly related to the soliton resolution conjecture of this energy-critical wave equation. We guess that all non-radiative solutions to this equation are solitary waves. But it is clearly out of our reach at this time. The author would like to mention that a very recent work by Côte-Laurent [2] shows that the set of small non-radiative solutions is a manifold whose tangent space at zero is exactly the space of all non-radiative free waves.

Notation. In this work the notation $A \lesssim B$ implies that there exists a constant c, such that the inequality $A \leq cB$ holds. A subscript to the notation $\lesssim$ means that the implicit constant c depends on nothing but the given subscript.

2. The characteristics of $P(R)$

In this section we give an explicit expression of the element in the space $P(R)$. We use spherical harmonics and follow a similar argument to Duyckaerts-Kenig-Merle [9]. Let us first give a brief review on some basic properties of spherical harmonics. We recall that the eigenfunctions of the Laplace-Beltrami operator on $\mathbb{S}^{d-1}$ are exactly the homogeneous harmonic polynomials of the variables $x_1, x_2, \cdots, x_d$. Such a polynomial Φ of degree ν satisfies

$$-\Delta_{\mathbb{S}^{d-1}}\Phi = \nu(\nu + d - 2)\Phi.$$

We choose a Hilbert basis $\{\Phi_k(\theta)\}_{k\geq 0}$ of the operator $-\Delta_{\mathbf{S}^{d-1}}$ on the sphere $\mathbb{S}^{d-1}$. Here we assume that the harmonic polynomial Φ_k is of degree ν_k. In particular we assume $\nu_0 = 0$ and $\nu_k > 0$ if $k \geq 1$. Next we give the statement of our first main result. We start by the odd dimensional case and then deal with the even dimensional case. Please note that a similar result for odd dimensions has been proved in Côte-Laurent [1] by the Radon transform. The novelty of our result includes

- We give an L^2 decay estimate of $\partial_r u_0$ near infinity in addition;
- The argument works for even dimensions as well, with minor modifications.

Our main result of this section is

Proposition 2.1. *Assume that $d \geq 3$ is an integer. Let $\mu, \tilde{\mu}$ to be two integers and ρ be a measure on $\mathbb{R}^+$ as given below*

$$(\mu, \tilde{\mu}) = \begin{cases} \left(\frac{d-1}{2}, \frac{d-1}{2}\right), & n \text{ odd}; \\ \left(\frac{d}{2}, \frac{d-2}{2}\right), & n \text{ even}. \end{cases} \qquad \mathrm{d}\rho(z) = \begin{cases} \mathrm{d}z, & n \text{ odd}; \\ z\mathrm{d}z, & n \text{ even}. \end{cases}$$

Then $(u_0, u_1) \in P(R)$ if and only if there exist two sequences of polynomials $\{P_k(z)\}_{k\geq 0}$ and $\{Q_k(z)\}_{k\geq 0}$ in the following form (A_{k,k_1}, B_{k,k_2} are constants)

$$P_k(z) = \sum_{1\leq k_1 \leq \lfloor \frac{\tilde{\mu}+\nu_k+1}{2} \rfloor} A_{k,k_1} z^{\tilde{\mu}+1+\nu_k-2k_1}; \qquad Q_k(z) = \sum_{1\leq k_2 \leq \lfloor \frac{\tilde{\mu}+\nu_k}{2} \rfloor} B_{k,k_2} z^{\tilde{\mu}+\nu_k-2k_2};$$

with

$$\sum_{k=0}^{\infty} \int_0^{1/R} \left(\nu_k(d-2+\nu_k)\,|P_k(z)|^2 + |zP_k'(z)|^2 + |Q_k(z)|^2\right) \mathrm{d}\rho(z) < +\infty;$$

such that (r, θ are the spherical coordinates)

$$u_0(r\theta) = \sum_{k=0}^{\infty} r^{-\mu} P_k(1/r)\Phi_k(\theta), \quad r > R; \quad u_1(r\theta) = \sum_{k=0}^{\infty} r^{-\mu-1} Q_k(1/r)\Phi_k(\theta), \quad r > R. \tag{1}$$

Here the first identity holds for every fixed $r > R$ in the sense of $L^2(\mathbb{S}^{d-1})$ convergence. The second one holds in the sense of $L^2(\{x : |x| > R\})$ convergence. In addition, we have

(i) *The derivative of u_0 can be given by*

$$\nabla_x u_0(r,\theta) = \sum_{k=0}^{\infty} r^{-\mu-1} \left\{ P_k(1/r)\nabla_\theta \Phi_k(\theta) - [\mu P_k(1/r) + (1/r)P_k'(1/r)]\Phi_k(\theta)\theta \right\}.$$

This identities holds in the sense of $L^2(\{x : |x| > R\})$ convergence. Here $\nabla_\theta \Phi_k(\theta)$ can be viewed as a vector in $\mathbb{R}^d$ in a natural way: We let $\varphi_k(x) = \Phi_k(x/|x|)$ be a function defined in $\mathbb{R}^d \setminus \{0\}$ and let $\nabla_\theta \Phi_k(\theta) = (\nabla_x \varphi_k)(\theta) \in \mathbb{R}^d$ for any $\theta \in \mathbb{S}^{d-1} \subset \mathbb{R}^d$.

(ii) *The norms of (u_0, u_1) can be determined by the integrals of $P_k(z)$ and $Q_k(z)$'s:*

$$\|\not\nabla u_0\|^2_{L^2(\{x:|x|>R\})} = \sum_{k=1}^{\infty} \nu_k(d-2+\nu_k) \int_0^{1/R} |P_k(z)|^2 \,\mathrm{d}\rho(z);$$

$$\|u_1\|^2_{L^2(\{x:|x|>R\})} = \sum_{k=0}^{\infty} \int_0^{1/R} |Q_k(z)|^2 \,\mathrm{d}\rho(z);$$

$$\|\partial_r u_0\|^2_{L^2(\{x:|x|>R\})} = \sum_{k=0}^{\infty} \int_0^{1/R} |zP_k'(z) + \mu P_k(z)|^2 \mathrm{d}\rho(z) < +\infty.$$

(iii) *The derivative $\partial_r u_0$ satisfies the following decay estimates ($R_1 \geq 2R$)*

$$\int_{|x|>R_1} |\partial_r u_0(x)|^2 \mathrm{d}x \lesssim (R/R_1) \int_{|x|>R} |\nabla u_0(x)|^2 \mathrm{d}x;$$

$$\int_{|x|>R_1} |\partial_r u_0^*(x)|^2 \mathrm{d}x \lesssim (R/R_1) \int_{|x|>R} |\not\nabla u_0(x)|^2 \mathrm{d}x.$$

Here u_0^* is the non-radial part of u_0 defined by $u_0^* = u_0 - r^{-\mu} P_0(1/r)\Phi_0$ and $\not\nabla u_0$ is the non-radial derivative of u_0 defined by

$$\not\nabla u_0 = \nabla u_0 - \left(\frac{x}{|x|} \cdot \nabla u_0\right) \frac{x}{|x|}.$$

Proof. The rest of this subsection is devoted to the proof of this proposition. The proof for odd and even dimensions follows the same procedure, with minor modifications in details. We give the proof for odd dimensions in details and brief explains the difference in even dimensions at the end of this chapter. When n is odd, we have $\mu = \tilde{\mu} = (d-1)/2$ and $\mathrm{d}\rho(z) = \mathrm{d}z$. For convenience we will substitute $\tilde{\mu}$ by μ in this case. The proof consists of three major steps. Step one, we first show that any element in $P(R)$ can be written as in (1). Step two, we show any initial data given by (1) is indeed contained in $P(R)$. Finally in Step three we prove the identities and inequalities in the proposition.

Step one Let us assume that $u = \mathbf{S}_L(u_0, u_1)$ is an R-weakly non-radiative free wave and show that the initial data (u_0, u_1) have a decomposition as in (1). We start by defining

$$w_k(r,t) = r^{-\nu_k} \int_{\mathbb{S}^{d-1}} u(r\theta, t)\Phi_k(\theta)\mathrm{d}\theta.$$

Let $\Box = \partial_t^2 - \partial_r^2 - \frac{d+2\nu_k-1}{r}\partial_r$. A straight-forward calculation shows

$$\begin{aligned}
\Box w_k &= (\Box r^{-\nu_k}) \int_{\mathbb{S}^{d-1}} u(r\theta,t)\Phi_k(\theta)\mathrm{d}\theta + r^{-\nu_k}\int_{\mathbb{S}^{d-1}} \Box u(r\theta,t)\Phi_k(\theta)\mathrm{d}\theta \\
&\qquad - 2\partial_r(r^{-\nu_k}) \int_{\mathbb{S}^{d-1}} \partial_r u(r\theta,t)\Phi_k(\theta)\mathrm{d}\theta \\
&= r^{-\nu_k}\int_{\mathbb{S}^{d-1}} \left(\partial_t^2 - \partial_r^2 - \frac{d-1}{r}\partial_r\right) u(r\theta,t)\Phi_k(\theta)\mathrm{d}\theta + \nu_k(d-2+\nu_k)r^{-2}w_k \\
&= r^{-\nu_k}\int_{\mathbb{S}^{d-1}} r^{-2}\Delta_{\mathbf{S}^{d-1}} u(r\theta,t)\Phi_k(\theta)\mathrm{d}\theta + \nu_k(d-2+\nu_k)r^{-2}w_k \\
&= r^{-\nu_k-2}\int_{\mathbb{S}^{d-1}} u(r\theta,t)\Delta_{\mathbf{S}^{d-1}}\Phi_k(\theta)\mathrm{d}\theta + \nu_k(d-2+\nu_k)r^{-2}w_k \\
&= 0.
\end{aligned}$$

Thus if w_k is viewed as a radial function defined on $\mathbb{R}^{d+2\nu_k}$, it satisfies the free wave equation

$$\partial_t^2 w_k - \Delta_{\mathbb{R}^{d+2\nu_k}} w_k = 0, \qquad |x| > 0.$$

In addition, we may apply an integration by parts and deduce

$$\begin{aligned}
&\int_{R+|t|}^{\infty} \left(|\partial_r w_k(r,t)|^2 + |\partial_t w_k(r,t)|^2\right) r^{d+2\nu_k-1}\mathrm{d}r \\
&\quad = \int_{R+|t|}^{\infty} \left(\left|\int_{\mathbb{S}^{d-1}} \partial_r u(r\theta,t)\Phi_k(\theta)\mathrm{d}\theta\right|^2 + \left|\int_{\mathbb{S}^{d-1}} \partial_t u(r\theta,t)\Phi_k(\theta)\mathrm{d}\theta\right|^2\right) r^{d-1}\mathrm{d}r \\
&\qquad + \int_{R+|t|}^{\infty} \nu_k(\nu_k+d-2)\left|\int_{\mathbb{S}^{d-1}} u(r\theta,t)\Phi_k(\theta)\mathrm{d}\theta\right|^2 r^{d-3}\mathrm{d}r
\end{aligned}$$

$$+ \nu_k (R+|t|)^{d-2} \left| \int_{\mathbb{S}^{d-1}} u((R+|t|)\theta, t)\Phi_k(\theta) \mathrm{d}\theta \right|^2$$

$$\lesssim \int_{R+|t|}^{\infty} \left(\int_{\mathbb{S}^{d-1}} |\partial_r u(r\theta, t)|^2 \mathrm{d}\theta + \int_{\mathbb{S}^{d-1}} |\partial_t u(r\theta,t)|^2 \mathrm{d}\theta + r^{-2} \int_{\mathbb{S}^{d-1}} |u(r\theta,t)|^2 \mathrm{d}\theta \right) r^{d-1} \mathrm{d}r$$

$$+ \nu_k (R+|t|)^{d-2} \int_{\mathbb{S}^{d-1}} |u((R+|t|)\theta, t)|^2 \mathrm{d}\theta$$

$$\lesssim \int_{|x|>R+|t|} \left(|\nabla_{x,t} u(x,t)|^2 + \frac{|u(x,t)|^2}{|x|^2} \right) \mathrm{d}x + \frac{1}{R+|t|} \int_{|x|=R+|t|} |u(x,t)|^2 \mathrm{d}S.$$

This upper bound converges to zero as t tends to infinity, by a combination of the non-radiative assumption of u, Hardy's inequality outside a ball and the following estimate (see, for instance, Lemma 7.1.1 in [19])

$$\frac{1}{r} \int_{|x|=r} |u(x,t)|^2 \mathrm{d}S \lesssim \int_{|x|>r} |\nabla u(x)|^2 \mathrm{d}x, \qquad \forall\, u \in \dot{H}^1(\mathbb{R}^d).$$

Therefore w_k is also a weakly non-radiative solution. According to the explicit expression of radial non-radiative solutions given in Proposition 1.1, there exist constants A_{k,k_1} and B_{k,k_2}, such that

$$w_k(r,0) = \sum_{1 \leq k_1 \leq \frac{\mu+\nu_k+1}{2}} A_{k,k_1} r^{-d-2\nu_k+2k_1} = r^{-\mu-\nu_k} P_k(1/r);$$

$$\partial_t w_k(r,0) = \sum_{1 \leq k_2 \leq \frac{\mu+\nu_k}{2}} B_{k,k_2} r^{-d-2\nu_k+2k_2} = r^{-\mu-\nu_k-1} Q_k(1/r).$$

Here $P_k(z)$ and $Q_k(z)$ are polynomials as given in Proposition 2.1. Therefore we have

$$\int_{\mathbb{S}^{d-1}} u_0(r\theta)\Phi_k(\theta)\mathrm{d}\theta = r^{-\mu} P_k(1/r); \tag{2}$$

$$\int_{\mathbb{S}^{d-1}} u_1(r\theta)\Phi_k(\theta)\mathrm{d}\theta = r^{-\mu-1} Q_k(1/r). \tag{3}$$

Next we show the polynomials satisfy the inequalities given in Proposition 2.1. We have

$$\int_{\mathbb{S}^{d-1}} \nabla_\theta u_0(r\theta) \nabla_\theta \Phi_k(\theta) \mathrm{d}\theta = - \int_{\mathbb{S}^{d-1}} u(r\theta, 0) \Delta_{\mathbb{S}^{d-1}} \Phi_k(\theta) \mathrm{d}\theta$$
$$= \nu_k (d-2+\nu_k) r^{-\mu} P_k(1/r).$$

Since $\nabla_\theta \Phi_k$ are orthogonal to each other with $L^2(\mathbb{S}^{d-1})$ norm $\sqrt{\nu_k(d-2+\nu_k)}$, we have

$$\sum_{k=1}^{\infty} \nu_k(d-2+\nu_k) r^{-2\mu} |P_k(1/r)|^2 \leq \|\nabla_\theta u_0(r\theta)\|^2_{L^2(\mathbb{S}^{d-1})}.$$

A combination of this with the inequality

$$\|\nabla_\theta u_0(r\theta)/r\|_{L^2\{x:|x|>R\}} = \|\not\nabla u_0\|_{L^2\{x:|x|>R\}} \leq \|\nabla u\|_{L^2\{x:|x|>R\}}$$

then yields

$$\sum_{k=1}^{\infty} \nu_k(d-2+\nu_k) \int_0^{1/R} |P_k(z)|^2 \, \mathrm{d}z \lesssim_d \|\not\nabla u_0\|^2_{L^2(\{x:|x|>R\})} < +\infty. \tag{4}$$

Similarly

$$\sum_{k=0}^{\infty} \int_0^{1/R} |Q_k(z)|^2 \, \mathrm{d}z = \|u_1\|^2_{L^2(\{x:|x|>R\})} < +\infty.$$

Next we differentiate (2) in r and obtain

$$\int_{\mathbb{S}^{d-1}} \partial_r u_0(r\theta) \Phi_k(\theta) \mathrm{d}\theta = r^{-\mu-1}(-\mu P_k(1/r) - (1/r) P_k'(1/r)).$$

Following the same argument as above, we obtain

$$\sum_{k=0}^{\infty} \int_0^{1/R} \left|\mu P_k(z) + z P_k'(z)\right|^2 \mathrm{d}z \leq \|\partial_r u_0\|^2_{L^2(\{x:|x|>R\})} < +\infty.$$

Combining this inequality with (4), we have

$$\sum_{k=0}^{\infty} \int_0^{1/R} \left(\nu_k(d-2+\nu_k) |P_k(z)|^2 + |z P_k'(z)|^2\right) \mathrm{d}z < +\infty.$$

Since $\Phi_k(\theta)$ is a Hilbert basis, we may finally write (u_0, u_1) in the following form by (2) and (3). (These infinite sums are understood as convergence in $L^2(\mathbb{S}^{d-1})$ and $L^2(\{x : |x| > R\})$ respectively.)

$$u_0(r\theta) = \sum_{k=0}^{\infty} r^{-\mu} P_k(1/r) \Phi_k(\theta) \qquad u_1(r\theta) = \sum_{k=0}^{\infty} r^{-\mu-1} Q_k(1/r) \Phi_k(\theta).$$

Step two Let us assume that $P_k(z)$ and $Q_k(z)$ satisfy the conditions given in the proposition and that the initial data (u_0, u_1) satisfy (1). In this step we need to show that $(u_0, u_1) \in P(R)$. We start by giving the norm $\|(u_0, u_1)\|_{\dot{H}^1 \times L^2(\{x:|x|>R\})}$. In fact the series in (1) must converge to an element in $\dot{H}^1 \times L^2(\{x : |x| > R\})$, whose norm can be expressed in terms of $P_k(z)$ and $Q_k(z)$'s. First of all, we may use the orthogonality and the identity $2\mu = d - 1$ to deduce

$$\begin{aligned}\int_R^\infty \int_{\mathbb{S}^{d-1}} \left| \sum_{k=N}^\infty r^{-\mu-1} Q_k(1/r)\Phi_k(\theta) \right|^2 r^{d-1} \mathrm{d}\theta \mathrm{d}r &= \int_R^\infty \left(\sum_{k=N}^\infty r^{-2\mu-2} |Q_k(1/r)|^2 \right) r^{d-1} \mathrm{d}r \\ &= \sum_{k=N}^\infty \int_R^\infty r^{-2} |Q_k(1/r)|^2 \mathrm{d}r \\ &= \sum_{k=N}^\infty \int_0^{1/R} |Q_k(z)| \mathrm{d}z.\end{aligned}$$

This implies that the second series

$$\sum_{k=0}^\infty r^{-\mu-1} Q_k(1/r)\Phi_k(\theta)$$

converges in the space $L^2(\{x : |x| > R\})$ and the sum u_1 satisfies

$$\|u_1\|^2_{L^2(\{x:|x|>R\})} = \sum_{k=0}^\infty \int_0^{1/R} |Q_k(z)|^2 \, \mathrm{d}z < +\infty. \tag{5}$$

Next we show that the sum of the first series is contained in $\dot{H}^1(\{x : |x| > R\})$. We need the following technical lemma, whose proof is put in the Appendix.

Lemma 2.2. *Let $L \geq 2l > 0$ and $P(z)$ be a polynomial of degree κ. Then we have*

$$\max_{z\in[0,L]} |P(z)|^2 \leq \frac{(\kappa+1)^2}{L} \int_0^L |P(z)|^2 \mathrm{d}z;$$

$$\int_0^l |zP'(z)|^2 \mathrm{d}z \leq \frac{2\kappa(\kappa+1)l}{L} \int_0^L |P(z)|^2 \mathrm{d}z.$$

As a result, we have

$$\left\| \sum_{k=N}^\infty r^{-\mu} P_k(1/r)\Phi_k(\theta) \right\|_{L^2(\mathbb{S}^{d-1})} = r^{-\mu} \left(\sum_{k=N}^\infty |P_k(1/r)|^2 \right)^{1/2}$$

$$\lesssim r^{-\mu} \left(\sum_{k=N}^{\infty} \frac{(\mu+\nu_k)^2 R}{2} \int_0^{1/R} |P_k(z)|^2 \mathrm{d}z \right)^{1/2}$$

converges to zero uniformly in $r \in [R, +\infty)$ as $N \to +\infty$. Thus the series

$$\sum_{k=0}^{\infty} r^{-\mu} P_k(1/r)\Phi_k(\theta)$$

converges to its sum u_0 in the space $C([R, \infty); L^2(\mathbb{S}^{d-1}))$. Next we show

$$\begin{aligned} \nabla_x u_0(r\theta) &= \sum_{k=0}^{\infty} \nabla_x \left(r^{-\mu} P_k(1/r)\Phi_k(\theta) \right) \\ &= \sum_{k=0}^{\infty} r^{-\mu-1} \left\{ P_k(1/r)\nabla_\theta \Phi_k(\theta) - [\mu P_k(1/r) + (1/r)P_k'(1/r)]\Phi_k(\theta)\theta \right\}. \end{aligned} \tag{6}$$

Our assumptions on $P_k(z)$, as well as the orthogonality of $\{\nabla_\theta \Phi_k\}_{k\geq 0}$ and $\{\Phi_k\}_{k\geq 0}$, guarantee that the series in the right hand side converges in $L^2([R, \infty) \times \mathbb{S}^{d-1}; r^{d-1}\mathrm{d}r\mathrm{d}\theta)$, or equivalently in $L^2(\{x : |x| > R\})$. Given any $\varphi \in C_0^\infty(\{x : |x| > R\})$, we have

$$\int_{|x|>R} \left(\sum_{k=0}^{N} r^{-\mu} P_k(1/r)\Phi_k(\theta) \right) \nabla_x \varphi(r,\theta)\mathrm{d}x = -\int_{|x|>R} \varphi(x) \sum_{k=0}^{N} \nabla_x \left(r^{-\mu} P_k(1/r)\Phi_k(\theta) \right) \mathrm{d}x.$$

By the convergence of series we make $N \to +\infty$ and obtain

$$\int_{|x|>R} u_0(x)\nabla_x \varphi(x)\mathrm{d}x = -\int_{|x|>R} \varphi(x) \sum_{k=0}^{\infty} \nabla_x \left(r^{-\mu} P_k(1/r)\Phi_k(\theta) \right) \mathrm{d}x.$$

This verifies (6) and gives the derivative of u_0 in the region $\{x : |x| > R\}$. Please note that we always have $\nabla_\theta \Phi_k \cdot \theta = 0$, thus (6) is actually an orthogonal decomposition. This immediately gives the norms of u_0 in the exterior region:

$$\|\partial_r u_0\|^2_{L^2(\{x:|x|>R\})} = \sum_{k=0}^{\infty} \int_0^{1/R} \left| \mu P_k(z) + z P_k'(z) \right|^2 \mathrm{d}z < +\infty; \tag{7}$$

$$\|\nabla\!\!\!/ u_0\|^2_{L^2(\{x:|x|>R\})} = \sum_{k=1}^{\infty} \nu_k(d-2+\nu_k) \int_0^{1/R} |P_k(z)|^2 \mathrm{d}z < +\infty. \tag{8}$$

Next we show that $\mathbf{S}_L(u_0, u_1)$ is a weakly non-radiative solution. First of all, if $1 \leq k_1 \leq \frac{\mu+\nu_k+1}{2}$ we may find constants $C_{k_1-1}, \cdots, C_1$ such that

$$f(r,t) \doteq r^{-d-\nu_k+2k_1} + C_{k_1-1} t^2 r^{-d-\nu_k+2k_1-2} + \cdots + C_1 t^{2(k_1-1)} r^{-d-\nu_k+2}$$

satisfies the equation $(\partial_t^2 - \partial_r^2 - \frac{d-1}{r}\partial_r) f(r,t) = -\frac{\nu_k(d+\nu_k-2)}{r^2} f(r,t)$. In fact these constants can be determined inductively. Therefore $v(r\theta,t) = f(r,t)\Phi_k(\theta)$ solves the equation

$$(\partial_t^2 - \Delta_x) v = \left(\partial_t^2 - \partial_r^2 - \frac{d-1}{r}\partial_r - \frac{\Delta_{\mathbf{S}^{d-1}}}{r^2}\right) v = 0, \qquad |x| = r > 0, \tag{9}$$

with initial data $(r^{-d-\nu_k+2k_1}\Phi_k(\theta), 0)$. A straight-forward calculation shows that

$$\lim_{t\to\pm\infty} \int_{|x|>|t|+R} |\nabla_{x,t} v(x,t)|^2 \mathrm{d}x = 0.$$

Similarly if $1 \leq k_2 \leq \frac{\mu+\nu_k}{2}$, we may construct an R-weakly non-radiative solution v to (9) with initial data $(0, r^{-d-\nu_k+2k_2}\Phi_k(\theta))$. By linearity, we may construct a non-radiative solution v_N to (9) with initial data $(v_{0,N}, v_{1,N})$ given by

$$v_{0,N}(r\theta) = \sum_{k=0}^{N} r^{-\mu} P_k(1/r)\Phi_k(\theta); \qquad v_{1,N}(r\theta) = \sum_{k=0}^{N} r^{-\mu-1} Q_k(1/r)\Phi_k(\theta).$$

By a standard centre cut-off technique and finite speed of propagation we obtain initial data $(u_{0,N}, u_{1,N}) \in \dot{H}^1 \times L^2$ and corresponding free wave $u_N = \mathbf{S}_L(u_{0,N}, u_{1,N})$ such that

$$u_{0,N}(r\theta) = \sum_{k=0}^{N} r^{-\mu} P_k(1/r)\Phi_k(\theta), \quad u_{1,N}(r\theta) = \sum_{k=0}^{N} r^{-\mu-1} Q_k(1/r)\Phi_k(\theta), \quad r > R;$$

and that

$$\lim_{t\to\pm\infty} \int_{|x|>R+|t|} |\nabla_{x,t} u_N(x,t)|^2 \mathrm{d}x = 0.$$

In addition, finite speed of energy propagation implies that $u = \mathbf{S}_L(u_0, u_1)$ satisfies

$$\limsup_{t\to\pm\infty} \int_{|x|>R+|t|} |\nabla_{x,t}(u(x,t) - u_N(x,t))|^2 \mathrm{d}x \leq \int_{|x|>R} \left(|\nabla u_0 - \nabla u_{0,N}|^2 + |u_1 - u_{1,N}|^2\right) \mathrm{d}x.$$

A combination of two limits given above immediately yields

$$\limsup_{t\to\pm\infty} \int_{|x|>R+|t|} |\nabla_{x,t} u(x,t)|^2 \mathrm{d}x \lesssim_1 \int_{|x|>R} \left(|\nabla u_0 - \nabla u_{0,N}|^2 + |u_1 - u_{1,N}|^2\right) \mathrm{d}x, \quad \forall N \geq 1.$$

Finally we make $N \to +\infty$ and conclude that $(u_0, u_1) \in P(R)$.

Step three Now we show that the identities and inequalities given in Proposition 2.1 hold. Part (i) and (ii) have been proved in step two, see (5), (6), (7) and (8). Now we consider part (iii). We have

$$\partial_r u_0^*(r\theta) = \sum_{k=1}^{\infty} r^{-\mu-1}(-\mu P_k(1/r) - (1/r)P_k'(1/r))\Phi_k(\theta).$$

Thus

$$\int_{|x|>R_1} |\partial_r u_0^*|^2 \mathrm{d}x = \sum_{k=1}^{\infty} \int_0^{1/R_1} |\mu P_k(z) + zP_k'(z)|^2 \mathrm{d}z \lesssim_d \sum_{k=1}^{\infty} \int_0^{1/R_1} \left(|P_k(z)|^2 + |zP_k'(z)|^2\right) \mathrm{d}z.$$

We then apply Lemma 2.2 and obtain

$$\int_{|x|>R_1} |\partial_r u_0^*|^2 \mathrm{d}x \lesssim_d \sum_{k=1}^{\infty} (\mu+\nu_k)^2 \frac{R}{R_1} \int_0^{1/R} |P_k(z)|^2 \mathrm{d}z \lesssim_d (R/R_1) \int_{|x|>R} |\nabla\!\!\!/ u_0(x)|^2 \mathrm{d}x.$$

In order to find an upper bound of $\|\partial_r u_0\|_{L^2}$, we also need to consider the radial part $r^{-\mu} P_0(1/r)\Phi_0(\theta)$. In this case $\nu_0 = 0$ and Φ_0 is simply a constant. We may follow the same argument above and obtain

$$\begin{aligned}
\int_{|x|>R_1} \left|\partial_r [r^{-\mu} P_0(1/r)\Phi_0(\theta)]\right|^2 \mathrm{d}x &= \int_0^{1/R_1} |\mu P_0(z) + zP_0'(z)|^2 \mathrm{d}z \\
&\leq \mu^2 \frac{R}{R_1} \int_0^{1/R} |\mu P_0(z) + zP_0'(z)|^2 \mathrm{d}z \\
&= \mu^2 \frac{R}{R_1} \int_{|x|>R} \left|\partial_r [r^{-\mu} P_0(1/r)\Phi_0(\theta)]\right|^2 \mathrm{d}x \\
&\lesssim_d \frac{R}{R_1} \int_{|x|>R} |\partial_r u_0(x)|^2 \mathrm{d}x.
\end{aligned}$$

Here we recall that $\mu P_0(z) + zP_0'(z)$ is a polynomial of degree $\mu - 1$ or less and apply Lemma 2.2. In summary, we may use orthogonality to conclude ($R_1 \geq 2R$)

$$\begin{aligned}
\int_{|x|>R_1} |\partial_r u_0|^2 \mathrm{d}x &= \int_{|x|>R_1} |\partial_r u_0^*|^2 \mathrm{d}x + \int_{|x|>R_1} \left|\partial_r [r^{-\mu} P_0(1/r)\Phi_0(\theta)]\right|^2 \mathrm{d}x \\
&\lesssim \frac{R}{R_1} \int_{|x|>R} |\nabla u_0(x)|^2 \mathrm{d}x.
\end{aligned}$$

This completes the proof of Proposition 2.1 in the odd dimensional case.

Even dimensions The proof in the even dimensions is almost the same as in the odd dimensions. The main difference is that we rely on a slightly modified version of the technical lemma about polynomials, which is given below and proved in the appendix.

Lemma 2.3. *Let $L \geq 2l > 0$ and $P(z)$ be a polynomial of degree κ. Then we have*

$$\max_{z\in[0,L]} z|P(z)|^2 \leq \frac{4(\kappa+1)^2}{L}\int_0^L z|P(z)|^2 \mathrm{d}z;$$

$$\int_0^l z|zP'(z)|^2 \mathrm{d}z \leq \frac{2\kappa(\kappa+2)l}{L}\int_0^L z|P(z)|^2 \mathrm{d}z.$$

In this case the measure $\mathrm{d}\rho(z) = z\mathrm{d}z$. We would like to explain where the additional z comes form, by considering an example. Let us calculate the norm $\|u_1\|_{L^2(\{x:|x|>R\})}$ in term of $Q_k(z)$. By following the same argument as in the odd dimensional case, we have (see (5) for the odd dimensional case)

$$\int_{|x|>R} |u_1(x)|^2 \mathrm{d}x = \int_R^\infty \left(\sum_{k=0}^\infty r^{-2\mu-2}|Q_k(1/r)|^2\right) r^{d-1}\mathrm{d}r.$$

Unlike the odd dimensional case, we have $2\mu = d$ in the even dimensional case. Therefore

$$\int_{|x|>R} |u_1(x)|^2 \mathrm{d}x = \sum_{k=0}^\infty \int_R^\infty r^{-3}|Q_k(1/r)|^2 \mathrm{d}r = \sum_{k=0}^\infty \int_0^{1/R} z|Q_k(z)|^2 \mathrm{d}z.$$

3. Non-linear non-radiative solutions

In this section we show that non-radiative solutions to a wide range of nonlinear wave equations in the three-dimensional case share the same asymptotic behaviour as non-radiative free waves, without the radial assumption.

Assumptions. We consider the energy-critical non-linear wave equation in $\mathbb{R}^3$

$$\partial_t^2 u - \Delta u = F(x,t,u), \qquad (x,t) \in \mathbb{R}^3 \times \mathbb{R}.$$

Here the nonlinear term $F(x,t,u)$ satisfies

$$|F(x,t,u)| \leq C|u|^5; \qquad |F(x,t,u_1) - F(x,t,u_2)| \leq C(|u_1|^4 + |u_2|^4)|u_1 - u_2|. \tag{10}$$

This covers both the defocussing ($F(x,t,u) = -|u|^4 u$) and focusing ($F(x,t,u) = |u|^4 u$) wave equations, which have been extensively studied in the past decades.

3.1. Preliminary results

We first give a few preliminary results and introduce a few notations.

Radiation fields Radiation field describes the asymptotic behaviour of free waves as time tends to infinity. In its earlier history radiation field was mainly a conception in mathematical physics. See Friedlander [11,12], for instance. The following modern version is given in [7].

Theorem 3.1 *(Radiation fields). Assume that* $d \geq 3$ *and let* u *be a solution to the free wave equation* $\partial_t^2 u - \Delta u = 0$ *with initial data* $(u_0, u_1) \in \dot{H}^1 \times L^2(\mathbb{R}^d)$. *Then*

$$\lim_{t\to\pm\infty} \int_{\mathbb{R}^d} \left(|\nabla\!\!\!/ u(x,t)|^2 + \frac{|u(x,t)|^2}{|x|^2} \right) \mathrm{d}x = 0$$

and there exist two functions $G_\pm \in L^2(\mathbb{R} \times \mathbb{S}^{d-1})$ *so that*

$$\lim_{t\to\pm\infty} \int_0^\infty \int_{\mathbb{S}^{d-1}} \left| r^{\frac{d-1}{2}} \partial_t u(r\theta, t) - G_\pm(r \mp t, \theta) \right|^2 \mathrm{d}\theta \mathrm{d}r = 0;$$

$$\lim_{t\to\pm\infty} \int_0^\infty \int_{\mathbb{S}^{d-1}} \left| r^{\frac{d-1}{2}} \partial_r u(r\theta, t) \pm G_\pm(r \mp t, \theta) \right|^2 \mathrm{d}\theta \mathrm{d}r = 0.$$

In addition, the maps $(u_0, u_1) \to \sqrt{2} G_\pm$ *are bijective isometries from* $\dot{H}^1 \times L^2(\mathbb{R}^d)$ *to* $L^2(\mathbb{R} \times \mathbb{S}^{d-1})$.

We call $G_\pm$ radiation profiles associated to the free wave u. Throughout this section we utilize the notations $\mathbf{T}_\pm$ for the linear map from the initial data (u_0, u_1) to the corresponding radiation profiles $G_\pm$. It immediately follows from Theorem 3.1 that

$$\lim_{t\to\pm\infty} \int_{R_1+|t|<|x|<R_2+|t|} |\nabla_{t,x} u(x,t)|^2 \mathrm{d}x = 2 \int_{R_1}^{R_2} \int_{\mathbb{S}^2} |G_\pm(s,\theta)|^2 \mathrm{d}\theta \mathrm{d}s, \quad -\infty \leq R_1 < R_2 \leq +\infty.$$

In addition, $G_\pm$ satisfy the following symmetric property

$$G_+(s,\theta) = \begin{cases} (-1)^{\frac{d-1}{2}} G_-(-s,-\theta), & d \text{ is odd;} \\ (-1)^{\frac{d}{2}} (\mathcal{H} G_-)(-s,-\theta), & d \text{ is even.} \end{cases}$$

Here $\mathcal{H}$ is the Hilbert transform with respect to the first variable. This symmetry can be verified in different methods. Please refer to Côte-Laurent [1], Duyckaerts-Kenig-Merle [5] and Li-Shen-Wei [17], for examples. As a result, the following identity holds for all odd dimensions $d \geq 3$:

$$\sum_\pm \lim_{t\to\pm\infty} \int_{|x|>R+|t|} |\nabla_{t,x} u(x,t)|^2 \mathrm{d}x = 2 \int_{|r|>R} \int_{\mathbb{S}^2} |G_-(s,\theta)|^2 \mathrm{d}\theta \mathrm{d}s. \tag{11}$$

As a result, the L^2 decay rate of radiation profile $G_-(s,\theta)$ near the infinity indicates to what extent the free wave u looks like a non-radiative solution. For convenience we define

Definition 3.2. Assume that $d \geq 3$ is odd. We call a linear free wave u with a finite energy, or equivalently its initial data, of non-radiative degree γ, if there exists $R > 0$, so that the radiation profile G_- associated to u satisfies

$$\|G_-\|_{L^2(\{s:|s|>r\}\times\mathbb{S}^{d-1})} \lesssim r^{-\gamma}, \qquad \forall r \geq R.$$

This is equivalent to saying (see (11))

$$\lim_{t\to\pm\infty} \int_{|x|>r+|t|} |\nabla_{t,x} u(x,t)|^2 \mathrm{d}x \lesssim r^{-2\gamma}, \qquad \forall r \geq R;$$

or equivalently

$$\lim_{t\to+\infty} \int_{||x|-|t||>r} |\nabla_{t,x} u(x,t)|^2 \mathrm{d}x \lesssim r^{-2\gamma}, \qquad \forall r \geq R.$$

Remark 3.3. The third equivalence given above means that the energy decays fast if we moves away from the main light cone $|x| = |t|$. In addition, the non-radiative degree can be defined equivalently by using the radiation profile G_+ in the positive time direction, as long as the space dimension is odd, since $G_+(s,\omega) = (-1)^{\frac{d-1}{2}} G_-(-s,-\omega)$.

Decay of linear non-radiative solutions Another important ingredient of our estimate on non-linear non-radiative solutions is the corresponding decay estimates of linear non-radiative solutions. We claim that given any constant $\kappa \in (0, 1/5)$, the following inequality holds

$$\|u\|_{L_t^5 L^{10}(\{x:|x|>r+|t|\})} \lesssim_\kappa (R/r)^\kappa E^{1/2}; \tag{12}$$

$$\|u\|_{L^5 L^{10}(\{t:|t|>t_1\}\times\mathbb{R}^3)} \lesssim_\kappa (R/t_1)^\kappa E^{1/2}; \tag{13}$$

for any $r > 0$, $t_1 > 0$ and R-weakly non-radiative linear wave u with a finite energy E, i.e. a finite-energy solution to the homogeneous linear wave equation $\partial_t^2 u - \Delta u = 0$ such that

$$\lim_{t\to\pm\infty} \int_{|x|>R+|t|} |\nabla_{t,x} u(x,t)|^2 \mathrm{d}x = 0.$$

In fact, it was prove in Li-Shen-Wang [18] that any R-weakly non-radiative linear wave u satisfies the following inequalities

$$\|u\|_{L_t^\infty L^6(\{x:|x|>r+|t|\})} \lesssim (R/r)^{1/3} E^{1/2}; \tag{14}$$

$$\|u(\cdot,t)\|_{L^6(\mathbb{R}^3)} \lesssim (R/|t|)^{1/3} E^{1/2}. \tag{15}$$

An interpolation between these inequalities and a regular Strichartz estimate (see Ginibre-Velo [14])

$$\|u\|_{L_t^p L_x^q(\mathbb{R}\times\mathbb{R}^3)} \lesssim_{p,q} E^{1/2}$$

with $p = 2^+$ and $q = \infty^-$ yields the inequalities (12) and (13) for any $\kappa \in (0, 1/5)$. For convenience we introduce the notation

$$\|u\|_{Y(r)} = \|u\|_{L_t^5(\mathbb{R};L^{10}(\{x:|x|>r+|t|\}))} = \left(\int_{\mathbb{R}} \left(\int_{|x|>r+|t|} |u(x,t)|^{10}\mathrm{d}x\right)^{1/2} \mathrm{d}t\right)^{1/5}.$$

Lemma 3.4. *Assume that $d = 3$ and $\kappa \in (0, 1/5)$. Let u be a linear free wave of non-radiative degree $\gamma > \kappa$ with initial data (u_0, u_1) whose radiation profile G satisfies*

$$\|G\|_{L^2(\{s:|s|>r\}\times\mathbb{S}^2)} \le Cr^{-\gamma}, \quad r \ge R > 1; \qquad \|G\|_{L^2(\mathbb{R}\times\mathbb{S}^2)} \le C.$$

Then we have

$$\|u\|_{Y(r)} + \|u\|_{L^5L^{10}(\{t:|t|>r\}\times\mathbb{R}^3)} \lesssim C(R/r)^\kappa.$$

If $\gamma > 1/3$, then we also have

$$\sup_{|t|>r} \|u(\cdot,t)\|_{L^6(\mathbb{R}^3)} + \sup_{t\in\mathbb{R}} \|u(\cdot,t)\|_{L^6(\{x:|x|>r+|t|\})} \lesssim C(R/r)^{1/3}.$$

If $\gamma > 1/2$, then

$$\int_{|x|>r} |u_0(x)|^2\mathrm{d}x \lesssim C^2R/r.$$

The implicit constants in the inequalities depend on κ, γ only.

Proof. First of all, it is not necessary to specify whether the radiation profile G is G_+ or G_- by the symmetry between them. The idea of the proof is to apply a dyadic decomposition. We write

$$G = \sum_{j=0}^{\infty} G_j, \qquad \text{in } L^2(\mathbb{R}\times\mathbb{S}^2);$$

with

$$G_0(s,\omega) = \begin{cases} G(s,\omega), & |s| \le R; \\ 0, & |s| > R; \end{cases} \quad G_j(s,\omega) = \begin{cases} G(s,\omega), & 2^{j-1}R < |s| \le 2^jR; \\ 0, & \text{otherwise}; \end{cases} \quad j \ge 1;$$

then let u_j and $(u_{0,j}, u_{1,j})$ be the corresponding free linear wave and initial data with the radiation profile G_j. We have

$$u=\sum_{j=0}^{\infty} u_j \qquad \text{in } L^5L^{10}(\mathbb{R}\times\mathbb{R}^3) \text{ and } C(\mathbb{R}_t; L^6(\mathbb{R}^3));$$

and

$$(u_0,u_1)=\sum_{j=0}^{\infty}(u_{0,j},u_{1,j}) \text{ in } \dot{H}^1\times L^2(\mathbb{R}^3).$$

Our assumption on G implies that the compactly supported functions G_j satisfy

$$\|G_0\|_{L^2(\mathbb{R}\times\mathbb{S}^2)}\le C; \qquad\qquad \|G_j\|_{L^2(\mathbb{R}\times\mathbb{S}^2)}\lesssim C(2^jR)^{-\gamma}.$$

Now we may apply the decay estimates (12) and (13) on linear weakly non-radiative free waves u_j to obtain

$$\begin{aligned}&\|u_0\|_{Y(r)}+\|u_0\|_{L^5L^{10}(\{t:|t|>r\}\times\mathbb{R}^3)}\lesssim C(R/r)^{\kappa};\\ &\|u_j\|_{Y(r)}+\|u_j\|_{L^5L^{10}(\{t:|t|>r\}\times\mathbb{R}^3)}\lesssim C(2^jR)^{-\gamma}(2^jR/r)^{\kappa}.\end{aligned}$$

Therefore we may take a sum

$$\begin{aligned}\|u\|_{Y(r)}+\|u\|_{L^5L^{10}(\{t:|t|>r\}\times\mathbb{R}^3)}&\lesssim C(R/r)^{\kappa}+\sum_{j=1}^{\infty}C(2^jR)^{-\gamma}(2^jR/r)^{\kappa}\\ &\lesssim C(R/r)^{\kappa}+\sum_{j=1}^{\infty}C(2^jR)^{\kappa-\gamma}r^{-\kappa}\\ &\lesssim C(R/r)^{\kappa}+CR^{-\gamma}(R/r)^{\kappa}\\ &\lesssim C(R/r)^{\kappa}.\end{aligned}$$

Similarly if $\gamma>1/3$, then we may utilize the decay estimates (14), (15) and obtain

$$\begin{aligned}\sup_{|t|>r}\|u(\cdot,t)\|_{L^6(\mathbb{R}^3)}+\sup_{t\in\mathbb{R}}\|u(\cdot,t)\|_{L^6(\{x:|x|>r+|t|\})}&\lesssim C(R/r)^{1/3}+\sum_{j=1}^{\infty}C(2^jR)^{-\gamma}(2^jR/r)^{1/3}\\ &\lesssim C(R/r)^{1/3}+\sum_{j=1}^{\infty}C(2^jR)^{1/3-\gamma}r^{-1/3}\\ &\lesssim C(R/r)^{1/3}.\end{aligned}$$

Finally we consider the L^2 estimate of the radial derivative $\partial_r u_0$ when $\gamma>1/2$. Given $j\ge 0$, if $r\ge 2^{j+1}R$, then we apply Proposition 2.1 and obtain

$$\int_{|x|>r}|\partial_r u_{0,j}(x)|^2\mathrm{d}x\lesssim\frac{2^jR}{r}\int_{|x|>2^jR}|\nabla u_{0,j}(x)|^2\mathrm{d}x\lesssim\frac{2^jR}{r}\|G_j\|_{L^2}^2.$$

On the other hand, if $r < 2^{j+1}R$, then we also have

$$\int_{|x|>r} |\partial_r u_{0,j}(x)|^2 \mathrm{d}x \lesssim \int_{\mathbb{R}^3} |\nabla u_{0,j}(x)|^2 \mathrm{d}x \lesssim \frac{2^j R}{r} \|G_j\|_{L^2}^2.$$

In summary, we may take a sum and finish the proof.

$$\begin{aligned}
\left(\int_{|x|>r} |\partial_r u_0(x)|^2 \mathrm{d}x\right)^{1/2} &\le \sum_{j=0}^{\infty} \left(\int_{|x|>r} |\partial_r u_{0,j}(x)|^2 \mathrm{d}x\right)^{1/2} \\
&\lesssim \sum_{j=0}^{\infty} (2^j R/r)^{1/2} \|G_j\|_{L^2} \\
&\lesssim C(R/r)^{1/2} + \sum_{j=1}^{\infty} C(2^j R)^{-\gamma} (2^j R/r)^{1/2} \\
&\lesssim C(R/r)^{1/2}. \quad \square
\end{aligned}$$

3.2. Statement and proof

Proposition 3.5. *Let u be an R-weakly non-radiative solution to the non-linear wave equation*

$$\begin{cases} \partial_t^2 u - \Delta u = F(x,t,u), & (x,t) \in \mathbb{R}^3 \times \mathbb{R}; \\ (u, u_t)|_{t=0} = (u_0, u_1) \in \dot{H}^1 \times L^2(\mathbb{R}^3). \end{cases}$$

Here the nonlinear term satisfies (10). *Then we have*

(a) *Given any $\kappa \in (0, 1/5)$, the initial data (u_0, u_1) are of non-radiative degree 5κ.*
(b) *The initial data u_0 satisfy the decay estimate*

$$\int_{|x|>r} |\partial_r u_0(x)|^2 \mathrm{d}x \lesssim r^{-1}, \qquad \forall r \gg 1.$$

(c) *We also the decay estimate*

$$\sup_{t\in\mathbb{R}} \int_{|x|>r+|t|} |u(x,t)|^6 \mathrm{d}x \lesssim r^{-2}, \qquad \forall r \gg 1.$$

Proof. Let us first introduce a notation for convenience. We define

$$S(r) = \|G_-\|_{L^2(\{s:|s|>r\}\times\mathbb{S}^2)} = \left(\int_{|s|>r} \int_{\mathbb{S}^2} |G_-(s,\omega)|^2 \mathrm{d}\omega \mathrm{d}s\right)^{1/2}.$$

Given any $r \gg r_1 \gg R$, we may break G_- into two parts

$$G_1(s,\omega) = \begin{cases} G_-(s,\omega), & |s| \le r_1; \\ 0, & |s| > r_1; \end{cases} \qquad G_2(s,\omega) = \begin{cases} 0, & |s| \le r_1; \\ G_-(s,\omega), & |s| > r_1. \end{cases}$$

Therefore we have

$$(u_0, u_1) = \mathbf{T}_-^{-1} G_1 + \mathbf{T}_-^{-1} G_2. \tag{16}$$

We also define $\chi_r(x,t)$ to be the characteristic function of the exterior region $\Omega(r) = \{(x,t) : |x| > |t| + r\}$ thus

$$\|u\|_{Y(r)} = \|\chi_r(x,t)u\|_{L^5 L^{10}(\mathbb{R}\times\mathbb{R}^3)}.$$

Next we give a reasonable upper bound of $\|\mathbf{S}_L(u_0,u_1)\|_{Y(r)}$:

$$\begin{aligned} \|\mathbf{S}_L(u_0,u_1)\|_{Y(r)} &\le \|\mathbf{S}_L \mathbf{T}_-^{-1} G_1\|_{Y(r)} + \|\mathbf{S}_L \mathbf{T}_-^{-1} G_2\|_{Y(r)} \\ &\lesssim (r_1/r)^\kappa \|G_1\|_{L^2} + \|G_2\|_{L^2} \\ &\lesssim (r_1/r)^\kappa + S(r_1). \end{aligned} \tag{17}$$

Here we utilize the fact that G_1 is supported in $[-r_1, r_1] \times \mathbb{S}^2$ thus the linear free wave $\mathbf{S}_L \mathbf{T}_-^{-1} G_1$ with radiation profile G_1 is an r_1-weakly non-radiative free wave. We then apply (12) on the G_1 part and the classic Strichartz estimate on the G_2 part. Now we consider a modified non-linear wave equation

$$\begin{cases} \partial_t^2 v - \Delta v = \chi_r(x,t) F(x,t,v), & (x,t) \in \mathbb{R}^3 \times \mathbb{R}; \\ (v, v_t)|_{t=0} = (u_0, u_1) \in \dot{H}^1 \times L^2(\mathbb{R}^3). \end{cases} \tag{18}$$

First of all, the following inequalities hold by our assumption on the nonlinear term F.

$$\|\chi_r F(x,t,v)\|_{L^1 L^2(\mathbb{R}\times\mathbb{R}^3)} \lesssim \|v\|_{Y(r)}^5;$$
$$\|\chi_r F(x,t,v_1) - \chi_r F(x,t,v_2)\|_{L^1 L^2(\mathbb{R}\times\mathbb{R}^3)} \lesssim (\|v_1\|_{Y(r)}^4 + \|v_2\|_{Y(r)}^4)\|v_1 - v_2\|_{Y(r)}.$$

We also recall the classic Strichartz estimate (see [14]): if w solves the 3D linear wave equation $\partial_t^2 w - \Delta w = F$ with initial data (w_0, w_1), then

$$\|w\|_{L^5 L^{10}(\mathbb{R}\times\mathbb{R}^3)} + \|(w, w_t)\|_{C(\mathbb{R}_t; \dot{H}^1 \times L^2)} \lesssim \|(w_0, w_1)\|_{\dot{H}^1 \times L^2} + \|F\|_{L^1 L^2(\mathbb{R}\times\mathbb{R}^3)}.$$

We may combine all these inequalities, apply a standard fixed-point argument of contraction map and conclude that as long as $\|\mathbf{S}_L(u_0,u_1)\|_{Y(r)}$ is sufficiently small, which holds under our assumption $r \gg r_1 \gg R$ by (17), the equation (18) always has a global-in-time solution v, so that

$$\|v\|_{Y(r)} \le 2\|\mathbf{S}_L(u_0,u_1)\|_{Y(r)}. \tag{19}$$

More details about the fixed-point argument of this kind can be found, for instance, in Pecher [20]. Furthermore, we may write v as a sum of two terms

$$v = v_1 + v_2.$$

They are the linear propagation part and the contribution of non-linear term, respectively:

$$v_1 = \mathbf{S}_L(u_0, u_1); \qquad v_2 = \int_0^t \frac{\sin(t-\tau)\sqrt{-\Delta}}{\sqrt{-\Delta}}(\chi_r F(\cdot, \tau, v(\cdot, \tau)))\mathrm{d}\tau. \tag{20}$$

The triangle inequality in L^2 space gives

$$\left(\int_{|x|>r+|t|} |\nabla_{t,x} v|^2 \mathrm{d}x\right)^{1/2} \geq \left(\int_{|x|>r+|t|} |\nabla_{t,x} v_1|^2 \mathrm{d}x\right)^{1/2} - \left(\int_{|x|>r+|t|} |\nabla_{t,x} v_2|^2 \mathrm{d}x\right)^{1/2}$$

for any given time t. A comparison of our modified non-linear wave equation (18) with the original one shows that $u(x,t) \equiv v(x,t)$ in the exterior region $\Omega(r)$ by finite speed of propagation. Therefore our non-radiative assumption on u also applies on v in the exterior region $\Omega(r)$. This gives

$$\lim_{t\to\pm\infty} \int_{|x|>r+|t|} |\nabla_{t,x} v|^2 \mathrm{d}x = 0.$$

Therefore we have

$$\liminf_{t\to\pm\infty} \int_{|x|>r+|t|} |\nabla_{t,x} v_2(x,t)|^2 \mathrm{d}x \geq \lim_{t\to\pm\infty} \int_{|x|>r+|t|} |\nabla_{t,x} v_1(x,t)|^2 \mathrm{d}x.$$

We then recall the property of radiation field and obtain

$$\sum_{\pm} \lim_{t\to\pm\infty} \int_{|x|>r+|t|} |\nabla_{t,x} v_1(x,t)|^2 \mathrm{d}x = 2 \int_{|s|>r} \int_{\mathbb{S}^2} |G_-(s,\omega)|^2 \mathrm{d}\omega \mathrm{d}s = 2S^2(r).$$

We may also find an upper bound of the integral about v_2 by Strichartz estimates

$$\begin{aligned}
\int_{|x|>r+|t|} |\nabla_{t,x} v_2(x,t)|^2 \mathrm{d}x &\leq \int_{\mathbb{R}^3} |\nabla_{t,x} v_2(x,t)|^2 \mathrm{d}x \\
&\leq \|\chi_r F(x,t,v)\|^2_{L^1 L^2(\mathbb{R}\times\mathbb{R}^3)} \\
&\lesssim \|v\|^{10}_{Y(r)}.
\end{aligned}$$

Combining these inequalities we obtain $S(r) \lesssim \|v\|^5_{Y(r)}$. We then utilize the upper bound given in (19) and obtain

$$S(r) \lesssim \|\mathbf{S}_L(u_0, u_1)\|^5_{Y(r)}, \qquad r \gg R. \tag{21}$$

A combination of this inequality with (17) immediately gives a recursion formula when $r \gg r_1 \gg R$.

$$S(r) \lesssim (r_1/r)^{5\kappa} + S^5(r_1).$$

We then apply Lemma A.3, whose statement and proof is postponed to the appendix, and conclude that given any $\beta \in (0, 4\kappa)$, the following estimate holds if $r \geq R_0(u, \kappa, \beta)$ is sufficiently large

$$S(r) \leq r^{-\beta}.$$

In other words, the initial data are of non-radiative degree β. Next we fix $\beta \in (\kappa, 4\kappa)$, apply Lemma 3.4 and obtain

$$\|\mathbf{S}_L(u_0, u_1)\|_{Y(r)} \lesssim r^{-\kappa}, \qquad r \gg R. \tag{22}$$

We then plug this upper bound in (21) and conclude that

$$S(r) \lesssim r^{-5\kappa}, \qquad r \gg R.$$

This finishes the proof of part (a). Part (b) immediately follows from Lemma 3.4 since we has verified that the initial data are of non-radiative degree γ for any $\gamma \in (0, 1)$. Finally we prove part (c). Again we apply Lemma 3.4 and obtain

$$\sup_{t\in\mathbb{R}} \left(\int_{|x|>r+|t|} |\mathbf{S}_L(u_0, u_1)(x, t)|^6 \mathrm{d}x \right)^{1/6} \lesssim r^{-1/3}.$$

Next we recall that if we let v solves (18) and define v_1, v_2 accordingly as in (20), then

$$u(x, t) = v(x, t) = v_1(x, t) + v_2(x, t)$$

holds in the exterior region $\{(x, t) : |x| > r + |t|\}$. Our argument above has already given L^6 upper bound of $v_1 = \mathbf{S}_L(u_0, u_1)$. It suffices to consider the upper bound of v_2. By the Strichartz estimates, we have

$$\sup_{t\in\mathbb{R}} \|v_2(\cdot, t)\|_{L^6(\mathbb{R}^3)} \lesssim \sup_{t\in\mathbb{R}} \|v_2(\cdot, t)\|_{\dot{H}^1(\mathbb{R}^3)} \lesssim \|\chi_r F(x, t, v)\|_{L^1L^2(\mathbb{R}\times\mathbb{R}^3)} \lesssim \|v\|_{Y(r)}^5$$

Finally we fix $\kappa \in (1/15, 1/5)$, recall (19), (22) and deduce $\|v\|_{Y(r)} \lesssim r^{-\kappa}$. Combining this with the inequality above we have

$$\sup_{t\in\mathbb{R}} \|v_2(\cdot, t)\|_{L^6(\mathbb{R}^3)} \lesssim r^{-5\kappa}, \qquad r \gg R.$$

We finally collect upper bounds of $v_1 = \mathbf{S}_L(u_0, u_1)$ and v_2 to conclude the proof of part (c). □

We can also give asymptotic behaviour of non-linear non-radiative solutions in the exterior regions.

Proposition 3.6. *Let u be an R-weakly non-radiative solution as in Proposition 3.5. Then there exists a radius $R_1 = R_1(u) > R$ and a linear free wave u^+, such that given any $\kappa \in (0, 1/5)$, the linear free wave u^+ is of non-radiative degree 5κ with*

$$\begin{aligned} \|u - u^+\|_{L_t^5([t_1,+\infty);L^{10}(\{x:|x|>R_1+t\}))} &\lesssim t_1^{-5\kappa}, && t_1 > 0;\\ \|\nabla_{t,x} u(\cdot, t_1) - \nabla_{t,x} u^+(\cdot, t_1)\|_{L^2(\{x:|x|>R_1+t_1\})} &\lesssim t_1^{-5\kappa}, && t_1 > 0;\\ \|u\|_{L_t^5([t_1,+\infty);L^{10}(\{x:|x|>R_1+t\}))} &\lesssim t_1^{-\kappa}, && t_1 > 0. \end{aligned}$$

Proof. Let us first choose a sufficiently large $R_1 > R$, such that

$$\|\mathbf{S}_L(u_0, u_1)\|_{Y(R_1)} \ll 1. \tag{23}$$

We then consider the solution v to the following modified non-linear wave equation, as we did in the proof of Proposition 3.5.

$$\begin{cases} \partial_t^2 v - \Delta v = \chi_{R_1}(x,t) F(x,t,v), & (x,t) \in \mathbb{R}^3 \times \mathbb{R};\\ (v, v_t)|_{t=0} = (u_0, u_1) \in \dot{H}^1 \times L^2(\mathbb{R}^3). \end{cases} \tag{24}$$

A fixed-point argument gives a global solution v that scatters in the positive time direction. Namely there exists a linear free wave u^+, so that

$$\lim_{t\to+\infty} \|\nabla_{t,x} v(\cdot, t) - \nabla_{t,x} u^+(\cdot, t)\|_{L^2(\mathbb{R}^3)} = 0. \tag{25}$$

Finite speed of propagation then gives $u(x,t) \equiv v(x,t)$ in the exterior region $\Omega_{R_1} = \{(x,t) : |x| > R_1 + |t|\}$. The proof of Proposition 3.5, as well as finite speed of propagation, has already given the upper bound

$$\|v\|_{Y(r)} \lesssim r^{-\kappa}, \qquad r \geq R_1. \tag{26}$$

Now we introduce a new notation for convenience

$$S_1(t_1) = \|v\|_{L_t^5([t_1,+\infty);L_x^{10}(\{x:|x|>R_1+|t|\}))} = \|\chi_{R_1} v\|_{L^5L^{10}([t_1,+\infty)\times\mathbb{R}^3)}, \qquad t_1 > 0.$$

Next we choose $t_2 > t_1 > R_1$ and write the solution v in the form of

$$v(x,t) = u_L(x,t) + v_1(x,t) + v_2(x,t), \qquad t \in (t_1, +\infty).$$

Here $u_L = \mathbf{S}_L(u_0, u_1)$. In addition, v_1, v_2 are defined by

$$\begin{cases} \partial_t^2 v_1 - \Delta v_1 = \chi_{R_1}(x,t) F(x,t,v), & (x,t) \in \mathbb{R}^3 \times (0, t_1);\\ \partial_t^2 v_1 - \Delta v_1 = 0, & (x,t) \in \mathbb{R}^3 \times [t_1, +\infty);\\ (v_1, \partial_t v_1)|_{t=0} = (0,0); \end{cases} \tag{27}$$

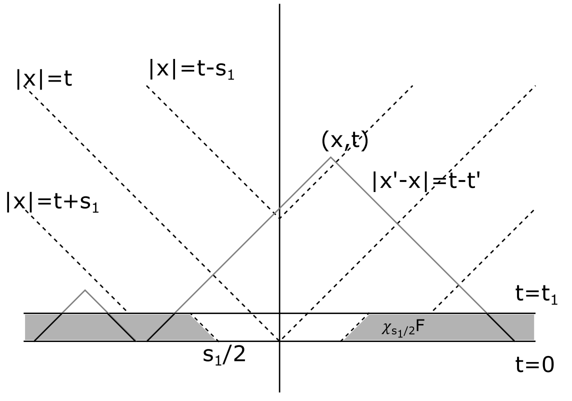

Fig. 1. Dependence of v_1 on $F(x,t,v)$.

and

$$\begin{cases} \partial_t^2 v_2 - \Delta v_2 = \chi_{R_1}(x,t)F(x,t,v), & (x,t) \in \mathbb{R}^3 \times (t_1,+\infty); \\ (v_2, \partial_t v_2)|_{t=t_1} = (0,0). \end{cases}$$

We observe that v_1 becomes a free linear wave when $t > t_1$. We claim that this linear free wave is of non-radiative degree 5κ. It is sufficient to show that

$$\lim_{t\to+\infty} \int_{||x|-t|>s_1} |\nabla_{t,x} v_1(x,t)|^2 \mathrm{d}x \lesssim s_1^{-10\kappa}, \qquad s_1 > 4t_1. \tag{28}$$

We consider another modified solution with additional cut-off in the inhomogeneous term:

$$\begin{cases} \partial_t^2 \tilde{v}_1 - \Delta \tilde{v}_1 = \chi_{s_1/2}(x,t)F(x,t,v), & (x,t) \in (0,t_1) \times \mathbb{R}; \\ \partial_t^2 \tilde{v}_1 - \Delta \tilde{v}_1 = 0, & (x,t) \in [t_1,+\infty) \times \mathbb{R}; \\ (\tilde{v}_1, \partial_t \tilde{v}_1)|_{t=0} = (0,0). \end{cases} \tag{29}$$

By strong Huygen's principle, the value of v_1 (or $\tilde{v}_1$) at the point (x,t) (with $t > t_1$) only depends on the values of inhomogeneous terms $\chi F(x,t,v)$ on the cone

$$\left\{(x',t') \in \mathbb{R}^3 \times \mathbb{R} : |x'-x| = t-t', 0 < t' < t_1\right\}.$$

If $|x| > t + s_1$ or $|x| < t - s_1$, then all the points on the cone given above satisfies

$$|x'| \geq |x| - |x-x'| > t + s_1 - t + t' = s_1 + t' > s_1/2 + t', \qquad \text{if } |x| > t + s_1;$$
$$|x'| \geq |x-x'| - |x| > t - t' - t + s_1 = s_1 - t' > s_1/2 + t', \qquad \text{if } |x| < t - s_1.$$

In other words, all these points are contained in the region $\Omega_{s_1/2} \subset \Omega_{R_1}$, as shown in Fig. 1. This implies that the difference of cut-off functions χ between the equations (27) and (29) will not affect the value of the solution at (x,t), i.e. $\tilde{v}_1(x,t) = v_1(x,t)$, as long as $|x| > t + s_1$ or $|x| < t - s_1$ holds. As a result, we have

$$\lim_{t\to+\infty}\int_{||x|-t|>s_1}|\nabla_{t,x}v_1(x,t)|^2\mathrm{d}x=\lim_{t\to+\infty}\int_{||x|-t|>s_1}|\nabla_{t,x}\tilde{v}_1(x,t)|^2\mathrm{d}x$$
$$\leq\lim_{t\to+\infty}\int_{\mathbb{R}^3}|\nabla_{t,x}\tilde{v}_1(x,t)|^2\mathrm{d}x.$$

We then apply the Strichartz estimates, use the upper bound (26) and obtain

$$\lim_{t\to+\infty}\int_{||x|-t|>s_1}|\nabla_{t,x}v_1(x,t)|^2\mathrm{d}x\leq\|\chi_{s_1/2}(x,t)F(x,t,v)\|^2_{L^1L^2((0,t_1)\times\mathbb{R}^3)}\lesssim\|v\|^{10}_{Y(s_1/2)}\lesssim s_1^{-10\kappa}.$$

This verifies the non-radiative degree of v_1. In addition, we may apply the Strichartz estimates and obtain ($t>t_1$)

$$\int_{\mathbb{R}^3}|\nabla_{t,x}v_1(x,t)|^2\mathrm{d}x\leq\|\chi_{R_1}(x,t)F(x,t,v)\|^2_{L^1L^2((0,t_1)\times\mathbb{R}^3)}\lesssim\|v\|^{10}_{Y(R_1)}\lesssim R_1^{-10\kappa}. \tag{30}$$

Please note that the implicit constants in both (28) and (30) are independent of t_1. We then apply Lemma 3.4 and obtain

$$\|v_1\|_{L^5L^{10}([t_2,+\infty)\times\mathbb{R}^3)}\lesssim(t_1/t_2)^\kappa.$$

Similarly we recall the non-radiative degree of the initial data (u_0,u_1) as given in Proposition 3.5 and obtain

$$\|u_L\|_{L^5L^{10}([t_2,+\infty)\times\mathbb{R}^3)}\lesssim t_2^{-\kappa}.$$

Next we apply the Strichartz estimates on v_2

$$\|v_2\|_{L^5L^{10}([t_1,+\infty)\times\mathbb{R}^3)}\lesssim\|\chi_{R_1}(x,t)F(x,t,v)\|_{L^1L^2([t_1,+\infty)\times\mathbb{R}^3)}\lesssim S_1^5(t_1).$$

Collecting the upper bounds given above, we obtain a recursion formula of S_1 if $t_2>t_1>R_1$.

$$S_1(t_2)\lesssim(t_1/t_2)^\kappa+S_1^5(t_1).$$

As a result, Lemma A.3 implies that there exists $t_0=t_0(u,R_1,\kappa)>R_1$, such that

$$S_1(t)\leq t^{-3\kappa/5},\qquad t>t_0.$$

Next we show that the linear free wave u^+ is of non-radiative degree 3κ. In fact, given any constant $s>5t_0$, we may recall (25) and obtain

$$\lim_{t\to+\infty}\int_{||x|-t|>s}|\nabla_{t,x}u^+(x,t)|^2\mathrm{d}x=\lim_{t\to+\infty}\int_{||x|-t|>s}|\nabla_{t,x}v(x,t)|^2\mathrm{d}x.$$

We then choose $t_1 = s/5$ and define v_1, v_2 accordingly as above. Since $v = u_L + v_1 + v_2$, it is clear that

$$\lim_{t\to+\infty} \int_{||x|-t|>s} |\nabla_{t,x} u^+(x,t)|^2 \mathrm{d}x \lesssim \lim_{t\to+\infty} \int_{||x|-t|>s} |\nabla_{t,x} u_L(x,t)|^2 \mathrm{d}x + \sum_{j=1,2} \lim_{t\to+\infty} \int_{||x|-t|>s} |\nabla_{t,x} v_j(x,t)|^2 \mathrm{d}x.$$

The right hand side comes with three terms. The upper bound of the term concerning v_1 has been given in (28). The linear wave u_L is of non-radiative degree 5κ, by Proposition 3.5. Finally the Strichartz estimates give

$$\sup_{t>s/5} \|\nabla_{t,x} v_2(\cdot,t)\|_{L^2(\mathbb{R}^3)} \le \|\chi_{R_1}(x,t) F(x,t,v)\|_{L^1L^2([s/5,+\infty))} \lesssim S_1^5(s/5).$$

As a result, we have

$$\lim_{t\to+\infty} \int_{||x|-t|>s} |\nabla_{t,x} u^+(x,t)|^2 \mathrm{d}x \lesssim s^{-10\kappa} + S_1^{10}(s/5). \tag{31}$$

We then plug in the upper bound of S_1 and conclude that u^+ is of non-radiative degree 3κ. This enable us to apply Lemma 3.4 and obtain a decay estimate of u^+

$$\|u^+\|_{L^5L^{10}(\{t:|t|>r\}\times\mathbb{R}^3)} \lesssim r^{-\kappa}.$$

This helps us obtain a stronger decay estimate of $S_1(t)$. In fact we have

$$S_1(t) \le \|v\|_{L^5L^{10}([t,+\infty)\times\mathbb{R}^3)} \le \|u^+\|_{L^5L^{10}([t,+\infty)\times\mathbb{R}^3)} + \|v-u^+\|_{L^5L^{10}([t,+\infty)\times\mathbb{R}^3)}.$$

The upper bound of the second term follows from Strichartz estimate

$$\begin{aligned}
\|v-u^+\|_{L^5L^{10}([t,+\infty)\times\mathbb{R}^3)} &= \lim_{t'\to+\infty} \|v-u^+\|_{L^5L^{10}([t,t']\times\mathbb{R}^3)} \\
&\lesssim \lim_{t'\to+\infty} \left(\|\nabla_{t,x} v(\cdot,t') - \nabla_{t,x} u^+(\cdot,t')\|_{L^2} + \|\chi_{R_1} F\|_{L^1L^2([t,t']\times\mathbb{R}^3)}\right) \\
&\lesssim 0 + \lim_{t'\to+\infty} \|\chi_{R_1} v\|^5_{L^5L^{10}([t,t']\times\mathbb{R}^3)} \\
&\lesssim S_1^5(t).
\end{aligned}$$

Thus we have

$$S_1(t) \lesssim t^{-\kappa} + S_1^5(t) \lesssim t^{-\kappa}, \qquad t > t_0.$$

We recall that $S_1(t) \le S_1(0) < +\infty$, thus the inequality $S_1(t) \lesssim t^{-\kappa}$ holds for all $t > 0$ with a possibly different explicit constant. We then plug this upper bound in (31) and conclude that u^+

is of non-radiative degree 5κ. The estimates of u then follow from the coincidence of u and v in the exterior region Ω_{R_1} and the upper bounds about v given above.

$$\|u\|_{L_t^5([t_1,+\infty);L^{10}(\{x:|x|>R_1+t\}))} \leq S_1(t_1) \lesssim t_1^{-\kappa};$$
$$\|u-u^+\|_{L_t^5([t_1,+\infty);L^{10}(\{x:|x|>R_1+t\}))} \leq \|v-u^+\|_{L^5L^{10}([t_1,+\infty)\times\mathbb{R}^3)} \lesssim S_1^5(t_1) \lesssim t_1^{-5\kappa};$$
$$\|\nabla_{t,x}u(\cdot,t_1)-\nabla_{t,x}u^+(\cdot,t_1)\|_{L^2(\{x:|x|>R_1+t_1\})} \leq \|\nabla_{t,x}v(\cdot,t_1)-\nabla_{t,x}u^+(\cdot,t_1)\|_{L^2(\mathbb{R}^3)} \lesssim t_1^{-5\kappa}.$$

The last inequality follows a similar argument to the one given above by the Strichartz estimates. □

Corollary 3.7. *Let u be an R-weakly non-radiative solution as in Proposition 3.5. Then there exists a radius $R_1 > R$, so that*

$$\int_{|x|>R_1+|t|} |u(x,t)|^6 \mathrm{d}x \lesssim |t|^{-2}, \qquad |t| \gg 1.$$

Proof. It suffices to consider the positive time direction $t > 0$. Let us fix $\kappa = (1/5)^-$. On one hand, the non-radiative degree of u^+ implies that $\|u^+(\cdot,t)\|_{L^6(\mathbb{R}^3)} \lesssim t^{-1/3}$, according to Lemma 3.4. On the other hand, the argument in the proof of Proposition 3.6 gives

$$\begin{aligned}\left(\int_{|x|>R_1+|t|} |u(x,t)-u^+(x,t)|^6 \mathrm{d}x\right)^{1/6} &= \left(\int_{|x|>R_1+|t|} |v(x,t)-u^+(x,t)|^6 \mathrm{d}x\right)^{1/6}\\ &\lesssim \left(\int_{\mathbb{R}^3} |\nabla v(x,t)-\nabla u^+(x,t)|^2 \mathrm{d}x\right)^{1/2}\\ &\lesssim t^{-5\kappa} \lesssim t^{-1/3}.\end{aligned}$$

Combining these two estimates, we finish the proof. □

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Appendix A

In this section we prove a few technical lemmata. The authors believe that these results are probably previously known. For completeness we still give their proof.

Polynomial estimates We start by Lemma 2.2. By change of variables $x = 2z/L - 1$, we may rewrite this technical lemma as below.

Lemma A.1. *Let* $0 < \delta \le 1$ *and* $P(x)$ *be a polynomial of degree* κ*. Then we have*

$$\max_{x\in[-1,1]} |P(x)|^2 \le \frac{(\kappa+1)^2}{2}\int_{-1}^{1} |P(x)|^2 \mathrm{d}x;$$

$$\int_{-1}^{-1+\delta} |(x+1)P'(x)|^2 \mathrm{d}x \le \kappa(\kappa+1)\delta \int_{-1}^{1} |P(x)|^2 \mathrm{d}x.$$

Proof. Let us recall Legendre polynomials P_n defined by

$$P_n(x) = \frac{1}{2^n n!}\frac{\mathrm{d}^n}{\mathrm{d}x^n}(x^2-1)^n.$$

It is well known that $\{P_n\}_{n=0,1,2,\cdots}$ are orthogonal to each other in $L^2([-1,+1])$ with norm $\|P_n\|_{L^2}^2 = \frac{2}{2n+1}$. In addition, these polynomials satisfy $|P_n(x)| \le 1, \forall |x| \le 1$ and the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[(1-x^2)\frac{\mathrm{d}}{\mathrm{d}x}P_n(x)\right] + n(n+1)P_n(x) = 0.$$

More details about the properties of Legendre polynomials can be found, for instance, in Folland [13]. We consider the orthogonal decomposition of $P(x)$:

$$P(x) = \sum_{n=0}^{\kappa} a_n P_n(x) \qquad \Rightarrow \qquad \int_{-1}^{1} |P(x)|^2 \mathrm{d}x = \sum_{n=0}^{\kappa} \frac{2|a_n|^2}{2n+1}.$$

This immediately gives

$$\max_{x\in[-1,1]} |P(x)|^2 \le \left(\sum_{n=0}^{\kappa} |a_n|\right)^2 \le \left(\sum_{n=0}^{\kappa} \frac{2n+1}{2}\right)\left(\sum_{n=0}^{\kappa} \frac{2|a_n|^2}{2n+1}\right) = \frac{(\kappa+1)^2}{2}\int_{-1}^{1} |P(x)|^2 \mathrm{d}x.$$

We also have

$$\int_{-1}^{-1+\delta} |(x+1)P'(x)|^2 \mathrm{d}x \le \delta \int_{-1}^{-1+\delta} (1-x^2)|P'(x)|^2 \mathrm{d}x \le \delta \int_{-1}^{1} (1-x^2)|P'(x)|^2 \mathrm{d}x.$$

We then integrate by parts, use the differential equation above and obtain

$$\int_{-1}^{1} (1-x^2)|P'(x)|^2 \mathrm{d}x = -\int_{-1}^{1} P(x)\cdot \frac{\mathrm{d}}{\mathrm{d}x}[(1-x^2)P'(x)]\mathrm{d}x$$

$$
\begin{aligned}
&= \int_{-1}^{1} \left(\sum_{n=0}^{\kappa} a_n P_n(x)\right)\left(\sum_{k=0}^{\kappa} n(n+1) a_n P_n(x)\right) \mathrm{d}x \\
&= \sum_{n=0}^{\kappa} \frac{2n(n+1)|a_n|^2}{2n+1} \\
&\leq \kappa(\kappa+1) \int_{-1}^{1} |P(x)|^2 \mathrm{d}x.
\end{aligned}
$$

Combining these two inequalities, we finish the proof. □

We also need a similar lemma, where $\mathrm{d}x$ is substituted by $(x+1)\mathrm{d}x$. This immediately gives Lemma 2.3 by a change of variables $x = 2z/L - 1$.

Lemma A.2. *Let $0 < \delta \leq 1$ and $P(x)$ be a polynomial of degree κ. Then we have*

$$
\max_{x\in[-1,1]} (x+1)|P(x)|^2 \leq 2(\kappa+1)^2 \int_{-1}^{1} (x+1)|P(x)|^2 \mathrm{d}x; \tag{32}
$$

$$
\int_{-1}^{-1+\delta} (x+1)^3 |P'(x)|^2 \mathrm{d}x \leq \kappa(\kappa+2)\delta \int_{-1}^{1} (x+1)|P(x)|^2 \mathrm{d}x. \tag{33}
$$

Proof. We define $Q_n(x)$ to be the modified Legendre polynomial of degree n:

$$
Q_n(x) = \frac{1}{2^{n+1}(n+1)!} \frac{\mathrm{d}^{n+1}}{\mathrm{d}x^{n+1}} [(x+1)^n (x-1)^{n+1}] = \frac{(2n+1)!}{2^{n+1} n!(n+1)!} x^n + \cdots .
$$

If $n \geq m$ are nonnegative integers, then we may apply integration by parts and obtain

$$
\int_{-1}^{1} (x+1) Q_n(x) Q_m(x) \mathrm{d}x = \frac{(-1)^{n+1}}{2^{n+1}(n+1)!} \int_{-1}^{1} (x+1)^n (x-1)^{n+1} \frac{\mathrm{d}^{n+1}}{\mathrm{d}x^{n+1}} [(x+1) Q_m(x)] \, \mathrm{d}x.
$$

A basic calculation shows

$$
\frac{\mathrm{d}^{n+1}}{\mathrm{d}x^{n+1}} [(x+1) Q_m(x)] = \begin{cases} \dfrac{(2n+1)!}{2^{n+1} n!}, & \text{if } m = n; \\ 0, & \text{if } m < n. \end{cases}
$$

Therefore $\{Q_n(x)\}_{n\geq 0}$ are orthogonal to each other in the Hilbert space $L^2([-1,1];(x+1)\mathrm{d}x)$ and the norms of these polynomials are given by

$$
\|Q_n\|^2_{L^2([-1,1];(x+1)\mathrm{d}x)} = \frac{1}{2(n+1)}.
$$

In addition, these polynomials satisfy a similar differential equation to Legendre polynomials.

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[(x+1)(1-x^2)\frac{\mathrm{d}}{\mathrm{d}x}Q_n(x)\right]+n(n+2)(x+1)Q_n(x)=0. \tag{34}$$

In order to prove this identity, we observe that $\frac{\mathrm{d}}{\mathrm{d}x}\left[(x+1)(x^2-1)\frac{\mathrm{d}}{\mathrm{d}x}Q_n(x)\right]$ is a polynomial of degree $n+1$ and contain a factor of $x+1$. Thus we may write

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[(x+1)(x^2-1)\frac{\mathrm{d}}{\mathrm{d}x}Q_n(x)\right]=\sum_{j=0}^{n}a_j(x+1)Q_j(x).$$

We multiply both sides by $Q_j(x)$, integrate from $x=-1$ to $x=1$ and apply integration by parts

$$\begin{aligned}\frac{a_j}{2(j+1)}&=\int_{-1}^{1}Q_j(x)\frac{\mathrm{d}}{\mathrm{d}x}\left[(x+1)(x^2-1)\frac{\mathrm{d}}{\mathrm{d}x}Q_n(x)\right]\mathrm{d}x\\&=\int_{-1}^{1}Q_n(x)\frac{\mathrm{d}}{\mathrm{d}x}\left[(x+1)(x^2-1)\frac{\mathrm{d}}{\mathrm{d}x}Q_j(x)\right]\mathrm{d}x\\&=\frac{(-1)^{n+1}}{2^{n+1}(n+1)!}\int_{-1}^{1}(x+1)^n(x-1)^{n+1}\frac{\mathrm{d}^{n+2}}{\mathrm{d}x^{n+2}}\left[(x+1)(x^2-1)\frac{\mathrm{d}}{\mathrm{d}x}Q_j(x)\right]\mathrm{d}x.\end{aligned}$$

A direct calculation shows

$$\frac{\mathrm{d}^{n+2}}{\mathrm{d}x^{n+2}}\left[(x+1)(x^2-1)\frac{\mathrm{d}}{\mathrm{d}x}Q_j(x)\right]=\begin{cases}\dfrac{n(n+2)\cdot(2n+1)!}{2^{n+1}n!}, & \text{if } j=n;\\ 0, & \text{if } j<n.\end{cases}$$

Thus we have $a_j=0$ if $j<n$ and $a_n=n(n+2)$. This gives (34). Now we are ready to prove Lemma A.2. We first prove the second inequality (33). Let $P(x)$ be a polynomial of degree κ. We may write

$$P(x)=\sum_{n=0}^{\kappa}a_nQ_n(x).$$

We have

$$\int_{-1}^{-1+\delta}(x+1)^3|P'(x)|^2\mathrm{d}x\le\delta\int_{-1}^{-1+\delta}(x+1)(1-x^2)|P'(x)|^2\mathrm{d}x\le\delta\int_{-1}^{1}(x+1)(1-x^2)|P'(x)|^2\mathrm{d}x.$$

We then integrate by parts, use the differential equation and orthogonality of $\{Q_n\}$.

$$\begin{aligned}\int_{-1}^{1}(x+1)(1-x^2)|P'(x)|^2\mathrm{d}x &= -\int_{-1}^{1}P(x)\frac{\mathrm{d}}{\mathrm{d}x}\left[(x+1)(1-x^2)P'(x)\right]\mathrm{d}x\\ &= \int_{-1}^{1}\left(\sum_{n=0}^{\kappa}a_n Q_n(x)\right)\left(\sum_{n=0}^{\kappa}n(n+2)a_n(x+1)Q_n(x)\right)\mathrm{d}x\\ &= \sum_{n=0}^{\kappa}\frac{n(n+2)|a_n|^2}{2(n+1)}\\ &\leq \kappa(\kappa+2)\int_{-1}^{1}(x+1)|P(x)|^2\mathrm{d}x.\end{aligned}$$

Combining these two inequalities, we finish the proof of (33). We then prove the first inequality (32). First of all, we have

$$\max_{x\in[0,1]}|P(x)|^2 \leq (\kappa+1)^2\int_{0}^{1}|P(x)|^2\mathrm{d}y \leq (\kappa+1)^2\int_{-1}^{1}(x+1)|P(x)|^2\mathrm{d}y. \tag{35}$$

Here we apply Lemma 2.2. This deals with the case $x\in[0,1]$. Next we observe that if $x\in(-1,0)$, then we may apply a translated-version of Lemma 2.2 and obtain

$$|P(x)|^2 \leq \max_{y\in[x,1]}|P(y)|^2 \leq \frac{(\kappa+1)^2}{1-x}\int_{x}^{1}|P(y)|^2\mathrm{d}y \leq \frac{(\kappa+1)^2}{1-x^2}\int_{x}^{1}(1+y)|P(y)|^2\mathrm{d}y.$$

This immediately gives

$$(1+x)|P(x)|^2 \leq \frac{(\kappa+1)^2}{1-x}\int_{x}^{1}(1+y)|P(y)|^2\mathrm{d}y \leq (\kappa+1)^2\int_{-1}^{1}(1+y)|P(y)|^2\mathrm{d}y, \quad x\in(-1,0).$$

Finally we combine this with the upper bound (35) for $x\in[0,1]$ to finish the proof of (32). □

Decay by recursion Finally we prove a lemma giving polynomial decay by a suitable recurrence formula.

Lemma A.3. *Assume that $l>1$ and $\alpha>0$ are constants. Let $S:[R,+\infty)\to[0,+\infty)$ be a function satisfying*

- *$S(r)\to 0$ as $r\to+\infty$;*
- *The recursion formula $S(r_2)\lesssim(r_1/r_2)^{\alpha}+S^{l}(r_1)$ holds when $r_2\gg r_1\gg R$.*

Then given any constant $\beta\in(0,(1-1/l)\alpha)$, the decay estimate $S(r)\leq r^{-\beta}$ holds as long as $r>R_0$ is sufficiently large.

Proof. Without loss of generality, we may assume the recursion formula

$$S(r_2) \le \frac{1}{2}(r_1/r_2)^\alpha + \frac{1}{2}S^l(r_1)$$

holds for $r_2 \gg r_1 \gg r$. Otherwise we may slightly reduce the values of l and α. We first find a small constant $\gamma > 0$ so that $S(r) \le r^{-\gamma}$ for large r, then plug this estimate back in the recursion formula and slightly enlarge the value of γ, finally iterate our argument to finish the proof. We start by recalling the assumption on the limit of $S(r)$ at the infinity and choosing a large constant $M > R$ so that

$$S(r) < 1/2, \qquad \forall r \in [M, M^l].$$

This implies that we may choose a sufficiently small constant $\gamma \in (0, (1-1/l)\alpha)$ so that

$$S(r) < r^{-\gamma}, \qquad \forall r \in [M, M^l].$$

Next we prove that $S(r) \le r^{-\gamma}$ holds for any $r \ge M$ by induction. It suffices to shows that this inequality holds for $r \in [M^{l^k}, M^{l^{k+1}}]$ if it holds for $r \in [M^{l^{k-1}}, M^{l^k}]$. In fact, if $r \in [M^{l^k}, M^{l^{k+1}}]$, then we have

$$S(r) \le \frac{1}{2}(r^{1/l}/r)^\alpha + \frac{1}{2}S^l(r^{1/l}) \le \frac{1}{2}r^{-(1-1/l)\alpha} + \frac{1}{2}r^{-\gamma} \le r^{-\gamma}.$$

Here we utilize induction hypothesis on $S(r^{1/l})$. Next we plug in $r_1 = r^{\alpha/(\alpha+\gamma l)}$ and $r_2 = r$ in the recursion formula, use the already known upper bound $S(r_1) \le r_1^{-\gamma}$, then obtain

$$S(r) \le \frac{1}{2}(r^{\alpha/(\alpha+\gamma l)}/r)^\alpha + \frac{1}{2}S^l(r^{\alpha/(\alpha+\gamma l)}) \le r^{-\alpha\gamma l/(\alpha+\gamma l)}, \qquad r \gg 1.$$

We may iterate this argument and conclude that

$$S(r) \le r^{-\gamma_k}, \qquad \forall r \ge r_k \gg 1.$$

Here $\gamma_k \in (0, (1-1/l)\alpha)$ are defined by the induction formula

$$\gamma_0 = \gamma; \qquad \gamma_{k+1} = \frac{\alpha\gamma_k l}{\alpha + \gamma_k l}, \quad k \ge 0.$$

In order to finish the proof, we only need to show $\gamma_k \to (1-1/l)\alpha$ as $k \to +\infty$. In fact, we may rewrite the induction formula in the form of

$$(1-1/l)\alpha - \gamma_{k+1} = \frac{\alpha}{\alpha + \gamma_k l} \cdot \left[(1-1/l)\alpha - \gamma_k\right].$$

Thus $\gamma_k \in (0, (1-1/l)\alpha)$ increases as k grows. This implies

$$(1-1/l)\alpha - \gamma_{k+1} \le \frac{\alpha}{\alpha + \gamma l} \cdot \left[(1-1/l)\alpha - \gamma_k\right] \qquad \Rightarrow \qquad (1-1/l)\alpha - \gamma_k \to 0^+,$$

which gives the desired limit. □

Data availability

No data was used for the research described in the article.

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