

Probability Distance Estimates Between Diffusion Processes and Applications to Singular McKean-Vlasov SDEs*

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Abstract

The L^k -Wasserstein distance \mathbb{W}_k ($k \geq 1$) and the probability distance \mathbb{W}_ψ induced by a concave function ψ , are estimated between different diffusion processes with singular coefficients. As applications, the well-posedness, probability distance estimates and the log-Harnack inequality are derived for McKean-Vlasov SDEs with multiplicative distribution dependent noise, where the coefficients are singular in time-space variables and $(\mathbb{W}_k + \mathbb{W}_\psi)$ -Lipschitz continuous in the distribution variable. This improves existing results derived in the literature under the \mathbb{W}_k -Lipschitz or derivative conditions in the distribution variable.

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1 Introduction

Let $T > 0$, and let Ξ be the space of (a, b) , where

$$b : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^d, \quad a : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^d \otimes \mathbb{R}^d$$

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are measurable, and for any $(t, x) \in [0, T] \times \mathbb{R}^d$, $a(t, x)$ is positive definite. For any $(a, b) \in \Xi$, consider the time dependent second order differential operator on \mathbb{R}^d :

$$L_t^{a,b} := \text{tr}\{a(t, \cdot) \nabla^2\} + b(t, \cdot) \cdot \nabla, \quad t \in [0, T].$$

Let $(a_i, b_i) \in \Xi, i = 1, 2$, such that for any $s \in [0, T]$, each $(L_t^{a_i, b_i})_{t \in [s, T]}$ generates a unique diffusion process $(X_{s,t}^{i,x})_{(t,x) \in [s,T] \times \mathbb{R}^d}$ on \mathbb{R}^d with $X_{s,s}^{i,x} = x$. Let

$$P_{s,t}^{i,x} := \mathcal{L}_{X_{s,t}^{i,x}}$$

be the distribution of $X_{s,t}^{i,x}$. When $s = 0$, we simply denote

$$X_{0,t}^{i,x} = X_t^{i,x}, \quad P_{0,t}^{i,x} = P_t^{i,x}.$$

If the initial value is random with distributions $\gamma \in \mathcal{P}$, where \mathcal{P} is the set of all probability measures on \mathbb{R}^d , we denote the diffusion process by $X_{s,t}^{i,\gamma}$, which has distribution

$$(1.1) \quad P_{s,t}^{i,\gamma} = \int_{\mathbb{R}^d} P_{s,t}^{i,x} \gamma(dx), \quad i = 1, 2, \quad 0 \leq s \leq t \leq T.$$

By developing the bi-coupling argument and using an entropy inequality due to [1], the relative entropy

$$\text{Ent}(P_{s,t}^{1,\gamma} | P_{s,t}^{2,\tilde{\gamma}}) := \int_{\mathbb{R}^d} \left(\log \frac{dP_{s,t}^{1,\gamma}}{dP_{s,t}^{2,\tilde{\gamma}}} \right) dP_{s,t}^{1,\gamma}, \quad 0 \leq s < t \leq T, \gamma, \tilde{\gamma} \in \mathcal{P}$$

is estimated in [13], and as an application, the log-Harlanck inequality is established for McKean-Vlasov SDEs with multiplicative distribution dependent noise, where the drift is Dini continuous in the spatial variable x , and the diffusion coefficient is Lipschitz continuous in x and the distribution variable with respect to \mathbb{W}_2 .

In this paper, we estimate a weighted variational distance between $P_t^{1,\gamma}$ and $P_t^{2,\tilde{\gamma}}$ for diffusion processes with singular coefficients, and apply to the study of singular McKean-Vlasov SDEs with multiplicative distribution dependent noise, so that existing results in the literature are considerably extended.

Consider the class

$$\mathcal{A} := \{\psi : [0, \infty) \rightarrow [0, \infty) \text{ is increasing and concave, } \psi(r) > 0 \text{ for } r > 0\}.$$

For any $\psi \in \mathcal{A}$, the ψ -continuity modulus of a function f on \mathbb{R}^d is

$$[f]_\psi := \sup_{x \neq y} \frac{|f(x) - f(y)|}{\psi(|x - y|)}.$$

Then

$$\mathcal{P}_\psi := \left\{ \mu \in \mathcal{P} : \|\mu\|_\psi := \int_{\mathbb{R}^d} \psi(|x|) \mu(dx) < \infty \right\}$$

is a complete metric space under the distance \mathbb{W}_ψ induced by ψ :

$$\mathbb{W}_\psi(\mu, \nu) := \sup_{[f]_\psi \leq 1} |\mu(f) - \nu(f)|,$$

where $\mu(f) := \int_{\mathbb{R}^d} f d\mu$ for $f \in L^1(\mu)$. In particular, $\mathbb{W}_\psi = \mathbb{W}_1$ is the L^1 -Wasserstein distance if $\psi(r) = r$, while \mathbb{W}_ψ with $\psi \equiv 2$ reduces to the total variational distance

$$\|\mu - \nu\|_{var} := \sup_{|f| \leq 1} |\mu(f) - \nu(f)|.$$

For any $k > 0$, the L^k -Wasserstein distance is

$$\mathbb{W}_k(\mu, \nu) := \inf_{\pi \in \mathcal{C}(\mu, \nu)} \left(\int_{\mathbb{R}^d \times \mathbb{R}^d} |x - y|^k \pi(dx, dy) \right)^{\frac{1}{1 \vee k}},$$

where $\mathcal{C}(\mu, \nu)$ is the set of couplings for μ and ν . Then

$$\mathcal{P}_k := \{\mu \in \mathcal{P} : \mu(|\cdot|^k) < \infty\}$$

is a Polish space under \mathbb{W}_k . Since ψ has at most linear growth, we have $\mathcal{P}_k \subset \mathcal{P}_\psi$, and \mathcal{P}_k is complete under $\mathbb{W}_\psi + \mathbb{W}_k$.

To characterize the singularity of coefficients in time-space variables, we recall some functional spaces introduced in [17]. For any $p \geq 1$, $L^p(\mathbb{R}^d)$ is the class of measurable functions f on \mathbb{R}^d such that

$$\|f\|_{L^p(\mathbb{R}^d)} := \left(\int_{\mathbb{R}^d} |f(x)|^p dx \right)^{\frac{1}{p}} < \infty.$$

For any $p, q > 1$ and a measurable function f on $[0, T] \times \mathbb{R}^d$, let

$$\|f\|_{\tilde{L}_q^p(s, t)} := \sup_{z \in \mathbb{R}^d} \left(\int_s^t \|1_{B(z, 1)} f_r\|_{L^p(\mathbb{R}^d)}^q dr \right)^{\frac{1}{q}},$$

where $B(z, 1) := \{x \in \mathbb{R}^d : |x - z| \leq 1\}$. When $s = 0$, we simply denote $\|\cdot\|_{\tilde{L}_q^p(t)} = \|\cdot\|_{\tilde{L}_q^p(0, t)}$. Let

$$\mathcal{K} := \left\{ (p, q) : p, q \in (2, \infty), \frac{d}{p} + \frac{2}{q} < 1 \right\}.$$

Let $\|\cdot\|_\infty$ be the uniform norm, and for any function f on $[0, T] \times \mathbb{R}^d$, let

$$\|f\|_{t, \infty} := \sup_{x \in \mathbb{R}^d} |f(t, x)|, \quad \|f\|_{r \rightarrow t, \infty} := \sup_{s \in [r, t]} \|f\|_{s, \infty}, \quad 0 \leq r \leq t \leq T.$$

We make the following assumptions for the coefficients $(a, b) \in \Xi$, where ∇ is the gradient operator on \mathbb{R}^d .

($A^{a,b}$) There exist constants $\alpha \in (0, 1]$, $K > 1$, $l \in \mathbb{N}$ and $\{(p_i, q_i)\}_{0 \leq i \leq l} \subset \mathcal{K}$ such that the following conditions hold.

(1) $\|a\|_\infty \vee \|a^{-1}\|_\infty \leq K$, and

$$(1.2) \quad \|a(t, x) - a(t, y)\| \leq K|x - y|^\alpha, \quad t \in [0, T], x, y \in \mathbb{R}^d.$$

Moreover, there exist $\{1 \leq f_i\}_{1 \leq i \leq l}$ with $\sum_{i=1}^l \|f_i\|_{\tilde{L}_{q_i}^{p_i}(T)} \leq K$, such that

$$\|\nabla a\| \leq \sum_{i=1}^l f_i.$$

(2) b has a decomposition $b = b^{(0)} + b^{(1)}$ such that

$$\sup_{t \in [0, T]} |b^{(1)}(t, 0)| + \|\nabla b^{(1)}\|_\infty + \|b^{(0)}\|_{\tilde{L}_{q_0}^{p_0}(T)} \leq K.$$

Let $\sigma(t, x) := \sqrt{2a(t, x)}$, and let W_t be a d -dimensional Brownian motion on a probability basis $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \in [0, T]}, \mathbb{P})$. By [11, Theorem 2.1] for $V(x) := 1 + |x|^2$, see also [17] or [19], under $(A^{a,b})$, for any $(s, x) \in [0, T] \times \mathbb{R}^d$, the SDE

$$(1.3) \quad dX_{s,t}^x = b(t, X_{s,t}^x)dt + \sigma(t, X_{s,t}^x)dW_t, \quad t \in [s, T]$$

is well-posed, so that $(L_t^{a,b})_{t \in [s, T]}$ generates a unique diffusion process. Moreover, for any $k \geq 1$, there exists a constant $c(k) > 0$ such that

$$(1.4) \quad \mathbb{E} \left[\sup_{t \in [s, T]} |X_{s,t}^x|^k \right] \leq c(k)(1 + |x|^k), \quad (s, x) \in [0, T] \times \mathbb{R}^d.$$

The associated Markov semigroup is given by

$$P_{s,t}^{a,b} f(x) := \mathbb{E}[f(X_{s,t}^x)], \quad 0 \leq s \leq t \leq T, x \in \mathbb{R}^d, f \in \mathcal{B}_b(\mathbb{R}^d).$$

Since $(p_0, q_0) \in \mathcal{K}$, we have

$$m_0 := \inf \left\{ m > 1 : \frac{(m-1)p_0}{m} \wedge \frac{(m-1)q_0}{m} > 1, \frac{dm}{p_0(m-1)} + \frac{2m}{q_0(m-1)} < 2 \right\} \in (1, 2).$$

For a $\mathbb{R}^d \otimes \mathbb{R}^d$ valued differentiable function $a = (a^{ij})_{1 \leq i, j \leq d}$, its divergence is an \mathbb{R}^d valued function defined as

$$(\operatorname{div} a)^i := \sum_{j=1}^d \partial_j a^{ij}, \quad 1 \leq i \leq d.$$

Our first result is the following.

Theorem 1.1. Assume $(A^{a,b})$ for $(a, b) = (a_i, b_i), i = 1, 2$. Then for any $m \in (m_0, 2)$, there exists a constant $c > 0$ depending only on m, K, d, T and $(p_i, q_i)_{0 \leq i \leq l}$, such that for any $\psi \in \mathcal{A}$ and $\gamma, \tilde{\gamma} \in \mathcal{P}$,

$$\begin{aligned}
\mathbb{W}_\psi(P_{s,t}^{1,\gamma}, P_{s,t}^{2,\tilde{\gamma}}) &\leq \frac{c\psi((t-s)^{\frac{1}{2}})}{\sqrt{t-s}} \mathbb{W}_1(\gamma, \tilde{\gamma}) + c \int_s^t \frac{\psi((t-r)^{\frac{1}{2}}) \|a_1 - a_2\|_{r,\infty}}{\sqrt{(r-s)(t-r)}} dr \\
(1.5) \quad &+ c \left(\int_s^t \left(\frac{\psi((t-r)^{\frac{1}{2}}) \|a_1 - a_2\|_{r,\infty}}{\sqrt{t-r}} \right)^m dr \right)^{\frac{1}{m}} \\
&+ c \int_s^t \frac{\psi((t-r)^{\frac{1}{2}})}{\sqrt{t-r}} \{ \|b_1 - b_2\|_{r,\infty} + \|\text{div}(a_1 - a_2)\|_{r,\infty} \} dr, \quad 0 \leq s < t \leq T, \quad \gamma, \tilde{\gamma} \in \mathcal{P}.
\end{aligned}$$

Moreover, for any $k \geq 1$, there exists a constant $C > 0$ depending only on k, K, d, T and $(p_i, q_i)_{0 \leq i \leq l}$, such that for any $\gamma, \tilde{\gamma} \in \mathcal{P}$ and $0 \leq s \leq t \leq T$,

$$(1.6) \quad \mathbb{W}_k(P_{s,t}^{1,\gamma}, P_{s,t}^{2,\tilde{\gamma}}) \leq C \left[\mathbb{W}_k(\gamma, \tilde{\gamma}) + \int_s^t \|b_1 - b_2\|_{r,\infty} dr + \left(\int_s^t \|a_1 - a_2\|_{r,\infty}^2 dr \right)^{\frac{1}{2}} \right].$$

Next, we consider the following distribution dependent SDE on \mathbb{R}^d :

$$(1.7) \quad dX_t = b_t(X_t, \mathcal{L}_{X_t}) dt + \sigma_t(X_t, \mathcal{L}_{X_t}) dW_t, \quad t \in [0, T],$$

where \mathcal{L}_{X_t} is the distribution of X_t , and for some $k \geq 1$,

$$b : [0, T] \times \mathbb{R}^d \times \mathcal{P}_k \rightarrow \mathbb{R}^d, \quad a : [0, T] \times \mathbb{R}^d \times \mathcal{P}_k \rightarrow \mathbb{R}^d \otimes \mathbb{R}^d$$

are measurable, each $a_t(x, \mu)$ is positive definite and $\sigma = \sqrt{2a}$.

Let $C_b^w([0, T]; \mathcal{P}_k)$ be the set of all weakly continuous maps $\mu : [0, T] \rightarrow \mathcal{P}_k$ such that

$$\sup_{t \in [0, T]} \mu_t(|\cdot|^k) < \infty.$$

We call the SDE (1.7) well-posed for distributions in \mathcal{P}_k , if for any initial value X_0 with $\mathcal{L}_{X_0} \in \mathcal{P}_k$ (correspondingly, any initial distribution $\nu \in \mathcal{P}_k$), the SDE has a unique solution (correspondingly, a unique weak solution) with $(\mathcal{L}_{X_t})_{t \in [0, T]} \in C_b^w([0, T]; \mathcal{P}_k)$. In this case, let $P_t^* \nu := \mathcal{L}_{X_t}$ for the solution with $\mathcal{L}_{X_0} = \nu$, and define

$$P_t f(\nu) := \int_{\mathbb{R}^d} f d(P_t^* \nu), \quad \nu \in \mathcal{P}_k, t \in [0, T], f \in \mathcal{B}_b(\mathbb{R}^d).$$

In particular, for $k = 2$, the following log-Harnack inequality

$$(1.8) \quad P_t \log f(\gamma) \leq \log P_t f(\tilde{\gamma}) + \frac{c}{t} \mathbb{W}_2(\mu, \nu)^2, \quad f \in \mathcal{B}_b^+(\mathbb{R}^d), t \in (0, T], \mu, \nu \in \mathcal{P}_2$$

for some constant $c > 0$ has been established and applied in [6, 8, 12, 14, 15] for $\sigma_t(x, \mu) = \sigma_t(x)$ not dependent on μ , see also [4, 5, 16] for extensions to the infinite-dimensional and

reflecting models. When the noise coefficient is also distribution dependent and is \mathbb{W}_2 -Lipschitz continuous, this inequality is established in the recent work [13] by using a bi-coupling method.

In the following, we consider more singular situation where $\sigma_t(x, \mu)$ may be not \mathbb{W}_2 -Lipschitz continuous in μ , and the drift is singular in the time-spatial variables. For any $\mu \in C_b^w([0, T]; \mathcal{P}_k)$, let

$$a^\mu(t, x) := a_t(x, \mu_t), \quad b^\mu(t, x) := b_t(x, \mu_t), \quad t \in [0, T], x \in \mathbb{R}^d.$$

Correspondingly to $(A^{a,b})$, we make the following assumption.

$(B^{a,b})$ Let $k \in [1, \infty)$ and $\psi \in \mathcal{A}$ with $\lim_{t \rightarrow 0} \psi(t) = 0$.

- (1) $(A^{a,b})$ holds for $(a, b) = (a^\mu, b^\mu)$ uniformly in $\mu \in C_b^w([0, T]; \mathcal{P}_k)$, with drift decomposition $b^\mu = (b^\mu)^{(0)} + (b^\mu)^{(1)}$.
- (2) There exists a constant $K > 0$ such that

$$\|a_t(\cdot, \gamma) - a_t(\cdot, \tilde{\gamma})\|_\infty \leq K(\mathbb{W}_\psi + \mathbb{W}_k)(\gamma, \tilde{\gamma}), \quad t \in [0, T], \gamma, \tilde{\gamma} \in \mathcal{P}_k.$$

- (3) There exist $p \geq 2$ and $1 \leq \rho \in L^p([0, T])$, where $p = 2$ if $\int_0^1 \frac{\psi(r)^2}{r} dr < \infty$ and $p > 2$ otherwise, such that for any $t \in [0, T]$ and $\gamma, \tilde{\gamma} \in \mathcal{P}_k$,

$$\|b_t(\cdot, \gamma) - b_t(\cdot, \tilde{\gamma})\|_\infty + \|\text{div}(a_t(\cdot, \gamma) - a_t(\cdot, \tilde{\gamma}))\|_\infty \leq \rho_t(\mathbb{W}_\psi + \mathbb{W}_k)(\gamma, \tilde{\gamma}).$$

Remark 1.2. We give a simple example satisfying $(B^{a,b})$ for some $\rho \in L^\infty([0, T])$, where b contains a locally integrable term $b^{(0)}$, and the dependence of b and σ in distribution is given by singular integral kernels. Let $\psi \in \mathcal{A}$ with $\lim_{t \rightarrow 0} \psi(t) = 0$, let

$$b_t(\cdot, \mu) = b_t^{(0)} + \int_{\mathbb{R}^d} \tilde{b}_t(\cdot, y) \mu(dy),$$

$$\sigma_t(\cdot, \mu) = \sqrt{\lambda I + \int_{\mathbb{R}^d} (\tilde{\sigma}_t \tilde{\sigma}^*)(\cdot, y) \mu(dy)}, \quad (t, \mu) \in [0, T] \times \mathcal{P}_k,$$

where $\lambda > 0$ is a constant, $b^{(0)} : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ satisfies $\|b^{(0)}\|_{\tilde{L}_{q_0}^{p_0}(T)} < \infty$ for some $(p_0, q_0) \in \mathcal{K}$, $\tilde{b} : [0, T] \times \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ is measurable such that

$$|\tilde{b}_t^{(1)}(x, y) - \tilde{b}_t^{(1)}(\tilde{x}, \tilde{y})| \leq K(|x - \tilde{x}| + \psi(|y - \tilde{y}|)), \quad x, \tilde{x}, y, \tilde{y} \in \mathbb{R}^d, t \in [0, T]$$

holds for some constant $K > 0$, and $\tilde{\sigma} : [0, T] \times \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}^{d \otimes d}$ is measurable and bounded such that for some constant $K > 0$,

$$\|\tilde{\sigma}_t(x, y) - \tilde{\sigma}_t(\tilde{x}, \tilde{y})\| \leq K(|x - \tilde{x}| + \psi(|y - \tilde{y}|)),$$

$$|\nabla \tilde{\sigma}_t(\cdot, y)(x) - \nabla \tilde{\sigma}_t(\cdot, \tilde{y})(x)| \leq K\psi(|y - \tilde{y}|), \quad (t, x, y) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^d.$$

We have the following result on the well-posedness and estimates on $(\mathbb{W}_\psi, \mathbb{W}_k)$ for P_t^* .

Theorem 1.3. *Assume $(B^{a,b})$. Then the following assertions hold.*

(1) *The SDE (1.7) is well-posed for distributions in \mathcal{P}_k . Moreover, for any $n \in \mathbb{N}$, there exists a constant $c > 0$ such that any solution satisfies*

$$(1.9) \quad \mathbb{E} \left[\sup_{t \in [0,T]} |X_t|^n \middle| \mathcal{F}_0 \right] \leq c(1 + |X_0|^n).$$

(2) *If ψ is a Dini function, i.e.*

$$(1.10) \quad \int_0^1 \frac{\psi(s)}{s} ds < \infty,$$

then there exists a constant $c > 0$ such that

$$(1.11) \quad \begin{aligned} \mathbb{W}_\psi(P_t^*\gamma, P_t^*\tilde{\gamma}) &\leq \frac{c\psi(t^{\frac{1}{2}})}{\sqrt{t}} \mathbb{W}_1(\gamma, \tilde{\gamma}) + c\mathbb{W}_k(\gamma, \tilde{\gamma}), \\ \mathbb{W}_k(P_t^*\gamma, P_t^*\tilde{\gamma}) &\leq c\mathbb{W}_k(\gamma, \tilde{\gamma}), \quad t \in (0, T], \quad \gamma, \tilde{\gamma} \in \mathcal{P}_k. \end{aligned}$$

Remark 1.4. *Theorem 1.3(1) improves existing well-posedness results for singular McKean-Vlasov SDEs where the coefficients are either $(\mathbb{W}_k + \mathbb{W}_\alpha)$ -Lipschitz continuous in distribution for some $\alpha \in (0, 1]$ and $k \geq 1$ (see [7, 3] and references therein), or satisfy some derivative conditions in distribution (see for instance [2]).*

To estimate $\mathbb{W}_\psi(P_t^*\gamma, P_t^*\tilde{\gamma})$ for worse ψ not satisfying (1.10), and to estimate the relative entropy $\text{Ent}(P_t^*\gamma | P_t^*\tilde{\gamma})$, we need the drift to be Dini continuous in the spatial variable.

Theorem 1.5. *Assume $(B^{a,b})$ with $\|\rho\|_\infty < \infty$ and $\int_0^1 \frac{\psi(r)^2}{r} dr < \infty$, and there exists $\phi \in \mathcal{A}$ satisfying (1.10) such that*

$$\sup_{\mu \in C_b^w([0,T]; \mathcal{P}_k)} \left\{ \|(b^\mu)^{(0)}\|_\infty + [(b^\mu)^0]_\phi + \|\nabla a^\mu\|_\infty \right\} < \infty.$$

Then the following assertions hold.

(1) *If $\psi(r)^2 \log(1 + r^{-1}) \rightarrow 0$ as $r \rightarrow 0$, then there exists a constant $c > 0$ such that (1.11) holds, and for any $t \in (0, T]$, $\gamma, \tilde{\gamma} \in \mathcal{P}_k$,*

$$(1.12) \quad \begin{aligned} \text{Ent}(P_t^*\gamma | P_t^*\tilde{\gamma}) &\leq \frac{c\mathbb{W}_2(\gamma, \tilde{\gamma})^2}{t} \\ &+ c\mathbb{W}_k(\gamma, \tilde{\gamma})^2 \left(\frac{1}{t} \int_0^t \frac{\psi(r)^2}{r} dr + \frac{\psi(t^{\frac{1}{2}})^2}{t} \log(1 + t^{-1}) \right). \end{aligned}$$

(2) If either $\|b\|_\infty < \infty$ or

$$(1.13) \quad \sup_{(t,\mu) \in [0,T] \times \mathcal{P}_k} (\|\nabla^i b_t(\cdot, \mu)\|_\infty + \|\nabla^i \sigma_t(\cdot, \mu)\|_\infty) < \infty, \quad i = 1, 2,$$

then there exists a constant $c > 0$ such that (1.11) holds, and

$$(1.14) \quad \text{Ent}(P_t^* \gamma | P_t^* \tilde{\gamma}) \leq \frac{c \mathbb{W}_2(\gamma, \tilde{\gamma})^2}{t} + \frac{c \mathbb{W}_k(\gamma, \tilde{\gamma})^2}{t} \int_0^t \frac{\psi(r)^2}{r} dr, \quad t \in (0, T], \gamma, \tilde{\gamma} \in \mathcal{P}_k.$$

Remark 1.6. When $k \leq 2$, (1.8) follows from (1.14) or (1.12). This improves [13, Theorem 1.2], where the \mathbb{W}_2 -Lipschitz condition on the coefficients (a, b) is relaxed as the $(\mathbb{W}_\psi + \mathbb{W}_k)$ -Lipschitz condition.

2 Proof of Theorem 1.1

We first present a lemma to bound \mathbb{W}_ψ by the total variation distance and \mathbb{W}_1 .

Lemma 2.1. For any $\psi \in \mathcal{A}$,

$$\mathbb{W}_\psi(\gamma, \tilde{\gamma}) \leq \sqrt{d} \psi(\sqrt{t}) \|\gamma - \tilde{\gamma}\|_{var} + \frac{d\psi(\sqrt{t})}{\sqrt{t}} \mathbb{W}_1(\gamma, \tilde{\gamma}), \quad \gamma, \tilde{\gamma} \in \mathcal{P}_1.$$

Proof. Since ψ is nonnegative and concave, we have

$$(2.1) \quad \psi(Rr) \leq R\psi(r), \quad r \geq 0, R \geq 1.$$

For any function f on \mathbb{R}^d with $[f]_\psi \leq 1$, let

$$f_t(x) := \mathbb{E}[f(x + B_t)], \quad t \geq 0, x \in \mathbb{R}^d,$$

where B_t is the standard Brownian motion on \mathbb{R}^d with $B_0 = 0$. We have $\mathbb{E}[|B_t|^2] = dt$. By $[f]_\psi \leq 1$, Jensen's inequality and (2.1), we obtain

$$|f_t(x) - f(x)| \leq \mathbb{E}[\psi(|B_t|)] \leq \psi(\mathbb{E}|B_t|) \leq \psi((dt)^{\frac{1}{2}}) \leq \sqrt{d}\psi(t^{\frac{1}{2}}), \quad t \geq 0, x \in \mathbb{R}^d.$$

So,

$$(2.2) \quad \sup_{[f]_\psi \leq 1} |\gamma(f_t - f) - \tilde{\gamma}(f_t - f)| \leq \sqrt{d}\psi(t^{\frac{1}{2}}) \|\gamma - \tilde{\gamma}\|_{var}, \quad t \geq 0.$$

Next, for $[f]_\psi \leq 1$, by Jensen's inequality, (2.1), $\mathbb{E}|B_t|^2 = dt$ and $\mathbb{E}|B_t| \leq \sqrt{dt}$, we obtain

$$|\nabla f_t(x)| = \left| \nabla_x \int_{\mathbb{R}^d} (2\pi t)^{-\frac{d}{2}} e^{-\frac{|x-y|^2}{2t}} (f(y) - f(z)) dy \right|_{z=x}$$

$$\begin{aligned}
&\leq (2\pi t)^{-\frac{d}{2}} \int_{\mathbb{R}^d} \frac{|x-y|}{t} |f(y) - f(x)| e^{-\frac{|x-y|^2}{2t}} dy \leq \frac{1}{t} \mathbb{E}[|B_t| \psi(|B_t|)] \\
&\leq \frac{\mathbb{E}|B_t|}{t} \psi\left(\frac{\mathbb{E}|B_t|^2}{\mathbb{E}|B_t|}\right) = \frac{\mathbb{E}|B_t|}{t} \psi\left(\frac{(d\mathbb{E}|B_t|^2)^{\frac{1}{2}}}{\mathbb{E}|B_t|} t^{\frac{1}{2}}\right) \leq dt^{-\frac{1}{2}} \psi(t^{\frac{1}{2}}), \quad t > 0.
\end{aligned}$$

Combining this with (2.2) and noting that

$$\mathbb{W}_1(\gamma, \tilde{\gamma}) = \sup_{\|\nabla g\| \leq 1} |\gamma(g) - \tilde{\gamma}(g)|,$$

we derive that for any f with $[f]_\psi \leq 1$,

$$\begin{aligned}
|\gamma(f) - \tilde{\gamma}(f)| &\leq |\gamma(f_t - f) - \tilde{\gamma}(f_t - f)| + |\gamma(f_t) - \tilde{\gamma}(f_t)| \\
&\leq \sqrt{d} \psi(t^{\frac{1}{2}}) \|\gamma - \tilde{\gamma}\|_{var} + dt^{-\frac{1}{2}} \psi(t^{\frac{1}{2}}) \mathbb{W}_1(\gamma, \tilde{\gamma}), \quad t > 0.
\end{aligned}$$

Then the proof is finished. \square

Next, we present a gradient estimate on $P_{s,t}^{a,b}$. All constants in the following only depend on T, K, d and $(p_i, q_i)_{0 \leq i \leq l}$.

Lemma 2.2. *Assume $(A^{a,b})$ without (1.2). Then there exists a constant $c > 0$ such that for any $\psi \in \mathcal{A}$,*

$$\sup_{[f]_\psi \leq 1} \|\nabla P_{s,t}^{a,b} f\|_\infty \leq c(t-s)^{-\frac{1}{2}} \psi((t-s)^{\frac{1}{2}}), \quad 0 \leq s < t \leq T.$$

Proof. (a) By [17, Theorem 1.1] or [15, Theorem 2.1], there exists a constant $c_1 > 0$ such that for any $0 \leq s < t \leq T$ and $x \in \mathbb{R}^d$, the Bismut formula

$$(2.3) \quad \nabla P_{s,t}^{a,b} f(x) = \mathbb{E}[f(X_{s,t}^x) M_{s,t}^x]$$

holds for some random variable $M_{s,t}^x$ on \mathbb{R}^d with

$$(2.4) \quad \mathbb{E}[M_{s,t}^x] = 0, \quad \mathbb{E}|M_{s,t}^x|^2 \leq c_1^2(t-s)^{-1}.$$

So, for any $z \in \mathbb{R}^d$ and a function f with $[f]_\psi \leq 1$,

$$|\nabla P_{s,t}^{a,b} f(x)| = \left| \mathbb{E}[\{f(X_{s,t}^x) - f(z)\} M_{s,t}^x] \right| \leq \mathbb{E}[\psi(|X_{s,t}^x - z|) |M_{s,t}^x|].$$

By Jensen's inequality for the weighted probability $\frac{|M_{s,t}^x|^\mathbb{P}}{\mathbb{E}|M_{s,t}^x|}$, we obtain

$$\begin{aligned}
|\nabla P_{s,t}^{a,b} f(x)| &\leq \mathbb{E}[|M_{s,t}^x|] \psi\left(\frac{\mathbb{E}[|X_{s,t}^x - z| \cdot |M_{s,t}^x|]}{\mathbb{E}[|M_{s,t}^x|]}\right) \\
&\leq \mathbb{E}[|M_{s,t}^x|] \psi\left(\frac{(\mathbb{E}[|M_{s,t}^x|]^2)^{\frac{1}{2}}}{\mathbb{E}[|M_{s,t}^x|]} (\mathbb{E}|X_{s,t}^x - z|^2)^{\frac{1}{2}}\right).
\end{aligned}$$

Combining this with (2.1) and (2.4), we obtain

$$(2.5) \quad \sup_{[f]_\psi \leq 1} |\nabla P_{s,t}^{a,b} f(x)| \leq c_1(t-s)^{-\frac{1}{2}} \inf_{z \in \mathbb{R}^d} \psi \left(\{\mathbb{E}|X_{s,t}^x - z|^2\}^{\frac{1}{2}} \right), \quad 0 \leq s < t \leq T, x \in \mathbb{R}^d.$$

(b) To estimate $\inf_{z \in \mathbb{R}^d} \mathbb{E}|X_{s,t}^x - z|^2$, we use Zvonkin's transform. By [19, Theorem 2.1], there exist constants $\beta \in (0, 1)$ and $\lambda, C > 0$ such that the PDE

$$(2.6) \quad (\partial_t + L_t^{a,b} - \lambda)u_t = -b^{(0)}(t, \cdot), \quad t \in [0, T], u_T = 0$$

for $u : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ has a unique solution satisfying

$$(2.7) \quad \|u\|_\infty + \|\nabla u\|_\infty + \sup_{x \neq y} \frac{|\nabla u_t(x) - \nabla u_t(y)|}{|x - y|^\beta} \leq \frac{1}{2},$$

$$(2.8) \quad \|\nabla^2 u\|_{\tilde{L}_{q_0}^{p_0}(T)} + \|(\partial_t + b^{(1)} \cdot \nabla)u\|_{\tilde{L}_{q_0}^{p_0}(T)} \leq C.$$

By Itô's formula, $Y_{s,t} := \Theta_t(X_{s,t}^x)$, where $\Theta_t(y) := y + u_t(y)$, solves the SDE

$$dY_{s,t} = \bar{b}(t, Y_{s,t})dt + \bar{\sigma}(t, Y_{s,t})dW_t, \quad t \in [s, T], Y_{s,s} = x + u_s(x),$$

where

$$(2.9) \quad \bar{b}(t, \cdot) := (\lambda u_t + b^{(1)}) \circ \Theta_t^{-1}, \quad \bar{\sigma}(t, \cdot) := \{(\nabla \Theta_t)\sigma_t\} \circ \Theta_t^{-1}.$$

By (2.7), we find a constant $c_1 > 0$ such that

$$(2.10) \quad |\bar{b}(t, y) - \bar{b}(t, z)| \leq c_1|y - z|, \quad \|\bar{\sigma}(t, y)\| \leq c_1, \quad t \in [s, T], y, z \in \mathbb{R}^d.$$

Let

$$\frac{d}{dt}\theta_{s,t} = \bar{b}(t, \theta_{s,t}), \quad t \in [s, T], \theta_{s,s} = Y_{s,s} = x + u_s(x).$$

By Itô's formula and (2.10), we find a constant $c_2 > 0$ and a martingale M_t such that

$$\begin{aligned} d|Y_{s,t} - \theta_{s,t}|^2 &= \left\{ 2\langle Y_{s,t} - \theta_{s,t}, \bar{b}(t, Y_{s,t}) - \bar{b}(t, \theta_{s,t}) \rangle + \|\bar{\sigma}(t, Y_{s,t})\|_{HS}^2 \right\} dt + dM_t \\ &\leq c_2 \{ |Y_{s,t} - \theta_{s,t}|^2 + 1 \} dt + dM_t, \quad t \in [s, T], |Y_{s,s} - \theta_{s,s}| = 0. \end{aligned}$$

Thus,

$$\mathbb{E}[|Y_{s,t} - \theta_{s,t}|^2] \leq c_2 e^{c_2 T} (t-s), \quad 0 \leq s \leq t \leq T.$$

Taking $z_{s,t} = \Theta_t^{-1}(\theta_{s,t})$ and noting that $\|\nabla \Theta^{-1}\|_\infty < \infty$ due to $\|\nabla u\|_\infty \leq \frac{1}{2}$ in (2.7), we find a constant $c_3 > 0$ such that

$$\mathbb{E}[|X_{s,t}^x - z_{s,t}|^2] = \mathbb{E}[|\Theta_t^{-1}(Y_{s,t}) - \Theta_t^{-1}(\theta_{s,t})|^2] \leq c_3(t-s), \quad 0 \leq s \leq t \leq T.$$

Combining this with (2.5) and (2.1), we finish the proof. \square

Moreover, we estimate $\nabla_y p_{s,t}^{a,b}(x, y)$, where ∇_y is the gradient in y and $p_{s,t}^{a,b}(x, \cdot)$ is the density function of $\mathcal{L}_{X_{s,t}^x}$. For any constant $\kappa > 0$, let

$$g_\kappa(r, z) := (\pi \kappa r)^{-\frac{d}{2}} e^{-\frac{|z|^2}{\kappa r}}, \quad r > 0, z \in \mathbb{R}^d$$

be the standard Gaussian heat kernel with parameter κ .

Lemma 2.3. *Assume $(A^{a,b})$. Then for any $m \in (m_0, 2)$ there exists a constant $c(m) > 0$ such that for any $t \in (0, T]$ and $0 \leq g_{\cdot,t} \in \mathcal{B}([0, t])$,*

$$(2.11) \quad \begin{aligned} & \int_s^t \frac{g_{r,t}}{\sqrt{t-r}} dr \int_{\mathbb{R}^d} |\nabla_y p_{s,r}^{a,b}(x, y)| dy \\ & \leq c(m) \int_s^t \frac{g_{r,t}}{\sqrt{(t-r)(r-s)}} dr + c(m) \left(\int_s^t \left(\frac{g_{r,t}}{\sqrt{t-r}} \right)^m dr \right)^{\frac{1}{m}}, \quad s \in [0, t]. \end{aligned}$$

Consequently, there exists a constant $c > 0$ such that

$$(2.12) \quad \int_s^t (t-r)^{-\frac{1}{2}} dr \int_{\mathbb{R}^d} |\nabla_y p_{s,r}^{a,b}(x, y)| dy \leq c, \quad 0 \leq s < t \leq T.$$

Proof. Let u_t be in (2.6). By $(A^{a,b})$, $\sigma = \sqrt{2a}$, (2.7) and (2.9), we find a constant $c_1 > 0$ such that

$$|\bar{b}(t, x) - \bar{b}(t, y)| \leq c_1 |x - y|, \quad \|\bar{\sigma}(t, x) - \bar{\sigma}(t, y)\| \leq c_1 |x - y|^{\alpha \wedge \beta}, \quad t \in [0, T], x, y \in \mathbb{R}^d.$$

Let $\bar{p}_{s,t}(x, y)$ be the density function of $\mathcal{L}_{Y_{s,t}}$. According to [10, Theorem 1.2], there exists a constant $\kappa \geq 1$ and some $\theta_{s,t} : \mathbb{R}^d \rightarrow \mathbb{R}^d$ such that

$$(2.13) \quad |\nabla_y^i \bar{p}_{s,t}(x, y)| \leq \kappa (t-s)^{-\frac{i}{2}} g_\kappa(t-s, \theta_{s,t}(x) - y), \quad 0 \leq s < t \leq T, x, y \in \mathbb{R}^d, i = 0, 1,$$

where $\nabla^0 f := f$. Noting that $X_{s,t}^x = \Theta_t^{-1}(Y_{s,t})$, we have

$$(2.14) \quad p_{s,t}^{a,b}(x, y) = \bar{p}_{s,t}(\Theta_s(x), \Theta_t(y)) |\det(\nabla \Theta_t(y))|.$$

Combining this with (2.7), (2.10) and (2.13), we find a constant $c_2 > 0$ such that

$$(2.15) \quad \begin{aligned} |\nabla_y p_{s,t}^{a,b}(x, y)| & \leq c_2 \kappa (t-s)^{-\frac{1}{2}} g_\kappa(t-s, \theta_{s,t}(\Theta_s(x)) - \Theta_t(y)) |\det(\nabla \Theta_t(y))| \\ & \quad + c_2 \|\nabla^2 u_t(y)\| p_{s,t}^{a,b}(x, y), \quad 0 \leq s < t, x, y \in \mathbb{R}^d. \end{aligned}$$

Since $(p_0, q_0) \in \mathcal{K}$, for any $m > m_0$, we have

$$(2.16) \quad \tilde{p} := \frac{(m-1)p_0}{m} > 1, \quad \tilde{q} := \frac{(m-1)q_0}{m} > 1, \quad \frac{d}{\tilde{p}} + \frac{2}{\tilde{q}} < 2.$$

By Krylov's estimate, see [19, Theorem 3.1], we find a constant $c > 0$ such that

$$(2.17) \quad \begin{aligned} & \int_s^t dr \int_{\mathbb{R}^d} \|\nabla^2 u_r(y)\|^{\frac{m}{m-1}} p_{s,r}^{a,b}(x,y) dy \\ &= \mathbb{E} \int_s^t \|\nabla^2 u_r\|^{\frac{m}{m-1}} (X_{s,r}^x) dr \leq c \|\|\nabla^2 u\|^{\frac{m}{m-1}}\|_{\tilde{L}_{\tilde{q}}^{\tilde{p}}(s,t)} = c (\|\nabla^2 u\|_{\tilde{L}_{q_0}^{p_0}(s,t)})^{\frac{m}{m-1}}. \end{aligned}$$

This together with (2.8), (2.14) and (2.15) implies that for any $m \in (m_0, 2)$, there exists a constant $c(m) > 0$ such that

$$\begin{aligned} & \int_s^t \frac{g_{r,t}}{\sqrt{t-r}} dr \int_{\mathbb{R}^d} |\nabla_y p_{s,r}^{a,b}(x,y)| dy \leq c_2 \kappa \int_s^t g_{r,t} (t-r)^{-\frac{1}{2}} (r-s)^{-\frac{1}{2}} dr \\ &+ c_2 \left(\int_s^t \left(\frac{g_{r,t}}{\sqrt{t-r}} \right)^m dr \right)^{\frac{1}{m}} \left(\int_s^t dr \int_{\mathbb{R}^d} \|\nabla^2 u_r(y)\|^{\frac{m}{m-1}} p_{s,r}^{a,b}(x,y) dy \right)^{\frac{m-1}{m}} \\ &\leq c(m) \int_s^t \frac{g_{r,t}}{\sqrt{(t-r)(r-s)}} dr + c(m) \left(\int_s^t \left(\frac{g_{r,t}}{\sqrt{t-r}} \right)^m dr \right)^{\frac{1}{m}}. \end{aligned}$$

So, (2.11) holds. Letting $g_{r,t} \equiv 1$ and $m = \frac{m_0+2}{2}$, we find a constant $c > 0$ such that (2.11) implies (2.12). \square

Proof of Theorem 1.1. By (1.1), it suffices to prove for $\gamma = \delta_x, \tilde{\gamma} = \delta_y, x, y \in \mathbb{R}^d$.

(a) We first consider $x = y$. Let $f \in C_b^2(\mathbb{R}^d)$ with $[f]_\psi \leq 1$. By Itô's formula we have

$$P_{s,t}^{a_2,b_2} f(x) = f(x) + \int_s^t P_{s,r}^{a_2,b_2} (L_r^{a_2,b_2} f)(x) dr, \quad 0 \leq s \leq t \leq T.$$

This implies the Kolmogorov forward equation

$$(2.18) \quad \partial_t P_{s,t}^{a_2,b_2} f = P_{s,t}^{a_2,b_2} (L_t f), \quad \text{a.e. } t \in [s, T].$$

On the other hand, for $(p, q) \in \mathcal{K}$ and $t \in (0, T]$, let $\tilde{W}_{1,q,b_2^{(1)}}^{2,p}(0, t)$ be the set of all maps $u : [0, t] \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ satisfying

$$\|u\|_{0 \rightarrow t, \infty} + \|\nabla u\|_{0 \rightarrow t, \infty} + \|\nabla^2 u\|_{\tilde{L}_q^p(t)} + \|(\partial_s + b_2^{(1)} \cdot \nabla) u\|_{\tilde{L}_q^p(t)} < \infty.$$

By [19, Theorem 2.1], the PDE

$$(2.19) \quad (\partial_s + L_s^{a_2,b_2}) u_s = -L_s^{a_2,b_2} f, \quad s \in [0, t], u_t = 0$$

has a unique solution in the class $\tilde{W}_{1,q,b_2^{(1)}}^{2,p}(0, t)$. So, by Itô's formula [19, Lemma 3.3],

$$du_r(X_{s,r}^{2,x}) = -L_r^{a_2,b_2} f(X_{s,r}^{2,x}) + dM_r, \quad r \in [s, t]$$

holds for some martingale M_r . This and (2.18) yield

$$\begin{aligned} 0 &= \mathbb{E} u_t(X_{s,t}^{2,x}) = u_s(x) - \int_s^t (P_{s,r}^{a_2,b_2} L_r^{a_2,b_2} f) dr \\ &= u_s(x) - \int_s^t \frac{d}{dr} (P_{s,r}^{a_2,b_2} f) dr = u_s(x) - P_{s,t}^{a_2,b_2} f + f, \quad 0 \leq s \leq t \leq T. \end{aligned}$$

Combining this with (2.19), we derive $P_{\cdot,t}^{a_2,b_2} f \in \tilde{W}_{1,q,b_2^{(1)}}^{2,p}(0,t)$ for $t \in (0,T]$ and the Kolmogorov backward equation

$$(2.20) \quad \partial_s P_{s,t}^{a_2,b_2} f = \partial_s u_s = -L_s^{a_2,b_2} (u_s + f) = -L_s^{a_2,b_2} P_{s,t}^{a_2,b_2} f, \quad 0 \leq s \leq t \leq T.$$

By Itô's formula to $P_{r,t}^{a_2,b_2} f(X_{s,r}^{1,x})$ for $r \in [s,t]$, see [19, Lemma 3.3], we derive

$$\begin{aligned} P_{s,t}^{a_1,b_1} f(x) - P_{s,t}^{a_2,b_2} f(x) &= \mathbb{E} \int_s^t (\partial_r + L_r^{a_1,b_1}) P_{r,t}^{a_2,b_2} f(X_{s,r}^{1,x}) dr \\ &= \int_s^t dr \int_{\mathbb{R}^d} p_{s,r}^{a_1,b_1}(x,y) (L_r^{a_1,b_1} - L_r^{a_2,b_2}) P_{r,t}^{a_2,b_2} f(y) dy. \end{aligned}$$

By the integration by parts formula, we obtain

$$\begin{aligned} &\left| \int_{\mathbb{R}^d} p_{s,r}^{a_1,b_1}(x,y) [\text{tr}\{(a_1 - a_2)(r,y) \nabla^2 P_{r,t}^{a_2,b_2} f(y)\}] dy \right| \\ &= \left| \int_{\mathbb{R}^d} \left\langle (a_1 - a_2)(r,y) \nabla_y p_{s,r}^{a_1,b_1}(x,y) + p_{s,r}^{a_1,b_1}(x,y) \text{div}(a_1 - a_2)(r,y), \nabla P_{r,t}^{a_2,b_2} f(y) \right\rangle dy \right|. \end{aligned}$$

Combining these with Lemma 2.2 and Lemma 2.3, for any $m \in (m_0, 2)$, we find constants $c_1, c_2 > 0$ such that

$$\begin{aligned} |P_{s,t}^{a_1,b_1} f(x) - P_{s,t}^{a_2,b_2} f(x)| &\leq c_1 \int_s^t \frac{\psi((t-r)^{\frac{1}{2}}) \|a_1 - a_2\|_{r,\infty}}{\sqrt{t-r}} dr \int_{\mathbb{R}^d} |\nabla_y p_{s,r}^{a_1,b_1}(x,y)| dy \\ &\quad + c_1 \int_s^t \frac{\psi((t-r)^{\frac{1}{2}})}{(t-r)^{\frac{1}{2}}} (\|b_1 - b_2\|_{r,\infty} + \|\text{div}(a_1 - a_2)\|_{r,\infty}) dr \\ &\leq c_2 \int_s^t \frac{\psi((t-r)^{\frac{1}{2}})}{\sqrt{t-r}} \left(\frac{\|a_1 - a_2\|_{r,\infty}}{\sqrt{r-s}} + \|b_1 - b_2\|_{r,\infty} + \|\text{div}(a_1 - a_2)\|_{r,\infty} \right) dr \\ &\quad + c_2 \left(\int_s^t \left(\frac{\psi((t-r)^{\frac{1}{2}}) \|a_1 - a_2\|_{r,\infty}}{\sqrt{t-r}} \right)^m dr \right)^{\frac{1}{m}} =: I_{s,t}. \end{aligned}$$

Therefore,

$$(2.21) \quad \mathbb{W}_\psi(P_{s,t}^{1,x}, P_{s,t}^{2,x}) \leq I_{s,t}, \quad 0 \leq s < t \leq T, \quad x \in \mathbb{R}^d.$$

(b) Let $x, y \in \mathbb{R}^d$ and $0 \leq s < t \leq T$. By the triangle inequality for \mathbb{W}_ψ , (2.21) and Lemma 2.1, we obtain

$$\begin{aligned} \mathbb{W}_\psi(P_{s,t}^{1,x}, P_{s,t}^{2,y}) &\leq \mathbb{W}_\psi(P_{s,t}^{1,x}, P_{s,t}^{2,x}) + \mathbb{W}_\psi(P_{s,t}^{2,x}, P_{s,t}^{2,y}) \\ (2.22) \quad &\leq I_{s,t} + \psi((t-s)^{\frac{1}{2}}) \|P_{s,t}^{2,x} - P_{s,t}^{2,y}\|_{var} + \frac{\psi((t-s)^{\frac{1}{2}})}{\sqrt{t-s}} \mathbb{W}_1(P_{s,t}^{2,x}, P_{s,t}^{2,y}). \end{aligned}$$

By [15, Theorem 2.1] or [17, Theorem 1.1], $(A^{a,b})$ for $(a, b) = (a_2, b_2)$ implies that for some constant $c_3 > 0$,

$$\mathbb{W}_1(P_{s,t}^{2,x}, P_{s,t}^{2,y}) \leq c_3 |x - y|, \quad \|P_{s,t}^{2,x} - P_{s,t}^{2,y}\|_{var} \leq \frac{c_3}{\sqrt{t-s}} |x - y|$$

holds for any $0 \leq s < t \leq T$ and $x, y \in \mathbb{R}^d$. Combining this with (2.22), we derive (1.5) for $\gamma = \delta_x$ and $\tilde{\gamma} = \delta_y$.

(c) It remains to prove (1.6). Let u be in (2.6) for $(a, b) = (a_1, b_1)$. Let $\Theta_t(y) := y + u_t(y)$, and

$$Y_{s,t}^{1,x} = \Theta_t(X_{s,t}^{1,x}), \quad Y_{s,t}^{2,y} = \Theta_t(X_{s,t}^{2,y}), \quad t \in [s, T].$$

By Itô's formula [19, Lemma 3.3], we obtain

$$\begin{aligned} dY_{s,t}^{1,x} &= \{b_1^{(1)}(t, \cdot) + \lambda u_t\}(X_{s,t}^{1,x}) dt + \{(\nabla \Theta_t)\sigma_1(t, \cdot)\}(X_{s,t}^{1,x}) dW_t, \\ dY_{s,t}^{2,y} &= \{b_1^{(1)}(t, \cdot) + \lambda u_t\}(X_{s,t}^{2,y}) dt + \{(\nabla \Theta_t)(b_2 - b_1) + \text{tr}[(a_2 - a_1)(t, \cdot) \nabla^2 u_t]\}(X_{s,t}^{2,y}) dt \\ &\quad + \{(\nabla \Theta_t)\sigma_2(t, \cdot)\}(X_{s,t}^{2,y}) dW_t, \quad t \in [s, T], \quad Y_{s,s}^{1,x} = \Theta_s(x), \quad Y_{s,s}^{2,y} = \Theta_s(y). \end{aligned}$$

For any non-negative function f on \mathbb{R}^d , let

$$\mathcal{M}f(x) := \sup_{r \in (0,1]} \frac{1}{|B(x, r)|} \int_{B(x, r)} f(y) dy, \quad x \in \mathbb{R}^d, B(x, r) := \{y \in \mathbb{R}^d : |y - x| < r\}.$$

By $(A^{a,b})$ for $a = a_i$, $\sigma_i = \sqrt{2a_i}$, (2.7), the maximal inequality in [17, Lemma 2.1], and Itô's formula, for any $k \geq 1$ we find a constant $c_1 > 1$ such that

$$(2.23) \quad c_1^{-1} |X_{s,t}^{1,x} - X_{s,t}^{2,y}|^{2k} \leq \xi_t := |Y_{s,t}^{1,x} - Y_{s,t}^{2,y}|^{2k} \leq c_1 |X_{s,t}^{1,x} - X_{s,t}^{2,y}|^{2k},$$

$$(2.24) \quad d\xi_t \leq c_1 \xi_t (1 + \eta_t) dt + c_1 \xi_t^{\frac{2k-1}{2k}} \gamma_t dt + c_1 \xi_t^{\frac{k-1}{k}} \|a_1 - a_2\|_{t,\infty}^2 dt + dM_t,$$

where M_t is a martingale and

$$\begin{aligned} \gamma_t &:= \|b_1 - b_2\|_{t,\infty} + \|a_1 - a_2\|_{t,\infty} \|\nabla^2 u_t\|(X_{s,t}^{2,y}), \\ \eta_t &:= \mathcal{M}(\|\nabla \sigma_1\|_{t,\infty}^2 + \|\nabla^2 u\|^2)(X_{s,t}^{1,x}) + \mathcal{M}(\|\nabla \sigma_1\|_{t,\infty}^2 + \|\nabla^2 u\|^2)(X_{s,t}^{2,y}). \end{aligned}$$

Note that for $q \in (\frac{2k-1}{2k}, 1)$,

$$\mathbb{E} \left\{ \left(\sup_{r \in [s,t]} \xi_r^q \right)^{\frac{2k-1}{2kq}} \int_s^t \|a_1 - a_2\|_{r,\infty} \|\nabla^2 u_r\|(X_{s,r}^{2,y}) dr \right\}$$

$$\begin{aligned}
&\leq \left(\mathbb{E} \sup_{r \in [s,t]} \xi_r^q \right)^{\frac{2k-1}{2kq}} \left(\mathbb{E} \left(\int_s^t \|a_1 - a_2\|_{r,\infty} \|\nabla^2 u_r\| (X_{s,r}^{2,y}) dr \right)^{\frac{2kq}{2kq-2k+1}} \right)^{\frac{2kq-2k+1}{2kq}} \\
&\leq \left(\mathbb{E} \sup_{r \in [s,t]} \xi_r^q \right)^{\frac{2k-1}{2kq}} \left(\int_s^t \|a_1 - a_2\|_{r,\infty}^m dr \right)^{\frac{1}{m}} \\
&\quad \times \left(\mathbb{E} \left(\int_s^t \|\nabla^2 u_r\|^{\frac{m}{m-1}} (X_{s,r}^{2,y}) dr \right)^{\frac{2(m-1)kq}{m(2kq-2k+1)}} \right)^{\frac{2kq-2k+1}{2kq}}, \quad m > 1.
\end{aligned}$$

So, by the stochastic Grownwall inequality [18, Lemma 2.8] for $q \in (\frac{2k-1}{2k}, 1)$, [17, Lemma 2.1], and the Krylov estimate in [19, Theorem 3.1] which implies the Khasminskii inequality in [18, Lemma 3.5], we find constants $c_2, c_3 > 0$ such that

$$\begin{aligned}
\left[\mathbb{E} \sup_{r \in [s,t]} \xi_r^q \right]^{\frac{1}{q}} &\leq c_2 |x - y|^{2k} + c_2 \mathbb{E} \int_s^t \left\{ \xi_r^{\frac{2k-1}{2k}} \gamma_r dr + \xi_r^{\frac{k-1}{k}} \|a_1 - a_2\|_{r,\infty}^2 dr \right\} dr \\
&\leq c_2 |x - y|^{2k} + c_2 \mathbb{E} \left[\left(\sup_{r \in [s,t]} \xi_r^q \right)^{\frac{2k-1}{2kq}} \int_s^t \gamma_r dr + \left(\sup_{r \in [s,t]} \xi_r^q \right)^{\frac{k-1}{kq}} \int_s^t \|a_1 - a_2\|_{r,\infty}^2 dr \right] \\
&\leq c_2 |x - y|^{2k} + \frac{1}{2} \left[\mathbb{E} \sup_{r \in [s,t]} \xi_r^q \right]^{\frac{1}{q}} + c_3 \left(\int_s^t \|a_1 - a_2\|_{r,\infty}^2 dr \right)^k + c_3 \left(\int_s^t \|b_1 - b_2\|_{r,\infty} dr \right)^{2k} \\
&\quad + c_3 \left(\int_s^t \|a_1 - a_2\|_{r,\infty}^m dr \right)^{\frac{2k}{m}} \left(\mathbb{E} \left(\int_s^t \|\nabla^2 u_r\|^{\frac{m}{m-1}} (X_{s,r}^{2,y}) dr \right)^{\frac{2(m-1)kq}{m(2kq-2k+1)}} \right)^{\frac{2kq-2k+1}{q}}, \quad m > 1.
\end{aligned}$$

Noting that [11, Theorem 2.1(3)] implies

$$\left[\mathbb{E} \sup_{r \in [s,t]} \xi_r^q \right] < \infty,$$

we obtain

$$\begin{aligned}
\left[\mathbb{E} \sup_{r \in [s,t]} \xi_r^q \right]^{\frac{1}{q}} &\leq 2c_2 |x - y|^{2k} + 2c_3 \left(\int_s^t \|a_1 - a_2\|_{r,\infty}^2 dr \right)^k \\
(2.25) \quad &\quad + 2c_3 \left(\int_s^t \|b_1 - b_2\|_{r,\infty} dr \right)^{2k} \\
&\quad + 2c_3 \left(\int_s^t \|a_1 - a_2\|_{r,\infty}^m dr \right)^{\frac{2k}{m}} \left(\mathbb{E} \left(\int_s^t \|\nabla^2 u_r\|^{\frac{m}{m-1}} (X_{s,r}^{2,y}) dr \right)^{\frac{2(m-1)kq}{m(2kq-2k+1)}} \right)^{\frac{2kq-2k+1}{q}}.
\end{aligned}$$

Recall that (\tilde{p}, \tilde{q}) is defined in (2.16). By (2.8), [19, Theorem 3.1] and [18, Lemma 3.5], we find a constant $c_4 > 0$ such that

$$\mathbb{E} \left(\int_s^t \|\nabla^2 u_r\|^{\frac{m}{m-1}} (X_{s,r}^{2,y}) dr \right)^{\frac{2(m-1)kq}{m(2kq-2k+1)}}$$

$$\leq c_4(\|\|\nabla^2 u\|^{\frac{m}{m-1}}\|_{\tilde{L}_{\tilde{q}}^{\tilde{p}}(s,t)})^{\frac{2(m-1)kq}{m(2kq-2k+1)}} = c_4(\|\nabla^2 u\|_{\tilde{L}_{q_0}^{p_0}(0,T)})^{\frac{2kq}{2kq-2k+1}} < \infty.$$

Combining this with (2.25), we find a constant $c_5 > 0$ such that

$$\begin{aligned} (\mathbb{E}|Y_{s,t}^{1,x} - Y_{s,t}^{2,y}|^k)^2 &\leq \left[\mathbb{E} \sup_{r \in [s,t]} \xi_r^q \right]^{\frac{1}{q}} \leq c_5|x - y|^{2k} + c_5 \left(\int_s^t \|b_1 - b_2\|_{r,\infty} dr \right)^{2k} \\ &\quad + c_5 \left(\int_s^t \|a_1 - a_2\|_{r,\infty}^m dr \right)^{\frac{2k}{m}} + c_5 \left(\int_s^t \|a_1 - a_2\|_{r,\infty}^2 dr \right)^k. \end{aligned}$$

Noting that (2.23) implies

$$\mathbb{W}_k(P_{s,t}^{1,x}, P_{s,t}^{2,y})^k \leq \sqrt{c_1} \mathbb{E}|Y_{s,t}^{1,x} - Y_{s,t}^{2,y}|^k,$$

by Jensen's inequality we derive (1.6) for some constant $C > 0$ and $\gamma = \delta_x, \tilde{\gamma} = \delta_y$. \square

3 Proof of Theorem 1.3

Once the well-posedness of (1.7) is proved, the proof of [7, (1.5)] implies (1.9) under $(B^{a,b})$. We skip the details to save space. So, in the following we only prove the well-posedness and estimate (1.11).

(a) Let X_0 be \mathcal{F}_0 -measurable with $\gamma := \mathcal{L}_{X_0} \in \mathcal{P}_k$. Let

$$\mathcal{C}_T^\gamma := \{\mu \in C([0, T]; \mathcal{P}_k) : \mu_0 = \gamma\}.$$

For any $\lambda \geq 0$, C_T^γ is a complete space under the metric

$$\rho_\lambda(\mu, \tilde{\mu}) := \sup_{t \in [0, T]} e^{-\lambda t} \{ \mathbb{W}_\psi(\mu_t, \tilde{\mu}_t) + \mathbb{W}_k(\mu_t, \tilde{\mu}_t) \}.$$

For any $\mu \in C([0, T]; \mathcal{P}_k)$, let

$$b_t^\mu(x) := b_t(x, \mu_t), \quad \sigma_t^\mu(x) = \sigma_t(x, \mu_t), \quad (t, x) \in [0, T] \times \mathbb{R}^d.$$

According to [11, Theorem 2.1], $(B^{a,b})$ implies that the SDE

$$dX_t^\mu = b_t^\mu(X_t^\mu)dt + \sigma_t^\mu(X_t^\mu)dW_t, \quad t \in [0, T], X_0^\mu = X_0$$

is well-posed, and

$$\mathbb{E} \left[\sup_{s \in [0, T]} |X_s^\mu|^k \right] < \infty.$$

So, we define a map

$$\Phi^\gamma : \mathcal{C}_T^\gamma \rightarrow \mathcal{C}_T^\gamma; \quad \mu \mapsto \{(\Phi^\gamma \mu)_t := \mathcal{L}_{X_t^\mu}\}_{t \in [0, T]}.$$

According to [9, Theorem 3.1], if Φ^γ has a unique fixed point in \mathcal{C}_T^γ , then (1.7) is well-posed for distributions in \mathcal{P}_k .

(b) Let $\tilde{\gamma} \in \mathcal{P}_k$ which may be different from γ , and let $\tilde{\mu} \in \mathcal{C}_T^{\tilde{\gamma}}$. We estimate the ρ_λ -distance between $\Phi^\gamma\mu$ and $\Phi^{\tilde{\gamma}}\tilde{\mu}$. By Theorem 1.1 and $(B^{a,b})$, for any $m \in (m_0, 2)$, there exist constants $c_1, c_2 > 0$ such that

$$\begin{aligned} & \mathbb{W}_\psi((\Phi^\gamma\mu)_t, (\Phi^{\tilde{\gamma}}\tilde{\mu})_t) + \mathbb{W}_k((\Phi^\gamma\mu)_t, (\Phi^{\tilde{\gamma}}\tilde{\mu})_t) \\ & \leq \frac{c_1\psi(t^{\frac{1}{2}})}{\sqrt{t}} \mathbb{W}_k(\gamma, \tilde{\gamma}) + c_1 \left(\int_0^t \|a^\mu - a^{\tilde{\mu}}\|_{r,\infty}^2 dr \right)^{\frac{1}{2}} \\ & \quad + c_1 \left(\int_0^t \left(\frac{\psi((t-r)^{\frac{1}{2}}) \|a^\mu - a^{\tilde{\mu}}\|_{r,\infty}}{\sqrt{t-r}} \right)^m dr \right)^{\frac{1}{m}} \\ & \quad + c_1 \int_0^t \frac{c_1\psi((t-r)^{\frac{1}{2}})}{\sqrt{t-r}} \left(\frac{\|a^\mu - a^{\tilde{\mu}}\|_{r,\infty}}{\sqrt{r}} + \|b^\mu - b^{\tilde{\mu}}\|_{r,\infty} + \|\operatorname{div}(a^\mu - a^{\tilde{\mu}})\|_{r,\infty} \right) dr \\ & \leq \frac{c_1\psi(t^{\frac{1}{2}})}{\sqrt{t}} \mathbb{W}_k(\gamma, \tilde{\gamma}) + c_2 \left(\int_0^t \left(\mathbb{W}_\psi(\mu_r, \tilde{\mu}_r) + \mathbb{W}_k(\mu_r, \tilde{\mu}_r) \right)^2 dr \right)^{\frac{1}{2}} \\ & \quad + c_2 \left(\int_0^t \left(\frac{\psi((t-r)^{\frac{1}{2}})(\mathbb{W}_\psi(\mu_r, \tilde{\mu}_r) + \mathbb{W}_k(\mu_r, \tilde{\mu}_r))}{\sqrt{t-r}} \right)^m dr \right)^{\frac{1}{m}} \\ & \quad + c_2 \int_0^t \frac{\psi((t-r)^{\frac{1}{2}})}{\sqrt{r(t-r)}} (1 + \sqrt{r}\rho_r) (\mathbb{W}_\psi(\mu_r, \tilde{\mu}_r) + \mathbb{W}_k(\mu_r, \tilde{\mu}_r)) dr \end{aligned}$$

Let $\gamma = \tilde{\gamma}$. We obtain

$$\rho_\lambda(\Phi^\gamma\mu, \Phi^\gamma\tilde{\mu}) \leq \delta(\lambda)\rho_\lambda(\mu, \tilde{\mu}),$$

where by $(B^{a,b})$ and $m \in (m_0, 2)$, as $\lambda \rightarrow \infty$ we have

$$\begin{aligned} \delta(\lambda) &:= c_2 \sup_{t \in [0, T]} \left[\int_0^t \frac{\psi((t-r)^{\frac{1}{2}}) e^{-\lambda(t-r)}}{\sqrt{t-r}} \left(\frac{1}{\sqrt{r}} + \rho_r \right) dr + \left(\int_0^t e^{-2\lambda(t-r)} dr \right)^{\frac{1}{2}} \right] \\ & \quad + c_2 \left(\int_0^t \left(\frac{\psi((t-r)^{\frac{1}{2}}) e^{-\lambda(t-r)}}{\sqrt{t-r}} \right)^m dr \right)^{\frac{1}{m}} \rightarrow 0. \end{aligned}$$

So, Φ^γ is ρ_λ -contractive on \mathcal{C}_T^γ for large $\lambda > 0$, and hence has a unique fixed point. This implies the well-posedness of (1.7) for distributions in \mathcal{P}_k .

(c) For $s \in [0, T]$, let $P_{s,t}^* \gamma = \mathcal{L}_{X_{s,t}^\gamma}$, where $X_{s,t}^\gamma$ solves (1.7) for $t \in [s, T]$ and $\mathcal{L}_{X_{s,s}^\gamma} = \gamma$. By (1.9) for s replacing 0, we have

$$\sup_{t \in [s, T]} (P_{s,t}^* \gamma)(|\cdot|^k) < \infty, \quad \gamma \in \mathcal{P}_k.$$

Since ψ has growth slower than linear, and (2.1) implies the boundedness of $\frac{r}{\psi(r)}$ for $r \in [0, T]$, this implies that for any $\gamma, \tilde{\gamma} \in \mathcal{P}_k$ and $s \in [0, T]$,

$$(3.1) \quad \sup_{r \in [s, t]} (\mathbb{W}_\psi + \mathbb{W}_k)(P_{s,r}^* \gamma, P_{s,r}^* \tilde{\gamma}) < \infty, \quad t \in [s, T],$$

$$(3.2) \quad \Gamma_{s,t} := \sup_{r \in [s, t]} \frac{\sqrt{r-s}}{\psi((r-s)^{\frac{1}{2}})} (\mathbb{W}_\psi + \mathbb{W}_k)(P_{s,r}^* \gamma, P_{s,r}^* \tilde{\gamma}) < \infty, \quad t \in [s, T].$$

Let

$$\begin{aligned} a_1(t, x) &:= a_t(x, P_{s,t}^* \gamma), \quad b_1(t, x) := b_t(x, P_{s,t}^* \gamma), \\ a_2(t, x) &:= a_t(x, P_{s,t}^* \tilde{\gamma}), \quad b_1(t, x) := b_t(x, P_{s,t}^* \tilde{\gamma}), \quad (t, x) \in [s, T] \times \mathbb{R}^d. \end{aligned}$$

Then $P_{s,t}^* \gamma = P_{s,t}^{1,\gamma}$, $P_{s,t}^* \tilde{\gamma} = P_{s,t}^{2,\tilde{\gamma}}$, and (1.1) implies

$$(3.3) \quad P_{s,t}^* \gamma = \int_{\mathbb{R}^d} P_{s,t}^{1,x} \gamma(dx), \quad P_{s,t}^* \tilde{\gamma} = \int_{\mathbb{R}^d} P_{s,t}^{2,x} \tilde{\gamma}(dx).$$

Thus, by Theorem 1.1 and $(B^{a,b})$, for any $m \in (m_0, 2)$, we find a constant $k_0 > 0$ such that

$$\begin{aligned} \mathbb{W}_\psi(P_{s,t}^* \gamma, P_{s,t}^* \tilde{\gamma}) &= \mathbb{W}_\psi(P_{s,t}^{1,\gamma}, P_{s,t}^{2,\tilde{\gamma}}) \leq \frac{k_0 \psi((t-s)^{\frac{1}{2}})}{\sqrt{t-s}} \mathbb{W}_1(\gamma, \tilde{\gamma}) \\ (3.4) \quad &+ k_0 \int_s^t \frac{\psi((t-r)^{\frac{1}{2}})}{\sqrt{(r-s)(t-r)}} \left(1 + \rho_r \sqrt{r-s}\right) (\mathbb{W}_\psi + \mathbb{W}_k)(P_{s,r}^* \gamma, P_{s,r}^* \tilde{\gamma}) dr \\ &+ k_0 \left(\int_s^t \left(\frac{\psi((t-r)^{\frac{1}{2}}) (\mathbb{W}_\psi + \mathbb{W}_k)(P_{s,r}^* \gamma, P_{s,r}^* \tilde{\gamma})}{\sqrt{t-r}} \right)^m dr \right)^{\frac{1}{m}}, \end{aligned}$$

$$\begin{aligned} \mathbb{W}_k(P_{s,t}^* \gamma, P_{s,t}^* \tilde{\gamma}) &= \mathbb{W}_k(P_{s,t}^{1,\gamma}, P_{s,t}^{2,\tilde{\gamma}}) \leq k_0 \mathbb{W}_k(\gamma, \tilde{\gamma}) \\ (3.5) \quad &+ k_0 \int_s^t \rho_r (\mathbb{W}_\psi + \mathbb{W}_k)(P_{s,r}^* \gamma, P_{s,r}^* \tilde{\gamma}) dr + k_0 \left(\int_s^t (\mathbb{W}_\psi + \mathbb{W}_k)^2(P_{s,r}^* \gamma, P_{s,r}^* \tilde{\gamma}) dr \right)^{\frac{1}{2}}. \end{aligned}$$

By combining these with the definition of $\Gamma_{s,t}$ in (3.2), we find a constant $k_1 > 0$ such that

$$\begin{aligned} \Gamma_{s,t} &\leq k_1 \mathbb{W}_k(\gamma, \tilde{\gamma}) + k_1 \Gamma_{s,t} h(t-s), \quad 0 \leq s < t \leq T, \\ h(t) &:= \sup_{(s,\theta) \in (0,t] \times [0,T-t]} \frac{\sqrt{s}}{\psi(s^{\frac{1}{2}})} \int_0^s \frac{\psi(r^{\frac{1}{2}}) \psi((s-r)^{\frac{1}{2}})}{\sqrt{r(s-r)}} \left(\frac{1}{\sqrt{r}} + \rho_{\theta+r} \right) dr \\ (3.6) \quad &+ \sup_{s \in (0,t]} \frac{\sqrt{s}}{\psi(s^{\frac{1}{2}})} \left(\int_0^s \left(\frac{\psi((s-r)^{\frac{1}{2}}) \psi(r^{\frac{1}{2}})}{\sqrt{r} \sqrt{s-r}} \right)^m dr \right)^{\frac{1}{m}} \\ &+ \left(\int_0^t \left(\frac{\psi(r^{\frac{1}{2}})}{\sqrt{r}} \right)^2 dr \right)^{\frac{1}{2}}, \quad t \in (0, T]. \end{aligned}$$

Note that

$$\begin{aligned}
& \frac{\sqrt{s}}{\psi(s^{\frac{1}{2}})} \int_0^s \frac{\psi(r^{\frac{1}{2}})\psi((s-r)^{\frac{1}{2}})}{r\sqrt{s-r}} dr \\
(3.7) \quad & \leq \frac{\sqrt{s}}{\psi(s^{\frac{1}{2}})} \left(\int_0^{\frac{s}{2}} \frac{\psi(s^{\frac{1}{2}})}{\sqrt{s/2}} \cdot \frac{\psi(r^{\frac{1}{2}})}{r} dr + \int_{\frac{s}{2}}^s \frac{\psi((s-r)^{\frac{1}{2}})}{s-r} \cdot \frac{\sqrt{s}\psi(s^{\frac{1}{2}})}{s/2} dr \right) \\
& \leq (2 + \sqrt{2}) \int_0^{\frac{s}{2}} \frac{\psi(r^{\frac{1}{2}})}{r} dr = 2(2 + \sqrt{2}) \int_0^{\sqrt{s/2}} \frac{\psi(r)}{r} dr.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
& \frac{\sqrt{s}}{\psi(s^{\frac{1}{2}})} \left(\int_0^s \left(\frac{\psi((s-r)^{\frac{1}{2}})\psi(r^{\frac{1}{2}})}{\sqrt{r}\sqrt{s-r}} \right)^m dr \right)^{\frac{1}{m}} \\
(3.8) \quad & \leq \sqrt{2} \left(\left(\int_0^{\frac{s}{2}} \left(\frac{\psi(r^{\frac{1}{2}})}{\sqrt{r}} \right)^m dr \right)^{\frac{1}{m}} + \left(\int_{\frac{s}{2}}^s \left(\frac{\psi((s-r)^{\frac{1}{2}})}{s-r} \right)^m dr \right)^{\frac{1}{m}} \right) \\
& \leq 2\sqrt{2} \left(\int_0^{\frac{s}{2}} \left(\frac{\psi(r^{\frac{1}{2}})}{\sqrt{r}} \right)^m dr \right)^{\frac{1}{m}}, \\
& \frac{\sqrt{s}}{\psi(s^{\frac{1}{2}})} \int_0^s \frac{\psi(r^{\frac{1}{2}})\psi((s-r)^{\frac{1}{2}})}{\sqrt{r(s-r)}} \rho_{\theta+r} dr \\
(3.9) \quad & = \frac{\sqrt{s}}{\psi(s^{\frac{1}{2}})} \left(\int_0^{\frac{s}{2}} \frac{\psi(s^{\frac{1}{2}})}{\sqrt{s/2}} \cdot \frac{\psi(r^{\frac{1}{2}})}{\sqrt{r}} \rho_{\theta+r} dr + \int_{\frac{s}{2}}^s \frac{\psi((s-r)^{\frac{1}{2}})}{\sqrt{s-r}} \cdot \frac{\sqrt{s}\psi(s^{\frac{1}{2}})}{s/\sqrt{2}} \rho_{\theta+r} dr \right) \\
& \leq 2\sqrt{2} \int_0^s \left(\frac{\psi(r^{\frac{1}{2}})}{\sqrt{r}} + \frac{\psi((s-r)^{\frac{1}{2}})}{\sqrt{s-r}} \right) \rho_{\theta+r} dr \leq 4\sqrt{2} \left(\int_0^s \frac{\psi(r^{\frac{1}{2}})^2}{r} dr \right)^{\frac{1}{2}} \left(\int_0^T \rho_r^2 dr \right)^{\frac{1}{2}}.
\end{aligned}$$

Combining these with (1.10), we conclude that $h(t)$ defined in (3.6) satisfies $h(t) \rightarrow 0$ as $t \rightarrow 0$. Letting $r_0 > 0$ such that $k_1 h(t) \leq \frac{1}{2}$ for $t \in [0, r_0]$, we deduce from (3.2) and (3.6) that

$$\frac{\sqrt{t-s}}{\psi((t-s)^{\frac{1}{2}})} (\mathbb{W}_\psi + \mathbb{W}_k)(P_{s,t}^* \gamma, P_{s,t}^* \tilde{\gamma}) \leq \Gamma_{s,t} \leq 2k_1 \mathbb{W}_k(\tilde{\gamma}, \gamma)$$

holds for all $s \in [0, T]$ and $t \in (s, (s+r_0) \wedge T]$. Consequently,

$$\begin{aligned}
& (\mathbb{W}_\psi + \mathbb{W}_k)(P_{s,t}^* \gamma, P_{s,t}^* \tilde{\gamma}) \leq \frac{2k_1 \psi((t-s)^{\frac{1}{2}})}{\sqrt{t-s}} \mathbb{W}_k(\gamma, \tilde{\gamma}), \\
& s \in [0, T], t \in (s, (s+r_0) \wedge T], \gamma, \tilde{\gamma} \in \mathcal{P}_k.
\end{aligned}$$

Combining this with the flow property

$$P_{s,t}^* = P_{r,t}^* P_{s,r}^*, \quad 0 \leq s \leq r \leq t \leq T,$$

we find a constant $k_2 > 0$ such that

$$(3.10) \quad (\mathbb{W}_\psi + \mathbb{W}_k)(P_{s,t}^* \gamma, P_{s,t}^* \tilde{\gamma}) \leq \frac{k_2 \psi((t-s)^{\frac{1}{2}})}{\sqrt{t-s}} \mathbb{W}_k(\gamma, \tilde{\gamma}), \quad t \in (s, T], \gamma, \tilde{\gamma} \in \mathcal{P}_k.$$

By the conditions on ψ in $(B^{a,b})(3)$ and (1.10), we have

$$\begin{aligned} \sup_{t \in (0, T]} \left\{ \int_0^t \frac{\psi(r^{\frac{1}{2}}) \psi((t-r)^{\frac{1}{2}})}{r \sqrt{t-r}} (1 + \rho_r \sqrt{r}) dr + \left(\int_0^t \left(\frac{\psi(r^{\frac{1}{2}})}{\sqrt{r}} \right)^2 dr \right)^{\frac{1}{2}} \right. \\ \left. + \left(\int_0^t \left(\frac{\psi((t-r)^{\frac{1}{2}}) \psi(r^{\frac{1}{2}})}{\sqrt{r} \sqrt{t-r}} \right)^m dr \right)^{\frac{1}{m}} \right\} < \infty. \end{aligned}$$

Therefore, substituting (3.10) into (3.4) and (3.5), we derive (1.11) for some constant $c > 0$.

4 Proof of Theorem 1.5

(a) We use the notations in step (c) in the proof of Theorem 1.3. By Pinsker's inequality, [13, (1.3)] and $(B^{a,b})$ with $\|\rho\|_\infty < \infty$, we find constants $\varepsilon \in (0, \frac{1}{2}]$, $c_1 > 0$ such that

$$\begin{aligned} \|P_{s,t}^{1,x} - P_{s,t}^{2,y}\|_{var} &\leq \sqrt{2 \text{Ent}(P_{s,t}^{1,x} | P_{s,t}^{2,y})} \\ &\leq \frac{c_1 |x - y|}{\sqrt{t-s}} + \frac{c_1}{\sqrt{t-s}} \left(\int_s^t (\mathbb{W}_\psi + \mathbb{W}_k)^2(P_{s,r}^* \gamma, P_{s,r}^* \tilde{\gamma}) dr \right)^{\frac{1}{2}} \\ &\quad + c_1 \sqrt{\log(1 + (t-s)^{-1})} \sup_{r \in [s+\varepsilon(t-s), t]} (\mathbb{W}_\psi + \mathbb{W}_k)^2(P_{s,r}^* \gamma, P_{s,r}^* \tilde{\gamma}) dr, \quad t \in [s, T]. \end{aligned}$$

Combining this with (3.3) and Lemma 2.1, we obtain

$$\begin{aligned} (4.1) \quad &\mathbb{W}_\psi(P_{s,t}^* \gamma, P_{s,t}^* \tilde{\gamma}) - \frac{\psi((t-s)^{\frac{1}{2}})}{\sqrt{t-s}} \mathbb{W}_1(P_{s,t}^{1,\gamma}, P_{s,t}^{2,\tilde{\gamma}}) \leq \psi((t-s)^{\frac{1}{2}}) \|P_{s,t}^{1,\gamma} - P_{s,t}^{2,\tilde{\gamma}}\|_{var} \\ &\leq \frac{\psi((t-s)^{\frac{1}{2}})}{\sqrt{t-s}} \left(\int_s^t (\mathbb{W}_\psi + \mathbb{W}_k)^2(P_{s,r}^* \gamma, P_{s,r}^* \tilde{\gamma}) dr \right)^{\frac{1}{2}} \\ &\quad + c_1 \psi((t-s)^{\frac{1}{2}}) \sqrt{\log(1 + (t-s)^{-1})} \sup_{r \in [s+\varepsilon(t-s), t]} (\mathbb{W}_\psi + \mathbb{W}_k)(P_{s,r}^* \gamma, P_{s,r}^* \tilde{\gamma}) \end{aligned}$$

for $t \in [s, T]$. On the other hand, since $b^{(0)}$ is bounded, $\|b^{(0)}\|_{\tilde{L}_{q_0}^{p_0}(T)} < \infty$ holds for any $p_0, q_0 > 2$, so that (1.6) holds for $m = 2$. Then there exists a constant $c_2 > 0$ such that

$$\begin{aligned} (4.2) \quad &\mathbb{W}_1(P_{s,t}^{1,\gamma}, P_{s,t}^{2,\tilde{\gamma}}) \leq \mathbb{W}_k(P_{s,t}^{1,\gamma}, P_{s,t}^{2,\tilde{\gamma}}) \\ &\leq c_2 \mathbb{W}_k(\gamma, \tilde{\gamma}) + c_2 \left(\int_s^t (\mathbb{W}_\psi + \mathbb{W}_k)^2(P_{s,r}^* \gamma, P_{s,r}^* \tilde{\gamma}) dr \right)^{\frac{1}{2}}. \end{aligned}$$

Combining this with (4.1), we find a constant $c_3 > 0$ such that instead of (3.6) we have

$$(4.3) \quad \begin{aligned} \Gamma_{s,t} &\leq c_3 \mathbb{W}_k(\gamma, \tilde{\gamma}) + c_2 h(t-s) \Gamma_{s,t}, \quad 0 \leq s \leq t \leq T, \\ h(t) &:= \left(\int_0^t \frac{\psi(s^{\frac{1}{2}})^2}{s} ds \right)^{\frac{1}{2}} + \sup_{r \in (0,t]} \psi(r^{\frac{1}{2}}) \sqrt{\log(1+r^{-1})}, \quad t > 0. \end{aligned}$$

Since $\int_0^1 \frac{\psi(r)^2}{r} dr < \infty$, we have $h(t) \rightarrow 0$ as $t \rightarrow 0$ if $\lim_{r \rightarrow 0} \psi(r)^2 \log(1+r^{-1}) = 0$, so that (1.11) follows as explained in step (c) in the proof of Theorem 1.3.

(b) Next, by (3.3), [13, (1.3)] and $(B^{a,b})$ with $\|\rho\|_\infty < \infty$, we find constants $\varepsilon \in (0, \frac{1}{2}]$, $c_1 > 0$ such that for any $\gamma, \tilde{\gamma} \in \mathcal{P}_k$,

$$\begin{aligned} \text{Ent}(P_t^* \gamma | P_t^* \tilde{\gamma}) &\leq \frac{\mathbb{W}_2(\gamma, \tilde{\gamma})^2}{t} + \frac{c_1}{t} \int_0^t (\mathbb{W}_\psi + \mathbb{W}_k)^2(P_r^* \gamma, P_r^* \tilde{\gamma}) dr \\ &\quad + c_1 \log(1+t^{-1}) \sup_{r \in [\varepsilon t, t]} (\mathbb{W}_\psi + \mathbb{W}_k)^2(P_r^* \gamma, P_r^* \tilde{\gamma}), \quad t \in (0, T]. \end{aligned}$$

Combining this with (1.11), we find a constant $c > 0$ such that (1.12) holds.

(c) If either $\|b\|_\infty < \infty$ or (1.13) holds, then we may apply [13, (1.4)] to delete the term $\log(1+(t-s)^{-1})$ from the above calculations, so that $h(t)$ in (4.3) becomes $\left(\int_0^t \frac{\psi(s^{\frac{1}{2}})^2}{s} ds \right)^{\frac{1}{2}}$ which goes to 0 as $t \rightarrow 0$. Therefore, (1.11) and (1.14) hold for some constant $c > 0$ as shown above.

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