CHORDAL LOEWNER CHAINS AND TEICHMÜLLER SPACES ON THE HALF-PLANE

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ABSTRACT. We consider a univalent analytic function f on the half-plane satisfying the condition that the supremum norm of its (pre-)Schwarzian derivative vanishes on the boundary. Under certain extra assumptions on f, we show that there exists a chordal Loewner chain initiated from f until some finite time, and that this Loewner chain defines a quasiconformal extension of f over the boundary such that its complex dilatation is given explicitly in terms of the (pre-)Schwarzian derivative in some neighborhood of the boundary. This can be regarded as the half-plane version of the corresponding result developed on the disk by Becker, and also as a generalization of the Ahlfors–Weill formula. As an application of this quasiconformal extension, we complete the characterization of an element of the VMO-Teichmüller space on the half-plane using the vanishing Carleson measure condition induced by the (pre-)Schwarzian derivative.

1. INTRODUCTION

Disks and half-planes in the complex plane are conformally equivalent. Most of the results in complex analysis show no essential difference between these cases, and thus one may easily overlook situations where a difference occurs when boundary conditions are imposed on functions defined on these domains; one boundary is compact and the other is non-compact. In active research areas of complex analysis interacting with each other, we can find these phenomena in recent studies. One is a Loewner chain and the other is a Teichmüller space. In this paper, we focus on a Loewner chain based on a point on a non-compact boundary, and consider its application to a Teichmüller space whose functions satisfy a certain vanishing condition on the non-compact boundary.

The idea of Loewner theory, first introduced by Loewner [30] in 1923 and later developed by Kufarev [26] in 1943 and Pommerenke [32] in 1965, is to embed a univalent function into a family of time-parametrized univalent functions (nowadays known as a Loewner chain) satisfying a suitable differential equation. Subsequently, the parametric representation method of univalent functions has been widely applied and developed in the theory of

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univalent functions as well as in other fields of mathematics. These include its stochastic analogue (Schramm–Loewner Evolution) discovered by Schramm [35] in 2000, which has attracted substantial attention in probability theory and conformal field theory. The most remarkable result in classical complex analysis is that the famous Bieberbach conjecture was solved by de Branges [18] in 1985 by extending Loewner's original approach.

In this paper, our interest is mainly devoted to an application of the Loewner differential equation of chordal type to quasiconformal extensions of univalent functions on the right half-plane $\mathbb{H} := \{z = x + iy \in \mathbb{C} : x > 0\}$. The chordal case was less well known until a decade ago, whereas the radial case has been the main focus of the classical Loewner theory.

Pommerenke successfully applied the radial Loewner differential equation to show univalence criteria for analytic functions on the unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ (see [33, Theorem 6.2]). In particular, it is worth mentioning that many sufficient conditions for univalence can be deduced in this way (see [11, Theorem 5.3]). Moreover, most of these univalence criteria can be refined to quasiconformal extension criteria.

We first recall the following well-known result prior to the application of Loewner theory. Let f be analytic in a disk D in the extended complex plane $\overline{\mathbb{C}}$, and assume that

$$\sup_{z\in D} \frac{1}{2}\rho_D^{-2}(z)|Sf(z)| \leq k. \tag{1.1}$$

Here, ρ_D denotes the hyperbolic density on D (with Gaussian curvature $\equiv -4$), which satisfies $\rho_D(\varphi(z))|\varphi'(z)| = 1/(2\mathrm{Re}\, z)$ for a conformal mapping φ of $\mathbb{H}$ onto D; $Sf = (Pf)' - (Pf)^2/2$ is the Schwarzian derivative of f given by the pre-Schwarzian derivative $Pf := f''/f'$. If $k = 1$ then f is univalent in D, which is a sharp univalence criterion proved by Nehari [31] in 1949; if $k < 1$ then f admits a k-quasiconformal extension to $\overline{\mathbb{C}}$, as Ahlfors and Weill [3] found in 1962 (see also [28, p.87]). In fact, when $D = \mathbb{D}$, the complex dilatation $\mu(z)$ of this extension has the form

$$\mu(1/\bar{z}) = -\frac{1}{2}\left(z/\bar{z}\right)^2(1-|z|^2)^2 Sf(z); \tag{1.2}$$

when $D = \mathbb{H}$, it has the form

$$\mu(-\bar{z}) = -\frac{1}{2}(2\mathrm{Re}\, z)^2 Sf(z). \tag{1.3}$$

These results can also be obtained by using the Loewner chain. In 1972, Becker [9] discovered a method for showing the quasiconformal extendibility of univalent functions in $\mathbb{D}$ by means of the radial Loewner differential equation: let $(f_t)_{t\geq 0}$ be a univalent solution of

$$\dot{f}_t(z) = zp(z,t)f_t'(z)$$

for all $z \in \mathbb{D}$ and almost every $t \geq 0$ with some Herglotz function $p(z,t)$. He showed that if the image domain $p(\mathbb{D}, t)$ lies in a closed disk

$$U(k) := \left\{ w \in \mathbb{H} : \left| w - \frac{1+k^2}{1-k^2} \right| \leq \frac{2k}{1-k^2} \right\}$$

for almost every $t \geq 0$ and for some $k < 1$ independent of t, then each function f_t has a quasiconformal extension to the whole plane $\mathbb{C}$. More recently, Gumenyuk and Hotta [23] developed this method in the chordal case by considering the corresponding equation

$$\dot{f}_t(z) = -p(z,t) f_t'(z),$$

and opened a way to quasiconformal extensions of analytic functions defined on $\mathbb{H}$. The advantage of Loewner theory is that these extensions are obtained explicitly by tracing the boundary values of the Loewner chain.

There is another advantage of using the Loewner chain. As the existence of a quasiconformal extension depends only on the boundary curve of the image domain, it is natural to consider the limit at the boundary corresponding to condition (1.1). In this case, one has to assume first that f is already known to be univalent and has a Jordan domain as its image. Towards this direction, we refer to Becker's work (see [10, Theorem 1] and [11, Theorem 5.4]). As mentioned above, he obtained a method for global quasiconformal extendibility. Moreover, this implies that if the time-variable t is taken only over an interval $[0, \tau)$, then the extension only to the disk $\{z \in \mathbb{C} : |z| < R\}$ for some $R > 1$ can be obtained. Based on that, the following sufficient condition for quasiconformal extendibility of univalent analytic functions was proved; see [12, Theorem 3].

Theorem 1.1 (Becker). *Let f be univalent and analytic in $\mathbb{D}$ such that $f(\mathbb{D})$ is a Jordan domain. If*

$$\lim_{t \to 0^+} \sup_{1-t \leq |z| < 1} \frac{1}{2}(1-|z|^2)^2 |Sf(z)| = 0, \tag{1.4}$$

then f admits a quasiconformal extension to the disk $\{z \in \mathbb{C} : |z| < R\}$ for some $R > 1$ whose complex dilatation μ has the form in (1.2).

We note that under the same assumption of the above theorem, f given on a smaller disk $\{z \in \mathbb{C} : |z| < R - \varepsilon\}$ for any $\varepsilon > 0$ can be further extended quasiconformally to $\overline{\mathbb{C}}$ by a general fact of quasiconformal extension as in [29, Theorem II.1.8].

In this paper, we will give, by means of the chordal Loewner differential equation, the half-plane analogue of Theorem 1.1, which is also the boundary-limit version in the half-plane case of the Ahlfors–Weill extension:

Theorem 1.2. *Let h be univalent and analytic in $\mathbb{H}$ with $\lim_{z\to\infty} h(z) = \infty$ such that $h(\mathbb{H})$ is a Jordan domain. Let h satisfy the condition*

$$\lim_{t \to 0^+} \sup_{0 < \operatorname{Re} z \leq t} \frac{1}{2}(2\operatorname{Re} z)^2\, |Sh(z)| = 0. \tag{1.5}$$

Suppose further that either

(A) *both h and h^{-1} are locally uniformly continuous in a neighborhood of ∞, or*
(B) *h is locally quasiconformally extendible to a neighborhood of ∞.*

Then the function

$$\hat{h}(z) = \begin{cases} h(z), & \text{if } \operatorname{Re} z \geq 0, \\ h(z^*) + \frac{(2\operatorname{Re} z)h'(z^*)}{1-(\operatorname{Re} z)Ph(z^*)} \quad (z^* := -\bar{z}), & \text{if } -\tau < \operatorname{Re} z < 0, \end{cases} \tag{1.6}$$

defines a quasiconformal extension of h *over* $i\mathbb{R}$ *to* $\mathbb{H}^*_{(0,\tau)} := \{z \in \mathbb{C} : -\tau < \operatorname{Re} z < 0\}$ *for some* $\tau > 0$ *such that its complex dilatation* μ *on* $\mathbb{H}^*_{(0,\tau)}$ *has the form*

$$\mu(z) = -\frac{1}{2}(2\operatorname{Re} z)^2 Sh(z^*). \tag{1.7}$$

We make a precise explanation of assumption (**A**): Saying that a function φ defined on an unbounded domain $\Omega \subset \mathbb{C}$ is locally uniformly continuous in a neighborhood of ∞ means that φ is uniformly continuous in $\Omega_M := \{z \in \Omega : |z| > M\}$ for some $M > 0$, i.e., for any $\varepsilon > 0$ there exists a constant $\delta > 0$ such that

$$|z_1 - z_2| < \delta \Longrightarrow |\varphi(z_1) - \varphi(z_2)| < \varepsilon$$

for $z_1, z_2 \in \Omega_M$. Assumption (**A**) is a practical assumption. For instance, if h satisfies the hydrodynamic normalization at ∞ uniformly in $\mathbb{H}$ (see [22]), then (**A**) follows.

Assumption (**B**) is an a priori assumption in the sense that if h admits such a quasiconformal extension as in the conclusion of the theorem, then this assumption must be satisfied.

We remark that if neither assumption (**A**) nor (**B**) is imposed, then the assertion of Theorem 1.2 does not hold. This will be verified by constructing an explicit example in Corollary 3.3. This, in particular, illustrates that the case of the half-plane shown in Theorem 1.2 represents a more complicated situation where we have to deal with compactness problems on the boundary that do not arise in the case of the disk. In Section 3, we develop the arguments for this example.

We demonstrate the proof of Theorem 1.2 in Section 4 by relying on Loewner theory. In Section 2, we collect several necessary definitions and results from this theory which will be used in the proofs, as well as a concise summary of the Loewner theory from a modern viewpoint.

Under the circumstances of Theorem 1.2, a family of analytic functions initiated from h is defined canonically by

$$h_t(z) = h(z+t) - \frac{2th'(z+t)}{1+tPh(z+t)}$$

for $z \in \mathbb{H}$ and for t within a limited period of time starting at 0. An important step for the proof of Theorem 1.2 is to show that h_t is univalent when $t \geq 0$ is sufficiently small, which turns out to be a Loewner chain over an interval $[0, \tau)$ for some $\tau > 0$. Gumenyuk and Hotta [23] carried this out under circumstances where the time t can be extended to $+\infty$, but their argument does not work for the case of a limited time period. We remark that, unlike the case of the half-plane, the corresponding argument for Theorem 1.1 is essentially the same as that for the case where the time t extends to $+\infty$. Our method of showing that h_t is univalent in a small interval of time involves novel techniques. This is achieved in Theorem 4.3 under assumption (**A**) and Theorem 4.5 under assumption (**B**).

For the proof, we need to show that condition (1.5) is equivalent to

$$\lim_{t\to 0^+} \sup_{0<\operatorname{Re} z\leq t} (2\operatorname{Re} z)\,|Ph(z)| = 0. \tag{1.8}$$

In Section 3, we prove this equivalence in Theorem 3.5, which seems natural but is missing in the literature. Moreover, by replacing condition (1.5) with (1.8), we can also obtain the analogous result to Theorem 1.2, as Theorem 5.1, which represents the complex dilatation μ in terms of Ph. This is stated briefly in Section 5.

As a special case of Theorem 1.2 and also of Theorem 5.1, the following form can be used conveniently for subspaces of the universal Teichmüller space because univalent analytic functions are already assumed to be quasiconformally extendible to $\overline{\mathbb{C}}$ in these settings. The importance of this result lies not in the quasiconformal extendibility, of course, but in the representation of the complex dilatation.

Corollary 1.3. *Let h be univalent and analytic in $\mathbb{H}$ that is quasiconformally extendible to $\mathbb{C}$ with $\lim_{z\to\infty} h(z) = \infty$. If h satisfies* (1.5)*, then the assertion in Theorem 1.2 holds, that is, it has the extension whose complex dilatation is of the form in* (1.7) *in a strip domain $\mathbb{H}^*_{(0,t)}$ for some $t > 0$ over $i\mathbb{R}$, and is further extendible to $\mathbb{C}$ while keeping this initial extension.*

Owing to the fact that the complex dilatation μ of the quasiconformal extension $\hat{h}$ in Theorem 1.2 is expressed explicitly in terms of the Schwarzian derivative Sh of h, and to the way of constructing $\hat{h}$ by means of the boundary values of a suitable Loewner chain, the above Corollary 1.3 can be neatly used to give an appropriate quasiconformal extension for those subspaces of the universal Teichmüller space that are smaller than the so-called little universal Teichmüller space on the half-plane. In this paper, we deal with the VMO-Teichmüller space as such an example.

The universal Teichmüller space T can be regarded as the set of all conformal mappings h (up to post-composition with a Möbius transformation) on $\mathbb{H}$ which can be extended to a quasiconformal homeomorphism of $\overline{\mathbb{C}}$. Subspaces of T are obtained by imposing some conditions on h in general. If this condition is conformally invariant, then there is no difference between the cases where we consider it on $\mathbb{D}$ and on $\mathbb{H}$. The Weil–Petersson Teichmüller space and the BMO-Teichmüller space are such examples, which have received much attention over the years (see [5, 14, 36, 37, 41, 42] and references therein).

In contrast, if the condition is given on the boundary, for instance, the vanishing condition of the supremum norm of the Schwarzian derivative as for the little universal Teichmüller space, Teichmüller spaces on $\mathbb{D}$ and on $\mathbb{H}$ are different. Becker and Pommerenke [13] worked on this for $\mathbb{D}$, and later Hu, Wu and Shen [25] considered this on $\mathbb{H}$.

The VMO-Teichmüller space is the subspace of the BMO-Teichmüller space defined by a certain vanishing condition on the boundary (see [37]). If this space is defined on the half-plane $\mathbb{H}$, it is the set of all $h \in T$ satisfying $\log h' \in \mathrm{VMOA}(\mathbb{H})$, the space of analytic functions in $\mathbb{H}$ of vanishing mean oscillation. In Section 6, as an application of Corollary 1.3, we will prove the following result. This fills in the missing part left in the study of various models of VMO-Teichmüller space on the half-plane (see [36, Theorem 2.2], [41]).

Theorem 1.4. *Let h be univalent and analytic in $\mathbb{H}$ such that h is quasiconformally extendible to $\overline{\mathbb{C}}$ with $h(\infty)=\infty$. If $\log h' \in \mathrm{VMOA}(\mathbb{H})$, then h admits a quasiconformal extension to $\overline{\mathbb{C}}$ such that its complex dilatation $\tilde{\mu}$ induces a vanishing Carleson measure*

$$\lambda_{\tilde{\mu}} := \frac{|\tilde{\mu}(z)|^2}{(-2\mathrm{Re}\, z)} dxdy \tag{1.9}$$

on the left half-plane $\mathbb{H}^* := \{z = x+iy \in \mathbb{C} : x<0\}$.

2. Preliminaries on the Loewner theory

In this section, we give several basic definitions and results on the (generalized) Loewner theory, proposed by Bracci, Contreras and Díaz-Madrigal [7, 8], which allows us to treat evolution families with inner fixed points (the radial case) and with boundary fixed points (the chordal case) at the same time. This unified theory relies partially on the theory of one-parameter semigroups, which is actually the autonomous version of the Loewner theory.

2.1. **Generalized Loewner theory.** Let $D \subset \overline{\mathbb{C}}$ be a simply connected domain conformally equivalent to $\mathbb{D}$ and $\mathbb{H}$. We denote the family of all analytic functions on D by $\mathrm{Hol}(D,\mathbb{C})$, and the family of all analytic self-maps of D by $\mathrm{Hol}(D)$.

Let $\mathcal{U} \subset \mathrm{Hol}(\mathbb{D})$ be a semigroup with respect to composition containing the identity map $\mathrm{id}_{\mathbb{D}}$. A family $(\phi_t)_{t\ge 0}$ in $\mathcal{U}$ such that $\phi_0 = \mathrm{id}_{\mathbb{D}}$, $\phi_t \circ \phi_s = \phi_{t+s}$ for any $s,t \ge 0$, and $\phi_t(z) \to z$ locally uniformly on $\mathbb{D}$ as $t \to 0$, is called a continuous one-parameter semigroup. It is known (see [15]) that for any such semigroup (ϕ_t) there exists a function $G \in \mathrm{Hol}(\mathbb{D},\mathbb{C})$ such that for each $z \in \mathbb{D}$ the function $\phi_t(z)$ is the unique solution of the initial value problem

$$\frac{dw(z,t)}{dt} = G(w(z,t)), \quad w(z,0) = z. \tag{2.1}$$

The function G is called the infinitesimal generator of (ϕ_t). A criterion for a function $G \in \mathrm{Hol}(\mathbb{D},\mathbb{C})$ to be an infinitesimal generator of some continuous one-parameter semigroup is the following Berkson–Porta representation:

$$G(z) = (\tau - z)(1-\bar{\tau}z)\, p(z) \tag{2.2}$$

for a point $\tau \in \overline{\mathbb{D}}$ and a function $p \in \mathrm{Hol}(\mathbb{D},\mathbb{C})$ with $\mathrm{Re}\, p(z) \ge 0$ for all $z \in \mathbb{D}$. Moreover, if $G \not\equiv 0$, then such a representation is unique. In fact, the point τ is the common Denjoy–Wolff point (see below for its definition) of all ϕ_t that are different from $\mathrm{id}_{\mathbb{D}}$. Such a correspondence between the continuous one-parameter semigroup (ϕ_t) and the infinitesimal generator G is one-to-one.

Now we introduce the Herglotz vector field in $\mathbb{D}$, which can be regarded as a time-dependent infinitesimal generator. One of its equivalent definitions is as follows (see [8, Theorem 4.8]). A Herglotz vector field in $\mathbb{D}$ is a map $G : \mathbb{D} \times [0, +\infty) \to \mathbb{C}$ of the form

$$G(z,t) = (\tau(t) - z)(1 - \overline{\tau(t)}z)\, p(z,t) \tag{2.3}$$

for all $z \in \mathbb{D}$ and almost every $t \in [0, +\infty)$, where $\tau : [0, +\infty) \to \overline{\mathbb{D}}$ is a measurable function and $p : \mathbb{D} \times [0, +\infty) \to \mathbb{C}$ is a Herglotz function defined as follows.

Definition. A Herglotz function on $\mathbb{D}$ is a map $p : \mathbb{D} \times [0, +\infty) \to \mathbb{C}$ satisfying the following conditions:

(HF1) $p(z, \cdot)$ is locally integrable on $[0, +\infty)$ for all $z \in \mathbb{D}$;
(HF2) $p(\cdot, t)$ is analytic on $\mathbb{D}$ for almost every $t \in [0, +\infty)$ and $\operatorname{Re} p(\cdot, t) \geq 0$.

Moreover, [8, Theorem 4.8] asserts that if two couples (p_1, τ_1) and (p_2, τ_2) generate the same $G(z,t)$ satisfying $G(\cdot, t) \not\equiv 0$ up to a set of measure zero on the t-axis, then $p_1(z,t) = p_2(z,t)$ for all $z \in \mathbb{D}$ and almost every $t \in [0, +\infty)$, and $\tau_1(t) = \tau_2(t)$ for almost every $t \in [0, +\infty)$.

The evolution family $(\varphi_{s,t})_{t \geq s \geq 0}$ is the unique solution of the following initial value problem for the generalized Loewner–Kufarev ODE driven by the Herglotz vector field G (see [17, Section 2.1], [27, Chapter 18]):

$$\frac{d\varphi_{s,t}(z)}{dt} = G(\varphi_{s,t}(z),\, t), \quad \varphi_{s,s}(z) = z \tag{2.4}$$

for any initial point $z \in \mathbb{D}$, any starting time $s \geq 0$ and almost every $t \geq s$. This equation establishes a one-to-one correspondence between evolution families $(\varphi_{s,t})$ and Herglotz vector fields G up to a set of measure zero on the t-axis (see [8, Theorem 1.1]). Comparing (2.1) and (2.4), we can regard an evolution family as a non-autonomous analogue of a continuous one-parameter semigroup.

An independent definition of an evolution family is given as follows:

Definition ([8])**.** An evolution family in $\mathbb{D}$ is a two-parameter family $(\varphi_{s,t})_{t \geq s \geq 0} \subset \operatorname{Hol}(\mathbb{D})$ satisfying:

(EF1) $\varphi_{s,s} = \mathrm{id}_{\mathbb{D}}$ for all $s \geq 0$;
(EF2) $\varphi_{s,t} = \varphi_{u,t} \circ \varphi_{s,u}$ whenever $0 \leq s \leq u \leq t < +\infty$;
(EF3) for each $z \in \mathbb{D}$ there exists a non-negative locally integrable function k_z on $[0, +\infty)$ such that $|\varphi_{s,u}(z) - \varphi_{s,t}(z)| \leq \int_u^t k_z(\xi) d\xi$ whenever $0 \leq s \leq u \leq t < +\infty$.

For all $0 \leq s \leq t < +\infty$, $\varphi_{s,t}$ is univalent in $\mathbb{D}$, which follows from uniqueness of the solution of (2.4) (see [8, Corollary 6.3]), even though the definition does not require the elements of an evolution family to be univalent.

The notion of a Loewner chain can be given in the same framework as an evolution family.

Definition ([16])**.** A family $(f_t)_{t \geq 0} \subset \operatorname{Hol}(\mathbb{D}, \mathbb{C})$ is called a Loewner chain if it satisfies the following conditions:

(LC1) each f_t is univalent for all $t \geq 0$;
(LC2) $f_s(\mathbb{D}) \subset f_t(\mathbb{D})$ whenever $0 \leq s \leq t < +\infty$;
(LC3) for any compact subset $K \subset \mathbb{D}$ and any $T > 0$ there exists a non-negative integrable function $k_{K,T}$ on $[0, T]$ such that $|f_s(z) - f_t(z)| \leq \int_s^t k_{K,T}(\xi)\, d\xi$ whenever $z \in K$ and $0 \leq s \leq t \leq T$.

Remark. In [8] and [16], the definitions of evolution families and Loewner chains include an integrability order $d \in [1, +\infty]$. We only need to consider the most general case $d = 1$.

For a given Loewner chain (f_t), the equation $\varphi_{s,t} := f_t^{-1} \circ f_s$ defines an evolution family. Differentiating both sides with respect to t yields

$$f_t'(\varphi_{s,t})\, \dot{\varphi}_{s,t} + \dot{f}_t(\varphi_{s,t}) = 0,$$

where $f_t'(z) := df_t(z)/dz$, $\dot{f}_t := df_t/dt$, and $\dot{\varphi}_{s,t} := d\varphi_{s,t}/dt$ (the dot on $\dot{\varphi}_{s,t}$ always denotes differentiation with respect to the second parameter). Combined with (2.4), this yields the generalized Loewner–Kufarev PDE:

$$\dot{f}_t(z) = -f_t'(z)\, G(z, t) \tag{2.5}$$

for all $z \in \mathbb{D}$ and almost every $t \geq 0$.

Conversely, given an evolution family $(\varphi_{s,t})$, or equivalently a Herglotz vector field G, one can obtain the corresponding Loewner chain (f_t) by solving (2.5), and it is unique up to post-composition with a conformal mapping of the domain $\bigcup_{t\geq 0} f_t(\mathbb{D})$. Moreover, the so-called standard Loewner chain introduced in [16] is uniquely determined; it is a Loewner chain (f_t) such that $f_0(0) = 0$ and $f_0'(0) - 1 = 0$ (notice that only f_0 is normalized), and $\bigcup_{t\geq 0} f_t(\mathbb{D})$ is either $\mathbb{C}$ or a disk centered at the origin.

Overall, we have seen the one-to-one correspondences among evolution families $(\varphi_{s,t})$, Herglotz vector fields G, couples (p, τ) of Herglotz functions and measurable functions, and Loewner chains (f_t) up to conformal mappings. In particular, the relationship between $(\varphi_{s,t})$ and (p, τ) is expressed by

$$\frac{d\varphi_{s,t}(z)}{dt} = (\tau(t) - \varphi_{s,t}(z))(1 - \overline{\tau(t)}\varphi_{s,t}(z))\, p(\varphi_{s,t}(z), t), \quad \varphi_{s,s}(z) = z \tag{2.6}$$

for all $z \in \mathbb{D}$ and almost every $t \geq s \geq 0$. Moreover, the relationship between (f_t) and (p, τ) is expressed by

$$\dot{f}_t(z) = -f_t'(z)\, (\tau(t) - z)(1 - \overline{\tau(t)}z)\, p(z, t) \tag{2.7}$$

for all $z \in \mathbb{D}$ and almost every $t \geq 0$. We point out that if $\tau(t) \equiv \text{const}$, then such a constant is the Denjoy–Wolff point of $\varphi_{s,t}$ for all $0 \leq s \leq t < +\infty$ different from $\mathrm{id}_{\mathbb{D}}$ (see [8, Corollary 7.2]). This agrees with the Berkson–Porta representation for continuous one-parameter semigroups.

We end this subsection with a brief introduction of the Denjoy–Wolff point mentioned above. The Julia–Wolff–Carathéodory theorem (see [19, Theorem 1.10]) asserts that for any $\varphi \in \mathrm{Hol}(\mathbb{D})$ different from $\mathrm{id}_{\mathbb{D}}$ there exists a unique fixed point $\tau \in \overline{\mathbb{D}}$ such that

$\varphi(\tau) = \tau$ and $|\varphi'(\tau)| \leq 1$. Such τ is called the Denjoy–Wolff point of φ. In the case $\tau \in \partial\mathbb{D}$, these notions should be understood as angular limits, denoted by $\angle\lim$, including the angular derivative. Namely, $\varphi(\tau) = \tau$ means $\angle\lim_{z\to\tau}\varphi(z) = \tau$, and this condition in fact implies that $\varphi'(\tau) := \angle\lim_{z\to\tau}(\varphi(z)-\tau)/(z-\tau)$ exists in $(0, +\infty]$ (see [34, Proposition 4.13]). Thus, if $\tau \in \partial\mathbb{D}$ is the Denjoy–Wolff point of φ, then $0 < \varphi'(\tau) \leq 1$. In the case $\tau \in \mathbb{D}$, the condition $|\varphi'(\tau)| \leq 1$ follows from $\varphi(\tau) = \tau$ by the Schwarz–Pick lemma.

The Denjoy–Wolff theorem asserts that, excluding the trivial cases where φ is $\mathrm{id}_{\mathbb{D}}$ or an elliptic automorphism of $\mathbb{D}$, the sequence $\{\varphi_n\}$ of iterates of φ, defined by $\varphi_n := \varphi \circ \varphi_{n-1}$ with $\varphi_1 = \varphi$, converges to the Denjoy–Wolff point τ uniformly on compact subsets of $\mathbb{D}$. This is a prototype of hyperbolic dynamics and has contributed much to modern studies of dynamical systems (see [1, Appendix G]).

2.2. **(Classical) radial Loewner theory.** Hereafter, we consider the case where $\tau(t)$ is a constant function, and that it is the common interior Denjoy–Wolff point τ of the evolution family $(\varphi_{s,t})$ in $\mathbb{D}$. When $\tau \in \mathbb{D}$, we can assume that τ is 0 by conjugating with an automorphism of $\mathbb{D}$ sending τ to 0.

Definition. An evolution family $(\varphi_{s,t})$ in $\mathbb{D}$ is said to be of radial type if, for all $0 \leq s \leq t < +\infty$, all elements $\varphi_{s,t}$ different from $\mathrm{id}_{\mathbb{D}}$ share the common interior Denjoy–Wolff point at 0. A Loewner chain (f_t) in $\mathbb{D}$ is said to be of radial type if the corresponding evolution family $(\varphi_{s,t}) := (f_t^{-1} \circ f_s)$ is of radial type.

In this case, we see from the equation $\varphi_{s,t} = f_t^{-1} \circ f_s$ that a radial Loewner chain (f_t) is actually a Loewner chain such that $f_t(0) = f_0(0)$ for all $t \geq 0$, and moreover, the Herglotz vector field is $G(z,t) = -zp(z,t)$ for all $z \in \mathbb{D}$ and almost every $t \geq 0$. In view of this, any radial evolution family $(\varphi_{s,t})$ and the corresponding Loewner chain (f_t) satisfy the following radial Loewner–Kufarev ODE and PDE, respectively:

$$\frac{d\varphi_{s,t}(z)}{dt} = -\varphi_{s,t}(z)p(\varphi_{s,t}(z), t), \quad \varphi_{s,s}(z) = z \tag{2.8}$$

for all $z \in \mathbb{D}$ and almost every $t \geq s \geq 0$, and

$$\dot{f}_t(z) = zp(z,t)f_t'(z) \tag{2.9}$$

for all $z \in \mathbb{D}$ and almost every $t \geq 0$ with some Herglotz function p determined uniquely up to a set of measure zero on the t-axis.

In the modern literature, the so-called classical radial evolution families and Loewner chains mean a special kind of evolution families and Loewner chains considered by Pommerenke (see [34, Chapter 6]), defined as follows.

Definition. A radial evolution family and a radial Loewner chain are said to be classical if they satisfy the extra normalization hypotheses $\varphi_{s,t}'(0) = e^{s-t}$ and $f_t(0) = 0$, $f_t'(0) = e^t$ for all $0 \leq s \leq t < +\infty$, respectively.

Accordingly, the Herglotz function p satisfies the extra condition $p(0,t) = 1$ for almost every $t \geq 0$. Moreover, the condition $f_t'(0) = e^t$ for all $t \geq 0$ ensures $\bigcup_{t\geq 0} f_t(\mathbb{D}) = \mathbb{C}$, which gives a one-to-one correspondence between the classical radial evolution family $(\varphi_{s,t})$ and the classical radial Loewner chain (f_t) that is standard. Moreover, (f_t) can be recovered from $(\varphi_{s,t})$ by the equation $f_s = \lim_{t\to+\infty} e^t\varphi_{s,t}$.

2.3. **Chordal Loewner theory.** Consider an evolution family $(\varphi_{s,t})$ in $\mathbb{D}$ such that, for all $0 \le s \le t$, all elements $\varphi_{s,t}$ different from $\mathrm{id}_{\mathbb{D}}$ share the common boundary Denjoy–Wolff point at τ. We can assume that τ is 1 by conjugating with a rotation of $\mathbb{D}$. Obviously, the associated Herglotz vector field $G(z,t)$ has the form $(1-z)^2p(z,t)$ for all $z \in \mathbb{D}$ and almost every $t \ge 0$ with some Herglotz function p.

In the literature, the case of boundary fixed points is usually treated in the half-plane model instead of the unit disk model because there the associated vector field assumes a simpler form (see (2.10) below). Passing from the unit disk $\mathbb{D}$ to the right half-plane $\mathbb{H}$ by the Cayley transform $\zeta = H(z) = (1+z)/(1-z)$ that sends 1 to ∞, we define a family $\tilde\varphi_{s,t} := H \circ \varphi_{s,t} \circ H^{-1} \in \mathrm{Hol}(\mathbb{H})$ and assume the point ∞ to be the Denjoy–Wolff point of $\tilde\varphi_{s,t}$. Then, the solution $\varphi_{s,t}(z)$ of the differential equation $dw/dt = (1-w)^2p(w,t)$ transforms into the solution $\tilde\varphi_{s,t}(\zeta)$ of $dw/dt = \tilde p(w,t)$ where $\tilde p(\zeta,t) = 2p(H^{-1}(\zeta),\, t)$.

We say that $(\tilde\varphi_{s,t})$ is an evolution family in $\mathbb{H}$ if $(\varphi_{s,t})$ is an evolution family in $\mathbb{D}$. Similarly, we say that $(\tilde f_t) := (H \circ f_t \circ H^{-1})$ is a Loewner chain in $\mathbb{H}$ if (f_t) is a Loewner chain in $\mathbb{D}$; and $\tilde p(\cdot,t) := p(H^{-1}(\cdot),t)$ is a Herglotz function in $\mathbb{H}$ if $p(\cdot,t)$ is a Herglotz function in $\mathbb{D}$.

Definition. An evolution family $(\varphi_{s,t})$ in $\mathbb{H}$ is said to be of chordal type if, for all $0 \le s \le t < \infty$, all elements $\varphi_{s,t}$ different from $\mathrm{id}_{\mathbb{H}}$ share the common boundary Denjoy–Wolff point at ∞. A Loewner chain (f_t) in $\mathbb{H}$ is said to be of chordal type if the corresponding evolution family $(\varphi_{s,t}) := (f_t^{-1} \circ f_s)$ is of chordal type.

We remark that under these circumstances the half-plane version of the Julia–Wolff–Carathéodory theorem implies that each $\varphi_{s,t}$ satisfies

$$\mathrm{Re}\,\varphi_{s,t}(z) \ge \varphi'_{s,t}(\infty)\,\mathrm{Re}\,z$$

for all $z \in \mathbb{H}$, where $\varphi'_{s,t}(\infty)$ is the Carathéodory angular derivative of $(\varphi_{s,t})$ defined by

$$\varphi'_{s,t}(\infty) := \angle\lim_{z\to\infty}\frac{\varphi_{s,t}(z)}{z} = \frac{1}{(H^{-1}\circ\varphi_{s,t}\circ H)'(1)} \ge 1.$$

Based on the above arguments, any chordal evolution family $(\varphi_{s,t})$ and the corresponding Loewner chain (f_t) satisfy the following chordal Loewner–Kufarev ODE and PDE, respectively:

$$\frac{d\varphi_{s,t}(z)}{dt} = p(\varphi_{s,t}(z),t), \quad \varphi_{s,s}(z) = z \tag{2.10}$$

for all $z \in \mathbb{H}$ and almost every $t \ge s \ge 0$, and

$$\dot f_t(z) = -p(z,t)f_t'(z) \tag{2.11}$$

for all $z \in \mathbb{H}$ and almost every $t \geq 0$ with some Herglotz function p determined uniquely up to a set of measure zero on the t-axis. We say that $(\varphi_{s,t})$ and (f_t) are the evolution family and the Loewner chain associated with the Herglotz function p, respectively.

3. Boundary condition of the Schwarzian derivative

In this section, we consider the situation where the norm of the Schwarzian derivative of a univalent analytic function f on the half-plane $\mathbb{H}$ vanishes at the boundary. Differently from the case of $\mathbb{D}$, this does not necessarily imply that f is quasiconformally extendible, even if $f(\mathbb{H})$ is a Jordan domain. We show this fact. We also prove that the vanishing condition of the norm of the Schwarzian derivative is equivalent to that of the pre-Schwarzian derivative.

We start with the following elementary but crucial observation for the arguments of this section.

Lemma 3.1. *Let $D := D(1,1)$ be the open disk in $\mathbb{H}$ with center 1 and radius 1. For any $\epsilon > 0$, there exists a horodisk D' $(\subset D)$ tangent to ∂D at 0 such that $\rho_D^{-1}(z)/(\operatorname{Re} z) \leq \epsilon$ for all $z \in D \setminus D'$.*

Proof. The hyperbolic density $\rho_D(z)$ of D is reciprocally comparable to the distance $d(z, \partial D)$ to the boundary as $(2\rho_D(z))^{-1} \leq d(z, \partial D) \leq \rho_D^{-1}(z)$. Then, we consider the ratio $d(z, \partial D)/(\operatorname{Re} z)$ instead of $\rho_D^{-1}(z)/(\operatorname{Re} z)$.

For a given constant $a > 1$, we define the curve

$$\Gamma_a := \left\{ z \in D : \ \frac{d(z, \partial D)}{\operatorname{Re} z} = \frac{1}{a} \right\}.$$

By computation, Γ_a is an ellipse whose points $z = x + iy$ satisfy

$$\frac{(x - a/(a+1))^2}{(a/(a+1))^2} + \frac{y^2}{(a-1)/(a+1)} = 1.$$

The ellipse Γ_a becomes larger as a increases and tends to ∂D as $a \to \infty$. Moreover, Γ_a is contained in $\overline{D_a}$, where

$$D_a := \{ z \in D : \ |z - a/(a+1)| < a/(a+1) \}$$

is a horodisk tangent to ∂D at 0. Then we see that $d(z, \partial D)/(\operatorname{Re} z) \leq 1/a$ for all $z \in D \setminus D_a$. Consequently, by taking $a = 2/\epsilon$ and setting $D' = D_{2/\epsilon}$, we have $\rho_D^{-1}(z)/(\operatorname{Re} z) \leq \epsilon$ for all $z \in D \setminus D'$. □

This lemma makes it possible to construct conformal mappings on $\mathbb{H}$ that satisfy the vanishing condition of their Schwarzian derivatives at the boundary. Here is a general method for this.

Proposition 3.2. *Let $g(\zeta)$ be a conformal mapping on $\mathbb{H}$. Let $D = D(1,1) \subset \mathbb{H}$ and let $\zeta = \varphi(z) = 2/(z+1)$ be a Möbius transformation that maps $\mathbb{H}$ onto D with $\varphi(\infty) = 0$.*

Then the conformal mapping $f := g \circ \varphi$ *on* $\mathbb{H}$ *satisfies*

$$\lim_{\mathrm{Re}\, z \to 0^+} (2\mathrm{Re}\, z)^2 |Sf(z)| = 0 \tag{3.1}$$

uniformly for $\mathrm{Im}\, z \in \mathbb{R}$.

Proof. Since g is univalent on $\mathbb{H}$, we have $(2\mathrm{Re}\,\zeta)^2|Sg(\zeta)| \le 6$ for all $\zeta \in \mathbb{H}$ (see [28, p. 60]), and thus for all $\zeta \in D$,

$$\rho_D^{-2}(\zeta)|Sg(\zeta)| = (2\mathrm{Re}\,\zeta)^2|Sg(\zeta)|\frac{\rho_D^{-2}(\zeta)}{(2\mathrm{Re}\,\zeta)^2} \le \frac{3}{2}\left(\frac{\rho_D^{-1}(\zeta)}{\mathrm{Re}\,\zeta}\right)^2.$$

According to Lemma 3.1, for any $\epsilon > 0$, there exists a horodisk $D' \subset D$ tangent to ∂D at 0 such that $\rho_D^{-2}(\zeta)|Sg(\zeta)| \le \epsilon$ for all $\zeta \in D \setminus D'$.

Set $H_\epsilon = \varphi^{-1}(D')$, which is a half-plane in $\mathbb{H}$. Since

$$(2\mathrm{Re}\, z)^2|Sf(z)| = \rho_D^{-2}(\zeta)|Sg(\zeta)|$$

for all $z \in \mathbb{H}$ by the invariance under the Möbius transformation φ, we conclude that $(2\mathrm{Re}\, z)^2|Sf(z)| \le \epsilon$ for all $z \in \mathbb{H} \setminus H_\epsilon$. □

As a particular application of this construction, we obtain a counterexample to the statement of Theorem 1.2 when the extra assumptions (**A**) and (**B**) are dropped.

Corollary 3.3. *There exists a conformal mapping* f *on* $\mathbb{H}$ *satisfying*

(i) *the image* $f(\mathbb{H})$ *is a Jordan domain, but not a quasidisk;*
(ii) $\lim_{\mathrm{Re}\, z \to 0^+}(2\mathrm{Re}\, z)^2|Sf(z)| = 0$ *uniformly for* $\mathrm{Im}\, z \in \mathbb{R}$*, that is,* (3.1)*; and*
(iii) f *is not locally uniformly continuous in a neighborhood of* ∞.

Proof. Consider a half horizontal parallel strip V in $\mathbb{H}$ with vertices $\pm i\pi/2$, that is,

$$V := \{w = u + iv \in \mathbb{H} : -\pi/2 < v < \pi/2\}.$$

By the Schwarz–Christoffel formula,

$$w = g(\zeta) := -i \arcsin\left(\frac{i}{\zeta}\right) = -\log\left(-\frac{1}{\zeta} + \sqrt{1 + \frac{1}{\zeta^2}}\right)$$

maps $\mathbb{H}$ conformally onto V with $g(0) = \infty$, $g(i) = -i\pi/2$ and $g(-i) = i\pi/2$, where we choose the branch of $\arcsin(i/\zeta)$ so that it takes $\pi/2$ at $\zeta = i$. From this, we see that the image of $D = D(1, 1)$ under g is a Jordan domain, but not a quasidisk, since its boundary has a cusp at ∞.

Let φ be the Möbius transformation as in Proposition 3.2. Define

$$f(z) := g \circ \varphi(z) = -i \arcsin\left(i(z + 1)/2\right).$$

Then $f(\mathbb{H}) = g(D)$ satisfies condition (i). Moreover, f satisfies condition (ii) by Proposition 3.2. Condition (iii) follows from the explicit representation of f. □

In the latter half of this section, we compare the boundary behavior of Schwarzian derivatives and pre-Schwarzian derivatives for univalent analytic functions on $\mathbb{H}$. This will be used later in the proof of Theorem 1.2.

First, we state a claim below obtained from the Cauchy integral formula. As in [24, Lemma 4], if h is an analytic function on a domain $D \subset \mathbb{C}$, then

$$\sup_{z\in D} |h'(z)| d(z, \partial D)^2 \leq 4 \sup_{z\in D} |h(z)| d(z, \partial D). \tag{3.2}$$

This yields the implication

$$\lim_{z\to\partial D} |h(z)| d(z, \partial D) = 0 \quad \Longrightarrow \quad \lim_{z\to\partial D} |h'(z)| d(z, \partial D)^2 = 0. \tag{3.3}$$

Indeed, by setting $r := d(z, \partial D)/2$ for a fixed $z \in D$, (3.3) easily follows from:

$$\begin{aligned} |h'(z)| &= \left| \frac{1}{2\pi i} \int_{|\zeta - z| = r} \frac{h(\zeta)}{(\zeta - z)^2} d\zeta \right| \leq \frac{1}{r} \sup_{|\zeta - z| = r} |h(\zeta)| \\ &\leq \frac{2}{d(z, \partial D)} \sup_{|\zeta - z| = r} (|h(\zeta)| d(\zeta, \partial D)) \sup_{|\zeta - z| = r} \frac{1}{d(\zeta, \partial D)} \\ &= \frac{4}{d(z, \partial D)^2} \sup_{|\zeta - z| = r} (|h(\zeta)| d(\zeta, \partial D)). \end{aligned}$$

Suppose that D is a simply connected domain with the hyperbolic density ρ_D. Since $\rho_D(z) \leq 1/d(z, \partial D) \leq 4\rho_D(z)$ (see [34, p.92]), it follows from (3.2) and (3.3) that

$$\sup_{z\in D} |h'(z)| \rho_D^{-2}(z) \leq 64 \sup_{z\in D} |h(z)| \rho_D^{-1}(z);$$

$$\lim_{z\to\partial D} |h(z)| \rho_D^{-1}(z) = 0 \quad \Longrightarrow \quad \lim_{z\to\partial D} |h'(z)| \rho_D^{-2}(z) = 0.$$

For an analytic and locally univalent function g on D, we can apply these facts to the pre-Schwarzian derivative Pg and the Schwarzian derivative $Sg = (Pg)' - (Pg)^2/2$ to obtain the following implications:

$$\begin{aligned} \sup_{z\in D} |Pg(z)| \rho_D^{-1}(z) =: \lambda \quad &\Longrightarrow \quad \sup_{z\in D} |Sg(z)| \rho_D^{-2}(z) \leq 64\lambda + \lambda^2/2; \\ \lim_{z\to\partial D} |Pg(z)| \rho_D^{-1}(z) = 0 \quad &\Longrightarrow \quad \lim_{z\to\partial D} |Sg(z)| \rho_D^{-2}(z) = 0. \end{aligned} \tag{3.4}$$

Next, we consider the converse implication. The following result is essentially contained in Becker [12].

Proposition 3.4. *Let f be a univalent analytic function on $\mathbb{D}$ and let $f(\mathbb{D})$ be a Jordan domain. Suppose further that f satisfies condition* (1.4). *For $t \in (0, 1)$, set*

$$\hat{\beta}(t) := \sup_{1-t\leq|z|<1} (1 - |z|^2) |Pf(z)|,$$

$$\hat{\sigma}(t) := \sup_{1-t\leq|z|<1} (1 - |z|^2)^2 |Sf(z)|.$$

Then there is some $t_0 \in (0,1)$ *such that*

$$\hat{\beta}(t^{1+\epsilon}) \leq 4\hat{\sigma}(t) + 8t^{\epsilon}$$

for $0 < t \leq t_0$*. Here,* $\epsilon > 0$ *can be taken as an arbitrary positive constant.*

Proof. Suppose (1.4) holds. Then, by Theorem 1.1, f has a quasiconformal extension to $\mathbb{C}$ with complex dilatation μ satisfying

$$\hat{k}(t) := \inf\{k \geq 0 : |\mu(z)| \leq k \text{ for a.e. } z \text{ with } 1 < |z| \leq 1+t\} \leq \hat{\sigma}(t)/2 \tag{3.5}$$

for any $0 < t \leq t_0$, where $t_0 < 1$ is some constant. By applying Lehto's majorization method, [12, Theorem 2] implies that

$$\hat{\beta}(t^{1+\epsilon}) \leq 8(\hat{k}(t) + t^{\epsilon}), \quad \epsilon > 0 \tag{3.6}$$

for any $0 < t < 1$. Combining (3.5) and (3.6), we obtain the assertion. □

By the combination of this proposition with Lemma 3.1, we can obtain the equivalence between the vanishing conditions of the norms for the Schwarzian derivative and the pre-Schwarzian derivative.

Theorem 3.5. *Let* f *be a univalent analytic function on* $\mathbb{H}$*. For* $t > 0$*, set*

$$\beta(t) := \sup_{0<\mathrm{Re}\, z \leq t} (2\mathrm{Re}\, z)|Pf(z)|,$$
$$\sigma(t) := \sup_{0<\mathrm{Re}\, z \leq t} (2\mathrm{Re}\, z)^2|Sf(z)|.$$

Then $\beta(t) \to 0$ *if and only if* $\sigma(t) \to 0$ *as* $t \to 0$*.*

Proof. If $\beta(t) \to 0$ as $t \to 0$, then $\sigma(t) \to 0$ as $t \to 0$ by (3.4). We will show the opposite implication by means of Proposition 3.4.

Now suppose $\sigma(t) \to 0$ as $t \to 0$. For any small constant $\varepsilon > 0$, we choose $t_0 \in (0, \varepsilon]$ such that $\sigma(t_0) \leq \varepsilon$. For an arbitrary boundary point $iy_0 \in i\mathbb{R}$, take the open disk $D := D(1 + iy_0, 1) \subset \mathbb{H}$ of radius 1 that is tangent to $i\mathbb{R}$ at iy_0. For any z in the domain $\{z \in D : 0 < \mathrm{Re}\, z \leq t_0\}$, obviously we have

$$\rho_D^{-2}(z)|Sf(z)| \leq (2\mathrm{Re}\, z)^2|Sf(z)| \leq \sigma(t_0). \tag{3.7}$$

Moreover, by Lemma 3.1, there exists a horodisk $D'\,(\subset D)$ tangent to ∂D at iy_0 such that $\rho_D^{-1}(z)/(\mathrm{Re}\, z) \leq (2\sigma(t_0)/3)^{1/2}$ for all $z \in D \setminus D'$. This yields that

$$\rho_D^{-2}(z)|Sf(z)| = (2\mathrm{Re}\, z)^2|Sf(z)| \left(\frac{\rho_D^{-1}(z)}{2\mathrm{Re}\, z}\right)^2 \leq 6 \left(\frac{\rho_D^{-1}(z)}{2\mathrm{Re}\, z}\right)^2 \leq \sigma(t_0) \tag{3.8}$$

for all $z \in D \setminus D'$.

Combining (3.7) and (3.8), we can define a function $t_1 := \lambda(t_0)\,(\leq t_0)$ tending to 0 monotonically and continuously as $t_0 \to 0$ such that

$$\hat{\sigma}_D(t_1) := \sup_{\substack{0<d(z,\partial D)\leq t_1 \\ z\in D}} \rho_D^{-2}(z)|Sf(z)| \leq \sigma(t_0). \tag{3.9}$$

Similarly, we define

$$\hat{\beta}_D(t_1) := \sup_{\substack{0<d(z,\partial D)\leq t_1 \\ z\in D}} \rho_D^{-1}(z)|Pf(z)|.$$

By Proposition 3.4 for $\epsilon = 1$, we have

$$\hat{\beta}_D(t_1^2) \leq 4\hat{\sigma}_D(t_1) + 8t_1 \tag{3.10}$$

for all sufficiently small $t_1 > 0$.

Now we consider the estimate only on the particular segment $z = x+iy_0$ with $0 < x < t_1$ in the annulus $\{z \in D : 0 < d(z, \partial D) < t_1\}$. On this segment, we have

$$(\operatorname{Re} z)|Pf(z)| = d(z, \partial D)|Pf(z)| \leq \rho_D^{-1}(z)|Pf(z)| \leq \hat{\beta}_D(t_1). \tag{3.11}$$

Since $iy_0 \in i\mathbb{R}$ is taken arbitrarily, combining (3.9), (3.10), and (3.11), we obtain

$$\beta(t_1^2) \leq 8\sigma(\lambda^{-1}(t_1)) + 16t_1 \leq 24\varepsilon$$

since $t_1 \leq t_0 \leq \varepsilon$. This completes the proof of Theorem 3.5. □

Remark. To the best of our knowledge, the equivalence of the conditions in Theorem 3.5 is known only under the extra assumption that f can be quasiconformally extended to $\overline{\mathbb{C}}$ (see [36, Theorem 6.2], [40, Theorem 5.1]). By a similar proof or by an application, we see that the statement of Theorem 3.5 is true also for $\mathbb{D}$. Namely, $\hat{\beta}(t) \to 0$ if and only if $\hat{\sigma}(t) \to 0$ as $t \to 0$ without any extra assumption on a univalent analytic function g on $\mathbb{D}$. This improves the result in [12].

4. Quasiconformal extensions by means of Schwarzian derivative (Proof of Theorem 1.2)

In this section, we prove Theorem 1.2. For this argument, the following chordal analogue of Becker's result [9] mentioned in the introduction plays an important role, which was obtained in Gumenyuk and Hotta [23, Theorem 3.5].

Theorem 4.1. *Let (f_t) be a chordal Loewner chain in $\mathbb{H}$ over the interval $0 \leq t < \tau$ with Herglotz function p. Suppose that there exists a constant $k \in [0, 1)$ such that*

$$p(z,t) \in U(k) = \left\{ w \in \mathbb{H} : \left| \frac{w-1}{w+1} \right| \leq k \right\} \tag{4.1}$$

for all $z \in \mathbb{H}$ and almost every $0 \leq t < \tau$. Then,

- (i) *f_t has a continuous extension to $i\mathbb{R}$ for all $0 \leq t < \tau$;*
- (ii) *f_t can be k-quasiconformally extended to $\{z \in \mathbb{C} : 0 \leq -\operatorname{Re} z < \tau - t\}$ by setting $f_t(z) = f_{t-\operatorname{Re} z}(i \operatorname{Im} z)$ for all $0 \leq t < \tau$;*
- (iii) *all elements of the evolution family $(\varphi_{s,t})$ associated with p are k-quasiconformally extendible to $\overline{\mathbb{C}}$ with $\varphi_{s,t}(\infty) = \infty$ for all $s \geq 0$ and $s \leq t < \tau$.*

We remark that the original statement of [23, Theorem 3.5] concerned the case $\tau = +\infty$, namely, the time variable t extends to $+\infty$ as usual in the Loewner theory. By examining its proof, however, we see that the above form of the statement is valid for any constant τ in the interval $(0, +\infty]$.

For the construction of the quasiconformal extension $\hat{h}$ of h in Theorem 1.2, a chordal Loewner chain (h_t) over the interval $[0, \tau)$ will be formed so that $h_0(z) = h(z)$ and the boundary values of $h_t(z)$ yield the extension $\hat{h}$, as shown in (ii) of Theorem 4.1. In the next lemma, we show that a canonical family of analytic functions (h_t) can be constructed as in [23, Proposition 5.5], and that it satisfies the chordal Loewner–Kufarev PDE with an appropriate Herglotz function. Thus, we see that (h_t) is a chordal Loewner chain over $[0, \tau)$ (see [16, Theorem 4.1]).

Lemma 4.2. *Let h be univalent and analytic in $\mathbb{H}$ with $\lim_{z\to\infty} h(z) = \infty$. Let h satisfy*

$$\lim_{t\to 0^+} \sup_{0<\operatorname{Re} z\le t} \frac{1}{2}(2\operatorname{Re} z)^2\, |Sh(z)| = 0. \tag{1.5}$$

Then, for any $0 < k < 1$, there exists a positive constant $\tau_0 > 0$ such that the functions on $\mathbb{H}$ defined by

$$h_t(z) = h(z+t) - \frac{2th'(z+t)}{1 + tPh(z+t)} \tag{4.2}$$

for $0 \le t \le \tau_0$ are analytic, and this family (h_t) satisfies $h_0 = h$ and the chordal Loewner–Kufarev PDE

$$\dot{h}_t(z) = -p(z,t)h_t'(z) \tag{4.3}$$

with Herglotz function $p(z,t)$ satisfying (4.1)*.*

Proof. In view of Theorem 3.5, (1.5) implies that

$$\lim_{t\to 0^+} \sup_{0<\operatorname{Re} z\le t} (2\operatorname{Re} z)\, |Ph(z)| = 0, \tag{1.8}$$

that is, for any $0 < k < 1$, there exists a positive constant t_0 such that

$$(2\operatorname{Re} z)\, |Ph(z)| \le k$$

for all $z \in \mathbb{H}$ with $0 < \operatorname{Re} z \le t_0$. It follows that

$$t|Ph(is+t)| = \operatorname{Re}(is+t)|Ph(is+t)| \le k/2 \tag{4.4}$$

for all $is \in i\mathbb{R}$ and all $0 < t \le t_0$.

Since h is univalent on the right half-plane $\mathbb{H}$, it holds that

$$(2\operatorname{Re} z)|Ph(z)| \le 6$$

for any $z \in \mathbb{H}$ (see [38, Theorem 5.3.1]), which implies

$$t|Ph(z+t)| \le \operatorname{Re}(z+t)|Ph(z+t)| \le 3$$

for a fixed $t \in (0, t_0]$. From this, we see that the analytic function $tPh(z+t)$ in $\mathbb{H}$ is bounded above. Since $tPh(z+t)$ extends continuously to $i\mathbb{R}$ and satisfies (4.4), the Lindelöf maximal principle (see [2, p.38]) implies that

$$t|Ph(z+t)| \leq k/2 \tag{4.5}$$

for any $z \in \mathbb{H}$.

Similarly, according to (1.5), there exists a positive constant τ_0 ($< t_0$) such that

$$\frac{1}{2}(2\mathrm{Re}\, z)^2\, |Sh(z)| \leq k$$

for all $z \in \mathbb{H}$ with $0 < \mathrm{Re}\, z \leq \tau_0$. From this it follows that

$$2t^2|Sh(iy+t)| = \frac{1}{2}\left(2\mathrm{Re}(iy+t)\right)^2 |Sh(iy+t)| \leq k$$

for all $iy \in i\mathbb{R}$ and all $0 < t \leq \tau_0$. A reasoning similar to (4.5) also gives that

$$2t^2|Sh(z+t)| \leq k, \qquad \text{for all } z \in \mathbb{H}. \tag{4.6}$$

We consider the family of functions given by (4.2). This is well defined and analytic in $\mathbb{H}$ for $0 \leq t \leq t_0$ by (4.5). Obviously, $h_0(z) = h(z)$. By computation, we have

$$\dot{h}_t(z) := \frac{\partial h_t(z)}{\partial t} = -h'(z+t)\frac{1 - 2t^2 Sh(z+t)}{\left(1 + tPh(z+t)\right)^2}, \tag{4.7}$$

and

$$h_t'(z) := \frac{\partial h_t(z)}{\partial z} = h'(z+t)\frac{1 + 2t^2 Sh(z+t)}{\left(1 + tPh(z+t)\right)^2}. \tag{4.8}$$

From these expressions (4.7) and (4.8), it follows that (h_t) satisfies the chordal Loewner–Kufarev PDE (4.3) with Herglotz function

$$p(z,t) = \frac{1 - 2t^2 Sh(z+t)}{1 + 2t^2 Sh(z+t)}. \tag{4.9}$$

Moreover, by (4.6) we deduce that

$$\left|\frac{p(z,t) - 1}{p(z,t) + 1}\right| = 2t^2|Sh(z+t)| \leq k < 1 \tag{4.10}$$

for all $z \in \mathbb{H}$ and all $0 \leq t \leq \tau_0$. Namely, $p(z,t) \in U(k)$. □

In the rest of this section, we will focus on proving the univalence of the analytic function h_t in $\mathbb{H}$ defined by (4.2) for each t in some interval. The arguments will be separated into the cases of assumptions (**A**) and (**B**).

Theorem 4.3. *Let h be univalent and analytic in $\mathbb{H}$ with $\lim_{z\to\infty} h(z) = \infty$ and let $h(\mathbb{H})$ be a Jordan domain. Let h satisfy*

$$\lim_{t\to 0^+} \sup_{0<\mathrm{Re}\, z \leq t} \frac{1}{2}(2\mathrm{Re}\, z)^2\, |Sh(z)| = 0. \tag{1.5}$$

Suppose (**A**)*: both h and h^{-1} are locally uniformly continuous in a neighborhood of ∞. Then, there is a positive constant $\tau \leq \tau_0$ such that the analytic function h_t on $\mathbb{H}$ defined by* (4.2) *is univalent for all $0 \leq t \leq \tau$. Here, τ_0 is the constant occurring in Lemma 4.2.*

Proof. The proof contains several claims. For all claims, as well as other explanations, the assumptions of the theorem are always in force. By Lemma 4.2, the family (h_t) defined by (4.2) for $0 \leq t \leq \tau_0$ satisfies the chordal Loewner–Kufarev PDE (4.3) with the Herglotz function $p(z,t)$ in (4.1). In particular, $|p(z,t)| \leq K$ for $K := (1+k)/(1-k)$. Here, k is the constant taken in the proof of Lemma 4.2, and τ_0 is the constant occurring in Lemma 4.2, depending on k.

Let $(\varphi_{s,t})$ be the evolution family associated with this Herglotz function p for all $0 \leq s \leq t \leq \tau_0$. Namely, $(\varphi_{s,t})$ satisfies the chordal Loewner–Kufarev ODE:

$$\dot{\varphi}_{s,t}(z) = p(\varphi_{s,t}(z), t), \quad 0 \leq s \leq t \leq \tau_0, \quad \varphi_{s,s}(z) = z, \tag{4.11}$$

where $\dot{\varphi}_{s,t} := d\varphi_{s,t}/dt$. By Theorem 4.1, $\varphi_{s,t}$ is k-quasiconformally extendible to $\overline{\mathbb{C}}$ with $\varphi_{s,t}(\infty) = \infty$. Combining (4.3) and (4.11), we obtain that for any fixed $z \in \mathbb{H}$,

$$\begin{aligned}\frac{d}{dt}(h_t \circ \varphi_{s,t}(z)) &= h_t'(\varphi_{s,t}(z))\dot{\varphi}_{s,t}(z) + \dot{h}_t(\varphi_{s,t}(z)) \\ &= h_t'(\varphi_{s,t}(z))\dot{\varphi}_{s,t}(z) - p(\varphi_{s,t}(z), t)h_t'(\varphi_{s,t}(z)) \\ &= h_t'(\varphi_{s,t}(z))(\dot{\varphi}_{s,t}(z) - p(\varphi_{s,t}(z), t)) = 0.\end{aligned}$$

Thus, $h_t \circ \varphi_{s,t}(z)$ does not depend on t. By taking $t = s$ we see that $h_t \circ \varphi_{s,t}(z) = h_s(z)$. The equations $\varphi_{0,t} = \varphi_{s,t} \circ \varphi_{0,s}$ and $h_s = h_t \circ \varphi_{s,t}$ will be frequently used below.

We note that the univalent analytic function h on $\mathbb{H}$ has a continuous and injective extension to $i\hat{\mathbb{R}} = i\mathbb{R} \cup \{\infty\}$ with $h(\infty) = \infty$ by the Carathéodory theorem (see [34, p.18]) since $h(\mathbb{H})$ is a Jordan domain.

Claim 0. *Let $z_0 \in \mathbb{H} \cup i\hat{\mathbb{R}}$. For any $z_0 \neq \infty$, $h_t(z)$ converges to $h(z_0)$ as $z \in \mathbb{H}$ tends to z_0 and $t \in [0, \tau_0]$ tends to 0, while $h_t(z)$ converges to ∞ $(= h(\infty))$ independently of t as $z \in \mathbb{H}$ tends to ∞.*

Proof. For any $z \in \mathbb{H}$ and $0 \leq t \leq t' \leq \tau_0$,

$$\begin{aligned}|\varphi_{t,t'}(z) - z| &= |\varphi_{t,t'}(z) - \varphi_{t,t}(z)| \\ &= \left| \int_t^{t'} \dot{\varphi}_{t,s}(z) ds \right| = \left| \int_t^{t'} p(\varphi_{t,s}(z), s) ds \right| \\ &\leq \int_t^{t'} |p(\varphi_{t,s}(z), s)| ds \leq K(t' - t)\end{aligned} \tag{4.12}$$

is satisfied. Here, $\dot{\varphi}_{t,s} = d\varphi_{t,s}/ds$. This is also true for $z \in i\mathbb{R}$ since each $\varphi_{t,t'}$ has the continuous extension to $i\mathbb{R}$. In particular,

$$|\varphi_{0,t}(z) - z| \leq Kt \tag{4.13}$$

for any $z \in \mathbb{H} \cup i\mathbb{R}$ and $0 \le t \le \tau_0$.

Let $\Omega(M) := \{z \in \mathbb{H} : |z| \le M\}$ for any $M > 0$. Then,

$$|\varphi_{t,\tau_0}(z)| \le |\varphi_{t,\tau_0}(z) - z| + |z| \le K\tau_0 + M \tag{4.14}$$

for any $z \in \Omega(M)$. This shows that the family $\{\varphi_{t,\tau_0}\}_{t\in[0,\tau_0]}$ is uniformly bounded on the compact set $\overline{\Omega}(M) = \{z \in \mathbb{H} \cup i\mathbb{R} : |z| \le M\}$ for any $M > 0$.

If the family of k-quasiconformal mappings $\varphi_{t,\tau_0} : \mathbb{C} \to \mathbb{C}$ is uniformly bounded on $\overline{\Omega}(M + K\tau_0)$, then it is also equicontinuous on $\overline{\Omega}(M + K\tau_0)$ (see [29, Theorem II.4.1]), and thus uniformly Hölder continuous with exponent $1/K$ and with multiplicative constant $C_1 > 0$ (see [29, Theorem II.4.3]). From this and (4.13), it follows that

$$\begin{aligned} |\varphi_{t,\tau_0}(z) - \varphi_{0,\tau_0}(z)| &= |\varphi_{t,\tau_0}(z) - \varphi_{t,\tau_0} \circ \varphi_{0,t}(z)| \\ &\le C_1|z - \varphi_{0,t}(z)|^{1/K} \le C_1(Kt)^{1/K} \end{aligned} \tag{4.15}$$

for any $z \in \Omega(M)$.

Since $p(z,t) \in U(k)$ for all $z \in \mathbb{H}$ and all $0 \le t \le \tau_0$, there exists a positive constant $C_2 > 0$ such that $\operatorname{Re} p(z,t) \ge C_2$. It follows that

$$\begin{aligned} \operatorname{Re}(\varphi_{0,t}(z)) - \operatorname{Re} z &= \int_0^t \operatorname{Re}(\dot{\varphi}_{0,s}(z))ds \\ &= \int_0^t \operatorname{Re}(p(\varphi_{0,s}(z), s))ds \ge C_2 t. \end{aligned}$$

This implies that $\varphi_{0,\tau_0}(\mathbb{H})$ is contained in $\{z \in \mathbb{H} : \operatorname{Re} z > C_2\tau_0\}$. Combining this with (4.14) and (4.15), we can take a compact convex subset W in $\mathbb{H}$ such that $\varphi_{t,\tau_0}(z)$ and $\varphi_{0,\tau_0}(z)$ are contained in W for all $z \in \Omega(M)$ and for all sufficiently small t.

Let $C_3 := \max_{w\in W} |h'_{\tau_0}(w)| < +\infty$. Then, for any two points w_1, $w_2 \in W$,

$$\begin{aligned} |h_{\tau_0}(w_1) - h_{\tau_0}(w_2)| &= \left| \int_0^1 \frac{d}{dt} h_{\tau_0}((1-t)w_1 + tw_2)dt \right| \\ &= |w_2 - w_1| \left| \int_0^1 h'_{\tau_0}((1-t)w_1 + tw_2)dt \right| \le C_3|w_2 - w_1|, \end{aligned}$$

from which we deduce that

$$\begin{aligned} |h_t(z) - h(z_0)| &\le |h_t(z) - h(z)| + |h(z) - h(z_0)| \\ &= |h_{\tau_0} \circ \varphi_{t,\tau_0}(z) - h_{\tau_0} \circ \varphi_{0,\tau_0}(z)| + |h(z) - h(z_0)| \\ &\le C_3|\varphi_{t,\tau_0}(z) - \varphi_{0,\tau_0}(z)| + |h(z) - h(z_0)| \end{aligned} \tag{4.16}$$

if $z \in \Omega(M)$. In the case where $z_0 \ne \infty$, by choosing M according to z_0, we see that $h_t(z) \to h(z_0)$ as $z \to z_0$ and $t \to 0$ by (4.15) and (4.16).

Finally, we assume that $z_0 = \infty$. By the equation $h_{\tau_0} \circ \varphi_{0,\tau_0} = h$, we have $\lim_{z\to\infty} h_{\tau_0}(z) = \lim_{z\to\infty} h \circ \varphi_{0,\tau_0}^{-1}(z) = \infty$. Moreover,

$$\begin{aligned} |\varphi_{t,\tau_0}(z)| &\geq |z| - |\varphi_{t,\tau_0}(z) - \varphi_{t,t}(z)| \\ &= |z| - \left| \int_t^{\tau_0} \dot{\varphi}_{t,s}(z) ds \right| \\ &\geq |z| - \int_t^{\tau_0} |p(\varphi_{t,s}(z), s)| ds \geq |z| - K\tau_0. \end{aligned}$$

Here, $\dot{\varphi}_{t,s} = d\varphi_{t,s}/ds$. Hence, $h_t(z) = h_{\tau_0} \circ \varphi_{t,\tau_0}(z) \to \infty \, (= h(\infty))$ as $z \to \infty$ (independently of t). □

Claim 1. *There exist positive constants τ_1 $(\leq \tau_0)$ and r such that for any $0 \leq t \leq \tau_1$, $h_t(z)$ is univalent on each square*

$$Q_{2r} := (0, 2r) \times (y_0 - r, y_0 + r) \subset \mathbb{H},$$

where $y_0 \in \mathbb{R}$ can be chosen arbitrarily.

Remark that the square Q_{2r}, as a quasidisk, satisfies the Schwarzian univalence criterion (see [24, Corollary 5], [28, p.126]). Precisely, if a function g is locally univalent and analytic in Q_{2r} and if

$$\sup_{z\in Q_{2r}} |Sg(z)| \rho_{Q_{2r}}^{-2}(z) \leq 1/2,$$

then g is univalent in Q_{2r}. Moreover, by (3.4), there exists a positive constant a such that if

$$\sup_{z\in Q_{2r}} |Pg(z)| \rho_{Q_{2r}}^{-1}(z) \leq a,$$

then g is univalent in Q_{2r}.

We also need the following result, which can be deduced easily from the proof of [36, Lemma 6.3].

Proposition 4.4. *Let ψ be an analytic function of $z = x + iy$ in $\mathbb{H}$ satisfying the condition $\lim_{x\to+\infty} \psi(x + iy) = 0$ uniformly for $y \in \mathbb{R}$. Then, for any constant $\alpha > 0$,*

$$\sup_{z\in\mathbb{H}} |\psi(z)| x^{\alpha} < \infty \quad \Longleftrightarrow \quad \sup_{z\in\mathbb{H}} |\psi'(z)| x^{\alpha+1} < \infty,$$

and both terms are comparable with comparison constants depending only on α. Moreover,

$$\lim_{x\to 0^+} |\psi(z)| x^{\alpha} = 0 \quad \Longleftrightarrow \quad \lim_{x\to 0^+} |\psi'(z)| x^{\alpha+1} = 0.$$

Proof of Claim 1. By (4.6) and (4.8), we see that $h_t'(z) \neq 0$ for all $z \in \mathbb{H}$ and all $0 \leq t \leq \tau_0$. A direct computation yields that

$$\begin{aligned} Ph_t(z) &= \frac{h_t''(z)}{h_t'(z)} = (\log h_t'(z))' \\ &= Ph(z+t) + \frac{2t^2 \left(Sh(z+t)\right)'}{1 + 2t^2 Sh(z+t)} - \frac{2t \left(Ph(z+t)\right)'}{1 + tPh(z+t)}. \end{aligned}$$

From this we have

$$(2\mathrm{Re}\, z)\,|Ph_t(z)| \le (2\mathrm{Re}(z+t))|Ph(z+t)| + \frac{(2\mathrm{Re}(z+t))^3|(Sh(z+t))'|}{1-(2\mathrm{Re}(z+t))^2|Sh(z+t)|} + \frac{(2\mathrm{Re}(z+t))^2|(Ph(z+t))'|}{1-(2\mathrm{Re}(z+t))|Ph(z+t)|}.$$

Then, applying (1.5) and (1.8), we conclude by Proposition 4.4 that there exists a positive constant τ_1 ($\le \tau_0$) such that

$$(2\mathrm{Re}\, z)\,|Ph_t(z)| \le a$$

for any $z \in \mathbb{H}$ and $t \ge 0$ with $0 < \mathrm{Re}(z+t) \le 3\tau_1$. In particular, this holds for all $z \in \mathbb{H}$ with $0 < \mathrm{Re}\, z \le 2r$ and all $0 \le t \le \tau_1$ by taking $r = \tau_1$. We fix this constant $r > 0$.

By the monotonicity principle for hyperbolic densities with respect to the inclusion of domains (see [28, p.6]), we have

$$\rho_{Q_{2r}}^{-1}(z)\,|Ph_t(z)| \le (2\mathrm{Re}\, z)\,|Ph_t(z)| \le a$$

for all $z \in Q_{2r}$, which implies that h_t is univalent in each Q_{2r} for any $0 \le t \le \tau_1$. □

Claim 2. *There exists a positive constant τ_2 ($\le \tau_0$) such that for each $0 \le t \le \tau_2$, h_t is univalent in the half-plane $\mathbb{H}_r := \{z \in \mathbb{H} : \mathrm{Re}\, z > r\}$, where r is the constant chosen in Claim 1.*

Proof. Since h and $\varphi_{0,t}$ are univalent in $\mathbb{H}$ for any $0 \le t \le \tau_0$ and since $h_t \circ \varphi_{0,t} = h$, we see that h_t is univalent in the domain $\varphi_{0,t}(\mathbb{H})$. Here, (4.13) in particular implies that $\varphi_{0,t}$ converges to $\mathrm{id}_{\mathbb{H}}$ uniformly with respect to $z \in \mathbb{H}$ as $t \to 0$. Then, there exists a positive constant τ_2 ($\le \tau_0$) such that $\mathbb{H}_r \subset \varphi_{0,t}(\mathbb{H})$ for each $0 \le t \le \tau_2$, and thus h_t is univalent in $\mathbb{H}_r$. □

Let us recall the well-known Koebe distortion theorem (see [34, Theorem 1.3]). It says that if g is analytic and univalent in $\mathbb{D}$ then for all $z \in \mathbb{D}$ it holds that

$$|g'(0)|\frac{|z|}{(1+|z|)^2} \le |g(z)-g(0)| \le |g'(0)|\frac{|z|}{(1-|z|)^2}.$$

Applying it to an analytic and univalent function g on a closed disk $\overline{D}(z_0+t, t)$ with center $z_0 + t$ and radius t by translation and dilation, we have that for all $z \in \overline{\mathbb{D}}$,

$$t|g'(z_0+t)|\frac{|z|}{(1+|z|)^2} \le |g(tz+(z_0+t)) - g(z_0+t)| \le t|g'(z_0+t)|\frac{|z|}{(1-|z|)^2}.$$

In particular, the left inequality for $z = -1$ yields that

$$|g(z_0) - g(z_0+t)| \ge \frac{t}{4}|g'(z_0+t)|. \tag{4.17}$$

Claim 3. *Suppose that assumption* (**A**) *holds. Namely, h is uniformly continuous in $\Omega_M := \{z \in \mathbb{H} : |z| > M\}$ and h^{-1} is uniformly continuous in $\Omega'_M := \{\zeta \in h(\mathbb{H}) : |\zeta| > M\}$ for some $M > 0$. Then, there exist positive constants τ_3 ($< \tau_0$) and $R > M$ such that if $z_1, z_2 \in \Omega_R$ with $|z_1 - z_2| \ge r$, then $h_t(z_1) \ne h_t(z_2)$ for all $0 \le t \le \tau_3$, where r is the constant chosen in Claim 1.*

Proof. Suppose that h is uniformly continuous in Ω_M and h^{-1} is uniformly continuous in Ω'_M for some $M > 0$. Let $\zeta_1 = h(z_1)$ and $\zeta_2 = h(z_2)$. As $h(\infty) = \infty$, there is a constant $R > M$ such that if $z_1, z_2 \in \Omega_R$ then $\zeta_1, \zeta_2 \in \Omega'_M$. Since h^{-1} is uniformly continuous in Ω'_M, for the given constant r, there exists some $\delta > 0$ such that

$$|\zeta_1 - \zeta_2| < \delta \quad \Longrightarrow \quad |z_1 - z_2| < r,$$

or equivalently,

$$|z_1 - z_2| \geq r \quad \Longrightarrow \quad |\zeta_1 - \zeta_2| \geq \delta \tag{4.18}$$

for $z_1, z_2 \in \Omega_R$. Moreover, since h is uniformly continuous in $\Omega_R \subset \Omega_M$, for this δ, there exists a positive constant τ_3 $(< \tau_0)$ such that

$$|z_1 - z_2| \leq \tau_3 \quad \Longrightarrow \quad |\zeta_1 - \zeta_2| \leq \delta/64 \tag{4.19}$$

for $z_1, z_2 \in \Omega_R$.

Assume that $z_1, z_2 \in \Omega_R$ with $|z_1 - z_2| \geq r$ and $0 \leq t \leq \tau_3$. By using (4.5), (4.17), (4.18), and (4.19) in this order, we have

$$\begin{aligned}
|h_t(z_1) - h_t(z_2)| &= \left| h(z_1 + t) - h(z_2 + t) - \frac{2th'(z_1 + t)}{1 + tPh(z_1 + t)} + \frac{2th'(z_2 + t)}{1 + tPh(z_2 + t)} \right| \\
&\geq |h(z_1 + t) - h(z_2 + t)| - 4t|h'(z_1 + t)| - 4t|h'(z_2 + t)| \\
&\geq |h(z_1 + t) - h(z_2 + t)| - 16|h(z_1 + t) - h(z_1)| - 16|h(z_2 + t) - h(z_2)| \\
&\geq \delta - \delta/4 - \delta/4 > 0.
\end{aligned}$$

This proves that $h_t(z_1) \neq h_t(z_2)$. □

Claim 4. *There exists a positive constant τ_4 $(\leq \tau_0)$ such that if $|z_1 - z_2| \geq r$ and if either $|z_1| \leq R$ or $|z_2| \leq R$, then $h_t(z_1) \neq h_t(z_2)$ for all $0 \leq t \leq \tau_4$. Here, r is the constant chosen in Claim 1 and R is the constant chosen in Claim 3.*

Proof. Suppose that there is no such τ_4. Then, there exist a sequence $\{t_n\}$ with $t_n \to 0$ as $n \to \infty$, and sequences $\{z_{1n}\}$, $\{z_{2n}\}$ in $\mathbb{H}$ satisfying $|z_{1n} - z_{2n}| \geq r$ and either $|z_{1n}| \leq R$ or $|z_{2n}| \leq R$, but $h_{t_n}(z_{1n}) = h_{t_n}(z_{2n})$ for all n. We may assume that $\{z_{1n}\}$ and $\{z_{2n}\}$ converge to some points z_1 and z_2 in $\mathbb{H} \cup i\hat{\mathbb{R}}$, respectively, by passing to subsequences.

By Claim 0, $h_{t_n}(z_{1n}) \to h(z_1)$ and $h_{t_n}(z_{2n}) \to h(z_2)$ as $n \to \infty$. Since $h_{t_n}(z_{1n}) = h_{t_n}(z_{2n})$, we have $h(z_1) = h(z_2)$. However, both the condition $|z_{1n} - z_{2n}| \geq r$ and the condition that either $|z_{1n}| \leq R$ or $|z_{2n}| \leq R$ imply that $z_1 \neq z_2$. This contradicts the fact that h is injective in $\mathbb{H} \cup i\hat{\mathbb{R}}$. □

We now finish the proof. Set $\tau := \min\{\tau_1, \tau_2, \tau_3, \tau_4\} > 0$. Let r be the constant chosen in Claim 1, and R the constant chosen in Claim 3. We show that h_t is univalent in $\mathbb{H}$ for $0 \leq t \leq \tau$. For any distinct points z_1 and z_2 in $\mathbb{H}$, one of the following three cases occurs:

- If $|z_1 - z_2| < r$, then $h_t(z_1) \neq h_t(z_2)$ by Claims 1 and 2.
- If $|z_1 - z_2| \geq r$ and if both $|z_1| > R$ and $|z_2| > R$, then $h_t(z_1) \neq h_t(z_2)$ by Claim 3.
- If $|z_1 - z_2| \geq r$ and if either $|z_1| \leq R$ or $|z_2| \leq R$, then $h_t(z_1) \neq h_t(z_2)$ by Claim 4.

In any case, we have $h_t(z_1) \neq h_t(z_2)$ and hence h_t is injective in $\mathbb{H}$. This completes the proof of Theorem 4.3. □

Next, we prove the univalence of h_t under assumption (**B**).

Theorem 4.5. *Let h be univalent and analytic in $\mathbb{H}$ with $\lim_{z\to\infty} h(z) = \infty$ and let $h(\mathbb{H})$ be a Jordan domain. Let h satisfy* (1.5). *Suppose* (**B**)*: h is locally quasiconformally extendible to a neighborhood of ∞. Then, there is a positive constant $\tau \leq \tau_0$ such that the analytic function h_t on $\mathbb{H}$ defined by* (4.2) *is univalent for all $0 \leq t \leq \tau$. Here, τ_0 is the constant occurring in Lemma 4.2.*

Proof. In the proof of Theorem 4.3, assumption (**A**) is used only in Claim 3; the other claims are applicable also to the proof of Theorem 4.5. Hence, it is enough to show the following parallel result, Claim 5, to Claim 3 and we then reset $\tau := \min\{\tau_1, \tau_2, \tau_4, \tau_5\}$ and R, where τ_5 and R are determined there. □

Claim 5. *Suppose that assumption* (**B**) *holds. Namely, h has a quasiconformal extension to $D_M := \{z \in \mathbb{C} : |z| > M\}$ for some $M > 0$. Then, there exist positive constants τ_5 ($\leq \tau_0$) and $R > M$ such that if $z_1, z_2 \in \Omega_R$ with $|z_1 - z_2| \geq r$, then $h_t(z_1) \neq h_t(z_2)$ for all $0 \leq t \leq \tau_5$, where r is the constant chosen in Claim 1.*

The key concept we use to prove Claim 5 is quasisymmetry, which was introduced by Beurling and Ahlfors [6] on the real line and formulated for general metric spaces by Tukia and Väisälä [39]. For our purpose we only consider it for the complex plane. We refer to [4, Definition 3.2.1].

Definition. Let $D \subset \mathbb{C}$ be an open subset and $f : D \to \mathbb{C}$ an orientation-preserving mapping. Let $\eta : [0, +\infty) \to [0, +\infty)$ be a homeomorphism. We say that f is η-quasisymmetric if for each triple z_0, z_1, $z_2 \in D$ we have

$$\frac{|f(z_0) - f(z_1)|}{|f(z_0) - f(z_2)|} \leq \eta\left(\frac{|z_0 - z_1|}{|z_0 - z_2|}\right).$$

If $f : \mathbb{C} \to \mathbb{C}$ is a k-quasiconformal homeomorphism of $\mathbb{C}$, then f is η-quasisymmetric where η depends only on k (see [4, Theorem 3.5.3]); conversely, if $f : D \to \mathbb{C}$ is an η-quasisymmetric mapping on a domain D then f is quasiconformal (see [4, Theorem 3.4.1]).

Proof of Claim 5. Suppose that h has a quasiconformal extension to D_M, which we still denote by h. Let $R = M + 1$. By an extension theorem (see [29, Theorem II.1.8]), there exists a quasiconformal homeomorphism F of the whole plane $\mathbb{C}$ that coincides with h in D_R. In particular, it coincides with h in Ω_R. The inverse F^{-1} of F is also a quasiconformal homeomorphism of $\mathbb{C}$, and then it is a quasisymmetric homeomorphism of $\mathbb{C}$. Thus, there exists a homeomorphism $\eta : [0, +\infty) \to [0, +\infty)$ such that

$$\frac{|F^{-1}(w_0) - F^{-1}(w_1)|}{|F^{-1}(w_0) - F^{-1}(w_2)|} \leq \eta\left(\frac{|w_0 - w_1|}{|w_0 - w_2|}\right)$$

for any $w_0, w_1, w_2 \in \mathbb{C}$. We take the inverse $\lambda := \eta^{-1}$.

For any $z_1, z_2 \in \Omega_R$ with $|z_1 - z_2| \geq r$ and any $t > 0$, we have

$$\begin{aligned}\frac{|h(z_1+t)-h(z_2+t)|}{|h(z_1)-h(z_1+t)|} &= \frac{|F(z_1+t)-F(z_2+t)|}{|F(z_1)-F(z_1+t)|} \\ &\geq \lambda\left(\frac{|(z_1+t)-(z_2+t)|}{|z_1-(z_1+t)|}\right) = \lambda\left(\frac{|z_1-z_2|}{t}\right) \geq \lambda\left(\frac{r}{t}\right).\end{aligned} \tag{4.20}$$

We may assume that $|h'(z_1+t)| \geq |h'(z_2+t)|$ by exchanging the roles of z_1 and z_2 if necessary. Then, by using (4.5), (4.17), and (4.20), we obtain that

$$\begin{aligned}|h_t(z_1) - h_t(z_2)| &= \left| h(z_1+t) - h(z_2+t) - \frac{2th'(z_1+t)}{1+tPh(z_1+t)} + \frac{2th'(z_2+t)}{1+tPh(z_2+t)} \right| \\ &\geq |h(z_1+t) - h(z_2+t)| - 4t|h'(z_1+t)| - 4t|h'(z_2+t)| \\ &\geq |h(z_1) - h(z_1+t)|\lambda\left(\frac{r}{t}\right) - 8t|h'(z_1+t)| \\ &\geq \frac{t}{4}|h'(z_1+t)|\lambda\left(\frac{r}{t}\right) - 8t|h'(z_1+t)| = \frac{t}{4}|h'(z_1+t)|\left(\lambda\left(\frac{r}{t}\right) - 32\right)\end{aligned}$$

for all $0 < t \leq \tau_0$. It can be seen from the monotonicity of λ that there exists a positive constant τ_4 ($\leq \tau_0$) such that $\lambda(r/t) > 32$ for all $0 < t \leq \tau_4$. This shows that $h_t(z_1) \neq h_t(z_2)$ for all $0 \leq t \leq \tau_4$. □

We have proved Theorems 4.3 and 4.5. Then, Theorem 1.2 follows from these theorems combined with Theorem 4.1 and Lemma 4.2.

Proof of Theorem 1.2. By Theorems 4.3 and 4.5, we see that the analytic function h_t defined by (4.2) is univalent in $\mathbb{H}$ for each $0 \leq t \leq \tau$. Then, by Lemma 4.2, we obtain that (h_t) is a chordal Loewner chain over the interval $[0, \tau)$ with the associated Herglotz function p satisfying (4.1). From Theorem 4.1, the assertion of Theorem 1.2 follows. Indeed, the extension $\hat{h}$ is defined by

$$\hat{h}(z) = h_{-\mathrm{Re}\, z}(i\mathrm{Im}\, z), \qquad -\tau < \mathrm{Re}\, z < 0,$$

for the chordal Loewner chain (h_t) over the interval $[0, \tau)$ given by (4.2). This yields the explicit formula of this extension in (1.6). From this, its complex dilatation can be computed directly as in (1.7). □

5. Quasiconformal extensions by means of pre-Schwarzian derivative

We have the following parallel result to Theorem 1.2 by replacing the Schwarzian derivative Sf of f with the pre-Schwarzian derivative Pf of f.

Theorem 5.1. *Let f be univalent and analytic in $\mathbb{H}$ with $\lim_{z\to\infty} f(z) = \infty$ such that $f(\mathbb{H})$ is a Jordan domain. Let f satisfy the condition*

$$\lim_{t\to 0^+} \sup_{0<\mathrm{Re}\, z \leq t} (2\mathrm{Re}\, z)\,|Pf(z)| = 0.$$

Suppose further that either (**A**) *or* (**B**) *holds. Then the function*

$$\hat{f}(z) = \begin{cases} f(z), & \text{if } \operatorname{Re} z \geq 0, \\ f(z^*) + (2\operatorname{Re} z) f'(z^*), \quad (z^* := -\bar{z}) & \text{if } -\tau < \operatorname{Re} z < 0 \end{cases}$$

defines a quasiconformal extension of f *over* $i\mathbb{R}$ *to* $\mathbb{H}^*_{(0,\tau)} := \{z \in \mathbb{C} : -\tau < \operatorname{Re} z < 0\}$ *for some* $\tau > 0$ *such that its complex dilatation* μ *on* $\mathbb{H}^*_{(0,\tau)}$ *has the form*

$$\mu(z) = -(2\operatorname{Re} z)\, Pf(z^*).$$

The proof of Theorem 5.1 is completely similar to that of Theorem 1.2. We only mention the difference. Instead of considering the family (h_t) defined by (4.2), we consider a simpler one

$$f_t(z) := f(z+t) - 2tf'(z+t), \qquad z \in \mathbb{H}, \quad 0 \leq t \leq \tau_0$$

for some $\tau_0 > 0$. It is associated with the Herglotz function

$$p(z,t) = \frac{1 + 2tPf(z+t)}{1 - 2tPf(z+t)}$$

in comparison with (4.9). This satisfies

$$\left| \frac{p(z,t) - 1}{p(z,t) + 1} \right| = 2t\,|Pf(z+t)| \leq k < 1$$

for all $z \in \mathbb{H}$ and all $0 \leq t \leq \tau_0$ in comparison with (4.10). Then, the corresponding statement to Lemma 4.2 is obtained. To prove that there is some positive constant τ $(\leq \tau_0)$ such that f_t is univalent for any $0 \leq t \leq \tau$, we repeat the arguments for Theorems 4.3 and 4.5. Because we deal with the simpler family (f_t), the argument becomes a bit simpler. We omit the details here.

6. Application to VMO-Teichmüller space (Proof of Theorem 1.4)

Before proceeding to the proof of Theorem 1.4, let us recall some basic definitions on Carleson measures and BMO functions (see [21, Chapter 6]).

We say that a positive measure λ on $\mathbb{H}$ is a Carleson measure if

$$\|\lambda\|_c := \sup_{I \subset i\mathbb{R}} \frac{\lambda((0,\,|I|) \times I)}{|I|} < \infty,$$

where the supremum is taken over all bounded intervals I in $i\mathbb{R}$. A Carleson measure λ is called a vanishing Carleson measure if

$$\lim_{|I| \to 0} \frac{\lambda((0,\,|I|) \times I)}{|I|} = 0$$

uniformly. We denote by $\mathrm{CM}(\mathbb{H})$ and $\mathrm{CM}_0(\mathbb{H})$ the sets of all Carleson measures and vanishing Carleson measures on $\mathbb{H}$, respectively. The spaces $\mathrm{CM}(\mathbb{H}^*)$ and $\mathrm{CM}_0(\mathbb{H}^*)$ are defined similarly.

A locally integrable complex-valued function u on $i\mathbb{R}$ is of BMO (denoted by $u \in \mathrm{BMO}(i\mathbb{R})$) if

$$\|u\|_* := \sup_{I\subset i\mathbb{R}} \frac{1}{|I|}\int_I |u(z)-u_I|\,|dz| < \infty,$$

where the supremum is taken over all bounded intervals I on $i\mathbb{R}$ and u_I denotes the integral mean of u over I. Moreover, u is of VMO (denoted by $u \in \mathrm{VMO}(i\mathbb{R})$) if in addition

$$\lim_{|I|\to 0} \frac{1}{|I|}\int_I |u(z)-u_I|\,|dz| = 0$$

uniformly.

Let $\mathrm{BMOA}(\mathbb{H})$ denote the space of all analytic functions ϕ on $\mathbb{H}$ that are Poisson integrals of BMO functions on $i\mathbb{R}$, and let $\mathrm{VMOA}(\mathbb{H})$ denote the subspace of $\mathrm{BMOA}(\mathbb{H})$ whose elements have boundary values in $\mathrm{VMO}(i\mathbb{R})$. By using arguments similar to those in the case of the unit disk (see [21, p.233]), we conclude that an analytic function ϕ on $\mathbb{H}$ belongs to $\mathrm{BMOA}(\mathbb{H})$ if and only if ϕ induces a Carleson measure

$$(2\mathrm{Re}\, z)\,|\phi'(z)|^2 dxdy \in \mathrm{CM}(\mathbb{H}),$$

and moreover, ϕ belongs to $\mathrm{VMOA}(\mathbb{H})$ if and only if

$$(2\mathrm{Re}\, z)\,|\phi'(z)|^2 dxdy \in \mathrm{CM}_0(\mathbb{H}).$$

We need the following result from [36, Proposition 7.4]. This is conceptually similar to Proposition 4.4.

Proposition 6.1. *Let ψ be an analytic function on $\mathbb{H}$ such that $\lim_{x\to+\infty}\psi(x+iy)=0$ uniformly for $y\in\mathbb{R}$. For $\alpha>0$ set*

$$\lambda_1 := |\psi(z)|^2 x^{\alpha} dxdy,$$
$$\lambda_2 := |\psi'(z)|^2 x^{\alpha+2} dxdy,$$

for $z = x+iy \in \mathbb{H}$. Then $\lambda_1 \in \mathrm{CM}(\mathbb{H})$ if and only if $\lambda_2 \in \mathrm{CM}(\mathbb{H})$, and $\|\lambda_1\|_c \asymp \|\lambda_2\|_c$ with comparison constants depending only on α. Moreover, $\lambda_1 \in \mathrm{CM}_0(\mathbb{H})$ if and only if $\lambda_2 \in \mathrm{CM}_0(\mathbb{H})$.

Proof of Theorem 1.4. Since h is analytic and univalent on $\mathbb{H}$ with $h(\infty)=\infty$, we have the well-known inequalities

$$(2\mathrm{Re}\, z)\,|Ph(z)| \le 6, \qquad (2\mathrm{Re}\, z)^2|Sh(z)| \le 6 \tag{6.1}$$

for all $z\in\mathbb{H}$. These in particular yield that $Ph(z)\to 0$ and $Sh(z)\to 0$ uniformly for $\mathrm{Im}\, z\in\mathbb{R}$ as $\mathrm{Re}\, z\to+\infty$. Since $\log h' \in \mathrm{VMOA}(\mathbb{H})$, we obtain by [36, Lemma 7.1] that

$$\lim_{\mathrm{Re}\, z\to 0^+} (2\mathrm{Re}\, z)\,|Ph(z)| = 0$$

uniformly for $\mathrm{Im}\, z\in\mathbb{R}$, and then by Theorem 3.5 that

$$\lim_{\mathrm{Re}\, z\to 0^+} (2\mathrm{Re}\, z)^2|Sh(z)| = 0$$

uniformly for $\operatorname{Im} z \in \mathbb{R}$. Thus h satisfies the condition of Corollary 1.3. In addition, the Loewner chain (h_t) over some interval $[0,\tau)$ produced by h in the form of (4.2) satisfies the condition of Theorem 4.1.

It follows from the equality $Sh = (Ph)' - (Ph)^2/2$ and (6.1) that

$$\begin{aligned}(2\operatorname{Re} z)^3|Sh(z)|^2 &\leq 2(2\operatorname{Re} z)^3|(Ph(z))'|^2 + \frac{1}{2}\big((2\operatorname{Re} z)|Ph(z)|\big)^2(2\operatorname{Re} z)|Ph(z)|^2\\ &\leq 2(2\operatorname{Re} z)^3|(Ph(z))'|^2 + 18(2\operatorname{Re} z)\,|Ph(z)|^2.\end{aligned} \tag{6.2}$$

Since $\log h' \in \mathrm{VMOA}(\mathbb{H})$, or equivalently,

$$(2\operatorname{Re} z)\,|Ph(z)|^2 dxdy \in \mathrm{CM}_0(\mathbb{H}), \tag{6.3}$$

we conclude by Proposition 6.1 and (6.2) that

$$(2\operatorname{Re} z)^3|Sh(z)|^2 dxdy \in \mathrm{CM}_0(\mathbb{H}). \tag{6.4}$$

We choose any $0 < t < \tau$ and fix it. It follows from (4.8) that

$$Ph_t(z) = (\log h_t'(z))' = Ph(z+t) + \frac{2t^2(Sh(z+t))'}{1+2t^2Sh(z+t)} - \frac{2t(Ph(z+t))'}{1+tPh(z+t)}.$$

By means of (4.5) and (4.6), $|1 + 2t^2Sh(z+t)|$ and $|1 + tPh(z+t)|$ are bounded below by a positive constant. Hence,

$$\begin{aligned}(2\operatorname{Re} z)|Ph_t(z)|^2 \leq & C\big((2\operatorname{Re}(z+t))|Ph(z+t)|^2\\ &+ (2\operatorname{Re}(z+t))^5|(Sh(z+t))'|^2 + (2\operatorname{Re}(z+t))^3|(Ph(z+t))'|^2\big)\end{aligned}$$

for some positive constant C. Making use of Proposition 6.1, we conclude from (6.3) and (6.4) that

$$(2\operatorname{Re} z)\,|Ph_t(z)|^2 dxdy \in \mathrm{CM}(\mathbb{H}). \tag{6.5}$$

This is equivalent to saying that $\log(h_t)'$ belongs to $\mathrm{BMOA}(\mathbb{H})$.

Since h admits quasiconformal extensions to $\mathbb{C}$ with $\lim_{z\to\infty} h(z) = \infty$ and so does $\varphi_{0,t}$ by Theorem 4.1 (iii), we see that $h_t = h \circ \varphi_{0,t}^{-1}$ can be quasiconformally extended to $\mathbb{C}$ with $h_t(\infty) = \infty$. Moreover, in virtue of property (6.5), this further implies that h_t admits such a particular quasiconformal extension $\widehat{h_t}$ that its complex dilatation μ_t on $\mathbb{H}^*$ induces a Carleson measure

$$|\mu_t(z)|^2/(-2\operatorname{Re} z)\,dxdy \in \mathrm{CM}(\mathbb{H}^*) \tag{6.6}$$

(see [36, Theorem 7.2]). For example, it is known that the variant of the Beurling–Ahlfors extension by heat kernel (see [20, Theorem 4.2], [41, Theorem 3.4]) realizes such an extension.

Mediated by the relation $\hat{h}(-t+iy) = h_t(iy)$, a map $\tilde{h}$ on $\mathbb{C}$ is defined by

$$\tilde{h}(z) := \begin{cases} h(z), & \text{if } \operatorname{Re} z \geq 0,\\ h_{-\operatorname{Re} z}(i\operatorname{Im} z) = h(z^*) + \frac{(2\operatorname{Re} z)h'(z^*)}{1-(\operatorname{Re} z)Ph(z^*)}, & \text{if } -t \leq \operatorname{Re} z < 0,\\ \widehat{h_t}(z+t), & \text{if } \operatorname{Re} z < -t.\end{cases}$$

This is well-defined, and yields a quasiconformal extension of h to $\mathbb{C}$ whose complex dilatation $\tilde{\mu}$ on $\mathbb{H}^*$ is given by

$$\tilde{\mu}(z) = \begin{cases} -\frac{1}{2}(2\mathrm{Re}\, z)^2 Sh(z^*), & \text{if } \ -t \le \mathrm{Re}\, z < 0, \\ \mu_t(z+t), & \text{if } \ \mathrm{Re}\, z < -t. \end{cases}$$

We now show that $\tilde{\mu}$ induces a vanishing Carleson measure on $\mathbb{H}^*$ as in (1.9). For an interval $I \subset i\mathbb{R}$ with $|I| \le t$, we see from (6.4) that

$$\frac{1}{|I|}\iint_{(-|I|,\,0)\times I} \frac{|\tilde{\mu}(z)|^2}{(-2\mathrm{Re}\, z)} dxdy = \frac{1}{4|I|}\iint_{(0,\,|I|)\times I} (2\mathrm{Re}\, z)^3 |Sh(z)|^2 dxdy,$$

which is uniformly bounded with respect to I, and tends to 0 uniformly as $|I| \to 0$.

For an interval $I \subset i\mathbb{R}$ with $|I| > t$, we have

$$\begin{aligned} \frac{1}{|I|}\iint_{(-|I|,\,0)\times I} \frac{|\tilde{\mu}(z)|^2}{(-2\mathrm{Re}\, z)} dxdy =& \frac{1}{4|I|}\iint_{(0,t)\times I} (2\mathrm{Re}\, z)^2 |Sh(z)|^2 dxdy \\ &+ \frac{1}{|I|}\iint_{(-|I|,-t]\times I} \frac{|\mu_t(z+t)|^2}{(-2\mathrm{Re}\, z)} dxdy. \end{aligned} \tag{6.7}$$

By (6.4) and (6.6), it is not difficult to see that this is uniformly bounded with respect to I. Indeed, for the estimate of the first term of the right-hand side of (6.7), we divide $(0,t)\times I$ into n congruent rectangles $(0,t) \times I_i$ $(i = 1, \dots, n)$ so that $nt \le |I| < (n+1)t$. For the estimate of the second term, we replace $(-|I|, -t] \times I$ with a larger square $(-|I|-t, -t] \times I$ and translate it by t along the x-axis. Consequently, $|\tilde{\mu}(z)|^2/(-2\mathrm{Re}\, z)\, dxdy$ is a vanishing Carleson measure on $\mathbb{H}^*$. □

Remark. As mentioned in the introduction, this result completes the characterization of the elements of the VMO Teichmüller space on the upper half-plane developed by Shen [36]. In addition, we can show that the image of the space of Beltrami coefficients inducing vanishing Carleson measures on $\mathbb{H}^*$ by the Schwarzian derivative map coincides with the intersection of the Bers embedding of the universal Teichmüller space T and the space of Schwarzian derivatives inducing vanishing Carleson measures on $\mathbb{H}$. This answers a question in [36, Remark 5.2].

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