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Explicit criteria for recurrence of Markovian regime-switching diffusion processes under arbitrary switching rates ☆

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ABSTRACT

The recurrent property of diffusion processes with regime-switching is quite complicated, which could be transient even when it is recurrent at every fixed environment. This work provides explicit criteria in terms of the coefficients of diffusion processes such that the studied processes are recurrent or transient under arbitrary switching rates. The obtained criteria are in the integral form, which are particularly effective for the regime-switching processes with coefficients vibrating periodically. Examples are constructed to illustrate the applications of these criteria.

1. Introduction

Stochastic processes with regime-switching have been widely applied to model the system living in a random environment characterized by a continuous-time Markov chain. See, for instance, applications in mathematical finance [1–4], in biology [5], in biochemistry [6]. See the manuscripts [7,8] for more introduction on various applications of such models. Compared with stochastic processes without regime-switching, the recurrent property of the processes with regime-switching is much more complicated. Especially, Pinsky and Scheutzow [9] have constructed examples in half line such that the process in each fixed environment is recurrent(or transient), but the process in the random environment could be transient(recurrent respectively). Moreover, there are many works to reveal the impact of the stationary distribution of the Markov chain on the recurrence on the stochastic processes with regime-switching, such as, [10,11] on Ornstein–Uhlenbeck process with regime-switching; [12] on geometric Brownian motion with regime-switching; [8, Chapter 3] on linearizable processes with regime-switching. The recurrence of these processes is usually dependent on the stationary distribution of the Markov chain.

From the viewpoint of applications, it is also of great meaning to find suitable criteria to ensure the recurrence of the studied system with regime-switching under arbitrary switching rates. This is relevant when the switching of the random environment is either unknown or too complicated to be useful. In the study of hybrid dynamical system, such problem has been widely studied to design suitable feedback control to make the system stable under arbitrary switching. see, e.g. [13, Chapter 2] and references therein. However, there is very limited investigation on this topic for diffusion processes with regime-switching. In the 1-dimensional setting, [14] provided a way to realize this purpose. Namely, when the corresponding 1-dimensional diffusion process at each fixed environment is strongly ergodic. Then the diffusion with regime-switching must be recurrent regardless of the switching rates.

In addition, if one wants to use the averaging type criteria established in [15] to realize this purpose, the challenge lies in the construction of a common Lyapunov function $V(x)$ to characterize the recurrent property of the studied system at each fixed

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environment. However, this is not an easy task for diffusion processes with regime-switching, especially when the recurrent behavior of this system varies acutely at different environment states. This can be seen from the study of linear dynamical systems with regime-switching, for which the problem of finding a quadratic common Lyapunov function amounts to solving a system of linear matrix inequalities (cf. [16]).

In this work we shall provide explicit criteria based on the coefficients of the studied system to justify the recurrence for arbitrary switching rate. We present two forms of criteria. For the first type, the constructed explicit conditions are independent of states of the switching process. For the second type, the constructed explicit conditions depend on the coefficients of the diffusion processes at each fixed switching state. The criteria of first type are easier to be verified than the second one, but they are usually less accurate than the second. Furthermore, these results are generalized to deal with diffusion processes with state-dependent regime-switching. Besides owning the characteristic of not relying on switching rate, it is worth pointing out that our criteria are also useful to study the recurrence of regime-switching processes with coefficients vibrating periodically. The existing criteria in the averaging form based on a common Lyapunov function cannot deal with these processes. See Example 4.2 and Example 4.3 in Section 4 for more details. For the criteria of second type, the basic idea of our method is to overcome the difficulty caused by the generator of Markov chain via the generator of the diffusion process.

This work is organized as follows. Section 2 is devoted to dealing with diffusion processes with Markovian regime-switching. The first part of Section 2 presents the results which are uniform in the jumping component, and the second part of Section 2 shows the criteria which are non-uniform in the jumping component. In Section 3, we deal with the diffusion processes with state-dependent regime-switching. Examples are constructed in Section 4 to illustrate the application of these criteria. In Section 5, we summarize briefly this work and point out a problem on the null recurrence in the non-uniform type still left by us.

2. Criteria for Markovian regime-switching processes

The diffusion processes with regime-switching are used to model a system living in a random environment, which is characterized by a jumping process in a finite state space. Let us consider the following diffusion process (X_t, Λ_t) satisfying

$$\mathrm{d}X_t = b(X_t, \Lambda_t)\mathrm{d}t + \sigma(X_t, \Lambda_t)\mathrm{d}B_t, \quad X_0 = x_0 \in \mathbb{R}^d,\ \Lambda_0 = i \in \mathcal{S}, \tag{2.1}$$

where (B_t) is d-dimensional Brownian motion, (Λ_t) is a continuous-time Markov chain on a finite state space $\mathcal{S} = \{1, 2, \dots, N\}$ with irreducible, conservative Q-matrix $(q_{ij})_{i,j\in\mathcal{S}}$, $b : \mathbb{R}^d \times \mathcal{S} \to \mathbb{R}^d$, $\sigma : \mathbb{R}^d \times \mathcal{S} \to \mathbb{R}^{d\times d}$. (X_t, Λ_t) is a diffusion process with Markovian regime-switching. In this situation, it is assumed that (B_t) and (Λ_t) are mutually independent as usual. As we are focused on the recurrent property of (X_t, Λ_t), we always assume the existence and uniqueness of strong solution $(X_t, \Lambda_t)_{t\geq 0}$ to SDE (2.1) for every initial value $(X_0, \Lambda_0) = (x_0, i) \in \mathbb{R}^d \times \mathcal{S}$. We refer the monographs [7,8] on the study of the wellposedness of (X_t, Λ_t).

The infinitesimal generator of (X_t, Λ_t) is given by

$$\begin{aligned}\mathscr{A} f(x,i) &= \mathscr{L}^{(i)} f(x,i) + Qf(x,i)\\ &= \sum_{k=1}^d b_k(x,i)\frac{\partial f}{\partial x_k}(x,i) + \frac{1}{2}\sum_{k,l=1}^d a_{kl}(x,i)\frac{\partial^2 f}{\partial x_k \partial x_l}(x,i) + \sum_{j\in\mathcal{S}} q_{ij}(f(x,j) - f(x,i))\end{aligned} \tag{2.2}$$

for $f \in C^2(\mathbb{R}^d \times \mathcal{S})$, where $(a_{kl}(x,i)) = (\sigma\sigma^*)(x,i)$ and σ^* denotes the transpose matrix of σ. Here $\mathscr{L}^{(i)}$ is an infinitesimal generator corresponding to the diffusion process $(X_t^{(i)})$ satisfying

$$\mathrm{d}X_t^{(i)} = b(X_t^{(i)}, i)\mathrm{d}t + \sigma(X_t^{(i)}, i)\mathrm{d}B_t, \quad X_0^{(i)} = x_0, \tag{2.3}$$

which describes the behavior of the studied process (X_t) in the fixed environment $i \in \mathcal{S}$.

In this work we shall use the following assumption.

(H1) There is a $c_0 > 0$ such that

$$\sum_{k,l=1}^d a_{kl}(x,i)\xi_k\xi_l \geq c_0|\xi|^2, \quad x \in \mathbb{R}^d, i \in \mathcal{S}, \xi = (\xi_1, \dots, \xi_d) \in \mathbb{R}^d.$$

Before introducing our results, let us first introduce some notations. Let

$$\begin{aligned}&A(x,i) = \sum_{k,l=1}^d a_{kl}(x,i)\frac{x_k x_l}{|x|^2}, \quad \text{where } |x|^2 = \sum_{k=1}^d x_k^2 \text{ for } x = (x_1, \dots, x_d) \in \mathbb{R}^d,\ x \neq 0,\\ &\tilde{A}(x,i) = \sum_{k=1}^d a_{kk}(x,i), \quad B(x,i) = \sum_{k=1}^d b_k(x,i)x_k.\end{aligned}$$

Let $\underline{\gamma}(r,i)$ and $\overline{\gamma}(r,i)$ be continuous function on $(0,\infty)\times\mathcal{S}$ satisfying

$$\begin{aligned}&\overline{\gamma}(r,i) \geq \sup_{|x|=r} \frac{\tilde{A}(x,i) - A(x,i) + 2B(x,i)}{A(x,i)} \quad \text{for } r > 0,\\ &\underline{\gamma}(r,i) \leq \inf_{|x|=r} \frac{\tilde{A}(x,i) - A(x,i) + 2B(x,i)}{A(x,i)} \quad \text{for } r > 0.\end{aligned} \tag{2.4}$$

Let $\overline{\alpha}(r,i)$ and $\underline{\alpha}(r,i)$ be continuous functions on $(0,\infty)\times\mathcal{S}$ such that

$$\overline{\alpha}(r,i)\geq\sup_{|x|=r}A(x,i),\qquad \underline{\alpha}(r,i)\leq\inf_{|x|=r}A(x,i),\tag{2.5}$$

and

$$\overline{\gamma}(r)=\max_{i\in\mathcal{S}}\overline{\gamma}(r,i),\qquad \underline{\gamma}(r)=\min_{i\in\mathcal{S}}\underline{\gamma}(r,i);\tag{2.6}$$

$$\overline{\alpha}(r)=\max_{i\in\mathcal{S}}\overline{\alpha}(r,i),\qquad \underline{\alpha}(r)=\min_{i\in\mathcal{S}}\underline{\alpha}(r,i).\tag{2.7}$$

For $r_0>0$, define

$$\overline{I}(r,i)=\int_{r_0}^{r}\frac{\overline{\gamma}(u,i)}{u}\mathrm{d}u,\qquad \underline{I}(r,i)=\int_{r_0}^{r}\frac{\underline{\gamma}(u,i)}{u}\mathrm{d}u,\tag{2.8}$$

$$\overline{I}(r)=\int_{r_0}^{r}\frac{\overline{\gamma}(u)}{u}\mathrm{d}u,\qquad \underline{I}(r)=\int_{r_0}^{r}\frac{\underline{\gamma}(u)}{u}\mathrm{d}u.\tag{2.9}$$

After these preparations, we can present our criteria on the recurrence of (X_t,Λ_t).

2.1. Criteria uniformly in the switching component

We begin with the criteria uniformly in the switching component, which means that these criteria do not depend on the state $i\in\mathcal{S}$. The construction of these sufficient conditions uses the idea of the remarkable works of Bhattacharya [17] and Friedman [18] among others for the multidimensional diffusion processes.

Theorem 2.1. *Assume (H1) holds.*

(i) *If* $\int_{r_0}^{\infty}\exp\big(-\overline{I}(r)\big)dr=\infty$ *for some* $r_0>0$*, then* (X_t,Λ_t) *is recurrent.*
(ii) *If* $\int_{r_0}^{\infty}\exp\big(-\underline{I}(r)\big)dr<\infty$ *for some* $r_0>0$*, then* (X_t,Λ_t) *is transient.*

Proof. The basic idea of the proof is to construct suitable Lyapunov functions associated with the generator $\mathscr{A}$ of (X_t,Λ_t). For $F\in C^2([r_0,\infty))$ for $r_0>0$, consider the function $f(x,i):=F(|x|)$ for $|x|\geq r_0$, and we calculate directly to deduce that

$$2\mathscr{A}f(x,i)=A(x,i)F''(|x|)+\big(\tilde{A}(x,i)-A(x,i)+2B(x,i)\big)\frac{F'(|x|)}{|x|},\quad |x|\geq r_0,i\in\mathcal{S}.\tag{2.10}$$

Define

$$\overline{F}(r)=\int_{r_0}^{r}\exp(-\overline{I}(s))\mathrm{d}s,\quad \underline{F}(r)=\int_{r_0}^{r}\exp(-\underline{I}(s))\mathrm{d}s.$$

Then, it holds

$$\overline{F}''(r)+\frac{\overline{\gamma}(r)}{r}\overline{F}'(r)=0,$$
$$\underline{F}''(r)+\frac{\underline{\gamma}(r)}{r}\underline{F}'(r)=0.$$

Replacing F in (2.10) with $\overline{F}$ and $\underline{F}$ respectively, we can obtain that

$$2\mathscr{A}\overline{F}(|x|)\leq A(x,i)\Big(\overline{F}''(|x|)+\frac{\overline{\gamma}(|x|)}{|x|}\overline{F}'(|x|)\Big)=0,\tag{2.11}$$

$$2\mathscr{A}\underline{F}(|x|)\geq A(x,i)\Big(\underline{F}''(|x|)+\frac{\underline{\gamma}(|x|)}{|x|}\underline{F}'(|x|)\Big)=0.\tag{2.12}$$

Let $\zeta_m=\inf\{t>0;\ |X_t|=m\}$ for $m\in\mathbb{N}$, and

$$\tau=\inf\{t>0;|X_t|=\beta\}\text{ for some }\beta>r_0.$$

Since (Λ_t) is recurrent in a finite state space $\mathcal{S}$, the process (X_t,Λ_t) is recurrent (or transient) if and only if the process (X_t) is recurrent (or transient, respectively). For $i\in\mathcal{S}$, $x_0\in\mathbb{R}^d$ with $|x_0|>\beta$, if

$$\mathbb{P}_{x_0,i}(\tau<\infty):=\mathbb{P}\big(\tau<\infty|(X_0,\Lambda_0)=(x_0,i)\big)=1,$$

then the process (X_t) and hence (X_t,Λ_t) is recurrent; if

$$\mathbb{P}_{x_0,i}\big(\tau=\infty\big)>0,$$

then (X_t) and hence (X_t,Λ_t) is transient.

(i) For $t > 0$, by Dynkin's formula and (2.11), it holds

$$\mathbb{E}_{x_0,i}\overline{F}\big(|X_{t\wedge\zeta_m\wedge\tau}|\big) = \overline{F}(|x_0|) + \mathbb{E}_{x_0,i}\int_0^{t\wedge\zeta_m\wedge\tau} \mathscr{A}\overline{F}(|X_s|)\mathrm{d}s \le \overline{F}(|x_0|).$$

Letting $t \to \infty$, this yields that

$$\overline{F}(m)\mathbb{P}_{x_0,i}(\tau > \zeta_m) + \overline{F}(\beta)\mathbb{P}_{x_0,i}(\tau \le \zeta_m) \le \overline{F}(|x_0|),$$

and further that

$$\mathbb{P}_{x_0,i}\big(\tau \le \zeta_m\big) \ge \frac{\overline{F}(m) - \overline{F}(|x_0|)}{\overline{F}(m) - \overline{F}(\beta)}. \tag{2.13}$$

Therefore, letting $m \to \infty$, as $\zeta_m \to \infty$ a.s. and $\overline{F}(m) \to \infty$ due to $\int_{r_0}^\infty \mathrm{e}^{-\overline{I}(r)}\mathrm{d}r = \infty$, we obtain from (2.13) that

$$\mathbb{P}_{x_0,i}(\tau < \infty) = 1,$$

and hence (X_t, Λ_t) is recurrent.

(ii) By virtue of Dynkin's formula, due to (2.12), we have

$$\mathbb{E}_{x_0,i}\underline{F}\big(|X_{t\wedge\zeta_m\wedge\tau}|\big) = \underline{F}(|x_0|) + \mathbb{E}_{x_0,i}\int_0^{t\wedge\zeta_m\wedge\tau} \mathscr{A}\underline{F}(|X_s|)\mathrm{d}s \ge \underline{F}(|X_0|). \tag{2.14}$$

Letting $t \to \infty$, we have

$$\underline{F}(m)\mathbb{P}_{x_0,i}(\tau > \zeta_m) + \underline{F}(\beta)\mathbb{P}_{x_0,i}(\tau \le \zeta_m) \ge \underline{F}(|x_0|),$$
$$\mathbb{P}_{x_0,i}(\tau > \zeta_m) \ge \frac{\underline{F}(|x_0|) - \underline{F}(\beta)}{\underline{F}(m) - \underline{F}(\beta)}.$$

Passing the limit as $m \to \infty$, we get

$$\mathbb{P}_{x_0,i}(\tau = \infty) \ge \frac{\underline{F}(|x_0|) - \underline{F}(\beta)}{\underline{F}(\infty) - \underline{F}(\beta)} > 0$$

by the condition $\underline{F}(\infty) = \int_{r_0}^\infty \mathrm{e}^{-\underline{I}(r)}\mathrm{d}r < \infty$. Thus, (X_t, Λ_t) is transient. The proof of this theorem is completed. □

Theorem 2.2. *Assume* (H1) *holds.*

(i) *If* $\displaystyle\int_{r_0}^\infty \frac{1}{\underline{\alpha}(s)}\exp\big(\overline{I}(s)\big)ds < \infty$ *for some* $r_0 > 0$*, then* (X_t, Λ_t) *is positive recurrent.*

(ii) *If there exists some* $r_0 > 0$ *such that*

$$\lim_{N\to\infty} \frac{\int_{r_0}^N \mathrm{e}^{-\overline{I}(s)}\big(\int_{r_0}^s \exp\big(\overline{I}(u)\big)/\overline{\alpha}(u)du\big)ds}{\int_{r_0}^N \exp\big(-\underline{I}(s)\big)ds} = \infty,$$

then (X_t, Λ_t) *is null recurrent.*

Proof. Similar to the argument of Theorem 2.1, we can find the desired Lyapunov function $f(x,i)$ using a similar construction of Lyapunov functions $F \in C^2(\mathbb{R})$ as in [17, Theorem 3.5]. Precisely, to prove (i), let

$$F(r) = -\int_{r_0}^r \mathrm{e}^{\overline{I}(s)}\Big(\int_s^\infty \frac{1}{\underline{\alpha}(u)}\mathrm{e}^{\overline{I}(u)}\mathrm{d}u\Big)\mathrm{d}s, \quad r \ge r_0,$$

and $f(x,i) = F(|x|)$ for $|x| \ge r_0$ and $i \in \mathcal{S}$. Then, it holds $2\mathscr{A}f(x,i) \ge 1$ for $|x| \ge r_0$, $i \in \mathcal{S}$. To prove (ii), let us take

$$G(r) = \int_{r_0}^r \mathrm{e}^{-\overline{I}(s)}\Big(\int_{r_0}^s \frac{1}{\overline{\alpha}(u)}\mathrm{e}^{\overline{I}(u)}\mathrm{d}u\Big)\mathrm{d}s, \quad r \ge r_0,$$

and $f(x,i) = G(|x|)$ for $|x| \ge r_0$ and $i \in \mathcal{S}$. Then, $2\mathscr{A}f(x,i) \le 1$ for $|x| \ge r_0$, $i \in \mathcal{S}$. Moreover, we have assumed the non-explosive of (X_t, Λ_t) throughout this work, so condition (3.25) in [17] is not needed in current setting. Then, one can follow the argument of [17, Theorem 3.5] to derive the desired results. More details are omitted. □

2.2. Criteria non-uniformly in the switching component

In this subsection we shall construct the desired Lyapunov function for each state of the switching process (Λ_t). The construction method will be different to that used in Section 2.1 or [17,18] in order to remove the impact of operator Q in the infinitesimal generator $\mathscr{A}$ of (X_t, Λ_t). A little more precisely, here we need to construct a solution to the differential equation in the form

$$F''(x,i) + h(x,i)F'(x) = q_i F(x,i) + c_i, \quad x \in \mathbb{R},\ i \in \mathcal{S},$$

instead of a differential equation in the form

$$F''(x) + g(x)F'(x) = 0, \quad x \in \mathbb{R},$$

which is needed in Section 2.1 and [17,18].

Theorem 2.3. *Assume (H1) holds. If there is a constant $r_0 > 0$ such that for each $i \in \mathcal{S}$*

$$\int_{r_0}^{\infty} \mathrm{e}^{-\underline{I}(s,i)}\Big(\int_{r_0}^{s} \frac{\exp\big(\underline{I}(u,i)\big)}{\underline{\alpha}(u,i)} du\Big) ds < \infty, \tag{2.15}$$

then (X_t, Λ_t) is transient.

Proof. To show this theorem, we shall construct a Lyapunov function $f(x,i)$ satisfying the following conditions, which implies the transience of (X_t, Λ_t):

$$\begin{cases} f(x,i) = 0, & \text{for } |x| = r_0,\ i \in \mathcal{S}; \\ 0 < f(x,i) < M, & \text{for } |x| > r_0,\ i \in \mathcal{S}, \text{ some } M > 0; \\ \mathscr{A} f(x,i) \geq 0, & \text{for } |x| \geq r_0,\ i \in \mathcal{S}. \end{cases} \tag{2.16}$$

Indeed, if such a function f exists, then it follows from Dynkin's formula that

$$\mathbb{E}_{x_0,i} f(X_{t\wedge\tau\wedge\zeta_m}, \Lambda_{t\wedge\tau\wedge\zeta_m}) - f(x_0,i) = \mathbb{E}_{x_0,i} \int_0^{t\wedge\tau\wedge\zeta_m} \mathscr{A} f(X_s, \Lambda_s) ds \geq 0,$$

where $\tau = \inf\{t > 0; |X_t| = r_0\}$ and $\zeta_m = \inf\{t > 0; |X_t| = m\}$ for $m \in \mathbb{N}$. Letting $t \to \infty$, as $f(X_\tau, \Lambda_\tau) = 0$, we obtain that

$$M\mathbb{P}_{x_0,i}(\tau > \zeta_m) \geq f(x_0,i) > 0.$$

Letting $m \to \infty$ yields that $\mathbb{P}_{x_0,i}(\tau = \infty) > 0$, which means that (X_t, Λ_t) is transient.

Define

$$g(r,i) = \int_{r_0}^{r} \mathrm{e}^{-\underline{I}(s,i)}\Big(\int_{r_0}^{s} \frac{2q_i \mathrm{e}^{\underline{I}(u,i)}}{\underline{\alpha}(u,i)} \mathrm{d}u\Big) \mathrm{d}s, \quad r \geq r_0,\ i \in \mathcal{S},$$

where $q_i = \sum_{j\neq i, j\in\mathcal{S}} q_{ij}$. Put $g^{(0)}(r,i) = 1$, and define iteratively, for $r \geq r_0$, $i \in \mathcal{S}$,

$$g^{(n)}(r,i) = \int_{r_0}^{r} \mathrm{e}^{-\underline{I}(s,i)}\Big(\int_{r_0}^{s} \frac{2q_i \mathrm{e}^{\underline{I}(u,i)}}{\underline{\alpha}(u,i)} g^{(n-1)}(u,i) \mathrm{d}u\Big) \mathrm{d}s, \quad n \geq 1. \tag{2.17}$$

Let

$$\varphi(r,i) = \sum_{n=1}^{\infty} g^{(n)}(r,i), \qquad r \geq r_0, i \in \mathcal{S}. \tag{2.18}$$

Note that $r \mapsto g(r,i)$ is a nonnegative, increasing function, then by (2.17),

$$\begin{aligned} g^{(2)}(r,i) &\leq \int_{r_0}^{r} \mathrm{e}^{-\underline{I}(s,i)}\Big(\int_{r_0}^{s} \frac{2q_i \mathrm{e}^{\underline{I}(u,i)}}{\underline{\alpha}(u,i)} \mathrm{d}u\Big) g(s,i) \mathrm{d}s \\ &\leq \int_{r_0}^{r} g'(s,i) g(s,i) \mathrm{d}s = \frac{g(r,i)^2}{2}. \end{aligned}$$

One can deduce similarly $g^{(n)}(r,i) \leq \big(g(r,i)\big)^n / n!$ by induction. Thus, $\varphi(r,i)$ is well-defined. Moreover, it satisfies

$$\varphi(r_0,i) = 0,\ \ \varphi'(r,i) > 0,\ \ \varphi(r,i) \leq \mathrm{e}^{g(r,i)} - 1, \quad r > r_0,\ \ i \in \mathcal{S},$$

and

$$\varphi''(r,i) + \frac{\gamma(r,i)}{r}\varphi'(r,i) = \frac{2q_i}{\underline{\alpha}(r,i)} + \frac{2q_i}{\underline{\alpha}(r,i)}\varphi(r,i), \qquad r > r_0,\ \ i \in \mathcal{S}. \tag{2.19}$$

Due to (2.15), for each $i \in \mathcal{S}$,

$$g(r,i) \leq \int_{r_0}^{\infty} \mathrm{e}^{-\underline{I}(s,i)}\Big(\int_{r_0}^{s} \frac{2q_i \mathrm{e}^{\underline{I}(u,i)}}{\underline{\alpha}(u,i)} \mathrm{d}u\Big) \mathrm{d}s =: \beta_i < \infty. \tag{2.20}$$

After these preparations, let

$$f(x,i) = \varphi(|x|,i) \qquad \text{for } |x| \geq r_0,\ i \in \mathcal{S}. \tag{2.21}$$

Then, $f(x,i) = 0$ for $|x| = r_0$, $i \in \mathcal{S}$;

$$0 < f(x,i) \leq \mathrm{e}^{g(|x|,i)} - 1 < \mathrm{e}^{\beta_i} < \infty, \quad |x| > r_0,\ i \in \mathcal{S}.$$

This yields that f is bounded as $\mathcal{S}$ is a finite state space. Moreover, by (2.19),

$$\begin{aligned}
&2\mathscr{A} f(x,i)\\
&= A(x,i)\Big(\varphi''(|x|,i)+\frac{\tilde{A}(x,i)-A(x,i)+2B(x,i)}{A(x,i)}\cdot\frac{\varphi'(|x|,i)}{|x|}\Big)\\
&\qquad +2\sum_{j\neq i} q_{ij}\big(\varphi(|x|,j)-\varphi(|x|,i)\big)\\
&\geq A(x,i)\Big(\varphi''(|x|,i)+\frac{\gamma(|x|,i)}{|x|}\varphi'(|x|,i)\Big)-2q_i\varphi(|x|,i)+2\sum_{j\neq i}q_{ij}\varphi(|x|,j)\\
&= A(x,i)\Big(\frac{2q_i}{\underline{\alpha}(|x|,i)}+\frac{2q_i}{\underline{\alpha}(|x|,i)}\varphi(|x|,i)\Big)-2q_i\varphi(|x|,i)+2\sum_{j\neq i}q_{ij}\varphi(|x|,j)\\
&\geq 2q_i+2\sum_{j\neq i}q_{ij}\varphi(|x|,j)>0.
\end{aligned}$$

Consequently, the constructed function $f(x,i)$ in (2.21) satisfies all conditions in (2.16). Thus, (X_t,Λ_t) is transient. □

Theorem 2.4. *Assume* (H1) *holds. If for some $r_0>0$,*

$$\int_{r_0}^{\infty} \mathrm{e}^{-\overline{I}(s,i)}\Big(\int_s^{\infty}\frac{\exp(\overline{I}(u,i))}{\underline{\alpha}(u,i)}du\Big)ds<\infty, \qquad i\in\mathcal{S}, \tag{2.22}$$

then (X_t,Λ_t) is positive recurrent.

Proof. As $\mathcal{S}$ is finite and Q is conservative,

$$\Theta := \max_{i\in\mathcal{S}} q_i = \max_{i\in\mathcal{S}}\sum_{j\neq i} q_{ij}<\infty, \quad \theta := \min_{i\in\mathcal{S}} q_i>0.$$

Introduce the auxiliary functions as follows. For $K\in\mathbb{N}$, put

$$g_K(r,i)=\int_r^K \mathrm{e}^{-\overline{I}(s,i)}\Big(\int_s^K\frac{2\Theta\exp\big(\overline{I}(u,i)\big)}{\underline{\alpha}(u,i)}\mathrm{d}u\Big)\mathrm{d}s, \qquad 0<r\leq K. \tag{2.23}$$

Let $g_K^{(0)}(r,i)\equiv 1$, and define inductively that

$$g_K^{(n)}(r,i)=\int_r^K \mathrm{e}^{-\overline{I}(s,i)}\Big(\int_s^K\frac{2\Theta\exp\big(\overline{I}(u,i)\big)}{\underline{\alpha}(u,i)}g_K^{(n-1)}(u,i)\mathrm{d}u\Big)\mathrm{d}s, \qquad 0<r\leq K,\ i\in\mathcal{S}.$$

Let

$$\psi_K(r,i)=-\sum_{n=0}^{\infty} g_K^{(n)}(r,i), \quad 0<r\leq K,\ i\in\mathcal{S}. \tag{2.24}$$

Due to (2.22), as

$$g_K^{(n)}(r,i)\leq\frac{1}{n!}\big(g_K(r,i)\big)^n, \quad n\geq 1,$$

we get $\psi_K(r,i)$ is well-defined. Furthermore, ψ_K admits the following properties:

$$-\mathrm{e}^{g_K(r,i)}\leq\psi_K(r,i)\leq-\big(1+g_K(r,i)\big), \quad \psi_K'(r,i)\geq 0, \quad \psi_K(r,i)\leq -1, \tag{2.25}$$

$$2\Theta\psi_K(r,i)=\underline{\alpha}(r,i)\Big(\psi_K''(r,i)+\frac{\overline{\gamma}(r,i)}{r}\psi_K'(r,i)\Big). \tag{2.26}$$

For $x\in\mathbb{R}^d$ with $|x|>r_0$, $i\in\mathcal{S}$, we have

$$\begin{aligned}
&2\mathscr{A}\psi_K(|x|,i)\\
&\leq A(x,i)\Big(\psi_K''(|x|,i)+\frac{\overline{\gamma}(|x|,i)}{|x|}\psi_K'(|x|,i)\Big)-2q_i\psi_K(|x|,i)+2\sum_{j\neq i}q_{ij}\psi_K(|x|,j)\\
&\leq 2(\Theta-q_i)\psi_K(|x|,i)+2\sum_{j\neq i}q_{ij}\psi_K(|x|,j)\\
&\leq -2\theta<0,
\end{aligned}$$

where in the second inequality we used (2.26), and in the third inequality we used the fact $\psi_K(r,i)\leq -1$ and $\Theta\geq q_i$. Thus, by Dynkin's formula, this yields

$$\begin{aligned}
&\mathbb{E}_{x_0,i}\psi_K(|X_{t\wedge\tau\wedge\zeta_m}|,\Lambda_{t\wedge\tau\wedge\zeta_m})-\psi_K(|x_0|,i)\\
&=\mathbb{E}_{x_0,i}\int_0^{t\wedge\tau\wedge\zeta_m}\mathscr{A}\psi_K(|X_s|,\Lambda_s)\mathrm{d}s\leq-\theta\,\mathbb{E}_{x_0,i}\big(t\wedge\tau\wedge\zeta_m\big).
\end{aligned} \tag{2.27}$$

Passing the limit $t \to \infty$,

$$\begin{aligned}\theta\,\mathbb{E}_{x_0,i}\big(\tau\wedge\zeta_m\big) &\le \psi_K(|x_0|,i) - \mathbb{E}_{x_0,i}\big[\psi_K(r_0,\Lambda_\tau)\mathbf{1}_{\tau<\zeta_m}\big] + \mathbb{P}_{x_0,i}(\zeta_m<\tau)\\ &\le \psi_K(|x_0|,i) + 1 - \min_{j\in\mathcal{S}}\psi_K(r_0,j)\\ &\le -\min_{j\in\mathcal{S}}\psi_K(r_0,j)\\ &\le \max_{j\in\mathcal{S}}\exp\big(g_K(r_0,j)\big),\end{aligned}$$

where in the last inequality we have used (2.25). Letting $m\to\infty$, by condition (2.22), we obtain that

$$\theta\,\mathbb{E}_{x_0,i}(\tau) \le \max_{j\in\mathcal{S}}\exp\Big(\int_{r_0}^{\infty} \mathrm{e}^{-\overline{I}(s,j)}\Big(\int_s^{\infty}\frac{2\Theta\exp(\overline{I}(u,j))}{\underline{\alpha}(u,j)}\mathrm{d}u\Big)\mathrm{d}s\Big) < \infty. \tag{2.28}$$

Consequently, we conclude that (X_t,Λ_t) is positive recurrent. □

Remark 2.5. Condition (2.22) in Theorem 2.4 is in the similar form as the condition (2.15). They are used to construct the desired Lyapunov functions based on the Eqs. (2.19) and (2.26), which play a crucial role to remove the impact of the operator

$$Qf(x,i) = \sum_{j\neq i} q_{ij}(f(x,j)-f(x,i)).$$

It is this difficulty that prevents us to construct a Lyapunov function to justify the null recurrence of (X_t,Λ_t) in the current stage.

3. Criteria for state-dependent regime-switching processes

In this section we shall consider the state-dependent regime-switching diffusion processes, and want to provide criteria of recurrence and transience independent of the switching rates.

Consider

$$\mathrm{d}X_t = b(X_t,\Lambda_t)\mathrm{d}t + \sigma(X_t,\Lambda_t)\mathrm{d}B_t, \quad X_0 = x_0\in\mathbb{R}^d, \Lambda_0 = i\in\mathcal{S}, \tag{3.1}$$

where (B_t) is a Brownian motion in $\mathbb{R}^d$, (Λ_t) is jumping process on $\mathcal{S}=\{1,2,\ldots,N\}$ satisfying

$$\mathbb{P}(\Lambda_{t+\delta}=j|\Lambda_t=i,\ X_t=x) = \begin{cases} q_{ij}(x)\delta + o(\delta), & i\neq j,\\ 1+q_{ii}(x)\delta+o(\delta), & i=j,\end{cases} \tag{3.2}$$

for $\delta>0$. We also assume the existence and uniqueness of a nonexplosive solution $(X_t,\Lambda_t)_{t\ge 0}$ to SDEs (3.1) and (3.2). The process (X_t,Λ_t) is still a Markov process with infinitesimal generator given by

$$\begin{aligned}\mathscr{A}f(x,i) &= \mathscr{L}^{(i)}f(x,i) + Q(x)f(x,i)\\ &= \sum_{k=1}^d b_k(x,i)\frac{\partial f(x,i)}{\partial x_k} + \frac{1}{2}\sum_{k,l=1}^d a_{kl}(x,i)\frac{\partial^2 f(x,i)}{\partial x_k\partial x_l} + \sum_{j\neq i} q_{ij}(x)(f(x,j)-f(x,i))\end{aligned} \tag{3.3}$$

for $f\in C^2(\mathbb{R}^d\times\mathcal{S})$. The recurrent property of state-dependent regime-switching processes is more complicated than the Markovian regime-switching processes. The monograph [8] focuses on the study of various properties of state-dependent regime-switching processes. Besides, based on the common Lyapunov function, [15] provided several recurrent criteria using nonsingular M-matrix theory and the Fredholm alternative theorem. However, the criteria given in Section 2 are independent of the switching rate matrix, and hence are easier to be extended to deal with state-dependent regime-switching diffusion processes.

Firstly, it is easy to check that Theorems 2.1 and 2.2 are still valid for the process (X_t,Λ_t) satisfying SDEs (3.1), (3.2).

Secondly, let us extend Theorems 2.3 and 2.4 to the state-dependent situation.

Proposition 3.1. *Assume (H1) holds, and further that*

$$q_i^* := \sup_{x\in\mathbb{R}^d} q_i(x) < \infty, \quad i\in\mathcal{S}. \tag{3.4}$$

If there exists $r_0>0$ such that for each $i\in\mathcal{S}$

$$\int_{r_0}^{\infty}\mathrm{e}^{-\underline{I}(s,i)}\Big(\int_{r_0}^{s}\frac{\exp\big(\underline{I}(u,i)\big)}{\underline{\alpha}(u,i)}du\Big)ds < \infty,$$

then the process (X_t,Λ_t) satisfying (3.1), (3.2) is transient.

Proof. The argument follows the same line as that of Theorem 2.3 with the following modifications:

- Change the definitions of $g(r,i)$ and $g^{(n)}(r,i)$ into

$$g(r,i) = \int_{r_0}^{r}\mathrm{e}^{-\underline{I}(s,i)}\Big(\int_{r_0}^{s}\frac{2q_i^*\mathrm{e}^{\underline{I}(u,i)}}{\underline{\alpha}(u,i)}\mathrm{d}u\Big)\mathrm{d}s, \quad r\ge r_0,\ i\in\mathcal{S},$$

and

$$g^{(n)}(r,i)=\int_{r_0}^r \mathrm{e}^{-\underline{I}(s,i)}\Big(\int_{r_0}^s \frac{2q_i^*\mathrm{e}^{\underline{I}(u,i)}}{\underline{\alpha}(u,i)}g^{(n-1)}(u,i)\mathrm{d}u\Big)\mathrm{d}s,\quad n\geq 1.$$

Then, still for $f(x,i)=\varphi(|x|,i)$, it holds

$$\begin{aligned}&2\mathscr{A}f(x,i)\\ &\geq A(x,i)\Big(\varphi''(|x|,i)+\frac{\gamma(|x|,i)}{|x|}\varphi'(|x|,i)\Big)-2q_i(x)\varphi(|x|,i)+2\sum_{j\neq i}q_{ij}(x)\varphi(|x|,j)\\ &\geq 2q_i^*+2q_i^*\varphi(|x|,i)-2q_i(x)\varphi(|x|,i)+2\sum_{j\neq i}q_{ij}(x)\varphi(|x|,j)\\ &\geq 2q_i^*+2\sum_{j\neq i}q_{ij}(x)\varphi(|x|,j)>0.\end{aligned}$$

We can get the desired conclusion by (2.16). □

Proposition 3.2. *Assume (H1) holds, and*

$$\Theta:=\sup_{x\in\mathbb{R}^d}\max_{i\in\mathcal{S}}q_i(x)<\infty,\quad \theta:=\inf_{x\in\mathbb{R}^d}\min_{i\in\mathcal{S}}q_i(x)>0. \tag{3.5}$$

If for some $r_0>0$,

$$\int_{r_0}^\infty \mathrm{e}^{-\overline{I}(s,i)}\Big(\int_s^\infty \frac{\exp\big(\overline{I}(u,i)\big)}{\underline{\alpha}(u,i)}du\Big)ds<\infty,\qquad i\in\mathcal{S}, \tag{3.6}$$

then (X_t,Λ_t) satisfying (3.1), (3.2) is positive recurrent.

This proposition can be proved in the same way as Theorem 2.4 using Θ and θ given by (3.5) to define the Lyapunov function $\psi_K(|x|,i)$. Note that when coping with Markovian regime-switching process in a finite state space, condition (3.5) holds naturally. But, for the state-dependent regime-switching diffusion processes, we need to assume that (3.5) holds. This is a limitation of our result, Proposition 3.2, to the processes with arbitrary switching.

4. Examples

We shall provide three examples to show the application of our results obtained in Section 2 and Section 3.

Example 4.1. Consider the process (X_t,Λ_t) satisfying

$$\mathrm{d}X_t=\beta_{\Lambda_t}b(X_t)\mathrm{d}t+\sqrt{\beta_{\Lambda_t}}\sigma(X_t)\mathrm{d}B_t,\quad X_0=x_0\in\mathbb{R},\ \Lambda_0=i\in\mathcal{S}, \tag{4.1}$$

where $b:\mathbb{R}\to\mathbb{R}$, $\sigma:\mathbb{R}\to(0,\infty)$, $\mathcal{S}=\{1,2\}$, $\beta:\mathcal{S}\to(0,\infty)$. (Λ_t) is a continuous-time Markov chain on $\mathcal{S}$. Suppose that there is a unique nonexplosive solution (X_t,Λ_t) satisfying SDE (4.1). Suppose that b is an odd function and σ is an even function.

Without loss of generality, suppose $\beta_2>\beta_1$. According to (2.6)–(2.9), direct calculation yields that

$$\begin{aligned}\overline{\gamma}(r,i)&=\overline{\gamma}(r)=\underline{\gamma}(r,i)=\underline{\gamma}(r)=\frac{b(r)r}{\sigma^2(r)},\\ I(r)=\overline{I}(r,i)&=\overline{I}(r)=\underline{I}(r,i)=\underline{I}(r)=\int_{r_0}^r\frac{\gamma(u)}{u}\mathrm{d}u,\\ \overline{\alpha}(r,i)&=\beta_i\sigma^2(r),\qquad \underline{\alpha}(r,i)=\beta_i\sigma^2(r),\\ \overline{\alpha}(r)&=\beta_2\sigma^2(r),\qquad \underline{\alpha}(r)=\beta_1\sigma^2(r).\end{aligned}$$

By virtue of Theorem 2.1, (X_t,Λ_t) is recurrent if $\int_{r_0}^\infty\exp(-I(r))\mathrm{d}r=\infty$ for some $r_0>0$; is transient if $\int_{r_0}^\infty\exp(-I(r))\mathrm{d}r<\infty$ for some $r_0>0$. This simple example tells us that the criteria obtained in Theorem 2.1 are sharp in certain sense.

Example 4.2. Let us consider the Ornstein–Uhlenbeck process with Markovian regime-switching (Λ_t) on a finite state space $\mathcal{S}$:

$$\mathrm{d}X_t=\theta_{\Lambda_t}X_t\mathrm{d}t+\mathrm{d}B_t,\quad X_0=x\in\mathbb{R},\ \Lambda_0=i\in\mathcal{S}. \tag{4.2}$$

Let (π_i) denote the invariant probability measure of (Λ_t). According to [19], (X_t,Λ_t) is ergodic if and only if $\sum_{i\in\mathcal{S}}\pi_i\theta_i<0$.

By (2.6), (2.9),

$$\overline{I}(r)=\frac{1}{2}\big(\max_{i\in\mathcal{S}}\theta_i\big)(r^2-r_0^2),\quad \underline{I}(r)=\frac{1}{2}\big(\min_{i\in\mathcal{S}}\theta_i\big)(r^2-r_0^2)$$

Hence, if $\max_{i\in\mathcal{S}}\theta_i<0$, we have $\int_{r_0}^\infty\mathrm{e}^{-\overline{I}(r)}\mathrm{d}r=\infty$, which implies that (X_t,Λ_t) is recurrent according to Theorem 2.1. Similarly, if $\min_{i\in\mathcal{S}}\theta_i>0$, (X_t,Λ_t) is transient by Theorem 2.1. This result also tells us an interesting conclusion. For the regime-switching process

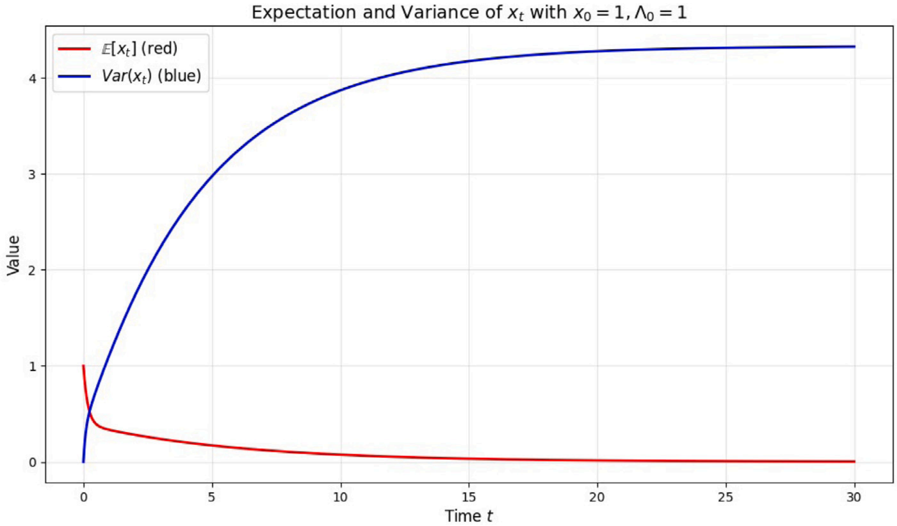

Fig. 1. OU process with $\theta_1 = 1/3$, $\theta_2 = -1/2$.

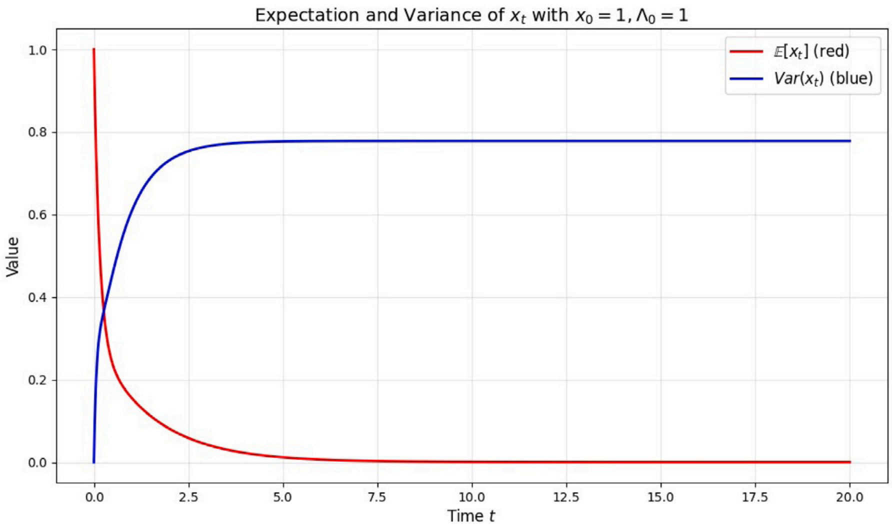

Fig. 2. OU process with $\theta_1 = -1$, $\theta_2 = -1/2$.

(X_t, Λ_t) satisfying (4.2), $\max_{i\in\mathcal{S}} \theta_i < 0$ means that the corresponding diffusion process at each fixed environment $i \in \mathcal{S}$ is recurrent, then (X_t, Λ_t) is recurrent regardless of Q. This result is meaningful by recalling the interesting examples constructed in [9], where there is an example of regime-switching diffusion process on the half line such that it is recurrent at each fixed environment $i \in \mathcal{S}$, but (X_t, Λ_t) could be transient by choosing suitable transition rate matrix Q of (Λ_t). Our example also indicates that the result on the strong ergodicity of regime-switching processes established in [14] has the room for improvement as the Ornstein–Uhlenbeck process is not strongly ergodic, but only exponentially ergodic.

Below, we illustrate the recurrent property of OU process (X_t) with regime-switching via numerical approximation. As the distribution of X_t is Gaussian, which is determined by its mean and variance, we plot the evolution of $\mathbb{E}[X_t]$ and $\mathrm{Var}(X_t) = \mathbb{E}[(X_t^2 - \mathbb{E}X_t)^2]$ to see its convergence or divergence.

First, let us take $\mathcal{S} = \{1,2\}$ with $Q = \begin{pmatrix} -2 & 2 \\ 1 & 1 \end{pmatrix}$ and $\theta_1 = 1/3$, $\theta_2 = -1/2$. Then, (X_t, Λ_t) is recurrent, which is also illustrated by Fig. 1.

Second, let us take $\mathcal{S} = \{1,2\}$ with $Q = \begin{pmatrix} -2 & 2 \\ 1 & 1 \end{pmatrix}$ and $\theta_1 = -1$, $\theta_2 = -1/2$. Then, (X_t, Λ_t) is recurrent, which is also illustrated by Fig. 2.

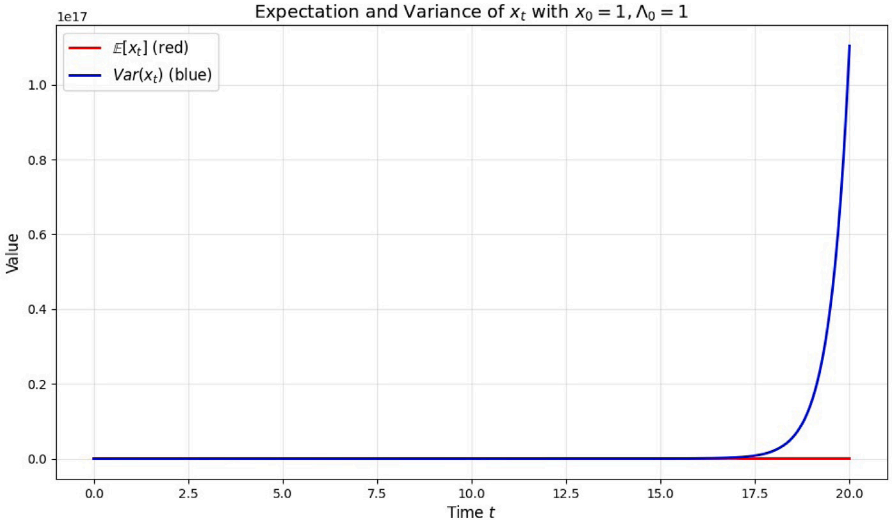

Fig. 3. OU process with $\theta_1 = 1$, $\theta_2 = 2$.

Third, let us take $\mathcal{S} = \{1, 2\}$ with $Q = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$ and $\theta_1 = 1$, $\theta_2 = 2$. Then, (X_t, Λ_t) is transient, which can be seen in certain sense via Fig. 3.

Example 4.3. Let us consider a diffusion process with Markovian regime-switching (X_t, Λ_t) satisfying:

$$\mathrm{d}X_t = b(X_t, \Lambda_t)\mathrm{d}t + \sigma(X_t)\mathrm{d}B_t, \quad X_0 = x \in \mathbb{R},\ \Lambda_0 = i \in \mathcal{S}, \tag{4.3}$$

where $\mathcal{S} = \{1, 2\}$, (Λ_t) is a continuous-time Markov chain on $\mathcal{S}$. b is an odd function, and

$$b(x, i) = \begin{cases} \beta_i n^2, & \text{if } x \in [2n, 2n+1), \\ -\beta_i/n, & \text{if } x \in [2n+1, 2n+2), \end{cases} \tag{4.4}$$

where β_1, β_2 are positive constants. σ is an even function, and

$$\sigma(x) = \sqrt{n} \quad \text{if } x \in [2n, 2n+2) \text{ for } n \geq 1, \quad \sigma(x) = \sqrt{2} \quad \text{if } x \in [0, 2]. \tag{4.5}$$

The vibration of the sign of $b(x, i)$ makes it hard to find a common Lyapunov function to apply the criteria given in [15] or [8].

Indeed, according to [15, Theorem 2.1], if we can find a common Lyapunov function V such that

$$V(x) > 0, \quad \mathscr{L}^{(i)}V(x) \leq \eta_i V(x), \quad |x| > r_0 \tag{4.6}$$

for some $r_0 > 0$, $\eta_i \in \mathbb{R}$, $i \in \mathcal{S}$. Assume that $\sum_{i\in\mathcal{S}} \pi_i \eta_i < 0$, where (π_i) denotes the unique invariant probability measure of (Λ_t). Then, (X_t, Λ_t) is transient if $\lim_{|x|\to\infty} V(x) = 0$, and is exponentially ergodic if $\lim_{|x|\to\infty} V(x) = \infty$. We can see from this criterion that the limit behavior of the common Lyapunov function $V(x)$ as $|x| \to \infty$ in (4.6) has been determined by the aim to establish the transience or ergodicity of (X_t, Λ_t). To be more precise, to prove the ergodicity of (X_t, Λ_t) with $X_t \in \mathbb{R}$ via [15, Theorem 2.1], one needs to find a $V(x)$ such that $\lim_{|x|\to\infty} V(x) = \infty$ satisfying condition (4.6), which leads to

$$\eta_i \geq \frac{b(x, i)V'(x)}{V(x)} + \frac{1}{2}\frac{\sigma^2(x, i)V''(x)}{V(x)}, \quad |x| \geq r_0 > 0. \tag{4.7}$$

The vibration of $b(x, i)$ and $\sigma(x, i)$ means that it is possible that on some intervals $x \in A_k$, $\mathscr{L}^{(i)}$ satisfies the dissipative condition (this means that one can get $\eta_i < 0$ for $x \in A_k$ satisfying (4.7)), but on other intervals $x \in B_k$, $\mathscr{L}^{(i)}$ does not satisfy the dissipative condition (this means that η_i satisfying (4.7) for $x \in B_k$ must be positive). In all, we can only find a positive η_i so that (4.7) holds for all $|x| \geq r_0$. Thus, the condition $\sum_{i\in\mathcal{S}} \pi_i \eta_i < 0$ cannot be satisfied due to the vibration of b and σ.

Now, we use the criteria established in this work to study the recurrent property of (X_t, Λ_t) given in this example. By (2.4)–(2.7), we have, for $r \geq 2$,

$$\underline{\alpha}(r) = \underline{\alpha}(r, i) = \overline{\alpha}(r) = \overline{\alpha}(r, i) = n, \quad \text{if } x \in [2n, 2n+2),$$

$$\overline{\gamma}(r, i) = \underline{\gamma}(r, i) = \begin{cases} 2\beta_i nr, & \text{if } r \in [2n, 2n+1), \\ -2\beta_i r/n^2, & \text{if } r \in [2n+1, 2n+2), \end{cases} \quad \text{for } n \geq 1.$$

Then, choosing $r_0 = 2$,

$$\overline{I}(s,i) = \underline{I}(s,i) = I(s,i)$$
$$= \begin{cases} \beta_i n(n-1) - 2\beta_i \sum_{k=1}^{n-1} \frac{1}{k^2} + 2\beta_i(s-2n), & \text{if } s \in [2n, 2n+1), \\ \beta_i n(n+1) - 2\beta_i \sum_{k=1}^{n-1} \frac{1}{k^2} - 2\beta_i(s-2n-1)/n^2, & \text{if } s \in [2n+1, 2n+2), \end{cases} \tag{4.8}$$

for $n \in \mathbb{N}$. This yields

$$\beta_i n(n-1) + 2\beta_i \geq I(s,i) \geq \beta_i n(n-1) - 2\beta\kappa_0, \quad s \in [2n, 2n+1),$$
$$\beta_i n(n+1) \geq I(s,i) \geq \beta_i n(n+1) - 2\beta_i\kappa_0 - 2\beta_i/n^2, \quad s \in [2n+1, 2n+2),$$

where $\kappa_0 = \sum_{k=1}^{\infty} 1/k^2$. Then, $I(s,i) \sim c_i s^2 + \tilde{c}$ with some constants $c_i > 0$ and $\tilde{c}$. Invoking

$$\lim_{s\to\infty} s^2\Big(\mathrm{e}^{-c_i s^2} \int_2^s \frac{\mathrm{e}^{c_i u^2}}{u}\mathrm{d}u\Big) = \frac{1}{2c_i}, \quad \text{and} \quad \int_2^{\infty} \frac{1}{s^2}\mathrm{d}s < \infty,$$

it holds

$$\int_2^{\infty} \mathrm{e}^{-\underline{I}(s,i)}\Big(\int_2^s \frac{\mathrm{e}^{\underline{I}(u,i)}}{\underline{\alpha}(u,i)}\mathrm{d}u\Big)\mathrm{d}s < \infty.$$

By virtue of Theorem 2.3, the process (X_t, Λ_t) is transient.

Example 4.4. Consider the state-dependent regime-switching process (X_t, Λ_t) satisfying

$$\mathrm{d}X_t = b(X_t, \Lambda_t)\mathrm{d}t + \sigma(X_t)\mathrm{d}B_t, \quad X_0 = x_0 \in \mathbb{R}, \ \Lambda_0 = i \in \mathcal{S}, \tag{4.9}$$

where $\mathcal{S} = \{1,2\}$, $b : \mathbb{R}\times\mathcal{S} \to \mathbb{R}$, $\sigma : \mathbb{R} \to (0,\infty)$. (Λ_t) satisfies

$$\mathbb{P}(\Lambda_{t+\delta} = j | \Lambda_t = i, X_t = x) = \begin{cases} q_{ij}(x)\delta + o(\delta), & i \neq j, \\ 1 + q_{ii}(x)\delta + o(\delta), & i = j, \end{cases} \tag{4.10}$$

provided $\delta > 0$, where $q_1(x) = q_{12}(x)$, $q_2(x) = q_{21}(x)$ are given by

$$q_{i(3-i)}(x) = \lambda_i \sum_{n=0}^{\infty} \mathbf{1}_{[2n,2n+1)}(|x|) + \tilde{\lambda}_i \sum_{n=0}^{\infty} \mathbf{1}_{[2n+1,2n+2)}(|x|), \quad i \in \mathcal{S} = \{1,2\} \tag{4.11}$$

with $\lambda_1, \lambda_2, \tilde{\lambda}_1, \tilde{\lambda}_2 > 0$. Such kind of diffusion processes with piecewise constant type state-dependent switching has been studied in [20]. One can use the method in [20] to establish the wellposedness of SDEs (4.9), (4.10). However, since $x \mapsto q_{ij}(x)$ vibrates infinite times in this example, the recurrence criteria given in [20] are not applicable in current setting.

Let us consider the following drift and diffusion coefficients: b is an odd function, and

$$b(x,i) = \begin{cases} -\beta_i n^2, & \text{if } x \in [2n, 2n+1), \\ \beta_i/n, & \text{if } x \in [2n+1, 2n+2), \end{cases} \tag{4.12}$$

where β_1, β_2 are positive constants. $\sigma(x) = \sigma(-x)$ for $x \in \mathbb{R}$ and

$$\sigma(x) = \sqrt{n} \quad \text{if } x \in [2n, 2n+2) \text{ for } n \geq 1, \quad \sigma(x) = \sqrt{2} \quad \text{if } x \in [0,2]. \tag{4.13}$$

Then, choosing $r_0 = 2$, for $r \geq 2$,

$$\underline{\alpha}(r) = \underline{\alpha}(r,i) = \overline{\alpha}(r) = \overline{\alpha}(r,i) = n, \quad \text{if } x \in [2n, 2n+2),$$
$$\overline{\gamma}(r,i) = \underline{\gamma}(r,i) = \begin{cases} -2\beta_i nr, & \text{if } r \in [2n, 2n+1), \\ 2\beta_i r/n^2, & \text{if } r \in [2n+1, 2n+2), \end{cases} \quad \text{for } n \geq 1.$$

Also,

$$\overline{I}(s,i) = \underline{I}(s,i) = I(s,i)$$
$$= \begin{cases} -\beta_i n(n-1) + 2\beta_i \sum_{k=1}^{n-1} \frac{1}{k^2} - 2\beta_i(s-2n), & \text{if } s \in [2n, 2n+1), \\ -\beta_i n(n+1) + 2\beta_i \sum_{k=1}^{n-1} \frac{1}{k^2} + 2\beta_i(s-2n-1)/n^2, & \text{if } s \in [2n+1, 2n+2) \end{cases} \tag{4.14}$$

for $n \in \mathbb{N}$. This yields

$$-\beta_i n(n-1) + 2\beta_i\kappa_0 \geq I(s,i) \geq -\beta_i n(n-1) - 2\beta\kappa_0, \quad s \in [2n, 2n+1),$$

$$-\beta_i n(n+1) + 2\beta_i\kappa_0 + 2\beta_i/n^2 \geq I(s,i) \geq -\beta_i n(n+1), \quad s \in [2n+1, 2n+2),$$

where $\kappa_0 = \sum_{k=1}^{\infty} 1/k^2$. Based on these estimates, we obtain that

$$\int_2^{\infty} \mathrm{e}^{-I(s,i)} \Big(\int_s^{\infty} \frac{\exp(I(u,i))}{\alpha(u,i)} \mathrm{d}u \Big) \mathrm{d}s < \infty, \quad i \in \mathcal{S},$$

and hence (X_t, Λ_t) is positive recurrent due to Theorem 2.4.

5. Conclusion

In this work we provide two types of criteria to justify the transience and recurrence of regime-switching diffusion processes with the aim of making these criteria independent of the switching rates. Two types of criteria are provided in this work: (i) uniform in the switching component; (ii) non-uniform in the switching component.

This work leaves an open problem: how to justify the null recurrence of the regime-switching processes in the non-uniform type similar to Theorem 2.3 for transience and Theorem 2.4 for positive recurrence. Heuristically, we construct the explicit solution to the differential equation

$$\mathscr{L}^{(i)} f(x,i) = \lambda_i f(x,i),$$

and use $\lambda_i f(x,i)$ to remove the impact of $Qf(x,i) = \sum_{j\neq i} q_{ij} f(x,j) - q_i f(x,i)$ in the construction of desired Lyapunov function

$$\mathscr{A} f(x,i) = \mathscr{L}^{(i)} f(x,i) + Qf(x,i) \leq -1.$$

However, to the null transience, we can only construct a Lyapunov function such that

$$\mathscr{L}^{(i)} f(x,i) \leq 1,$$

which is not enough to offset the impact of the operator Q.

CRediT authorship contribution statement

Jianrui Li: Writing – review & editing, Methodology, Conceptualization. **Jinghai Shao:** Writing – review & editing, Writing – original draft, Methodology, Conceptualization.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Data availability

No data was used for the research described in the article.

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