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WEAK CONVERGENCE OF PATH-DEPENDENT SDES WITH IRREGULAR COEFFICIENTS

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Dedicated to Professor George Yin on the Occasion of his 70th Birthday

Abstract. In this paper we develop via Girsanov's transformation a perturbation argument to investigate the weak convergence of Euler-Maruyama (EM) schemes for path-dependent SDEs with Hölder continuous drifts. This approach is available to other scenarios, e.g., truncated EM schemes for non-degenerate SDEs with finite memory or infinite memory. Also, such a trick can be applied to study the weak convergence of truncated EM schemes for a range of stochastic Hamiltonian systems with irregular coefficients and memory. Moreover, the weak convergence of path-dependent SDEs under integrability condition is investigated by establishing, via the dimension-free Harnack inequality, exponential integrability of irregular drifts w.r.t. the invariant probability measure constructed explicitly in advance.

1. **Introduction.** The strong/weak convergence of numerical schemes for SDEs with regular coefficients has been investigated extensively; see e.g. [3, 10, 11, 16, 30] and references therein. Meanwhile, strong approximations of solutions to SDEs with irregular coefficients have received much attention in the past few years; see e.g. [8, 13, 14, 15, 22, 23, 24, 25, 26, 27, 28] and references within. Also, there is considerable literature on the strong convergence of various numerical schemes (e.g., truncated/tamed EM scheme) for path-dependent SDEs (which, in terminology, are also named as functional SDEs or SDEs with delays) under regular conditions; see, for instance, [7, 18] and references within. So far, the weak convergence for SDEs with irregular terms has also gained much attention; see e.g. [11, 20] with the smooth payoff function.

In contrast to the strong convergence of numerical algorithms for path-dependent SDEs, the analysis of weak convergence is scarce. As far as path-dependent SDEs are concerned, the weak convergence of numerical methods was initiated in [11], whereas the rigorous justification of their statements was unavailable. With regard to the weak convergence of the EM scheme and its variants, we refer to [5] for a class of semi-linear path-dependent SDEs via the so-called "lift-up" approach, [6] for path-dependent SDEs with distributed delays by means of the duality trick, and [4] for path-dependent SDEs with point delays with the help of Malliavin calculus and the tamed Itô formula. In references [4, 6], as for the drift term b and the diffusion term σ, the assumptions that $b,\sigma\in C_b^\infty(\mathbb{R}^d)$ and the payoff function $f\in C_b^3(\mathbb{R}^d)$

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were imposed. Subsequently, by the aid of Malliavin calculus, [38] extended [4, 6] in a certain sense that the payoff function $f \in \mathscr{B}_b(\mathbb{R}^d)$, while $b, \sigma \in C_b^\infty(\mathbb{R}^d)$ therein. It is worthy to point out that the approaches adopted in [4, 6, 38] are applicable merely for path-dependent SDEs with regular coefficients. In the literature [4, 38], the tamed Itô formula plays a crucial role in investigating the weak convergence of EM schemes. Nevertheless, the tamed Itô formula barely works for path-dependent SDEs with distributed delays or point delays so that it seems hard to extend [4, 38] to path-dependent SDEs with general delays. To study the weak convergence of numerical schemes for path-independent SDEs with regular coefficients, the approach on the Kolmogorov backward equation is one of the more powerful methods. However, concerning path-dependent SDEs, the Kolmogrov backward equation is in general unavailable so that it cannot be adopted to handle the weak convergence of numerical schemes. As we stated above, concerning path-dependent SDEs, the Malliavin calculus is an effective tool to cope with the weak convergence; see, for example, [4, 6, 38]. Furthermore, slightly strong assumptions are imposed therein and the proof is not succinct in a certain sense. Moreover, Zvonkin's transformation [40] is one of the more powerful tools for investigating the strong convergence of EM schemes for path-independent SDEs with singular coefficients; see e.g. [28]. Nevertheless, such a trick no longer works provided the delay terms are irregular. On account of the motivations above, in this work we aim to develop a perturbation approach (see e.g. [12, 34]) to study the weak convergence of an EM scheme for path-dependent SDEs with additive noise, which allows for the drift terms to be irregular (e.g., Hölder continuous drifts and integrability drifts) and even the diffusion coefficients to be degenerate.

We point out that the dimension-free Harnack inequality plays an important role in investigating the weak error analysis for path-dependent SDEs under integrability conditions.

The content of this paper is arranged as follows. In Section 2, we investigate the weak convergence of EM schemes for a class of non-degenerate SDEs with memory and reveal the weak convergence rate. In Section 3, we apply the approach adopted in Section 2 to other scenarios, e.g., truncated EM schemes for non-degenerate SDEs with finite memory or infinite memory. In Section 4, we focus on the weak convergence order of truncated EM schemes for a range of stochastic Hamiltonian systems with singular drifts and memory. In the last section, we are interested in the weak convergence of EM schemes for path-dependent SDEs under integrability conditions, which allow the drift terms to have super-linear growth and be singular at some points.

Before proceeding further, a few words about the notation are in order. Throughout this paper, $c > 0$ stands for a generic constant which might change from occurrence to occurrence, and depends on the time parameters.

2. **Weak convergence: Non-degenerate case.** Let $(\mathbb{R}^d, \langle \cdot, \cdot \rangle, |\cdot|)$ be the d-dimensional Euclidean space with the inner product $\langle \cdot, \cdot \rangle$ which induces the Euclidean norm $|\cdot|$. Let $\mathbb{M}_{\rm non}^d$ be the set of all non-singular $d \times d$-matrices with real entries. A^* means the transpose of the matrix A. For a sub-interval $\mathbb{U} \subseteq \mathbb{R}$, denote by $C(\mathbb{U}; \mathbb{R}^d)$ the family of all continuous functions $f : \mathbb{U} \to \mathbb{R}^d$. Let $\tau > 0$ be a fixed number, and $\mathscr{C} = C([-\tau, 0]; \mathbb{R}^d)$, which is endowed with the uniform norm $\|f\|_\infty := \sup_{-\tau \le \theta \le 0} |f(\theta)|$. For $f \in C([-\tau, \infty); \mathbb{R}^d)$ and fixed $t \ge 0$, let $f_t \in \mathscr{C}$ be defined by $f_t(\theta) = f(t + \theta), \theta \in [-\tau, 0]$. In our terminology, $(f_t)_{t \ge 0}$ is called the segment (or window) process corresponding to $(f(t))_{t \ge -\tau}$. For $a \ge 0$, $\lfloor a \rfloor$ stipulates

the integer part of a. Let $\mathscr{B}_b(\mathbb{R}^d)$ be the collection of all bounded measurable functions $f:\mathbb{R}^d\to\mathbb{R}$ endowed with the uniform norm $\|\|f\|\|_\infty:=\sup_{x\in\mathbb{R}^d}|f(x)|$. Let $\mathbf{0}\in\mathbb{R}^d$ be the zero vector and $\xi_0(\theta)\equiv\mathbf{0}$ for any $\theta\in[-\tau,0]$.

In this section, we are interested in the following path-dependent SDE

$$\mathrm{d}X(t)=\big\{b\,(X(t))+Z(X_t)\big\}\mathrm{d}t+\sigma\,\mathrm{d}W(t),\ t>0,\ X_0=\xi\in\mathscr{C}, \tag{1}$$

where $b:\mathbb{R}^d\to\mathbb{R}^d$, $Z:\mathscr{C}\to\mathbb{R}^d$, $\sigma\in\mathbb{M}^d_{\text{non}}$, and $(W(t))_{t\geq 0}$ is a d-dimensional Brownian motion on the filtered probability space $(\Omega,\mathscr{F},(\mathscr{F}_t)_{t\geq0},\mathbb{P})$. We assume that:

(**A1**) b is Lipschitz with the Lipschitz constant L_1, i.e., $|b(x)-b(y)|\leq L_1|x-y|,\ x,y\in\mathbb{R}^d$, and there exist constants $C>0$ and $\beta\in\mathbb{R}$ such that

$$2\langle x,b(x)\rangle\leq C+\beta|x|^2,\ x\in\mathbb{R}^d; \tag{2}$$

(**A2**) Z is Hölder continuous with the Hölder exponent $\alpha\in(0,1]$ and the Hölder constant L_2, i.e., $|Z(\xi)-Z(\eta)|\leq L_2\|\xi-\eta\|_\infty^\alpha,\ \xi,\eta\in\mathscr{C}$;

(**A3**) The initial value $\xi\in\mathscr{C}$ is Lipschitz continuous with the Lipschitz constant $L_3>0$, i.e., $|\xi(t)-\xi(s)|\leq L_3|t-s|,\ s,t\in[-\tau,0]$.

Under (**A1**) and (**A2**), (1) enjoys a unique weak solution $(X^\xi(t))_{t\geq0}$ with the initial datum $X_0^\xi=\xi\in\mathscr{C}$; see Lemma 2.3 below for more details. Evidently, (2) holds with $\beta>0$ whenever b obeys the global Lipschitz condition. It is worthy to emphasize that β in (2) need not to be positive, which may allow the time horizon T to be much bigger as Lemma 2.4 below manifests. Moreover, (**A3**) is just imposed for the sake of continuity of the displacement of the segment process. For further details, please refer to Lemma 2.7 below.

For existence and uniqueness of strong solutions to path-dependent SDEs with regular coefficients, we refer to e.g. [17, 21, 31] and references therein. Recently, path-dependent SDEs with irregular coefficients have also received much attention; see e.g. [1] on existence and uniqueness of strong solutions, [2] for the strong Feller property of the semigroup generated by the functional solution (i.e., the segment process associated with the solution process), and [34] for regularity estimates on the density of invariant probability measures.

To treat the weak convergence of the EM scheme (5) with the singular coefficient Z, in this work we shall exploit a perturbation approach; see e.g. [33, 34] on regularity estimates of the density for invariant probability measures for SDEs under integrability conditions. To achieve this goal, we introduce the following reference SDE on $\mathbb{R}^d$

$$\mathrm{d}Y(t)=b(Y(t))\mathrm{d}t+\sigma\,\mathrm{d}W(t),\ t>0,\ Y(0)=x\in\mathbb{R}^d. \tag{3}$$

Under (**A1**), (3) has a unique strong solution $(Y^x(t))_{t\geq0}$ with the initial value $Y(0)=x$; see, for example, [21, Theorem 2.1, p.34]. Now, we extend $Y^x(t)$ from $[0,\infty)$ into $[-\tau,\infty)$ in the manner below:

$$Y^\xi(t):=\xi(t)\mathbf{1}_{[-\tau,0)}(t)+Y^{\xi(0)}(t)\mathbf{1}_{[0,\infty)}(t),\ t\in[-\tau,\infty),\ \xi\in\mathscr{C}. \tag{4}$$

We write $(Y_t^\xi)_{t\geq0}$ as the segment process corresponding to $(Y^\xi(t))_{t\geq-\tau}$.

Our main result in this section is stated as follows, which particularly reveals the weak convergence rate of the EM algorithm (5) associated with (1), which nevertheless allows the drift term to be path-dependent and Hölder continuous.

Let $\delta\in(0,1)$ be the step size given by $\delta=\tau/M$ for some $M\in\mathbb{N}$ sufficiently large. Given the step size $\delta\in(0,1)$, the continuous-time EM scheme associated with (1) is defined as below

$$\mathrm{d}X^{(\delta)}(t)=\big\{b(X^{(\delta)}(t_\delta))+Z(\widehat{X}^{(\delta)}_{t_\delta})\big\}\mathrm{d}t+\sigma\,\mathrm{d}W(t),\ t>0 \tag{5}$$

with the initial value $X^{(\delta)}(\theta)=X(\theta)=\xi(\theta),\theta\in[-\tau,0]$. Herein, $t_\delta:=\lfloor t/\delta\rfloor\delta$ and, for any $k\in\mathbb{N}$, $\widehat{X}^{(\delta)}_{k\delta}\in\mathscr{C}$ is defined by

$$\widehat{X}^{(\delta)}_{k\delta}(\theta)=\frac{\theta+(1+i)\delta}{\delta}X^{(\delta)}((k-i)\delta)-\frac{\theta+i\delta}{\delta}X^{(\delta)}((k-i-1)\delta) \tag{6}$$

whenever $\theta\in[-(i+1)\delta,-i\delta]$ for $i\in\mathbf{S}:=\{0,1,\cdots,M-1\}$, that is, the $\mathscr{C}$-valued process $(\widehat{X}^{(\delta)}_{k\delta})_{k\in\mathbb{N}}$ is constructed by the linear interpolations between the points on the gridpoints.

Theorem 2.1. *Let* (**A1**), (**A2**), *and* (**A3**) *hold. Then, for any* $\kappa\in(0,\alpha/2)$ *with* $\alpha\in(0,1]$ *given in* (**A2**) *and* $T>0$ *such that*

$$2\,\|\sigma\|^2_{\mathrm{HS}}\|\sigma^{-1}\|^2_{\mathrm{HS}}\big\{(4L_1^2+L_2^2)\mathbf{1}_{\{\alpha=1\}}+L_1^2\mathbf{1}_{\{\alpha\in(0,1)\}}\big\}<\mathrm{e}^{-(1+\beta T)}/T^2, \tag{7}$$

there exists a constant $C_{1,T}>0$ (*dependent on* $|||f|||_\infty$) *such that*

$$|\mathbb{E}f(X(t))-\mathbb{E}f(X^{(\delta)}(t))|\le C_{1,T}\,\delta^\kappa,\ \ f\in\mathscr{B}_b(\mathbb{R}^d),\ t\in[0,T]. \tag{8}$$

Remark 2.2. For the path-independent SDE (1) with Hölder continuous drift, [27, Theorem 2.6] revealed the weak convergence order is $\frac{\alpha}{2}\wedge\frac{1}{4}$, where $\alpha\in(0,1)$ is the Hölder exponent. In Theorem 2.1, we demonstrate that the weak convergence rate is $\alpha/2$. So, Theorem 2.1 is new even for path-independent SDEs with irregular drifts. For path-dependent SDEs with point delays or distributed delays, [4, 6] investigated the weak convergence under the regular assumption $Z\in C_b^\infty$ and with the payoff function $f\in C_b^3$. Nevertheless, in the present work, we might allow the drift Z to be unbounded and even Hölder continuous, and most importantly the payoff function f to be non-smooth. Hence, Theorem 2.1 improves e.g. [4, 6, 27] in a certain sense. Last but not least, the approach employed to prove Theorem 2.1 is universal in a sense that it is applicable to the other scenarios as shown in Sections 3 and 4.

Before we move forward to complete the proof of Theorem 2.1, we first prepare some warm-up lemmas. The following lemma addresses the existence and uniqueness of weak solutions to (1).

Lemma 2.3. *Under* (**A1**) *and* (**A2**), (1) *admits a unique weak solution.*

Proof. First of all, we show the existence of a weak solution to (1). Set

$$R_1^\xi(t):=\exp\bigg(\int_0^t\big\langle\sigma^{-1}Z(Y_s^\xi),\mathrm{d}W(s)\big\rangle-\frac{1}{2}\int_0^t\big|\sigma^{-1}Z(Y_s^\xi)\big|^2\mathrm{d}s\bigg),\ t\ge0,$$

and $\mathrm{d}\mathbb{Q}_1^\xi:=R_1^\xi(T)\mathrm{d}\mathbb{P}$, where $T>0$ satisfies $\|\sigma\|^2_{\mathrm{op}}\|\sigma^{-1}\|^2_{\mathrm{op}}L_2^2<\mathrm{e}^{-(1+\beta T)}/T^2$ for the setup of the Hölder exponent $\alpha=1$ and $T>0$ is arbitrary for $\alpha\in(0,1)$. Moreover, let

$$W_1^\xi(t)=W(t)-\int_0^t\sigma^{-1}Z(Y_s^\xi)\mathrm{d}s,\ t\ge0. \tag{9}$$

According to Lemma 2.4 below, we infer that

$$\mathbb{E}\mathrm{e}^{\frac{1}{2}\int_0^T|\sigma^{-1}Z(Y_t^\xi)|^2\mathrm{d}t}<\infty,$$

that is, the Novikov condition holds true. Whence, the Girsanov theorem implies that $(W_1^\xi(t))_{t\in[0,T]}$ is a Brownian motion under the weighted probability measure $\mathbb{Q}_1^\xi$. Note that (3) can be reformulated as

$$\mathrm{d}Y^\xi(t)=\big\{b(Y^\xi(t))+Z(Y_t^\xi)\big\}\mathrm{d}t+\sigma\mathrm{d}W_1^\xi(t),\ t\in[0,T],\ Y_0^\xi=\xi.$$

So, $(Y^\xi(t),W_1^\xi(t))_{t\in[0,T]}$ is a weak solution to (1) w.r.t. the probability space $(\Omega,\mathscr{F},(\mathscr{F}_t)_{t\geq0},\mathbb{Q}_1^\xi)$. Analogously, we can show inductively that (1) admits a weak solution on $[T,2T],[2T,3T],\cdots$. Hence, (1) admits a global weak solution.

Now we proceed to justify uniqueness of weak solutions to (1). In the following, it is sufficient to show the weak uniqueness on the time interval $[0,T]$ since it can be done analogously on $[T,2T],[2T,3T],\cdots$. Let $(X^{(i),\xi}(t),W^{(i)}(t))_{t\in[0,T]}$ be the weak solution to (1) w.r.t. the probability space $(\Omega^{(i)},\mathscr{F}^{(i)},(\mathscr{F}_t^{(i)})_{t\geq0},\mathbb{P}_i^\xi)$, $i=1,2$. In terms of [9, Proposition 2.1, p169, & Corollary, p206], it remains to show that

$$\mathbb{E}_{\mathbb{P}_1^\xi}f\big(X^{(1),\xi}([0,T]),W^{(1)}([0,T])\big)=\mathbb{E}_{\mathbb{P}_2^\xi}f\big(X^{(2),\xi}([0,T]),W^{(2)}([0,T])\big)\tag{10}$$

for any $f\in C_b(C([0,T];\mathbb{R}^d)\times C([0,T];\mathbb{R}^d);\mathbb{R})$, where $\mathbb{E}_{\mathbb{P}_i^\xi}$ means the expectation w.r.t. $\mathbb{P}_i^\xi$. However, (10) can be done exactly by following the argument of [34, Theorem 2.1 (2)]. We therefore omit the corresponding proof. □

The lemma below examines the exponential integrability of functionals for segment processes.

Lemma 2.4. *Assume that* (**A1**) *holds. Then, for any* $T>0$,

$$\mathbb{E}\mathrm{e}^{\lambda\int_0^T\|Y_t^\xi\|_\infty^2\mathrm{d}t}<\infty,\ \ \lambda<\frac{\mathrm{e}^{-(1+\beta T)}}{2\|\sigma\|_{\mathrm{op}}^2T^2}.\tag{11}$$

Proof. Applying Jensen's inequality and using the fact that $\|Y_t^\xi\|_\infty\leq\|\xi\|_\infty\vee\sup_{0\leq s\leq t}|Y^{\xi(0)}(s)|$, we have for all $T>0$,

$$\begin{aligned}\mathbb{E}\mathrm{e}^{\lambda\int_0^T\|Y_t^\xi\|_\infty^2\mathrm{d}t}&\leq\frac{1}{T}\int_0^T\mathbb{E}\mathrm{e}^{\lambda T\|Y_t^\xi\|_\infty^2}\mathrm{d}t\\&\leq\frac{\mathrm{e}^{\lambda T\|\xi\|_\infty^2}}{T}\int_0^T\mathbb{E}\Big(\sup_{0\leq s\leq t}\mathrm{e}^{\lambda T|Y^{\xi(0)}(s)|^2}\Big)\mathrm{d}t.\end{aligned}\tag{12}$$

Next, by Itô's formula, it follows from (**A1**) that there exists a constant $c>0$ such that for any $\gamma>0$,

$$\begin{aligned}\mathrm{d}\big(\mathrm{e}^{-\gamma t}\big|Y^{\xi(0)}(t)\big|^2\big)&=\mathrm{e}^{-\gamma t}\big\{-\gamma\big|Y^{\xi(0)}(t)\big|^2+2\big\langle Y^{\xi(0)}(t),b(Y^{\xi(0)}(t))\big\rangle+\|\sigma\|_{\mathrm{HS}}^2\big\}\mathrm{d}t\\&\quad+2\mathrm{e}^{-\gamma t}\big\langle\sigma^*Y^{\xi(0)}(t),\mathrm{d}W(t)\big\rangle\\&\leq\mathrm{e}^{-\gamma t}\big\{c-(\gamma-\beta)\big|Y^{\xi(0)}(t)\big|^2\big\}\mathrm{d}t\\&\quad+2\mathrm{e}^{-\gamma t}\big\langle\sigma^*Y^{\xi(0)}(t),\mathrm{d}W(t)\big\rangle.\end{aligned}\tag{13}$$

Once more, via Itô's formula, we deduce from (13) that for any $\varepsilon>0$,

$$\begin{aligned}\mathrm{d}\mathrm{e}^{\varepsilon\,\mathrm{e}^{-\gamma t}|Y^{\xi(0)}(t)|^2}&\leq-\varepsilon\big(\gamma-\beta-2\|\sigma\|_{\mathrm{op}}^2\varepsilon\big)\mathrm{e}^{-\gamma t}\mathrm{e}^{\varepsilon\mathrm{e}^{-\gamma t}|Y^{\xi(0)}(t)|^2}\big|Y^{\xi(0)}(t)\big|^2\mathrm{d}t\\&\quad+c\varepsilon\mathrm{e}^{-\gamma t}\mathrm{e}^{\varepsilon\mathrm{e}^{-\gamma t}|Y^{\xi(0)}(t)|^2}\mathrm{d}t\\&\quad+2\varepsilon\mathrm{e}^{-\gamma t}\mathrm{e}^{\varepsilon\mathrm{e}^{-\gamma t}|Y^{\xi(0)}(t)|^2}\big\langle\sigma^*Y^{\xi(0)}(t),\mathrm{d}W(t)\big\rangle,\end{aligned}\tag{14}$$

which implies that, for any $\gamma > \beta + 2\|\sigma\|_{\rm HS}^2\varepsilon$, by Gronwall's inequality,

$$\mathbb{E}e^{\varepsilon e^{-\gamma t}|Y^{\xi(0)}(t)|^2} \le e^{\varepsilon(c/\gamma+|\xi(0)|^2)}. \tag{15}$$

so that

$$\begin{aligned}\varepsilon\big(\gamma-\beta-2\|\sigma\|_{\rm HS}^2\varepsilon\big)\int_0^t e^{-\gamma s}\mathbb{E}\big(e^{\varepsilon e^{-\gamma s}|Y^{\xi(0)}(s)|^2}|Y^{\xi(0)}(s)|^2\big)ds\\ \le \Big(1+\frac{c\,\varepsilon}{\gamma}e^{\frac{c\varepsilon}{\gamma}}\Big)e^{\varepsilon|\xi(0)|^2}.\end{aligned} \tag{16}$$

Making use of BDG's inequality and Jensen's inequality, in the case of $\gamma > \beta + 2\|\sigma\|_{\rm op}^2\varepsilon$, we derive from (14) and (15) that

$$\begin{aligned}&\mathbb{E}\Big(\sup_{0\le s\le t} e^{\varepsilon e^{-\gamma s}|Y^{\xi(0)}(s)|^2}\Big)\\ &\le \Big(1+\frac{c\,\varepsilon}{\gamma}e^{\frac{c\,\varepsilon}{\gamma}}\Big)e^{\varepsilon\,|\xi(0)|^2}\\ &\quad+8\sqrt{2}\,\varepsilon\mathbb{E}\Big(\int_0^t e^{-2\gamma s}e^{2\varepsilon\,e^{-\gamma\,s}|Y^{\xi(0)}(s)|^2}\big|\sigma^*Y^{\xi(0)}(s)\big|^2ds\Big)^{1/2}\\ &\le \Big(1+\frac{c\,\varepsilon}{\gamma}e^{\frac{c\varepsilon}{\gamma}}\Big)e^{\varepsilon|\xi(0)|^2}+\frac{1}{2}\mathbb{E}\Big(\sup_{0\le s\le t} e^{\varepsilon e^{-\gamma s}|Y^{\xi(0)}(s)|^2}\Big)\\ &\quad+64\|\sigma\|_{\rm op}^2\varepsilon^2\int_0^t e^{-\gamma s}\mathbb{E}\big(e^{\varepsilon e^{-\gamma s}|Y^{\xi(0)}(s)|^2}\big|Y^{\xi(0)}(s)\big|^2\big)ds.\end{aligned} \tag{17}$$

So, plugging (16) back into (17) yields that

$$\mathbb{E}\Big(\sup_{0\le t\le T} e^{\varepsilon e^{-\gamma T}|Y^{\xi(0)}(t)|^2}\Big) < \infty \tag{18}$$

as long as $\gamma > \beta + 2\|\sigma\|_{\rm op}^2\varepsilon$. Note that

$$\sup_{\varepsilon>0}\Big(\varepsilon e^{-(\beta+2\|\sigma\|_{\rm op}^2\,\varepsilon)T}\Big) = \lambda_T := \frac{1}{2\|\sigma\|_{\rm op}^2 T}e^{-(1+\beta T)}.$$

Consequently, from (18), we arrive at

$$\mathbb{E}\Big(\sup_{0\le t\le T} e^{\lambda_0|Y^{\xi(0)}(t)|^2}\Big) < \infty,\ \ \lambda_0\in(0,\lambda_T) \tag{19}$$

In the end, (11) follows from (12) and (19) in case of $\lambda T < \lambda_T$. □

Remark 2.5. In terms of Lemma 2.4, (11) holds for small $T > 0$ provided that (3) is non-dissipative (i.e., in (2), $\beta \ge 0$). Also, (11) is satisfied with large $T > 0$ in the case that (3) is dissipative (i.e., in (2), $\beta < 0$).

For notation brevity, we set

$$h_1^\xi(t) := \sigma^{-1}\big\{b(Y^\xi(t)) - b(Y^\xi(t_\delta)) - Z(\widehat{Y}^\xi_{t_\delta})\big\},\ \ t\ge 0,\ \ \xi\in\mathscr{C}, \tag{20}$$

where $\widehat{Y}^\xi_\cdot$ is defined exactly as in (6) with $X^{(\delta)}$ replaced by Y^ξ.

The lemma below plays an important role in checking the Novikov condition so that the Girsanov theorem is applicable and in investigating the weak error analysis.

Lemma 2.6. *Suppose that* (**A1**) *and* (**A2**) *hold. Then,*

$$\mathbb{E}e^{\lambda\int_0^T|\sigma^{-1}Z(Y_t^\xi)|^2dt} < \infty \tag{21}$$

whenever $\lambda, T > 0$, *such that*

$$\lambda < \frac{\mathrm{e}^{-(1+\beta T)}}{2\,\|\sigma\|_{\rm op}^2\|\sigma^{-1}\|_{\rm op}^2\{L_2^2\mathbf{1}_{\{\alpha=1\}}+0\mathbf{1}_{\{\alpha\in(0,1)\}}\}T^2},$$

where we set $\frac{1}{0}=\infty$. *Moreover,*

$$\mathbb{E}\mathrm{e}^{\lambda\int_0^T|h_1^\xi(t)|^2\mathrm{d}t}<\infty \tag{22}$$

provided that $\lambda, T > 0$, *such that*

$$\lambda < \frac{\mathrm{e}^{-(1+\beta T)}}{4\,\|\sigma\|_{\rm op}^2\|\sigma^{-1}\|_{\rm op}^2\{(4L_1^2+L_2^2)\mathbf{1}_{\{\alpha=1\}}+L_1^2\mathbf{1}_{\{\alpha\in(0,1)\}}\}T^2}.$$

Proof. From (**A2**), it is obvious to see that

$$|Z(\xi)|\le|Z(\xi_0)|+L_2\|\xi\|_\infty^\alpha,\ \xi\in\mathscr{C}, \tag{23}$$

which, in addition to Young's inequality, implies that

$$\big|\sigma^{-1}Z(Y_t^\xi)\big|^2\le c_\varepsilon+\big\|\sigma^{-1}\big\|_{\rm op}^2\big\{(1+\varepsilon)L_2^2\mathbf{1}_{\{\alpha=1\}}+\varepsilon\mathbf{1}_{\{\alpha\in(0,1)\}}\big\}\big\|Y_t^\xi\big\|_\infty^2,\ \varepsilon>0 \tag{24}$$

for some constant $c_\varepsilon>0$. As a consequence, (21) holds true from (24) and by taking advantage of (11) followed by choosing $\varepsilon\in(0,1)$ sufficiently small.

By the definition of $\widehat{Y}_\cdot^\xi$ (see (6) with $X^{(\delta)}$ being replaced by Y^ξ for more details), a straightforward calculation shows that

$$\begin{aligned}\big\|\widehat{Y}_{t_\delta}^\xi\big\|_\infty&=\sup_{-\tau\le\theta\le0}\big|\widehat{Y}_{t_\delta}^\xi(\theta)\big|\\&\le\max_{k\in\mathbf{S}}\sup_{-(k+1)\delta\le\theta\le-k\delta}\bigg(\frac{\theta+(1+k)\delta}{\delta}|Y^\xi(t_\delta-k\delta)|\\&\qquad\qquad-\frac{\theta+k\delta}{\delta}|Y^\xi(t_\delta-(k+1)\delta)|\bigg)\\&\le\big\|Y_t^\xi\big\|_\infty\vee\big\|Y_{t-\tau}^\xi\big\|_\infty,\ t\ge0\end{aligned} \tag{25}$$

due to the fact that $(\theta+(1+k)\delta)/\delta-(\theta+k\delta)/\delta=1$, where we set $Y^\xi(t):=\xi^\xi(-\tau)$ whenever $t\in[-2\tau,-\tau)$. Subsequently, (25), together with (**A1**) as well as (23), yields that

$$\big|h_1^\xi(t)\big|^2\le\mu_\varepsilon+\nu_\varepsilon\big(\big\|Y_t^\xi\big\|_\infty^2\vee\big\|Y_{t-\tau}^\xi\big\|_\infty^2\big),\ \varepsilon>0,\ t\ge0 \tag{26}$$

for some $\mu_\varepsilon>0$ and

$$\nu_\varepsilon:=2\big\|\sigma^{-1}\big\|_{\rm op}^2\big\{\big(4L_1^2+(1+\varepsilon)L_2^2\big)\mathbf{1}_{\{\alpha=1\}}+L_1^2(1+\varepsilon)\mathbf{1}_{\{\alpha\in(0,1)\}}\big\}$$

Thereby, (22) follows from (19) and (26) and by noting that

$$\int_0^T\mathrm{e}^{\lambda(\|Y_t^\xi\|_\infty^2\vee\|Y_{t-\tau}^\xi\|_\infty^2)}\mathrm{d}t\le\tau\mathrm{e}^{\|\xi\|_\infty^2}+2\int_0^T\mathrm{e}^{\lambda\|Y_t^\xi\|_\infty^2}\mathrm{d}t,\ \lambda>0.$$

□

Next we intend to show that the displacement of segment process is continuous in the L^p-norm sense.

Lemma 2.7. *Under* (**A1**) *and* (**A3**), *for any* $p>2$ *and* $T>0$, *there exists a constant* $C_{p,T}>0$ *such that*

$$\sup_{0\le t\le T}\mathbb{E}\big\|Y_t^\xi-\widehat{Y}_{t_\delta}^\xi\big\|_\infty^p\le C_{p,T}\delta^{(p-2)/2}. \tag{27}$$

Proof. By invoking [17, Theorem 4.4, p61], for any $p>0$ and $T>0$, there exists a constant $\widehat{C}_{p,T}>0$ such that

$$\mathbb{E}\Big(\sup_{-\tau\le t\le T}\big|Y^\xi(t)\big|^p\Big)\le \widehat{C}_{p,T}\big(1+\|\xi\|_\infty^p\big). \tag{28}$$

By utilizing Hölder's inequality and BDG's inequality, it follows from (**A1**) and (28) that

$$\begin{aligned}&\mathbb{E}\Big(\sup_{k\delta\le t\le(k+2)\delta}\big|Y^\xi(t)-Y^\xi(k\delta)\big|^p\Big)\\&\le c\Big\{\delta^{p-1}\int_{k\delta}^{(k+2)\delta}\mathbb{E}\big|b(Y^\xi(t))\big|^p\mathrm{d}t+\mathbb{E}\Big(\sup_{0\le t\le 2\delta}|W(t)|^p\Big)\Big\}\\&\le c\Big\{\delta^{p-1}\int_{k\delta}^{(k+2)\delta}\big(1+\mathbb{E}|Y^\xi(t)|^p\big)\mathrm{d}t+\delta^{p/2}\Big\}\\&\le c\delta^{p/2},\ p>2,\ k\in\mathbb{N}.\end{aligned} \tag{29}$$

Trivially, there exists an integer $k_0\ge 0$ such that $t\in[k_0\delta,(k_0+1)\delta]$. So, for any $p>2$,

$$\begin{aligned}&\mathbb{E}\big\|Y_t^\xi-\widehat{Y}_{t_\delta}^\xi\big\|_\infty^p\\&\le M\max_{k\in\mathbf{S}}\mathbb{E}\Big(\sup_{-(k+1)\delta\le\theta\le-k\delta}\big|Y^\xi(t+\theta)-\widehat{Y}^\xi_{k_0\delta}(\theta)\big|^p\Big)\\&\le cM\max_{k\in\mathbf{S}}\mathbb{E}|Y^\xi((k_0-k)\delta)-Y^\xi((k_0-k-1)\delta)|^p\\&\quad+cM\max_{k\in\mathbf{S}}\mathbb{E}\Big(\sup_{(k_0-k-1)\delta\le s\le(k_0-k+1)\delta}\big|Y^\xi(s)-Y^\xi((k_0-k-1)\delta)\big|^p\Big).\end{aligned}$$

In the case of $k\le k_0-1$, we find from (29) that (27) holds. On the other hand, if $k=k_0$, from (**A3**), (29), and $M\delta=\tau$, then one gets that (27) holds. Moreover, for $k\ge 1+k_0$, (27) is still true due to (**A3**). The proof is therefore complete. □

With the previous lemmas in hand, we are now in the position to complete the following proof.

Proof of Theorem 2.1. Let

$$W_2^\xi(t)=W(t)+\int_0^t h_1^\xi(s)\mathrm{d}s,\ t\ge 0, \tag{30}$$

where h_1^ξ was introduced in (20). Define

$$R_2^\xi(t)=\exp\Big(-\int_0^t\big\langle h_1^\xi(s),\mathrm{d}W(s)\big\rangle-\frac{1}{2}\int_0^t\big|h_1^\xi(s)\big|^2\mathrm{d}s\Big),\ t\ge 0$$

and $\mathrm{d}\mathbb{Q}_2^\xi=R_2^\xi(T)\mathrm{d}\mathbb{P}$, where $T>0$ satisfies (7). Due to (7) and (22), the Girsanov theorem implies that $(W_2^\xi(t))_{t\in[0,T]}$ is a Brownian motion under the probability measure $\mathbb{Q}_2^\xi$. Thus, (3) can be rewritten in the form

$$\mathrm{d}Y^\xi(t)=\big\{b(Y^\xi(t_\delta))+Z(\widehat{Y}^\xi_{t_\delta})\big\}\mathrm{d}t+\sigma\,\mathrm{d}W_2^\xi(t),\ t>0 \tag{31}$$

with the initial value $Y^\xi(\theta)=\xi(\theta),\theta\in[-\tau,0]$ so that $(Y^\xi(t),W_2^\xi(t))_{t\in[0,T]}$ is a weak solution to (5) under $\mathbb{Q}_2^\xi$. Obviously, (5) has a unique strong solution so that the weak solution is unique. Since, by (7) and (21), $(Y^\xi(t),W_1^\xi(t))_{t\in[0,T]}$ is a weak solution to (1) under $\mathbb{Q}_1^\xi$ and $(Y^\xi(t), W_2^\xi(t))_{t\in[0,T]}$ is a weak solution to (5) under $\mathbb{Q}_2^\xi$, we deduce from the weak uniqueness due to Lemma 2.3 and Hölder's inequality that

$$
\begin{aligned}
&\big|\mathbb{E}f(X(T)) - \mathbb{E}f(X^{(\delta)}(T))\big| \\
&= \Big|\mathbb{E}_{\mathbb{Q}_1^\xi} f(Y^\xi(T)) - \mathbb{E}_{\mathbb{Q}_2^\xi} f(Y^\xi(T))\Big| \\
&= \big|\mathbb{E}\big(\big(R_1^\xi(T) - R_2^\xi(T)\big) f(Y^\xi(T))\big)\big| \\
&\le |\!|\!| f |\!|\!|_\infty\, \mathbb{E}\big|R_1^\xi(T) - R_2^\xi(T)\big| \\
&\le |\!|\!| f |\!|\!|_\infty\, \mathbb{E}\bigg(\big(R_1^\xi(T) + R_2^\xi(T)\big)\bigg(\Big|\int_0^T \langle \sigma^{-1}Z(Y_s^\xi) + h_1^\xi(s), \mathrm{d}W(s)\rangle\Big| \\
&\qquad\qquad + \frac{1}{2}\int_0^T \big|\,|h_1^\xi(s)|^2 - |\sigma^{-1}Z(Y_s^\xi)|^2\big|\mathrm{d}s\bigg)\bigg) \qquad (32) \\
&\le |\!|\!| f |\!|\!|_\infty \Big(\big(\mathbb{E}\big(R_1^\xi(T)\big)^q\big)^{1/q} + \big(\mathbb{E}\big(R_2^\xi(T)\big)^q\big)^{1/q}\Big) \\
&\quad \times \bigg\{\bigg(\mathbb{E}\bigg(\Big|\int_0^T \langle \sigma^{-1}Z(Y_s^\xi) + h_1^\xi(s), \mathrm{d}W(s)\rangle\Big|^p\bigg)\bigg)^{1/p} \\
&\quad + \frac{1}{2}\int_0^T \big(\mathbb{E}\big|\,|h_1^\xi(s)|^2 - |\sigma^{-1}Z(Y_s^\xi)|^2\big|^p\big)^{1/p}\mathrm{d}s\bigg\} \\
&=: |\!|\!| f |\!|\!|_\infty\, \Gamma(T)\big\{\Theta_1(T) + \Theta_2(T)\big\},\ t\in[0,T]
\end{aligned}
$$

for $1/p + 1/q = 1, p, q > 1$, where in the second inequality we utilized the fundamental inequality

$$|\mathrm{e}^x - \mathrm{e}^y| \le (\mathrm{e}^x + \mathrm{e}^y)|x - y|,\ x, y \in \mathbb{R},$$

and, in the last two inequalities, employed the Hölder inequality followed by the Minkowski inequality. For notation brevity, let

$$M_1(t) = \int_0^t \big\langle \sigma^{-1}Z(Y_s^\xi), \mathrm{d}W(s)\big\rangle \quad\text{and}\quad M_2(t) = -\int_0^t \big\langle h_1^\xi(s), \mathrm{d}W(s)\big\rangle,\ t \ge 0.$$

For any $q > 1$, using Hölder's inequality and the fact that $\mathrm{e}^{2qM_i(t) - 2q^2\langle M_i\rangle(t)}, i = 1, 2,$ is an exponential martingale leads to

$$
\begin{aligned}
\mathbb{E}\big(R_1^\xi(T)\big)^q + \mathbb{E}\big(R_2^\xi(T)\big)^q &= \mathbb{E}\mathrm{e}^{qM_1(T) - \frac{q}{2}\langle M_1\rangle(T)} + \mathbb{E}\mathrm{e}^{qM_2(T) - \frac{q}{2}\langle M_2\rangle(T)} \\
&\le \big(\mathbb{E}\mathrm{e}^{(2q^2 - q)\langle M_1\rangle(T)}\big)^{1/2} + \big(\mathbb{E}\mathrm{e}^{(2q^2 - q)\langle M_2\rangle(T)}\big)^{1/2} \\
&= \bigg(\mathbb{E}\exp\bigg((2q^2 - q)\int_0^T |\sigma^{-1}Z(Y_t^\xi)|^2\mathrm{d}t\bigg)\bigg)^{1/2} \\
&\quad + \bigg(\mathbb{E}\exp\bigg((2q^2 - q)\int_0^T |h_1^\xi(t)|^2\mathrm{d}t\bigg)\bigg)^{1/2}.
\end{aligned}
$$

Whence, by taking $q \downarrow 1$ and exploiting (7), (21), and (22), one has for some $\widetilde{C}_{q,T} > 0$,

$$\Gamma(T) \le \widetilde{C}_{q,T}. \qquad (33)$$

In view of (**A1**) and (**A2**), in addition to $|Y^\xi(t) - Y^\xi(t_\delta)| \le \|Y_t^\xi - \widehat{Y}_{t_\delta}^\xi\|_\infty$, it holds that

$$\begin{aligned}\big|\sigma^{-1}Z(Y_t^\xi)+h_1^\xi(t)\big| &\le \|\sigma^{-1}\|_{\rm op}\big\{\big|b(Y^\xi(t))-b(Y^\xi(t_\delta))\big|+\big|Z(Y_t^\xi)-Z(\widehat{Y}_{t_\delta}^\xi)\big|\big\}\\ &\le \|\sigma^{-1}\|_{\rm op}\big\{L_1\big|Y^\xi(t)-Y^\xi(t_\delta)\big|+L_2\big\|Y_t^\xi-\widehat{Y}_{t_\delta}^\xi\big\|_\infty^\alpha\big\}\\ &\le \|\sigma^{-1}\|_{\rm op}\big\{\big\|Y_t^\xi-\widehat{Y}_{t_\delta}^\xi\big\|_\infty+\big\|Y_t^\xi-\widehat{Y}_{t_\delta}^\xi\big\|_\infty^\alpha\big\}.\end{aligned}\tag{34}$$

This, with BDG's inequality followed by Hölder's inequality, yields that for $p>2/\alpha$,

$$\begin{aligned}\Theta_1(T) &\le c\Big(\int_0^T\mathbb{E}\big|\sigma^{-1}Z(Y_t^\xi)+h_1^\xi(t)\big|^p{\rm d}t\Big)^{1/p}\\ &\le c\Big(\int_0^T\big\{\mathbb{E}\big\|Y_t^\xi-\widehat{Y}_{t_\delta}^\xi\big\|_\infty^p+\mathbb{E}\big\|Y_t^\xi-\widehat{Y}_{t_\delta}^\xi\big\|_\infty^{p\alpha}\big\}{\rm d}t\Big)^{1/p}\\ &\le c\delta^{\frac{\alpha}{2}-\frac{1}{p}},\end{aligned}\tag{35}$$

where we utilized (27) in the last line. On the other hand, applying Hölder's inequality and combining (**A1**) with (**A2**) and (34) enables us to obtain that, for any $p>1/\alpha$,

$$\begin{aligned}\Theta_2(T) &\le \frac{1}{2}\int_0^T 2\big\{\mathbb{E}\big|h_1^\xi(t)-\sigma^{-1}Z(Y_t^\xi)\big|^{2p}\mathbb{E}\big|\sigma^{-1}Z(Y_t^\xi)+h_1^\xi(t)\big|^{2p}\big\}^{1/2p}{\rm d}t\\ &\le c\int_0^T\big\{\big(1+\mathbb{E}\big\|Y_t^\xi\big\|_\infty^{2p}+\mathbb{E}\big\|\widehat{Y}_{t_\delta}^\xi\big\|_\infty^{2p}\big)\\ &\qquad\times\big(\mathbb{E}\big\|Y_t^\xi-\widehat{Y}_{t_\delta}^\xi\big\|_\infty^{2p}+\mathbb{E}\big\|Y_t^\xi-\widehat{Y}_{t_\delta}^\xi\big\|_\infty^{2p\alpha}\big)\big\}^{1/2p}{\rm d}t\\ &\le c\int_0^T\big\{\mathbb{E}\big\|Y_t^\xi-\widehat{Y}_{t_\delta}^\xi\big\|_\infty^{2p}+\mathbb{E}\big\|Y_t^\xi-\widehat{Y}_{t_\delta}^\xi\big\|_\infty^{2p\alpha}\big\}^{1/2p}{\rm d}t\\ &\le c\delta^{\frac{\alpha}{2}-\frac{1}{2p}},\end{aligned}\tag{36}$$

where we used (25) and (28) in the penultimate procedure and exploited (27) in the last step. Consequently, substituting (33), (35), and (36) into (32) and taking $p>2/\alpha$ sufficiently large (so that $q\downarrow 1$), yields the assertions in (8). □

3. **Extensions to other scenarios.** In this section, we intend to extend the approach to derive Theorem 2.1 and investigate the weak convergence of the other kinds of numerical schemes for path-dependent SDEs with irregular coefficients.

3.1. **Extension to truncated EM scheme.** In this subsection we are still interested in (1). Rather than the EM scheme (5), we introduce the following truncated EM scheme (see e.g. [32]) associated with (1)

$${\rm d}X^{(\delta)}(t)=\big\{b\big(X^{(\delta)}(t_\delta)\big)+Z\big(\widehat{X}_t^{(\delta)}\big)\big\}{\rm d}t+\sigma\,{\rm d}W(t),\ t>0\tag{37}$$

with the initial value $X^{(\delta)}(\theta)=X(\theta)=\xi(\theta),\theta\in[-\tau,0]$, where $\widehat{X}_t^{(\delta)}\in\mathscr{C}$ is defined in the following way:

$$\widehat{X}_t^{(\delta)}(\theta):=X^{(\delta)}\big((t+\theta)\wedge t_\delta\big),\theta\in[-\tau,0].$$

As for the truncated EM scheme (37), the main result in this subsection is stated as below.

Theorem 3.1. *Let* (**A1**) *and* (**A2**) *hold. Then, for any* $T>0$ *such that*

$$2\,\|\sigma\|_{\rm op}^2\|\sigma^{-1}\|_{\rm op}^2\{(4L_1^2+L_2^2)\mathbf{1}_{\{\alpha=1\}}+L_1^2\mathbf{1}_{\{\alpha\in(0,1)\}}\}<{\rm e}^{-(1+\beta T)}/T^2,$$

there exists a constant $C_{2,T} > 0$ (*dependent on* $\|\|f\|\|_\infty$) *such that*

$$\big|\mathbb{E}f(X(t)) - \mathbb{E}f\big(X^{(\delta)}(t)\big)\big| \le C_{2,T}\,\delta^{\alpha/2},\ f \in \mathscr{B}_b(\mathbb{R}^d),\ t \in [0,T]. \tag{38}$$

Proof. Herein, we just outline some dissimilarities since the argument of Theorem 3.1 parallels that of Theorem 2.1. Set

$$h_2^\xi(t) := \sigma^{-1}\big\{b(Y^\xi(t)) - b(Y^\xi(t_\delta)) - Z(\widehat{Y}_t^\xi)\big\},\ t \ge 0,\ \xi \in \mathscr{C}$$

with

$$\widehat{Y}_t^\xi(\theta) := Y^\xi\big((t+\theta)\wedge t_\delta\big),\ \theta \in [-\tau, 0].$$

It is easy to see that

$$\big\|\widehat{Y}_t^\xi\big\|_\infty = \sup_{t-\tau\le s\le t}\big|Y^\xi(s\wedge t_\delta)\big| \le \big\|Y_t^\xi\big\|_\infty$$

so that Lemma 2.6 still holds with h_1^ξ being replaced by h_2^ξ by virtue of Lemma 2.4. On the other hand, by (**A1**) and (28), we infer from Hölder's inequality and BDG's inequality that

$$\begin{aligned}
\mathbb{E}\big\|Y_t^\xi - \widehat{Y}_t^\xi\big\|_\infty^p &= \mathbb{E}\Big(\sup_{t-\tau\le s\le t}\big|Y^\xi(s) - Y^\xi(s\wedge t_\delta)\big|^p\Big)\\
&= \mathbb{E}\Big(\sup_{t-\tau\le s\le t}|Y^\xi(s) - Y^\xi(t_\delta)|^p\mathbf{1}_{\{s\ge t_\delta\}}\Big)\\
&= \mathbb{E}\Big(\sup_{t-\tau\le s\le t}\Big|\int_{t_\delta}^s b(Y^\xi(u)\mathrm{d}u + \int_{t_\delta}^s \sigma\mathrm{d}W(u)\Big|^p\mathbf{1}_{\{s\ge t_\delta\}}\Big)\\
&\le c\Big\{\delta^{p-1}\int_{t_\delta}^t \mathbb{E}\big|b(Y^\xi(u)\big|^p\mathrm{d}u + \mathbb{E}\Big(\sup_{t_\delta\le s\le t}\Big|\int_{t_\delta}^s\sigma\mathrm{d}W(u)\Big|^p\Big)\Big\}\\
&\le c\delta^{p/2},\ p\ge 1.
\end{aligned} \tag{39}$$

Having Lemma 2.6, writing h_2^ξ in lieu of h_1^ξ, and (39) in hand, the proof of Theorem 3.1 is therefore complete by inspecting the argument of Theorem 2.1. □

Remark 3.2. In terms of Theorems 2.1 and 3.1, we conclude that the truncated EM scheme (37) enjoys a better weak convergence rate than the EM scheme (5). On the other hand, with regard to the truncated EM scheme, we drop the assumption (**A3**) in Theorem 3.1. Furthermore, we point out that the EM scheme (5) established via interpolation works merely for path-dependent SDEs with finite memory since the linear interpolation therein relies on the length of memory. The truncated EM scheme (37) is still available for path-dependent SDEs with infinite memory as the following subsection demonstrates.

3.2. **Extension to path-dependent SDEs with infinite memory.** As we depicted in Remark 3.2, one of the advantages of the truncated EM scheme (37) is that it is applicable to path-dependent SDEs with infinite memory. To proceed, we first introduce some additional notations. For a fixed number $r \in (0,\infty)$, let

$$\mathscr{C}_r = \Big\{\phi \in C((-\infty,0];\mathbb{R}^d) : \|\phi\|_r := \sup_{-\infty<\theta\le 0}(\mathrm{e}^{r\theta}|\phi(\theta)|) < \infty\Big\},$$

which is a Polish space under the metric induced by $\|\cdot\|_r$.

In this subsection, we focus on the path-dependent SDE with infinite memory

$$\mathrm{d}X(t) = \big\{b(X(t)) + Z(X_t)\big\}\mathrm{d}t + \sigma\mathrm{d}W(t),\ t > 0,\ X_0 = \xi \in \mathscr{C}_r, \tag{40}$$

in which

($\mathbf{A2'}$) $Z:\mathscr{C}_r\to\mathbb{R}^d$ is Hölder continuous, i.e., there exist $\alpha\in(0,1]$ and $L_4>0$ such that

$$|Z(\xi)-Z(\eta)|\le L_4\|\xi-\eta\|_r^\alpha,\ \ \xi,\eta\in\mathscr{C}_r,$$

and the other quantities are stipulated exactly as in (1). Similar to (37), we define the truncated EM scheme associated with (40) by

$$\mathrm{d}X^{(\delta)}(t)=\big\{b\big(X^{(\delta)}(t_\delta)\big)+Z\big(\widehat{X}_t^{(\delta)}\big)\big\}\mathrm{d}t+\sigma\,\mathrm{d}W(t),\ \ t>0 \tag{41}$$

with the initial datum $X^{(\delta)}(\theta)=X(\theta)=\xi(\theta),\theta\in(-\infty,0]$, in which $\widehat{X}_t^{(\delta)}\in\mathscr{C}_r$ is designed by

$$\widehat{X}_t^{(\delta)}(\theta):=X^{(\delta)}\big((t+\theta)\wedge t_\delta\big),\theta\in(-\infty,0].$$

The main result in this subsection is presented as follows.

Theorem 3.3. *Assume the assumptions of Theorem 3.1 hold with* ($\mathbf{A2}$) *replaced by* ($\mathbf{A2'}$). *Then, there exists a constant $C_{3,T}>0$ (dependent on $|\!|\!|f|\!|\!|_\infty$) such that*

$$\big|\mathbb{E}f(X(t))-\mathbb{E}f\big(X^{(\delta)}(t)\big)\big|\le C_{3,T}\,\delta^{\alpha/2},\ \ f\in\mathscr{B}_b(\mathbb{R}^d),\ \ t\in[0,T] \tag{42}$$

provided that the step size $\delta\in(0,1)$ is sufficiently small.

Proof. Since

$$\big\|Y_t^\xi\big\|_r\le\|\xi\|_r+\sup_{0\le s\le t}\big|Y^\xi(s)\big|,$$

Lemma 2.4 still holds with $\|\cdot\|_\infty$ replaced by $\|\cdot\|_r$. Also, (21) holds under assumptions ($\mathbf{A1}$) and ($\mathbf{A2'}$) so that (40) has a unique weak solution by following the argument of Lemma 2.3. Let

$$h_3^\xi(t)=\sigma^{-1}\big\{b\big(Y^\xi(t)\big)-b\big(Y^\xi(t_\delta)\big)-Z\big(\widehat{Y}_t^\xi\big)\big\},\ \ t\ge0,\ \ \xi\in\mathscr{C}_r,$$

where

$$\widehat{Y}_t^\xi(\theta):=Y^\xi\big((t+\theta)\wedge t_\delta\big),\ \ \theta\in(-\infty,0].$$

Clearly, we have

$$\big\|\widehat{Y}_t^\xi\big\|_r=\mathrm{e}^{-rt}\sup_{-\infty<s\le t}\big(\mathrm{e}^{rs}|Y^\xi(s\wedge t_\delta)|\big)\le\mathrm{e}^{r\delta}\mathrm{e}^{-rt}\sup_{-\infty<s\le t}\big(\mathrm{e}^{rs}\big|Y^\xi(s)\big|\big)=\mathrm{e}^{r\delta}\big\|Y_t^\xi\big\|_r.$$

So, (22), writing $h_3^\xi(t)$ instead of $h_1^\xi(t)$, remains true whenever the step size $\delta\in(0,1)$ is sufficiently small. Moreover, by virtue of ($\mathbf{A1}$), (28) and Hölder's inequality as well as BDG's inequality, it follows that

$$\begin{aligned}\mathbb{E}\big\|Y_t^\xi-\widehat{Y}_t^\xi\big\|_r^p&=\mathrm{e}^{-prt}\mathbb{E}\Big(\sup_{-\infty<s\le t}\big(\mathrm{e}^{prs}\big|Y^\xi(s)-Y^\xi(s\wedge t_\delta)\big|^p\big)\Big)\\&\le\mathbb{E}\Big(\sup_{t_\delta<s\le t}\Big(\Big|\int_{t_\delta}^s b(Y^{\xi(0)}(s))\mathrm{d}s+\sigma(W(s)-W(t_\delta))\Big|^p\Big)\Big)\\&\le c\delta^{p/2},\ \ p\ge2.\end{aligned}$$

Afterwards, carrying out a similar argument to derive Theorem 2.1, we obtain the desired assertion (42). □

Remark 3.4. To the best of our knowledge, Theorem 3.3 is the first result upon the weak convergence for path-dependent SDEs with infinite memory and irregular drifts. For path-dependent SDEs with finite memory, Theorems 2.1 and 3.1 show that the weak convergence order can be achieved for any $\delta\in(0,1)$. However, concerning path-dependent SDEs with infinite memory, the weak convergence rate can only be available whenever the step size $\delta\in(0,1)$ is sufficiently small. This illustrates one of the essential features between SDEs with finite memory and SDEs infinite memory. Moreover, Theorem 3.3 further shows the superiority of the truncated EM scheme (37) with contrast to the EM scheme established by interpolations at discrete-time points.

4. **Weak convergence: Degenerate case.** In the previous sections, we investigated the weak convergence of EM schemes and its variants for non-degenerate path-dependent SDEs with Hölder continuous drifts. In this section, we are still interested in the same topic, but concerned with a class of degenerate SDEs on $\mathbb{R}^{2d} := \mathbb{R}^d \times \mathbb{R}^d$

$$\begin{cases} \mathrm{d}X(t) = \{X(t) + Y(t)\}\mathrm{d}t \\ \mathrm{d}Y(t) = \{b(X(t), Y(t)) + Z(X_t, Y_t)\}\mathrm{d}t + \sigma \mathrm{d}W(t),\ t \geq 0 \end{cases} \tag{43}$$

with the initial datum $(X_0, Y_0) = (\xi, \eta) \in \mathscr{C}^2$, where $b : \mathbb{R}^{2d} \to \mathbb{R}^d$, $Z : \mathscr{C}^2 \to \mathbb{R}^d$, $\sigma \in \mathbb{M}^d_{\text{non}}$, and $(W(t))_{t\geq 0}$ is a d-dimensional Brownian motion on the probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P})$. (43) is the so-called stochastic Hamiltonian system which has been investigated considerably in [19, 29, 35, 37, 39], to name a few.

Throughout this section, we assume that:

(H1) b is Lipschitz continuous, that is, there exists a constant $K_1 > 0$ such that

$$|b(x_1, y_1) - b(x_2, y_2)| \leq K_1(|x_1 - y_1| + |y_1 - y_2|),\ (x_1, y_1), (x_2, y_2) \in \mathbb{R}^{2d} \tag{44}$$

and there exist constants $\alpha, \beta, \lambda, C > 0$ and $\gamma \in (-\alpha\beta, \alpha\beta)$ such that

$$\langle \alpha x + \gamma y, x + y\rangle + \langle \beta y + \gamma x, b(x, y)\rangle \leq C - \lambda(|x|^2 + |y|^2),\ (x, y) \in \mathbb{R}^{2d}. \tag{45}$$

(H2) Z is Hölder continuous, i.e., there exist constants $\alpha \in (0, 1]$ and $K_2 > 0$ such that

$$|Z(\xi_1, \eta_1) - Z(\xi_2, \eta_2)| \leq K_2\big(\|\xi_1 - \xi_2\|^\alpha_\infty + \|\eta_1 - \eta_2\|^\alpha_\infty\big),\ (\xi_1, \eta_1), (\xi_2, \eta_2) \in \mathscr{C}^2.$$

By carrying out a similar argument to derive Lemma 2.3 and taking advantage of Lemma 4.3 below, (43) has a unique weak solution under (**H1**) and (**H2**). Under (44), the following reference SDE

$$\begin{cases} \mathrm{d}U(t) = \{U(t) + V(t)\}\mathrm{d}t \\ \mathrm{d}V(t) = b\,(U(t), V(t))\mathrm{d}t + \sigma \mathrm{d}W(t),\ t \geq 0 \end{cases} \tag{46}$$

with the initial data $(U(0), V(0)) = (u, v) \in \mathbb{R}^{2d}$ is well-posed. To emphasize the initial value $(u, v) \in \mathbb{R}^{2d}$, we shall write $(U^{u,v}(t), V^{u,v}(t))$ instead of $(U(t), V(t))$. Analogous to (4), we can respectively extend $U(t)$ and $V(t)$ in the following ways:

$$U^{\xi,\eta}(t) = \xi(t)\mathbf{1}_{[-\tau,0)}(t) + U^{\xi(0),\eta(0)}(t)\mathbf{1}_{[0,\infty)}(t),\ t \in [-\tau, \infty),\ (\xi, \eta) \in \mathscr{C}^2$$

and

$$V^{\xi,\eta}(t) = \eta(t)\mathbf{1}_{[-\tau,0)}(t) + V^{\xi(0),\eta(0)}(t)\mathbf{1}_{[0,\infty)}(t),\ t \in [-\tau, \infty),\ (\xi, \eta) \in \mathscr{C}^2.$$

Let $U^{\xi,\eta}_t$ and $V^{\xi,\eta}_t$ be the segment process associated with $U^{\xi,\eta}(t)$ and $V^{\xi,\eta}(t)$, respectively. Next, the truncated EM scheme corresponding to (43) is given by

$$\begin{cases} \mathrm{d}X^{(\delta)}(t) = \big\{X^{(\delta)}(t) + Y^{(\delta)}(t)\big\}\mathrm{d}t \\ \mathrm{d}Y^{(\delta)}(t) = \big\{b\big(X^{(\delta)}(t_\delta), Y^{(\delta)}(t_\delta)\big) + Z\big(\widehat{X}^{(\delta)}_t, \widehat{Y}^{(\delta)}_t\big)\big\}\mathrm{d}t + \sigma \mathrm{d}W(t) \end{cases}$$

with the initial value $(X^{(\delta)}(\theta), Y^{(\delta)}(\theta)) = (X(\theta), Y(\theta)) = (\xi(\theta), \eta(\theta)) \in \mathbb{R}^{2d}$, $\theta \in [-\tau, 0]$, where

$$\widehat{X}^{(\delta)}_t(\theta) := X^{(\delta)}\big((t + \theta) \wedge t_\delta\big) \text{ and } \widehat{Y}^{(\delta)}_t(\theta) := Y^{(\delta)}\big((t + \theta) \wedge t_\delta\big),\ \theta \in [-\tau, 0].$$

Observe that

$$\mathrm{d}X^{(\delta)}(t) = \big\{X^{(\delta)}(t) + Y^{(\delta)}(0) + b\big(X^{(\delta)}(0), Y^{(\delta)}(0)\big)t + \Lambda(t) + \sigma W(t)\big\}\mathrm{d}t,\ t \in [0,\delta]$$

where, for any $\theta \in [-\tau, 0]$,

$$\Lambda(t) := \int_0^t Z\big(\widetilde{X}_s^{(\delta)}, \widetilde{Y}_s^{(\delta)}\big)\mathrm{d}s \ \text{ with } \widetilde{X}_s^{(\delta)}(\theta) := X\big((s+\theta)\wedge 0\big),\ \widetilde{Y}_s^{(\delta)}(\theta) := Y\big((s+\theta)\wedge 0\big).$$

Thus, $(X^{(\delta)}(t))_{t\in[0,\delta]}$ can be obtained explicitly via the variation-of-constants formula. Inductively, $X^{(\delta)}(t)$ enjoys an explicit formula.

In the following, for α, β, γ such that we have (45), consider the following Lyapunov function:

$$\mathbb{W}(x,y) := \frac{\alpha}{2}|x|^2 + \frac{\beta}{2}|y|^2 + \gamma\langle x, y\rangle,\ x, y \in \mathbb{R}^d.$$

For $\gamma \in (-\alpha\beta, \alpha\beta)$, it is easy to see that

$$\kappa_2(|x|^2 + |y|^2) \le \mathbb{W}(x,y) \le \kappa_1(|x|^2 + |y|^2),\ x, y \in \mathbb{R}^d, \tag{47}$$

in which $\kappa_1 := (1+\alpha)(1+\beta)/2$ and

$$\kappa_2 := \frac{1}{2}\bigg\{\bigg(\alpha - \frac{1}{2}(\alpha/|\gamma| + |\gamma|/\beta)\bigg) \wedge \Big(\beta - \frac{2|\gamma|}{\alpha/|\gamma| + |\gamma|/\beta}\Big)\bigg\}. \tag{48}$$

The main result in this section is presented as follows.

Theorem 4.1. *Assume* (**H1**) *and* (**H2**) *hold. Then, for any $T > 0$ such that*

$$2\,\kappa_3\,\|\sigma\|_{\rm op}^2\|\sigma^{-1}\|_{\rm op}^2\big\{(4K_1^2 + K_2^2)\mathbf{1}_{\{\alpha=1\}} + 2K_1^2\mathbf{1}_{\{\alpha\in(0,1)\}}\big\}T^2 < \kappa_2\,\mathrm{e}^{\lambda\kappa_2 T - 1}$$

there exists $C_{4,T} > 0$ (dependent on $|||f|||_\infty$) such that

$$\big|\mathbb{E}f(X(t), Y(t)) - \mathbb{E}f\big(X^{(\delta)}(t), Y^{(\delta)}(t)\big)\big| \le C_{4,T}\delta^{\alpha/2},\ f \in \mathscr{B}_b(\mathbb{R}^{2d}),\ t \in [0,T]. \tag{49}$$

Remark 4.2. The dissipative condition (45) is imposed to guarantee that the time horizon $T > 0$ in Theorem 4.1 is large in certain situations. Nevertheless, in case of $\lambda < 0$, (49) remains true, but for a small time horizon. Moreover, we can also investigate the weak convergence of the EM scheme via interpolation for (43), but with an additional assumption put on the initial value. Also, we point out that whenever the numerical scheme of the second component is established by interpolation, the algorithm for the first component is much more explicit compared to the truncated EM scheme.

The proof of Theorem 4.1 is based on several lemmas below. The following lemma shows exponential integrability of the segment process.

Lemma 4.3. Assume (**H1**) and (**H2**) hold. Then, for any $T > 0$,

$$\mathbb{E}\exp\bigg(\lambda\int_0^T\big(\|U_t^{\xi,\eta}\|_\infty^2 + \|V_t^{\xi,\eta}\|_\infty^2\big)\mathrm{d}t\bigg) < \infty,\ \lambda < \frac{\kappa_2\,\mathrm{e}^{\lambda\kappa_2 T - 1}}{\kappa_3\,\|\sigma\|_{\rm op}^2 T^2} \tag{50}$$

where $\kappa_3 := \gamma^2 \vee \beta^2$.

Proof. For notation simplicity, in what follows we write $U(t)$ and $V(t)$ in lieu of $U^{\xi,\eta}(t)$ and $V^{\xi,\eta}(t)$, respectively. By a close inspection of the proof for Lemma 2.4, to verify (50) it is sufficient to show that, for any $\varepsilon > 0$ and $\gamma > -\lambda\,\kappa_2 + \kappa_3\,\|\sigma\|_{\rm op}^2\,\varepsilon$,

$$\mathbb{E}\bigg(\sup_{0\le t\le T}\mathrm{e}^{\varepsilon\,\kappa_2\,\mathrm{e}^{-\gamma\,T}(|U(t)|^2 + |V(t)|^2)}\bigg) < \infty, \tag{51}$$

where κ_2 was given in (48) and $\kappa_3 := \gamma^2 \vee \beta^2$. By Itô's formula, it follows from (45) and (47) that

$$\begin{aligned}
\mathrm{d}\big(\mathrm{e}^{-\gamma t}\mathbb{W}(U(t),V(t))\big) &= \mathrm{e}^{-\gamma t}\Big\{-\gamma\mathbb{W}(U(t),V(t)) + \big\langle \alpha U(t)+\gamma V(t), U(t)+V(t)\big\rangle \\
&\quad + \big\langle \gamma U(t)+\beta V(t), b(U(t),V(t))\big\rangle + (C+\|\sigma\|_{\mathrm{HS}}^2/2)\Big\}\mathrm{d}t \\
&\quad + \mathrm{e}^{-\gamma t}\big\langle \sigma^*(\gamma U(t)+\beta V(t)), \mathrm{d}W(t)\big\rangle \\
&\le \mathrm{e}^{-\gamma t}\Big\{-(\gamma+\lambda\kappa_2)\mathbb{W}(U(t),V(t)) + (C+\|\sigma\|_{\mathrm{HS}}^2/2)\Big\}\mathrm{d}t \\
&\quad + \mathrm{e}^{-\gamma t}\big\langle \sigma^*(\gamma U(t)+\beta V(t)), \mathrm{d}W(t)\big\rangle.
\end{aligned}$$

This implies via Itô's formula that

$$\begin{aligned}
&\mathrm{d}\mathrm{e}^{\varepsilon \mathrm{e}^{-\gamma t}\mathbb{W}(U(t),V(t))} \\
&\le -\varepsilon\big(\gamma+\lambda\kappa_2-\kappa_3\|\sigma\|_{\mathrm{op}}^2\varepsilon\big)\mathrm{e}^{-\gamma t}\mathrm{e}^{\varepsilon \mathrm{e}^{-\gamma t}\mathbb{W}(U(t),V(t))}\mathbb{W}(U(t),V(t))\mathrm{d}t \\
&\quad + \varepsilon \mathrm{e}^{-\gamma t}\mathrm{e}^{\varepsilon \mathrm{e}^{-\gamma t}\mathbb{W}(U(t),V(t))}\big\langle \sigma^*(\gamma U(t)+\beta V(t)), \mathrm{d}W(t)\big\rangle, \\
&\quad + c\varepsilon \mathrm{e}^{\varepsilon \mathrm{e}^{-\gamma t}\mathbb{W}(U(t),V(t))}\mathrm{d}t, \quad \varepsilon>0
\end{aligned} \tag{52}$$

for some constant $c>0$. For any $\gamma > -\lambda\kappa_2 + \kappa_3\|\sigma\|_{\mathrm{op}}^2\varepsilon$, Gronwall's inequality, in addition to (47), yields that

$$\mathbb{E}\mathrm{e}^{\varepsilon\, \mathrm{e}^{-\gamma t} W(U(t),V(t))} \le \mathrm{e}^{\frac{c\,\varepsilon}{\gamma}}\mathrm{e}^{\varepsilon\kappa_1(|\xi(0)|^2+|\eta(0)|^2)}, \tag{53}$$

which, together with (52), further leads to

$$\begin{aligned}
&\varepsilon\big(\gamma+\lambda\kappa_2-\kappa_3\|\sigma\|_{\mathrm{op}}^2\varepsilon\big)\int_0^t \mathrm{e}^{-\gamma s}\mathbb{E}\mathrm{e}^{\varepsilon\,\mathrm{e}^{-\gamma s}\mathbb{W}(U(s),V(s))}\mathbb{W}(U(s),V(s))\mathrm{d}s \\
&\le \Big(1+\frac{c\varepsilon}{\gamma}\mathrm{e}^{\frac{c\,\varepsilon}{\gamma}}\Big)\mathrm{e}^{\varepsilon\kappa_1(|\xi(0)|^2+|\eta(0)|^2)}.
\end{aligned} \tag{54}$$

Subsequently, by means of BDG's inequality, we derive from (47) and (54) that

$$\begin{aligned}
&\mathbb{E}\bigg(\sup_{0\le s\le t}\int_0^s \mathrm{e}^{-\gamma u}\mathrm{e}^{\varepsilon\,\mathrm{e}^{-\gamma u}\mathbb{W}(U(u),V(u))}\langle \sigma^*(\gamma U(u)+\beta V(u)), \mathrm{d}W(u)\rangle\bigg) \\
&\le 4\sqrt{2}\mathbb{E}\bigg(\int_0^t \mathrm{e}^{-2\gamma s}\mathrm{e}^{2\varepsilon\,\mathrm{e}^{-\gamma s}\mathbb{W}(U(s),V(s))}|\sigma^*(\gamma U(s)+\beta V(s))|^2\mathrm{d}s\bigg)^{1/2} \\
&\le \frac{1}{2}\mathbb{E}\bigg(\sup_{0\le s\le t}\mathrm{e}^{\varepsilon\,\mathrm{e}^{-\gamma t}\mathbb{W}(U(t),V(t))}\bigg) \\
&\quad + c\int_0^t \mathrm{e}^{-\gamma s}\mathbb{E}\mathrm{e}^{\varepsilon \mathrm{e}^{-\gamma s}\mathbb{W}(U(s),V(s))}\mathbb{W}(U(s),V(s))\mathrm{d}s \\
&\le \frac{1}{2}\mathbb{E}\bigg(\sup_{0\le s\le t}\mathrm{e}^{\varepsilon\,\mathrm{e}^{-\gamma t}\mathbb{W}(U(t),V(t))}\bigg) + c\,(1+\mathrm{e}^{c_\varepsilon t})\mathrm{e}^{\varepsilon\kappa_1(|\xi(0)|^2+|\eta(0)|^2)}.
\end{aligned} \tag{55}$$

With (52)-(55) in hand, we arrive at

$$\mathbb{E}\bigg(\sup_{0\le t\le T}\mathrm{e}^{\varepsilon \mathrm{e}^{-\gamma\,T}\mathbb{W}(U(t),V(t))}\bigg) < \infty.$$

This, combined with (47), yields (51). □

For notation brevity, we set

$$h^{\xi,\eta}(t) := \sigma^{-1}\big\{b\big(U^{\xi,\eta}(t),V^{\xi,\eta}(t)\big) - b\big(U^{\xi,\eta}(t_\delta),V^{\xi,\eta}(t_\delta)\big) - Z\big(\widehat{U}_t^{\xi,\eta},\widehat{V}_t^{\xi,\eta}\big)\big\}.$$

Lemma 4.4. *Assume* (**H1**) *and* (**H2**). *Then,*

$$\mathbb{E}e^{\lambda \int_0^T |\sigma^{-1}Z(U_t^{\xi,\eta},V_t^{\xi,\eta})|^2 dt} < \infty \tag{56}$$

for any $\lambda, T > 0$, *such that*

$$\lambda < \frac{\kappa_2\, e^{\lambda\kappa_2 T-1}}{2\kappa_3\|\sigma\|_{\rm op}^2\|\sigma^{-1}\|_{\rm op}^2\{K_2^2\mathbf{1}_{\{\alpha=1\}}+0\mathbf{1}_{\{\alpha\in(0,1)\}}\}T^2}$$

Furthermore,

$$\mathbb{E}e^{\lambda \int_0^T |h^{\xi,\eta}(t)|^2 dt} < \infty,\ \lambda < \frac{\kappa_2\, e^{\lambda\kappa_2 T-1}}{4\kappa_3\|\sigma\|_{\rm op}^2\|\sigma^{-1}\|_{\rm op}^2(4K_1^2+K_2^2)T^2} \tag{57}$$

for any $\lambda, T > 0$, *such that*

$$\lambda < \frac{\kappa_2\, e^{\lambda\kappa_2 T-1}}{4\kappa_3\|\sigma\|_{\rm op}^2\|\sigma^{-1}\|_{\rm op}^2\{(4K_1^2+K_2^2)\mathbf{1}_{\{\alpha=1\}}+2K_1^2\mathbf{1}_{\{\alpha\in(0,1)\}}\}T^2}.$$

Proof. From (**A2**), it holds that there exists some constant $c_\varepsilon > 0$ such that, for any $\varepsilon > 0$,

$$\begin{aligned}\big|\sigma^{-1}Z\big(U_t^{\xi,\eta},V_t^{\xi,\eta}\big)\big|^2 \le c_\varepsilon + &\big\{2K_2^2\|\sigma^{-1}\|_{\rm op}^2(1+\varepsilon)\mathbf{1}_{\{\alpha=1\}}+\varepsilon\mathbf{1}_{\{\alpha\in(0,1)\}}\big\}\\ &\times\big(\|U_t^{\xi,\eta}\|_\infty^2+\|V_t^{\xi,\eta}\|_\infty^2\big).\end{aligned} \tag{58}$$

Thus, (56) follows from (58) and Lemma 4.3.

Next, with the aid of (44), and (**H2**) and due to the facts that $\|\widehat{U}_t^{\xi,\eta}\|_\infty \le \|U_t^{\xi,\eta}\|_\infty$ and $\|\widehat{V}_t^{\xi,\eta}\|_\infty \le \|V_t^{\xi,\eta}\|_\infty$, it follows that

$$\big|h^{\xi,\eta}(t)\big|^2 \le c_\varepsilon + 4\big\|\sigma^{-1}\big\|_{\rm op}^2\big(4K_1^2+K_2^2(1+\varepsilon)\big)\big(\big\|U_t^{\xi,\eta}\big\|_\infty^2+\big\|V_t^{\xi,\eta}\big\|_\infty^2\big) \tag{59}$$

for some $c_\varepsilon > 0$ and

$$\nu_\varepsilon := 4\big\|\sigma^{-1}\big\|_{\rm HS}^2\big\{\big(4\,K_1^2+K_2^2(1+\varepsilon)\big)\mathbf{1}_{\{\alpha=1\}}+2K_1^2\mathbf{1}_{\{\alpha\in(0,1)\}}\big\}$$

Therefore, by virtue of (59) and Lemma 4.3, (57) holds true. □

Now, we proceed to finish the proof of Theorem 4.1.

Proof of Theorem 4.1. Under Assumption (**H1**), it is standard to show that

$$\mathbb{E}\Big(\sup_{-\tau\le t\le T}\big(\big|U^{\xi,\eta}(t)\big|^p+\big|V^{\xi,\eta}(t)\big|^p\big)\Big) \le C_{p,T}\big(\|\xi\|_\infty^p+\|\eta\|_\infty^p\big).$$

This, combined Hölder's inequality with BDG's inequality, leads to

$$\sup_{0\le t\le T}\mathbb{E}\big\|U_t^{\xi,\eta}-\widehat{U}_t^{\xi,\eta}\big\|_\infty^p+\mathbb{E}\big\|V_t^{\xi,\eta}-\widehat{V}_t^{\xi,\eta}\big\|_\infty^p \le c\delta^{p\,\alpha/2}. \tag{60}$$

Thus, mimicking the argument of Theorem 2.1, we obtain the desired assertion from (60) and Lemma 4.4. □

5. **Weak convergence: Integrability conditions.** In the previous sections, we investigated the weak convergence of EM schemes for path-dependent SDEs, where the irregular drifts are at most of linear growth. In this section, we still focus on the topic of the weak convergence but for path-dependent SDEs under integrability conditions, which might allow the irregular drifts to be non-globally Lipschitz.

We start with some additional notations. Denote by $C^2(\mathbb{R}^d)$ the set of all continuously twice differentiable functions $f:\mathbb{R}^d\to\mathbb{R}$ and $C_0^\infty(\mathbb{R}^d)$ the family of arbitrarily often differentiable functions $f:\mathbb{R}^d\to\mathbb{R}$ with compact support. Let ∇ and ∇^2 represent the gradient operator and the Hessian operator, respectively. Let $\mathscr{P}(\mathbb{R}^d)$ stand for the collection of all probability measures on $\mathbb{R}^d$. For $\sigma\in\mathbb{M}^d_{\rm non}$ and $V\in C^2(\mathbb{R}^d)$ with $\mathrm{e}^{-V}\in L^1(\mathrm{d}x)$ and $\mu_0(\mathrm{d}x):=C_V\mathrm{e}^{-V(x)}\mathrm{d}x\in\mathscr{P}(\mathbb{R}^d)$, where C_V is the normalization, set $Z_0:\mathbb{R}^d\to\mathbb{R}^d$ by

$$Z_0(x):=-(\sigma\sigma^*)\nabla V(x),\ x\in\mathbb{R}^d. \tag{61}$$

Thus, by the integration by parts formula, the operator

$$\mathscr{L}_0f(x):=\frac{1}{2}\mathrm{tr}\big((\sigma\sigma^*)\nabla^2f\big)(x)+\langle Z_0(x),\nabla f(x)\rangle,\ x\in\mathbb{R}^d,\ f\in C_0^\infty(\mathbb{R}^d)$$

is symmetric on $L^2(\mu_0)$, i.e., for any $f,g\in C_0^\infty(\mathbb{R}^d)$,

$$\mathscr{E}_0(f,g):=\langle f,\mathscr{L}_0g\rangle_{L^2(\mu_0)}=\langle g,\mathscr{L}_0f\rangle_{L^2(\mu_0)}=-\langle\sigma^*\nabla f,\sigma^*\nabla g\rangle_{L^2(\mu_0)}.$$

Let $H^{1,2}_\sigma$ be the completion of $C_0^\infty(\mathbb{R}^d)$ under the Sobolev norm

$$\|f\|_{H^{1,2}_\sigma}:=\big(\mu_0(|f|^2+|\sigma^*f|^2)\big)^{1/2}.$$

Then, $(\mathscr{E}_0,H^{1,2}_\sigma)$ is a symmetric Dirichlet form on $L^2(\mu_0)$, and the associated Markov process can be constructed as the solution to the reference SDE

$$\mathrm{d}Y(t)=Z_0(Y(t))\mathrm{d}t+\sigma\mathrm{d}W(t),\ t>0,\ Y(0)=x, \tag{62}$$

where $W(t)$ is a d-dimensional Brownian motion defined on the probability space $(\Omega,\mathscr{F},\mathbb{P})$ with the filtration $(\mathscr{F}_t)_{t\ge0}$. Assume that

(C1) $Z_0:\mathbb{R}^d\to\mathbb{R}^d$ is Lipschitz continuous, i.e., there exists an $L_0>0$ such that

$$|Z_0(x)-Z_0(y)|\le L_0|x-y|\ x,y\in\mathbb{R}^d,$$

and there exist constants $C>0$ and $\beta\in\mathbb{R}$ such that

$$2\langle x,Z_0(x)\rangle\le C+\beta|x|^2,\ x\in\mathbb{R}^d.$$

Under (**C1**), (62) has a unique solution $(Y^x(t))_{t\ge0}$ with initial value $Y^x(0)=x$. Observe that μ_0 is the invariant probability measure of the Markov semigroup $P_tf(x):=\mathbb{E}f(Y^x(t))$, $f\in\mathscr{B}_b(\mathbb{R}^d)$.

In this section, we consider the following path-dependent SDE

$$\mathrm{d}X(t)=\bigg\{Z_0(X(t))+\int_{-\tau}^0Z(X(t+\theta))\rho(\mathrm{d}\theta)\bigg\}\mathrm{d}t+\sigma\mathrm{d}W(t),\ t\ge0,\ X_0=\xi, \tag{63}$$

where $\rho(\cdot)$ is a probability measure on $[-\tau,0]$. Under assumption (65) below, (63) admits a unique weak solution by following exactly the argument of Lemma 2.3. The EM scheme associated with (63) is given by

$$\mathrm{d}X^{(\delta)}(t)=\bigg\{Z_0(X^{(\delta)}(t_\delta))+\int_{-\tau}^0Z\big(\widehat{X}_t^{(\delta)}(\theta)\big)\rho(\mathrm{d}\theta)\bigg\}\mathrm{d}t+\sigma\mathrm{d}W(t) \tag{64}$$

with initial value $X^{(\delta)}(\theta)=X(\theta)=\xi(\theta),\theta\in[-\tau,0]$, where

$$\widehat{X}_t^{(\delta)}(\theta):=X^{(\delta)}\big((t+\theta)\wedge t_\delta\big),\ \theta\in[-\tau,0].$$

Analogously, we define

$$\widehat{Y}_t^{\xi}(\theta)=Y^{\xi}\big((t+\theta)\wedge t_\delta\big),\ \theta\in[-\tau,0],$$

where Y^ξ was extended as in (4). Moreover, we set

$$h_4^\xi(t):=\sigma^{-1}\bigg\{Z_0(Y^\xi(t))-Z_0(Y^\xi(t_\delta))-\int_{-\tau}^0 Z\big(\widehat{Y}_t^\xi(\theta)\big)\rho(\mathrm{d}\theta)\bigg\}.$$

Our main result in this section is as follows, which reveals the weak convergence order of the EM scheme for path-dependent SDEs under an integrability condition.

Theorem 5.1. *Assume* (**C1**) *holds, and suppose further that there exists a constant* $\kappa>0$ *such that*

$$\mu_0\big(\mathrm{e}^{\kappa|Z(\cdot)|^2}\big)<\infty \tag{65}$$

and that there exist constants $m\geq 1,\alpha\in(0,1]$, *and* $C>0$ *such that*

$$|Z(x)-Z(y)|\leq C\big(1+|x|^m+|y|^m\big)|x-y|^\alpha\ x,y\in\mathbb{R}^d. \tag{66}$$

Then, there exists $C_{5,T}>0$ *(dependent on* $\|\|f\|\|_\infty$*) such that*

$$\big|\mathbb{E}f(X^\xi(t))-\mathbb{E}f\big(X^{(\delta)}(t)\big)\big|\leq C_{1,T}\delta^\alpha,\ \xi\in\mathscr{C},\ f\in\mathscr{B}_b(\mathbb{R}^d),\ t\in[0,T], \tag{67}$$

where $T>0$ *satisfies*

$$1<\frac{\kappa}{2(2\vee d)\|\sigma^{-1}\|_{\rm op}^2T^2}\wedge\frac{\mathrm{e}^{-(1+\beta T)}}{32\,\|\sigma\|_{\rm op}^2\|\sigma^{-1}\|_{\rm op}^2T^2}\wedge\frac{\kappa}{(1\vee\frac{d}{2})T}. \tag{68}$$

Proof. From $\xi\in\mathscr{C}$ and (66), we infer from Lemma 5.3 below that (65) is available so that

$$\mathbb{E}\mathrm{e}^{(1+\varepsilon)\int_0^T|\int_{-\tau}^0 Z(Y^\xi(t+\theta))\rho(\mathrm{d}\theta)|^2\mathrm{d}t}+\mathbb{E}\mathrm{e}^{(1+\varepsilon)\int_0^T|h_4^\xi(t)|^2\mathrm{d}t}<\infty \tag{69}$$

for some $\varepsilon\in(0,1)$ sufficiently small and $T>0$ such that (68) is satisfied. Next, exploiting Hölder's inequality and taking advantage of (28), (39), and (66) enables us to obtain that

$$\begin{aligned}
&\int_{-\tau}^0\mathbb{E}\big|Z(Y^\xi(t+\theta))-Z(\widehat{Y}_t^\xi(\theta))\big|^p\rho(\mathrm{d}\theta)\\
&\leq\int_{-\tau}^0\mathbb{E}\bigg(\sup_{t_\delta\leq s\leq t}\big|Z(Y^\xi(s))-Z(Y^\xi(t_\delta))\big|^p\mathbf{1}_{\{t+\theta\geq t_\delta\}}\bigg)\rho(\mathrm{d}\theta)\\
&\leq c\mathbb{E}\bigg(\sup_{t_\delta\leq s\leq t}\big(1+\big|Z(Y^\xi(s))\big|^{pm}+\big|Y^\xi(t_\delta)\big|^{pm}\big)\big|Y^\xi(s)-Y^\xi(t_\delta)\big|^{p\alpha}\bigg)\\
&\leq c\delta^{p\alpha/2}.
\end{aligned} \tag{70}$$

With (71) and (79) in hand, the proof of Theorem 5.1 can be done by following the reasoning of Theorem 2.1. □

Remark 5.2. The integrability condition (65) is explicit and verifiable since the density of μ_0 is given in advance. If μ_0 is a Gaussian measure (e.g., $V(x)=c\,|x|^2$ for some constant $c>0$) and $Z:\mathbb{R}^d\to\mathbb{R}^d$ is Hölder continuous with the Hölder exponent $\alpha\in(0,1)$, then (65) holds definitely for any $\kappa>0$. Moreover, the linear growth of Z imposed in Lemma 2.4 is an essential ingredient, whereas the integrability condition (65) might allow Z to be non-globally Lipschitz. Last but not least, Z might be singular at a certain setup, e.g., $Z(x) = (\log \frac{1}{|x|^\alpha})\mathbf{1}_{\{|x|\le 1\}} + x\mathbf{1}_{\{|x|>1\}}$, $x \in \mathbb{R}$, for some $\alpha \in (0,1)$.

Via the dimension-free Harnack inequality (see e.g. [36]), we can establish the following exponential integrability under an integrability condition, which is an essential ingredient in analyzing weak convergence.

Lemma 5.3. *Assume that (***C1***) and (65) hold. Then,*

$$\mathbb{E}e^{\lambda\int_0^T|\int_{-\tau}^0 Z(Y^\xi(t+\theta))\rho(\mathrm{d}\theta)|^2\mathrm{d}t} < \infty \tag{71}$$

and

$$\mathbb{E}\,e^{\lambda\int_0^T|h_4^\xi(t)|^2\mathrm{d}t} < \infty \tag{72}$$

whenever $\lambda, T > 0$ such that

$$\lambda < \frac{\kappa}{2(2\vee d)\|\sigma^{-1}\|_{\mathrm{op}}^2T^2} \wedge \frac{e^{-(1+\beta T)}}{32\,\|\sigma\|_{\mathrm{op}}^2\|\sigma^{-1}\|_{\mathrm{op}}^2L_0^2T^2}.$$

Proof. By Hölder's inequality and Jensen's inequality, it follows that

$$\begin{aligned}
&\mathbb{E}e^{\lambda\int_0^T|\int_{-\tau}^0 Z(Y^\xi(t+\theta))\rho(\mathrm{d}\theta)|^2\mathrm{d}t}\\
&\le \mathbb{E}e^{\lambda\int_0^T\int_{-\tau}^0|Z(Y^\xi(t+\theta))|^2\rho(\mathrm{d}\theta)\mathrm{d}t}\\
&\le \frac{1}{T}\int_0^T\int_{-\tau}^0 \mathbb{E}e^{\lambda T|Z(Y^\xi(t+\theta))|^2}\rho(\mathrm{d}\theta)\mathrm{d}t\\
&\le \frac{1}{T}\Big\{\int_{-\tau}^0 e^{\lambda T|Z(\xi(\theta))|^2}\mathrm{d}\theta + \int_0^T \mathbb{E}e^{\lambda T|Z(Y^{\xi(0)}(t))|^2}\mathrm{d}t\Big\}.
\end{aligned} \tag{73}$$

If for any $\gamma > 0$ and $p > (1\vee d/2)$ with $p\gamma < \kappa$ there exists a continuous positive function $x \mapsto \Lambda_p(x)$ such that

$$\mathbb{E}\,e^{\gamma|Z(X^x(t))|^2} \le \Lambda_p(x)\big(1-e^{-L_0t}\big)^{-d/2p}\big(e^{\kappa|Z(\cdot)|^2}\big)^{1/p}, \tag{74}$$

then (71) holds true due to the facts that $1-e^{-L_0t} \sim L_0t$ as $t\to 0$ and

$$\int_0^t s^{-d/2p}\mathrm{d}s < \infty \quad \text{for} \quad p > \frac{d}{2}.$$

In what follows, it remains to verify that (74) holds. According to [36, Theorem 1.1], the following dimension-free Harnack inequality

$$(P_tf(x))^p \le P_tf^p(y)\exp\left(\frac{pL_0|x-y|^2}{2(p-1)(1-e^{-L_0t})}\right) \tag{75}$$

holds for any $x, y \in \mathbb{R}^d$, $f \in \mathscr{B}_b(\mathbb{R}^d)$, and $p > 1$. For any $n, \gamma > 0$, and $p > (1\vee d/2)$ with $p\gamma < \kappa$, applying the Harnack inequality (75) to the function $\mathbb{R}^d \ni x \mapsto e^{\gamma|Z(x)|^2}\wedge n \in \mathscr{B}_b(\mathbb{R}^d)$ yields that

$$\exp\left(-\frac{pL_0|x-y|^2}{2(p-1)(1-e^{-L_0t})}\right)\Big(\mathbb{E}\Big(e^{\gamma|Z(Y^x(t))|^2}\wedge n\Big)\Big)^p \le \mathbb{E}\Big(e^{p\gamma|Z(Y^y(t))|^2}\wedge n^p\Big).$$

Thereby, integrating w.r.t. $\mu_0(\mathrm{d}y)$ on both sides and taking the invariance of μ_0 and (65) into consideration leads to

$$\begin{aligned}
&\exp\Big(-\frac{pL_0}{2(p-1)}\Big)\int_{|x-y|^2\le 1-e^{-L_0t}}\mu_0(\mathrm{d}y)\Big(\mathbb{E}e^{\gamma|Z(Y^x(t))|^2}\wedge n\Big)^p\\
&\le \int_{\mathbb{R}^d}\mathbb{E}\Big(e^{p\gamma|Z(Y^y(t))|^2}\wedge n^p\Big)\mu_0(\mathrm{d}y)
\end{aligned}$$

$$\leq \mu_0\Big(\mathrm{e}^{p\gamma|Z(\cdot)|^2}\wedge n^p\Big)\leq \mu_0\big(\mathrm{e}^{\kappa|Z(\cdot)|^2}\big)<\infty,\ x\in\mathbb{R}^d,\ p\gamma<\kappa.$$

So, by the dominated convergence theorem, we arrive at

$$\begin{aligned}&\Big(\int_{|x-y|^2\leq 1-\mathrm{e}^{-L_0t}}\mu_0(\mathrm{d}y)\Big)^{1/p}\mathbb{E}\mathrm{e}^{\gamma|Z(Y^x(t))|^2}\\&\leq\Big(\mu_0(\mathrm{e}^{\kappa|Z(\cdot)|^2})\Big)^{1/p}\exp\Big(\frac{L_0}{2(p-1)}\Big).\end{aligned}\tag{76}$$

Next, from $\mu_0(\mathrm{d}y)=C_V\mathrm{e}^{-V(y)}\mathrm{d}y$ and Taylor's expansion, we deduce that

$$\begin{aligned}\int_{|x-y|^2\leq 1-\mathrm{e}^{-L_0t}}\mu_0(\mathrm{d}y)&=C_V\int_{|x-y|^2\leq 1-\mathrm{e}^{-L_0t}}\mathrm{e}^{-V(y)}\mathrm{d}y\\&\geq C_V\mathrm{e}^{-V(x)}\int_{|z|^2\leq 1-\mathrm{e}^{-L_0t}}\mathrm{e}^{-\int_0^1|\nabla V(x+\theta z)|\cdot|z|\mathrm{d}\theta}\mathrm{d}z\\&\geq C_V\mathrm{e}^{-V(x)}\inf_{|y|\leq 1+|x|}\mathrm{e}^{-|\nabla V|(y)}\int_{|z|^2\leq 1-\mathrm{e}^{-L_0t}}\mathrm{e}^{-|z|}\mathrm{d}z\\&\geq\frac{C_V\pi^{d/2}}{\Gamma(1+d/2)}\mathrm{e}^{-(1+V(x))}\\&\quad\times\inf_{|y|\leq 1+|x|}\mathrm{e}^{-|\nabla V|(y)}\big(1-\mathrm{e}^{-L_0t}\big)^{d/2},\end{aligned}\tag{77}$$

where $\Gamma(\cdot)$ is the Gamma function. Whence, inserting (77) back into (76) gives (74).

A direct calculation shows from (**C1**) and Hölder's inequality that

$$\big|h_4^\xi(t)\big|^2\leq 2\big\|\sigma^{-1}\big\|_{\mathrm{HS}}^2\Big\{2L_0^2\big(\big|Y^\xi(t)\big|^2+\big|Y^\xi(t_\delta)\big|^2\big)+\int_{-\tau}^0\big|Z(\widehat{Y}_t^\xi(\theta))\big|^2\rho(\mathrm{d}\theta)\Big\}.$$

Thus, Hölder's inequality implies that

$$\begin{aligned}\mathbb{E}\mathrm{e}^{\lambda\int_0^T|h_4^\xi(t)|^2\mathrm{d}t}&\leq\Big(\mathbb{E}\mathrm{e}^{16\lambda L_0^2\|\sigma^{-1}\|_{\mathrm{HS}}^2\int_0^T\|Y_t^\xi\|_\infty^2\mathrm{d}t}\Big)^{1/2}\\&\quad\times\Big(\mathbb{E}\mathrm{e}^{4\lambda\|\sigma^{-1}\|_{\mathrm{HS}}^2\int_0^T\int_{-\tau}^0|Z(\widehat{Y}_t^\xi(\theta))|^2\rho(\mathrm{d}\theta)\mathrm{d}t}\Big)^{1/2}\\&=:\sqrt{I_1(T)}\times\sqrt{I_2(T)}.\end{aligned}$$

On the one hand, in view of (11), it holds that

$$I_1(T)<\infty\ \text{ if }\ \lambda<\frac{\mathrm{e}^{-(1+\beta T)}}{32\,\|\sigma\|_{\mathrm{op}}^2\|\sigma^{-1}\|_{\mathrm{op}}^2L_0^2T^2}.\tag{78}$$

On the other hand, Hölder's inequality and Jensen's inequality show that, for any $\lambda>0$,

$$\begin{aligned}&\mathbb{E}\mathrm{e}^{\lambda\int_0^T\int_{-\tau}^0|Z(\widehat{Y}_t^\xi(\theta))|^2\rho(\mathrm{d}\theta)\mathrm{d}t}\\&\leq\frac{1}{T}\int_{-\tau}^0\int_0^T\mathbb{E}\mathrm{e}^{\lambda T|Z(\widehat{Y}_t^\xi(\theta))|^2}\mathrm{d}t\rho(\mathrm{d}\theta)\\&=\frac{1}{T}\int_{-\tau}^0\int_0^T\mathbb{E}\Big\{\mathrm{e}^{\lambda T|Z(Y^\xi(t+\theta))|^2}\mathbf{1}_{\{t+\theta\leq t_\delta\}}+\mathrm{e}^{\lambda T|Z(Y^\xi(t_\delta))|^2}\mathbf{1}_{\{t+\theta>t_\delta\}}\Big\}\mathrm{d}t\rho(\mathrm{d}\theta)\\&\leq\frac{1}{T}\Big\{\int_{-\tau}^0\mathrm{e}^{\lambda T|Z(\xi(\theta))|^2}\mathrm{d}\theta\end{aligned}$$

$$+ \mathrm{e}^{\lambda T|Z(\xi(0))|^2} + \int_0^T \mathbb{E}\mathrm{e}^{\lambda T|Z(Y^\xi(t))|^2}\mathrm{d}t + \int_\delta^T \mathbb{E}\,\mathrm{e}^{\lambda T|Z(Y^\xi(t_\delta))|^2}\mathrm{d}t\Big\}$$

so that, by virtue of (74),

$$I_2(T) < \infty \quad \text{if} \quad \lambda < \frac{\kappa}{2(2\vee d)\big\|\sigma^{-1}\big\|_{\mathrm{op}}^2 T^2}. \tag{79}$$

Thus, (72) follows immediately from (78) and (79). □

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