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Viscosity Solutions to HJB Equations with Hölder Continuous Coefficients

Jianrui Li[1] · **Jinghai Shao**[1] · **Hui Zhao**[2]

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Abstract
This work studies the optimal control problem for diffusion processes with Hölder continuous coefficients. Based on Meyer–Tanaka's formula, we establish an estimate on the local times of controlled diffusion processes, which enables us to provide a new estimate on the regularity of the value function. Then, the value function can be characterized as a unique viscosity solution of certain HJB equation.

Keywords Hölder continuous · Viscosity solutions · HJB equations · Comparison principle

Mathematics Subject Classification 93E20 · 60J60 · 35F50

1 Introduction

In this work we are interested in establishing the existence and uniqueness of viscosity solutions to Hamilton–Jacobi–Bellman (HJB) equations associated with the optimal control problem for the stochastic processes with Hölder continuous coefficients. In applications, there are many mathematical models characterized by stochastic differential equations with merely Hölder continuous coefficients, which contain, for instance, the extensively applied models Cox–Ingersoll–Ross (CIR) process [4] and

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✉ Jianrui Li
jianruili@tju.edu.cn

Jinghai Shao
shaojh@tju.edu.cn

Hui Zhao
zhaohuimath@tju.edu.cn

1 Center for Applied Mathematics, Tianjin University, Tianjin, China

2 Nankai-Taikang College of Insurance and Actuarial Science, Nankai University, Tianjin 300350, China

CKLS process [3] in the research field of mathematical finance. Therefore, besides studying the optimal control problem for stochastic processes with regular coefficients, it is also necessary to study the control problem for stochastic processes with Hölder continuous coefficients. Nevertheless, as being pointed out in the work [19], there are some new difficulties appeared in solving the optimal control problem for stochastic processes with Hölder continuous coefficients.

A little precisely, in [19], we applied the transformation introduced by Flandoli-Gubinelli-Priola [8] to investigate the optimal control problem for stochastic processes (may be multidimensional) with Hölder continuous drifts and Lipschitz continuous diffusion coefficients. Furthermore, based on [27, Theorem 4.2] and the estimate of local time [22], we can deal with the optimal control problem for 1-dimensional stochastic processes with Hölder continuous drifts and diffusion coefficients simultaneously. However, [19] can only deal with the stochastic processes whose diffusion coefficients are Hölder continuous with exponent q satisfying $q > 3/4$ at most (corresponding to the case $p = 1$, $\ell > 1/2$ and $q = 1 + \ell$ in [19, Theorem 4.2]).

It is a theoretical challenging task and of great meaning in applications to extend the range of exponent q, which is our main purpose of this work. The crucial point is to establish the uniqueness of viscosity solutions to the HJB equations with irregular Hamiltonian. We can extend the range q to the case $q > 1/\sqrt{2}$, which is obtained based on the improvements of the estimate of local times for diffusion processes in [22] and the result in [27] about the continuous dependence on the initial values. The case $q = 1/2$ corresponds to the optimal control problem for the CIR process, which is also of great interesting. But, our method is infeasible to deal with the HJB equations corresponding to the case $q = 1/2$; see Remark 2.1 below for details. Our result is also useful to the study of the optimal portfolio problem for stochastic volatility models with general objective functions (cf. [9, 16, 17, 21, 25, 26] and references therein), which will be investigated in our future work.

The theory of viscosity solutions to HJB equations with regular coefficients has been well developed and extensively applied in various kinds of optimal control problems; see, for instance, the books [6, 10, 23] and references therein. There are also some studies on viscosity solutions to HJB equations with irregular coefficients. In this direction, Caffarelli et al. [1] developed the concept of L_p-viscosity solution to deal with the HJB equations with L_p-integrable coefficients. The existence of viscosity solutions has been established there. The works [7, 14, 15], etc. studied the comparison principle of L_p-viscosity solution under certain regular conditions. In [6, Section 5], Crandall–Ishii–Lions considered the HJB equations with the Hamiltonian satisfying certain Hölder continuous condition. The strategy in [6] to establish the uniqueness of viscosity solution depends both on the structure condition on the Hamiltonian and the modulus of the continuity of the viscosity solution. To this end, for the optimal control problem for stochastic processes with Hölder continuous coefficients, we have to improve the estimate of the local time at the point 0 to investigate the modulus of the continuity of the corresponding value function.

To establish the uniqueness of the viscosity solution is not a trivial task. Notice the work of Ishii [12], which showed by constructing an example that the uniqueness of viscosity solution may fail for the HJB equations with Hölder continuous coefficients. Besides the non-Lipschitz coefficients, the degeneration of the diffusion coefficient also has important influence on the uniqueness of viscosity solutions (cf. Crandall and Huan [5]).

This work is organized as follows. In Sect. 2, we introduce the framework of the optimal control problem studied in this work. The explicit conditions on the coefficients are given in this part. Then we present our main result on the existence and uniqueness of viscosity solution. The arguments of the main result is given in Sect. 3.

2 Framework

Consider the SDE

$$\mathrm{d}X_t = b(X_t, \theta_t)\mathrm{d}t + \sigma(X_t, \theta_t)\mathrm{d}W_t, \tag{2.1}$$

where (W_t) is 1-dimensional Brownian motion, $b : \mathbb{R} \times U \to \mathbb{R}$, $\sigma : \mathbb{R} \times U \to \mathbb{R}$ are measurable functions, U is a Borel measurable set in $\mathbb{R}^k$ for some $k \in \mathbb{N}$. In this work we only consider the finite horizon optimal control problem. So we fix a $T > 0$ throughout this work.

Assume the following conditions on the coefficients and the objective functions.

(A1) The drift $b(x, \theta) = b_1(x, \theta) + b_2(x, \theta)$, and there exists $C_1 > 0$, $\ell \in (0, 1)$ such that

$$|b_1(x, \theta) - b_1(y, \theta)| \le C_1|x - y|^{\ell}, \quad x \mapsto b_1(x, \theta) \text{ is nonincreasing},$$
$$|b_2(x, \theta) - b_2(y, \theta)| \le C_1|x - y|, \quad |b(x, \theta)| \le C_1(1 + |x|)$$

for all $x, y \in \mathbb{R}$, $\theta \in U$.

(A2) There exist $C_2 > 0$, $\gamma \in (\frac{1}{2}, 1]$ such that

$$|\sigma(x, \theta) - \sigma(y, \theta)| \le C_2|x - y|^{\gamma}, \quad |\sigma(x, \theta)|^2 \le C_2(1 + x^2)$$

for all $x, y \in \mathbb{R}, \theta \in U$.

(A3) Let $f : [0, T] \times \mathbb{R} \times U \to \mathbb{R}$, $g : \mathbb{R} \to \mathbb{R}$ be measurable functions satisfying that for some $p \in (0, 1]$ and $C_3 > 0$

$$|f(t, x, \theta) - f(s, y, \theta)| + |g(x) - g(y)| \le C_3\big(|t - s|^{\frac{p}{2}} + |x - y|^p\big)$$

for any $s, t \in [0, T], x, y \in \mathbb{R}, \theta \in U$.

Remark 2.1 Since we shall consider to maximize the objective functions over the set of admissible controls, we need to guarantee the finiteness of the value function and then to characterize it via certain HJB equation. There are examples in [16] to show the possibility of infiniteness of the value function in the study of portfolio problem with Heston's model. According to [19], conditions (A1)–(A3) can guarantee the finiteness of the value function; see Proposition 3.5 below.

Definition 2.2 (Admissible control) For $s \in [0, T)$, term $\Theta = (\Omega, \mathscr{F}, \mathbb{P}, \mathscr{F}_\cdot, W_\cdot, X_\cdot, \theta_\cdot)$ is called an admissible control, if

(i) $W_\cdot$ is a 1-dimensional Brownian motion adapted to the filtration $\{\mathscr{F}_t\}_{t \geq 0}$.
(ii) $\theta : [s, T] \times \Omega \to U$ is $\mathscr{F}_t$-progressively measurable such that for any initial value $X_s = x \in \mathbb{R}$ SDE (2.1) admits a weak solution $(X_t)_{t \in [s,T]}$.

The set of all admissible controls for a given $s \in [0, T)$ is denoted by Π_s.

The objective function is defined by

$$J(s, x; \Theta) = \mathbb{E}_x \Big[\int_s^T f(r, X_r, \theta_r) \mathrm{d}r + g(X_T) \Big] = \mathbb{E}\Big[\int_s^T f(r, X_r, \theta_r) \mathrm{d}r + g(X_T) \Big| X_s = x \Big].$$

Correspondingly, the value function is defined by

$$V(s, x) = \sup \big\{ J(s, x; \Theta); \Theta \in \Pi_s \big\}. \tag{2.2}$$

If it holds that $V(s, x) = J(s, x; \Theta^*)$ for some $\Theta^* \in \Pi_s$, then Θ^* is called an optimal control.

In order to provide readers with a more intuitive understanding, we present a concrete class of admissible controls as follows, which is a specific kind of feedback control.

Example 2.1 Consider the feedback control strategies in the form $\theta_t = F(X_t)$ satisfying $F : \mathbb{R} \to \mathbb{R}$ being Lipschitz continuous. Suppose that the coefficients b, σ are Lipschitz continuous in θ and Hölder continuous in x, then under this feedback strategy $b(x, \theta) = b(x, F(x))$ and $\sigma(x, \theta) = \sigma(x, F(x))$ are Hölder continuous, thus the controlled SDE (2.1) admits a weak solution for all $t \geq 0$.

Moreover, it is worth noticing that the Lipschitz continuity of $(x, \theta) \mapsto b(x, \theta)$ and $(x, \theta) \mapsto \sigma(x, \theta)$ together with the adaptness of $t \mapsto \theta_t$ and the integrability $\int_0^T \theta_r^2 \mathrm{d}r$ are not enough to ensure the existence of weak solution to the controlled SDE. To be precise, let us see the following example.

Example 2.2 Consider SDE (2.1) with the coefficients $\sigma(x, \theta) = 0$, $b(x, \theta) = \theta$. Clearly, b and σ are Lipschitz continuous in x, θ. Let $\theta_t = \mathrm{sgn}(X_t)$, where $\mathrm{sgn}(x) = 1$ if $x \geq 0$ and $\mathrm{sgn}(x) = -1$ if $x < 0$. Then θ_t is bounded, which yields that θ_t is $\mathscr{F}_t$-adapted and $\int_0^T \theta_r^2 \mathrm{d}r < \infty$. In this case, the SDE (2.1) turns into

$$\mathrm{d}X_t = -\mathrm{sgn}(X_t)\mathrm{d}t, \quad X_0 = 0. \tag{2.3}$$

However, according to [2, Example 1.16], SDE (2.3) does not have a weak solution.

Next, we provide another example on the optimal portfolio problem with stochastic volatility models, which are considerably popular in financial applications (cf. e.g. [18] and [24]).

Consider a money market account (Y_t) and a stock (S_t) governed by the SDEs:

$$\begin{aligned} \mathrm{d}Y_t &= rY_t\mathrm{d}t, \\ \mathrm{d}S_t &= S_t(r+\lambda_t)\mathrm{d}t + S_t\lambda_t^{\gamma}\mathrm{d}W_t, \end{aligned}$$

in which volatility (λ_t) is characterized by the CKLS model

$$\mathrm{d}\lambda_t = \kappa(\theta-\lambda_t)\mathrm{d}t + \sigma\lambda_t^{\gamma}\mathrm{d}\widetilde{W}_t \tag{2.4}$$

with $\gamma \in (\frac{1}{2}, 1]$, where $r, \gamma, \kappa, \theta, \sigma$ are all positive constants. (W_t) and $(\widetilde{W}_t)$ are two 1-dimensional standard Brownian motions with a correlation coefficient $\rho \in [-1, 1]$. A trading strategy is a stochastic processes denoted by π_t. The dollar amount invested in the risk-free asset at time t is $X_t - \pi_t$. The wealth process for the problem then follows

$$\mathrm{d}X_t = (X_t-\pi_t)\frac{\mathrm{d}Y_t}{Y_t} + \pi_t\frac{\mathrm{d}S_t}{S_t} = (rX_t+\pi_t\lambda_t)\mathrm{d}t + \pi_t\lambda_t^{\gamma}\mathrm{d}W_t, \tag{2.5}$$

with the initial wealth $X_s = x \geq 0$ and $s \in [0, T)$. Obviously, the diffusion coefficient of (X_t) is γ-Hölder continuous and (A1), (A2) hold. In particular, taking $\pi_t = \pi \in U$ for $t \in [0, T]$, SDEs (2.4), (2.5) admit a unique strong solution (cf. [11, Chapter IV, Theorem 3.2]), which can be extended as in [20, Theorem 4.7] to SDEs with Markovian regime-switching. Then, for any filtered probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, \mathbb{P})$ and any $\mathscr{F}_t$-adapted Brownian motion $B.$, $\Theta = (\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, \mathbb{P}, B., \pi.)$ belongs to Π_s.

The infinitesimal generator $\mathscr{L}_\theta$ of the controlled process (X_t) is given by

$$\mathscr{L}_\theta h(t,x) = b(x,\theta)\partial_x h(t,x) + \frac{1}{2}\sigma(x,\theta)^2\partial^2_{xx}h(t,x)$$

for $h \in C^{1,2,2}([0, T]\times\mathbb{R}\times\mathbb{R})$. Consider the following HJB equation:

$$\begin{cases} -\partial_t v(t,x) - \sup_{\theta\in U}\left\{\mathscr{L}_\theta v(t,x) + f(t,x,\theta)\right\} = 0, & (t,x)\in[0,T)\times\mathbb{R}, \\ v(T,x) = g(x), & x\in\mathbb{R}. \end{cases} \tag{2.6}$$

Definition 2.3 Let $v : [0, T]\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$ be a continuous function.

(i) v is called a viscosity subsolution to (2.6) if $v(T, x) = g(x)$ and

$$-\partial_t v(t,x) - \sup_{\theta\in U}\left\{\mathscr{L}_\theta v(t,x) + f(t,x,\theta)\right\} \leq 0$$

for all $\varphi \in C^{1,2,2}([0, T]\times\mathbb{R}\times\mathbb{R})$ and all $(s_0, x_0, y_0) \in [0, T)\times\mathbb{R}\times\mathbb{R}$ being a maximum point of $v - \varphi$.

(ii) v is called a viscosity supersolution to (2.6) if $v(T, x) = g(x)$ and

$$-\partial_t v(t,x) - \sup_{\theta\in U}\left\{\mathscr{L}_\theta v(t,x) + f(t,x,\theta)\right\} \geq 0$$

for all $\varphi \in C^{1,2,2}([0,T]\times\mathbb{R}\times\mathbb{R})$ and all $(s_0,x_0,y_0)\in[0,T)\times\mathbb{R}\times\mathbb{R}$ being a minimum point of $v-\varphi$.

(iii) If v is both a viscosity subsolution and viscosity supersolution to (2.6), v is called a viscosity solution to (2.6).

Via the dynamic programming principle, we can show the value function V is a viscosity solution to (2.6). After proving the regularity of V, it is standard to establish the uniqueness of the viscosity solution (cf. [6, Section 5.A] and [7, 13, 14]). We include the full proof for the sake of completeness and readers' convenience.

Theorem 2.4 (Existence and uniqueness of viscosity solution) *Assume that* (A1)–(A3) *hold, and further that* $(\gamma+\frac{2\gamma-1}{2\gamma}p)\mathbf{1}_{\{2\gamma-1\le\ell\}}+(\ell+\frac{2\ell}{1+\ell}p)\mathbf{1}_{\{2\gamma-1>\ell\}}>1$. *Then the value function V defined in* (2.2) *is a unique viscosity solution to the HJB equation* (2.6) *satisfying the regularity properties*

$$|v(t,x)-v(s,y)|\le C\Big(|x-y|^{\frac{2((2\gamma-1)\wedge\ell)}{1+(2\gamma-1)\wedge l}p}+(1+x^p\vee y^p)|t-s|^{\frac{p}{2}}\Big), \tag{2.7}$$

$$|v(t,x)|\le C(1+x),\quad \forall s,t\in[0,T),x,y\in\mathbb{R},|x-y|<1 \tag{2.8}$$

for some constant $C>0$.

The argument of this theorem is given in next section after some necessary preparations. As a simple application of Theorem 2.4, when the drift and the objective functions are regular, the uniqueness of viscosity solution to HJB equation (2.6) holds for γ-Hölder diffusion coefficient with $\gamma>1/\sqrt{2}$.

Corollary 2.5 *Assume that* (A1)–(A3) *hold with $b_1=0$ and $p=1$, which means that the drift b and the objective functions f, g are Lipschitz continuous. Suppose $\gamma>1/\sqrt{2}$. Then the value function V defined in* (2.2) *is a unique viscosity solution to the HJB equation* (2.6) *satisfying the properties* (2.7) *and* (2.8).

3 Argument of Theorem 2.4

To study the regularity property of the value function V, we need to investigate the continuous dependence on the initial value of the solution (X_t) to SDE (2.1), for which much attention is paid to the Hölder continuous coefficient $\sigma(\cdot)$. Without the Lipschitz condition, we shall apply Meyer–Tanaka's formula for 1-dimensional diffusion processes to estimate $\mathbb{E}|X_t^{x_1}-X_t^{x_2}|$, where (X_t^x) denotes the solution to SDE (2.1) with initial value $X_s^x=x$. The key point is the estimate on the local time of the process $(X_t^{x_1}-X_t^{x_2})$ at the point 0. Based on an inequality on local time given in [22], we improve the estimate given in [27], which was used in our work [19]. This leads us to improving our result in [19] so that now we can deal with γ-Hölder continuous coefficients with $\gamma>1/\sqrt{2}$.

Lemma 3.1 *Let $(Z_t)_{t\ge s\ge 0}$ be a continuous semimartingale on $\mathbb{R}_+$ with $Z_s=z>0$. For any given $\varepsilon\in(0,z)$ define a double sequence of stopping times (α_n,β_n) by*

$$\alpha_1=s,\ \beta_1=\inf\{t>\alpha_1;\ Z_t=\varepsilon\},$$

$$\alpha_n = \inf\{t > \beta_{n-1};\, Z_t = 0\},\ \ \beta_n = \inf\{t > \alpha_n;\, Z_t = \varepsilon\},\ \ n \geq 2.$$

Let $\theta_r(Z) = \mathbf{1}_{\{\alpha_1 < r \leq \beta_1, Z_r \geq \varepsilon\}} + \sum_{n=2}^{\infty} \mathbf{1}_{\{\alpha_n < r \leq \beta_n, 0 < Z_r < \varepsilon\}}$, $U_t(Z) = \sup\{n \geq 1;\, \beta_n < t\}$, $n(t) = t \wedge \alpha_{U_t(Z)+1}$. *Define the local time of* $(Z_t)_{t \geq s}$ *at the point* 0 *by*

$$\ell^0_{s,t}(Z) = \lim_{\delta \to 0} \frac{1}{2\delta} \int_s^t \mathbf{1}_{\{-\delta < Z_r < \delta\}} d\langle Z \rangle(r), \quad t > s.$$

Then it holds that for any $F \in C^2(\mathbb{R}_+)$

$$\begin{aligned} &\ell^0_{s,t}(Z)\big(F(\varepsilon) - F(0) - \varepsilon F'(0)\big) \\ &= \big\{2\varepsilon F(z) - 2zF(\varepsilon) + 2F(0)(z-\varepsilon)\big\}\mathbf{1}_{\{U_t(Z) \geq 1\}} \\ &\quad + 2(F(\varepsilon) - F(0))\big(Z_t^+ - Z_{n(t)}^+\big) - 2\varepsilon\big(F(Z_t^+) - F(Z_{n(t)}^+)\big) \\ &\quad - 2\int_s^t \big(F(\varepsilon) - F(0) - \varepsilon F'(Z_r^+)\big)\theta_r(Z)dZ_r + \int_s^t \varepsilon F''(Z_r^+)\theta_r(Z)d\langle Z \rangle(r). \end{aligned} \tag{3.1}$$

Proof This lemma can be proved along the line of [22, Lemma 4.1] with a generalization to removing the condition $F(0) = 0$. Notice that the initial value $Z_0 = z$ equals to 0 or not has significant impact on the expression of $\theta_r(Z)$. In view this observation, the corresponding expression $\zeta_Z(s)$ in [27, Theorem 4.1] needs some modifications. □

Based on Lemma 3.1, we can derive the following estimate of the local time $\ell^0_{s,t}(Z)$.

Lemma 3.2 *Adopting the notations and conditions of Lemma* 3.1, *for* $\varepsilon \in (0, z)$, *it holds*

$$\begin{aligned} \ell^0_{s,t}(Z) \leq &\left\{\frac{2z(z-\varepsilon)}{z+\varepsilon}\right\}\mathbf{1}_{\{U_t(Z) \geq 1\}} + 2\varepsilon \\ &+ 2\int_s^t \Big(1 - \frac{2\varepsilon^2}{(\varepsilon + Z_r^+)^2}\Big)\theta_r(Z)dZ_r + \int_s^t \frac{4\varepsilon^2}{(\varepsilon + Z_r^+)^3}\theta_r(Z)d\langle Z \rangle(r). \end{aligned} \tag{3.2}$$

Proof Due to the definition of $U_t(Z)$, $U_t(Z) = 0$ if $t \in [s, \beta_1]$, and $U_t(Z) = n$ if $t \in (\beta_n, \beta_{n+1}]$. Here, $\sup \emptyset := 0$ as convention and we put $\beta_0 = s$. Hence, $\beta_{U_t(Z)} \leq t \leq \beta_{U_t(Z)+1}$ for all $t \geq s$. Furthermore,

if $\alpha_1 \leq t \leq \beta_1$, then $n(t) = s$ and $Z_t \geq \varepsilon$; if $\beta_{U_t(Z)} < t \leq \alpha_{U_t(Z)+1}$, then$n(t) = t$; if $\alpha_{U_t(Z)+1} \leq t \leq \beta_{U_t(Z)+1}$, then $n(t) = \alpha_{U_t(Z)+1}$ and $Z_{n(t)} = 0,\ Z_t \leq \varepsilon$.

Therefore, $Z_t^+ - Z_{n(t)}^+ \leq \varepsilon$ if $U_t(Z) \geq 1$. Besides, $\ell^0_{s,t}(Z) = 0$ for $t > s$ satisfying $U_t(Z) = 0$, since the process (Z_t) cannot hit 0 before it hits ε.

Let us take $F(x) = \frac{1}{\varepsilon + x}$ for $x \in [0, \infty)$. Then, it is clear that

$$F \in C^2(\mathbb{R}_+), \quad F'(x) = \frac{-1}{(\varepsilon + x)^2}, \quad F''(x) = \frac{2}{(\varepsilon + x)^3}.$$

Applying (3.1) to this function F, invoking the observation that $\ell^0_{s,t}(Z) = \ell^0_{s,t}(Z)\mathbf{1}_{\{U_t(Z)\geq 1\}}$, we can get the estimate (3.2) by direct calculation. □

Lemma 3.3 *Denote by* $(X^x_t)_{t\geq s}$ *the solution to SDE* (2.1) *with initial value* $X^x_s = x \in \mathbb{R}$. *Suppose* (A1) *and* (A2) *hold. Then there exists a positive constant* K_1 *depending on* T *such that*

$$\mathbb{E}|X^{x_1}_t - X^{x_2}_t| \leq K_1|x_1 - x_2|^{\frac{2((2\gamma-1)\wedge\ell)}{1+(2\gamma-1)\wedge l}}, \quad x_1, x_2 \in \mathbb{R},\ |x_1 - x_2| < 1. \tag{3.3}$$

Proof Without loss of generality, assume $x_1 > x_2 \geq 0$ satisfying $x_1 - x_2 < 1$. According to the comparison theorem for 1-dimensional diffusion process (cf. [11, Theorem VI−1.1, p. 437]), it holds $\mathbb{P}(X^{x_1}_t \geq X^{x_2}_t, t \geq 0) = 1$ as $x_1 > x_2$. Hence, due to (A1), we deduce from the nonincreasing property of $x \mapsto b_1(x, \theta)$ that

$$\mathrm{sgn}(X^{x_1}_r - X^{x_2}_r)(b_1(X^{x_1}_r, \theta_r) - b_1(X^{x_2}_r, \theta_r)) \leq 0, \quad \text{a.s.}$$

By Meyer–Tanaka's formula,

$$\begin{aligned}|X^{x_1}_t - X^{x_2}_t| &= x_1 - x_2 + \int_s^t \mathrm{sgn}(X^{x_1}_r - X^{x_2}_r)\mathrm{d}(X^{x_1}_r - X^{x_2}_r) + \ell^0_{s,t}(X^{x_1} - X^{x_2})\\ &\leq x_1 - x_2 + \int_s^t (b_2(X^{x_1}_r, \theta_r) - b_2(X^{x_2}_r, \theta_r))\mathrm{d}r + \ell^0_{s,t}(X^{x_1} - X^{x_2})\\ &\quad + \int_s^t (\sigma(X^{x_1}_r, \theta_r) - \sigma(X^{x_2}_r, \theta_r))\mathrm{d}W_r.\end{aligned}$$

Due to (A1),

$$\mathbb{E}|X^{x_1}_t - X^{x_2}_t| \leq x_1 - x_2 + C_1 \int_s^t \mathbb{E}|X^{x_1}_r - X^{x_2}_r|\mathrm{d}r + \mathbb{E}\,\ell^0_{s,t}(X^{x_1} - X^{x_2}). \tag{3.4}$$

We shall apply Lemma 3.2 to the process $Z_t := X^{x_1}_t - X^{x_2}_t$ with initial value $z := x_1 - x_2$ and $\varepsilon = z^p$ for some $p > 1$ to be determined later. Due to (3.2) and conditions (A1), (A2), we have

$$\begin{aligned}\mathbb{E}\,\ell^0_{s,t}(X^{x_1} - X^{x_2}) &\leq \frac{2z^2 + 2z^{2p}}{z + z^p} + 2C_1Tz^{lp} + 4C_2^2Tz^{(2\gamma-1)p} + 2C_1\int_s^t \mathbb{E}|Z_r|\mathrm{d}r\\ &\leq z^{2-p} + z^p + 2C_1Tz^{lp} + 4C_2^2Tz^{(2\gamma-1)p} + 2C_1\int_s^t \mathbb{E}|Z_r|\mathrm{d}r.\end{aligned} \tag{3.5}$$

Noting that $z < 1$, we take $p > 1$ such that $2 - p = \big((2\gamma - 1) \wedge l\big)p$, that is, $p = \frac{2}{1+(2\gamma-1)\wedge l}$, then there is a constant $C > 0$ such that

$$\mathbb{E}\,\ell^0_{s,t}(X^{x_1} - X^{x_2}) \leq Cz^{\frac{2(\tilde{\alpha}\wedge l)}{1+\tilde{\alpha}\wedge l}} + 2C_1\int_s^t \mathbb{E}|Z_r|\mathrm{d}r,$$

where $\tilde{\alpha} = 2\gamma - 1$. Inserting this estimate into (3.4), as $z \in (0, 1)$ and $\frac{2(\tilde{\alpha}\wedge l)}{1+\tilde{\alpha}\wedge l} \le 1$, there is some constant $\tilde{C} > 0$ such that

$$\mathbb{E}|X_t^{x_1} - X_t^{x_2}| \le \tilde{C}(x_1 - x_2)^{\frac{2(\tilde{\alpha}\wedge l)}{1+\tilde{\alpha}\wedge l}} + 3C_1 \int_s^t \mathbb{E}|X_r^{x_1} - X_r^{x_2}|\mathrm{d}r,$$

which yields that desired estimate (3.3) immediately by Gronwall's inequality. □

Lemma 3.4 *Suppose* (A1) *and* (A2) *hold. Let* (X_t^x) *be a solution to SDE* (2.1) *with initial value* $X_s^x = x \in \mathbb{R}$. *Then there exists* $K_2 > 0$ *such that*

$$\mathbb{E}|X_{t_2}^x - X_{t_1}^x| \le K_2|t_2 - t_1|^{\frac{1}{2}}, \quad t_1, t_2 \in [s, T].$$

Proof It is clear that (A1) and (A2) imply that b and σ satisfy the linear growth condition. Suppose $t_1 < t_2$. By Burkholder–Gundy–Davis's inequality, we have

$$\begin{aligned}\mathbb{E}|X_{t_2}^x - X_{t_1}^x| &\le \mathbb{E}\Big|\int_{t_1}^{t_2} b(X_r^x, \theta_r)\mathrm{d}r\Big| + \mathbb{E}\Big|\int_{t_1}^{t_2} \sigma(X_r^x, \theta_r)\mathrm{d}W_r\Big| \\ &\le C\int_{t_1}^{t_2} \mathbb{E}(1 + |X_r^x|)\mathrm{d}r + C\Big(\int_{t_1}^{t_2} \mathbb{E}(1 + |X_r^x|^2)\mathrm{d}r\Big)^{1/2} \le C|t_1 - t_2|^{1/2}\end{aligned}$$

for some constant $C > 0$ depending on T. □

Proposition 3.5 *Suppose* (A1)–(A3) *hold. Then the value function* V *given in* (2.2) *satisfies that there exists* $K_3 > 0$ *such that*

$$|V(t, x) - V(t', y)| \le K_3\big((1 + x^p \vee y^p)|t - t'|^{\frac{p}{2}} + |x - y|^{\frac{2((2\gamma-1)\wedge\ell)}{1+(2\gamma-1)\wedge l}p}\big), \tag{3.6}$$

$$|V(t, x)| \le K_3(1 + x^p), \quad t, t' \in [s, T],\ x, y \in \mathbb{R},\ |x - y| \le 1, \tag{3.7}$$

where $p \in (0, 1]$ *is the Hölder exponent of the objective functions* f *and* g *given in* (A3).

Proof By virtue of Lemma 3.4, this proposition can be proved completely along the line of [19, Proposition 2.5]. Hence, we omit the details to save space. □

Proposition 3.6 (Dynamic programming principle) *Suppose* (A1)–(A3) *hold, then the value function given in* (2.2) *satisfies the dynamic programming principle, that is, for* $s \in [0, T)$, $x \in \mathbb{R}$,

$$V(s, x) = \sup_{\Theta\in\Pi_s} \Big\{\mathbb{E}\Big[\int_s^\tau f(r, X_r^x, \theta_r)dr + V(\tau, X_\tau^x)\Big| X_s^x = x\Big]\Big\}$$

for every $\mathscr{F}_t$*-stopping time* τ *satisfying* $s \le \tau \le T$.

The argument of this proposition is quite standard; see [6, 10] or [19, Proposition 3.1] for its proof.

Proof of Theorem 2.4 *Existence of viscosity solution* Based on the dynamic programming principle, the assertion that the value function $V(s,x)$ satisfies the HJB equation (2.6) as a viscosity solution can be proved by the method of [19, Theorem 3.2]. In fact, our current case is even simpler than that in [19] as we don't need to deal with the Markov chain (Λ_t) any more.
Uniqueness of viscosity solution The essential part of Theorem 2.4 is to prove the uniqueness of viscosity solution of HJB equation (2.6) under the Hölder continuous condition.

Assume V is a viscosity supersolution and $\widetilde{V}$ a viscosity subsolution to the HJB equation (2.6), and both V and $\widetilde{V}$ satisfy the properties (3.6) and (3.7). It is easy to see that there is $K > 0$ such that for every $\eta > 0$, the function $u_\eta(t,x) = \widetilde{V}(t,x) - \eta \mathrm{e}^{K(T-t)}(1+|x|^2)$ is a viscosity subsolution of (2.6) and a function $v_\eta(t,x) = V(t,x) + \eta \mathrm{e}^{K(T-t)}(1+|x|^2)$ is a viscosity supersolution of (2.6). Then

$$\lim_{|x|\vee|y|\to\infty} (u_\eta(t,x) - v_\eta(t,x)) = -\infty \quad \text{uniformly for } t \in (0,T].$$

To show that $\widetilde{V} \le V$, it is enough to prove that $u_\eta \le v_\eta$ for every $\eta > 0$. Suppose there exists a point $(\bar{t},\bar{x}) \in [0,T) \times \mathbb{R}$ such that

$$u_\eta(\bar{t},\bar{x}) > v_\eta(\bar{t},\bar{x}). \tag{3.8}$$

If $\bar{t} = 0$, by the continuity of v_η and u_η, from (3.8) one can find $t' \in (0,T)$ such that

$$u_\eta(t',\bar{x}) > v_\eta(t',\bar{x}).$$

Hence, we may always choose $\bar{t} > 0$ such that (3.8) holds.

Introduce an auxiliary function on $[0,T] \times \mathbb{R} \times \mathbb{R}$

$$\Phi(t,x,y) := u_\eta(t,x) - v_\eta(t,y) - \frac{\lambda}{t} - \frac{1}{2\delta}|x-y|^2, \tag{3.9}$$

where $\lambda, \delta \in (0,1)$ are parameters. Since

$$\limsup_{|x|\vee|y|\to\infty} \Phi(t,x,y) = -\infty, \quad \limsup_{t\to 0} \Phi(t,x,y) = -\infty,$$

there exists a maximum point $(t_0,x_0,y_0) \in (0,T] \times \mathbb{R} \times \mathbb{R}$, depending on the parameters $\eta,\ \beta,\ \lambda,\ \delta$, such that

$$\Phi(t_0,x_0,y_0) = \sup\left\{\Phi(t,x,y);\ (t,x,y) \in (0,T] \times \mathbb{R} \times \mathbb{R}\right\}. \tag{3.10}$$

It follows from (3.10) that

$$\Phi(t_0,x_0,y_0) \ge \Phi(T,0,0) = -\frac{\lambda}{T} - 2\eta.$$

Combining this with the growth condition (3.7), there exists $K_4 > 0$ such that

$$\begin{aligned} &\frac{1}{2\delta}(x_0 - y_0)^2 + \frac{\lambda}{t_0} - 2\eta - \frac{\lambda}{T} \\ &\le \widetilde{V}(t_0, x_0) - V(t_0, y_0) - \eta e^{K(T-t)}(2 + |x_0|^2 + |y_0|^2) \\ &\le K_4(1 + x_0 + y_0) - \eta e^{K(T-t)}(2 + |x_0|^2 + |y_0|^2). \end{aligned} \tag{3.11}$$

This yields that

$$\eta(x_0^2 + y_0^2) \le K_4(1 + x_0 + y_0)$$

and further

$$x_0 + y_0 \le R \quad \text{for some } R \text{ independent of } \lambda, \delta. \tag{3.12}$$

Moreover, (3.11) and (3.12) yield that

$$|x_0 - y_0|^2 \le 2K_4(1 + R)\delta \longrightarrow 0, \quad \text{as } \delta \to 0. \tag{3.13}$$

Using (3.10) again, we get

$$2\Phi(t_0, x_0, y_0) \ge \Phi(t_0, x_0, x_0) + \Phi(t_0, y_0, y_0),$$

which yields that

$$\begin{aligned} \frac{1}{2\delta}|x_0 - y_0|^2 &\le |u_\eta(t_0, x_0) - u_\eta(t_0, y_0)| + |v_\eta(t_0, x_0) - v_\eta(t_0, y_0)| \\ &\le C(R)\big(|x_0 - y_0|^p + |x_0 - y_0|^{\frac{2((2\gamma-1)\wedge\ell)}{1+(2\gamma-1)\wedge l}p}\big). \end{aligned} \tag{3.14}$$

Invoking (3.13), we get

$$\text{for any } \tilde{\gamma} > 2 - \frac{2((2\gamma - 1) \wedge \ell)}{1 + (2\gamma - 1) \wedge l}p, \quad \lim_{\delta \to 0} \frac{1}{\delta}|x_0 - y_0|^{\tilde{\gamma}} = 0. \tag{3.15}$$

Next, we shall prove this theorem in two different cases according to $t_0 = T$ or not. *Case 1*. For some sequence $(\eta, \lambda, \delta) \to 0$, the corresponding maximum points (t_0, x_0, y_0) satisfy $t_0 = T$. For every fixed $\eta \in (0, 1)$, the points (t_0, x_0, y_0) should locate in the bounded set $(0, T] \times B_0(R) \times B_0(R)$, where $B_0(R)$ denotes the ball in $\mathbb{R}$ centered at 0 with radius R. By (3.13), $|x_0 - y_0| \to 0$ as $\delta \to 0$, and there exists a subsequence of (t_0, x_0, y_0) with $t_0 = T$ converging to a limit point in the form $(T, \bar{x}_0, \bar{x}_0)$ as $\delta \to 0$. Thanks to (3.10),

$$\Phi(\bar{t}, \bar{x}, \bar{x}) \le \Phi(T, \bar{x}_0, \bar{x}_0),$$

which implies

$$u_\eta(\bar{t}, \bar{x}) - v_\eta(\bar{t}, \bar{x}) - \frac{\lambda}{\bar{t}} \le u_\eta(T, \bar{x}_0) - v_\eta(T, \bar{x}_0) = -2\eta(1 + |\bar{x}_0|^2).$$

Letting $\lambda \to 0$, this yields

$$u_\eta(\bar{t}, \bar{x}) - v_\eta(\bar{t}, \bar{x}) < 0,$$

which contradicts (3.8).

Case 2. For any $(\eta, \delta, \lambda) \in (0, 1)$, the corresponding maximum points (t_0, x_0, y_0) all satisfy $t_0 < T$. Define

$$\varphi(t, x, y) = \frac{1}{2\delta}|x - y|^2 + \frac{\lambda}{t}, \tag{3.16}$$

then $\varphi \in C^{1,2,2}((0, T) \times \mathbb{R} \times \mathbb{R})$, and

$$\Phi(t, x, y) = u_\eta(t, x) - v_\eta(t, y) - \varphi(t, x, y).$$

Since $\Phi(t, x, y)$ attains a maximum at (t_0, x_0, y_0), according to Crandall-Ishii's lemma (cf. [6, Theorem 8.3]), for any $\varepsilon > 0$ there exist $\tilde{q},\ \hat{q} \in \mathbb{R}$, A, $B \in \mathbb{R}$ such that

$$\begin{aligned}
&(\tilde{q}, \partial_x \varphi(t_0, x_0, y_0), A) \in \mathscr{P}^{2,+} u_\eta(t_0, x_0),\\
&(\hat{q}, -\partial_y \varphi(t_0, x_0, y_0), B) \in \mathscr{P}^{2,-} v_\eta(t_0, y_0),
\end{aligned}$$

satisfying

$$\begin{aligned}
&\tilde{q} - \hat{q} = \partial_t \varphi(t_0, x_0, y_0) = -\frac{\lambda}{t_0^2},\\
&-(\frac{1}{\varepsilon} + \|X\|)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \le \begin{pmatrix} A & 0 \\ 0 & -B \end{pmatrix} \le \frac{1}{\delta}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} + \frac{\varepsilon}{\delta^2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}^2.
\end{aligned}$$

Here $\mathscr{P}^{2,+} v_\eta$ and $\mathscr{P}^{2,-} v_\eta$ denote respectively the parabolic superjets and subjets of v_η. Taking $\varepsilon = \delta$, we obtain that

$$\begin{aligned}
&\sigma(x_0, \theta)^2 A - \sigma(y_0, \theta)^2 B\\
&= \operatorname{tr}\left(\begin{pmatrix} Z(x_0)^2 & Z(x_0)Z(y_0) \\ Z(y_0)Z(x_0) & Z(y_0)^2 \end{pmatrix}\begin{pmatrix} A & 0 \\ 0 & -B \end{pmatrix}\right)\\
&\le \frac{3}{\delta}|Z(x_0) - Z(y_0)|^2,
\end{aligned} \tag{3.17}$$

where $Z(x_0) = \sigma(x_0, \theta)$, $Z(y_0) = \sigma(y_0, \theta)$. Since u_η, v_η are respectively viscosity subsolution and viscosity supersolution to (2.6), it holds

$$\begin{aligned} &-\tilde{q}-\mathcal{H}(t_0,x_0,\partial_x\varphi(t_0,x_0),A)\le 0,\\ &-\hat{q}-\mathcal{H}(t_0,y_0,-\partial_y\varphi(t_0,y_0),B)\ge 0, \end{aligned} \tag{3.18}$$

where

$$\mathcal{H}(t,x,p,A)=\sup_{\theta\in U}\Big\{b(x,\theta)p+\frac{1}{2}\sigma(x,\theta)^2A+f(t,x,\theta)\Big\}.$$

Subtracting these two inequalities yields that

$$\begin{aligned} \frac{\lambda}{t_0^2}&\le \mathcal{H}(t_0,x_0,\partial_x\varphi(t_0,x_0),A)-\mathcal{H}(t_0,y_0,-\partial_y\varphi(t_0,y_0),B)\\ &\le \sup_{\theta\in U}\bigg\{\frac{1}{\delta}(x_0-y_0)(b(x_0,\theta)-b(y_0,\theta))\\ &\qquad+\frac{1}{2}\Big(\sigma(x_0,\theta)^2A-\sigma(y_0,\theta)^2B\Big)+f(t_0,x_0,\theta)-f(t_0,y_0,\theta)\bigg\}\\ &=:\sup_{\theta\in U}\Big\{(\mathrm{I})+(\mathrm{II})+(\mathrm{III})\Big\}. \end{aligned} \tag{3.19}$$

Next, we go to estimate the terms (I), (II), (III) under the conditions (A1)–(A3).

$$(\mathrm{I})=\frac{1}{\delta}(x_0-y_0)(b(x_0,\theta)-b(y_0,\theta))\le\frac{C_1}{\delta}\Big(|x_0-y_0|^{1+\ell}+|x_0-y_0|^2\Big). \tag{3.20}$$

Due to (3.17),

$$(\mathrm{II})=\frac{1}{2}\Big(\sigma(x_0,\theta)^2A-\sigma(y_0,\theta)^2B\Big)\le\frac{3C_2^2}{2\delta}|x_0-y_0|^{2\gamma}. \tag{3.21}$$

It follows immediately from (A3) that

$$(\mathrm{III})=f(t_0,x_0,\theta)-f(t_0,y_0,\theta)\le C_3|x_0-y_0|^p. \tag{3.22}$$

Inserting (3.20), (3.21), (3.22) into (3.19), we get

$$\frac{\lambda}{t_0^2}\le\frac{C_1}{\delta}\Big(|x_0-y_0|^{1+\ell}+|x_0-y_0|^2\Big)+\frac{3C_2^2}{2\delta}|x_0-y_0|^{2\gamma}+C_3|x_0-y_0|^p. \tag{3.23}$$

As $(\gamma+\frac{2\gamma-1}{2\gamma}p)\mathbf{1}_{\{2\gamma-1\le\ell\}}+(\ell+\frac{2\ell}{1+\ell}p)\mathbf{1}_{\{2\gamma-1>\ell\}}>1$,

$$1+\ell>2-\frac{2((2\gamma-1)\wedge\ell)}{1+(2\gamma-1)\wedge l}p,\quad 2\gamma>2-\frac{2((2\gamma-1)\wedge\ell)}{1+(2\gamma-1)\wedge l}p.$$

By (3.13) and (3.15), letting $\delta\to 0$ in (3.23), we get

$$\frac{\lambda}{t_0^2}\le 0,$$

which contradicts the assumption $\lambda > 0$. Consequently, the assertion (3.8) is false, and we complete the proof of Theorem 2.4. □

4 Conclusions

Motivated by the portfolio problems associated with financial models, we studied the optimal control problem for stochastic differential equations with Hölder coefficients. Since the objective function studied in this work contains both the terminal reward and the running reward, the explicit construction of the value function via verification theorem usually does not work. Therefore, we characterize the value function as the viscosity solution to the associated HJB equation under irregular conditions, which is our primary contribution of this work. Moreover, under Hölder continuous conditions, studying the regular property of the value function is a challenging task, which is overcome by using Meyer–Tanaka's formula and improving the estimate of local times. In the current work, we do not study the optimal strategy such as its existence and explicit expression, which is left for further study.

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Declarations

Conflict of interest The authors have no relevant financial or non-financial interests to disclose.

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