SIAM J. CONTROL OPTIM.
Vol. 63, No. 5, pp. 3112–3138

OPTIMAL CONTROL PROBLEM FOR REFLECTED McKEAN–VLASOV SDEs*

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Abstract. This work investigates the optimal control problem for reflected McKean–Vlasov SDEs and the viscosity solutions to Hamilton–Jacobi–Bellman (HJB) equations on the Wasserstein space in terms of intrinsic derivative. It follows from the flow property of reflected McKean–Vlasov SDEs that the dynamic programming principle holds. Applying the decoupling method and the heat kernel estimates for parabolic equations, we show that the value function is a viscosity solution to an appropriate HJB equation on the Wasserstein space, where the characterization of absolutely continuous curves on the Wasserstein space by the continuity equations plays an important role. To establish the uniqueness of the viscosity solution, we generalize the construction of a distance-like function initiated in Burzoni et al. [*SIAM J. Control Optim.*, 58 (2020), pp. 1676–1699] to the Wasserstein space over multidimensional space and show its effectiveness for coping with HJB equations in terms of intrinsic derivative on the Wasserstein space.

Key words. viscosity solution, Wasserstein space, reflected McKean–Vlasov, feedback control

MSC codes. 60H10, 35Q93, 49L25

DOI. 10.1137/23M1602206

1. Introduction.

Let $\mathscr{O}$ be a bounded convex domain in $\mathbb{R}^d$ including the origin and with $C^{1,1}$ boundary. $\bar{\mathscr{O}}$ denotes the closure of $\mathscr{O}$ and $\partial\mathscr{O}$ its boundary. Let $\vec{\mathbf{n}}(\cdot)$ be the unit outward normal of $\mathscr{O}$. Consider the following reflected McKean–Vlasov equation:

$$\begin{cases} \mathrm{d}X_t = b(t, X_t, \mathscr{L}_{X_t}, \alpha_t)\mathrm{d}t + \sigma \mathrm{d}B_t - \vec{\mathbf{n}}(X_t)\mathrm{d}k_t, \\ k_t = \int_0^t \mathbf{1}_{\partial\mathscr{O}}(X_s)\mathrm{d}k_s, \end{cases} \tag{1.1}$$

where $\mathscr{L}_{X_t}$ denotes the distribution of X_t. (B_t) is a d-dimensional Brownian motion. The coefficients b, σ will be detailed later. The term α_t represents the control strategy imposed on (X_t). The solution of (1.1) is a pair (X_t, k_t), and the process (X_t) will stay always in $\bar{\mathscr{O}}$. (k_t) is called the local time of (X_t) on $\partial\mathscr{O}$, which is a continuous process and increases only when X_t hits the boundary $\partial\mathscr{O}$.

In this work, we shall investigate the finite horizon optimal control problem for the reflected process (X_t). The associated value function will be defined on the Wasserstein space $\mathscr{P}(\bar{\mathscr{O}})$, the space of all probability measures over $\bar{\mathscr{O}}$. The admissible controls considered in this work are of feedback control form and contain the set of deterministic controls used as in [8]. This adds a new difficulty to verifying the value function as a viscosity supersolution. The dynamic programming principle is established following from the flow property of the solutions to reflected McKean–Vlasov SDEs. Then, using the intrinsically differential structure for functions on $\mathscr{P}(\bar{\mathscr{O}})$, and taking advantage of the characterization of absolutely continuous curves on $\mathscr{P}(\bar{\mathscr{O}})$ in terms of continuity equations (cf. [3]), the value function is proved to be a viscosity

*Received by the editors September 17, 2023; accepted for publication (in revised form) June 24, 2025; published electronically September 2, 2025.
https://doi.org/10.1137/23M1602206
Funding: This work was supported in part by the National Key R&D Program of China (2022YFA1006000) and NNSF of China (12271397).
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solution to an appropriate HJB equation on $\mathscr{P}(\bar{\mathscr{O}})$. In the argument, we use the decoupling method and the heat kernel estimates for parabolic equations to overcome the difficulty caused by the feedback controls. The intrinsically differential structure on $\mathscr{P}(\bar{\mathscr{O}})$ is closely related to the study of the optimal transport map problem (cf. [3, 32]) and Monge–Ampère equations (cf. [9, 31, 32]), which can provide rich geometric structure on $\mathscr{P}(\bar{\mathscr{O}})$.

As is well known in the study of HJB equations over the Wasserstein space, one of the main challenges is to prove the uniqueness of the viscosity solution. The theory of viscosity solutions for HJB equations in infinite dimensional space was initiated by Crandall and Lions [15] on Hilbert space or certain Banach space. However, the recent study of the optimal control problem for McKean–Vlasov SDEs and mean field games has motivated a lot of research interest on HJB equations on the Wasserstein space; cf., e.g., [8, 11, 12, 25, 28, 29, 30].

The Wasserstein space has various differential structures, and correspondingly various HJB equations have been established on it. For example, Ambrosio and Feng [2] and Gangbo and Swiech [19] used a metric derivative to study viscosity solutions of (first order) Hamilton–Jacobi equations on the Wasserstein space. Lions [26] lifted functions defined on the Wasserstein space to functions on an appropriate L^2 space and used the well-developed viscosity solution theory for HJB equations on the Hilbert space to study HJB equations on the Wasserstein space. Gangbo, Nguyen, and Tudorascu [18] and Gangbo and Tudorascu [20] exploited the isometry between a quotient space of L^2 space to the Wasserstein space at length, and made inferences on partial differential equations in the latter space. Pham and Wei [28, 29] studied HJB equations on the Wasserstein space by using Lions' lifting to solve the optimal control problem for McKean–Vlasov SDEs (with common noise). Burzoni et al. [8] investigated the viscosity solutions to HJB equations using a linear functional derivative on the Wasserstein space. They raised a distance-like function on the Wasserstein space over $\mathbb{R}$, whose linear functional derivative can be controlled by itself. The construction of this distance-like function is quite subtle.

Our strategy to establish the uniqueness of the viscosity solution is based on two observations: (1) The L^2-Wasserstein distance $\mathbb{W}_2$ is only intrinsically differentiable at the probability measures satisfying a certain regular property, and its derivative cannot satisfy the smoothness condition of the HJB equation established by solving the optimal control problem associated with (1.1). (2) The distance-like function constructed in [8] is also intrinsically differentiable, and its intrinsic derivative is smooth enough to be used as the smooth test function of the value function. Thus, we generalize the construction of [8] for $\mathscr{P}(\mathbb{R})$ to the Wasserstein space $\mathscr{P}(\bar{\mathscr{O}})$ over multidimensional space and make use of the weak compactness of $\mathscr{P}(\bar{\mathscr{O}})$ to establish the comparison principle for the viscosity sub/supersolutions to our established HJB equations on $\mathscr{P}(\bar{\mathscr{O}})$. A technical restriction of this method, like in [8], is that the drift b can only depend on the finite order moments of μ and is independent of x.

This work is organized as follows. In section 2, we present the framework of the optimal control problem for (1.1) and study the continuity of the value function and establish the dynamic programming principle. In section 3, under the intrinsic differential structure of $\mathscr{P}(\bar{\mathscr{O}})$, the law μ_t of the controlled process X_t is shown to be an absolutely continuous curve in $\mathscr{P}(\bar{\mathscr{O}})$, whose velocity v_t can be characterized. Furthermore, the value function is shown to be a viscosity solution to an appropriate HJB equation on $\mathscr{P}(\bar{\mathscr{O}})$. In section 4, we study the regularity of L^2-Wasserstein distance $\mathbb{W}_2$ based on the regularity of the solution of the Monge–Ampère equation.

Then, the uniqueness of the viscosity solution is established under the generalization of [8]'s construction of a distance-like function.

2. Framework. Let $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, \mathbb{P})$ be a complete filtered probability space. Let $\mathscr{P}(\bar{\mathscr{O}})$ be the space of all probability measures over $\bar{\mathscr{O}}$. Let (α_t) be an $\mathscr{F}_t$-adapted process. Let U be a compact set in $\mathbb{R}^m$ for some $m \in \mathbb{N}$. We shall study the finite horizon optimal control problem, so let $T > 0$ be given and fixed in this work. As $\mathscr{O}$ is bounded, all the probability measures in $\mathscr{P}(\bar{\mathscr{O}})$ own finite pth moments for all $p \geq 1$. Let

$$\mathscr{P}^r(\bar{\mathscr{O}}) = \left\{\mu \in \mathscr{P}(\bar{\mathscr{O}}); \mathrm{d}\mu(x) \ll \mathrm{d}x \text{ and } \rho(\cdot) := \frac{\mathrm{d}\mu}{\mathrm{d}x} \in C^1(\bar{\mathscr{O}}),\ \rho(\cdot) > 0\right\}. \tag{2.1}$$

Definition 2.1. *A pair (X_t, k_t) is called a solution to* (1.1) *if (X_t) is an adapted continuous process on $\bar{\mathscr{O}}$, (k_t) is an adapted continuous increasing process such that $\mathbb{P}$-a.e.*

$$\int_0^t \big(|b(r, X_r, \mathscr{L}_{X_r}, \alpha_r)| + \|\sigma\|^2\big)\mathrm{d}r < \infty, \quad t \geq 0,$$

and $k_t = \int_0^t \mathbf{1}_{\partial\mathscr{O}}(X_s)\mathrm{d}k_s$,

$$X_t = X_0 + \int_0^t b(r, X_r, \mathscr{L}_{X_r}, \alpha_r)\mathrm{d}r + \sigma B_t - \int_0^t \vec{\mathbf{n}}(X_r)\mathrm{d}k_r.$$

A triple $(X_t, k_t, B_t)_{t\geq 0}$ is called a weak solution to (1.1) *if (B_t) is a d-dimensional Brownian motion under a probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, \mathbb{P})$, and (X_t, k_t) solves* (1.1) *with initial value $X_0 = \xi \in \mathscr{F}_0$:*

$$X_t = \xi + \int_0^t b(r, X_r, \mathscr{L}_{X_r}, \alpha_r)\mathrm{d}r + \sigma B_t - \int_0^t \vec{\mathbf{n}}(X_r)\mathrm{d}k_r.$$

If for any two weak solutions $(X_t, k_t, B_t)_{t\geq 0}$ under probability $\mathbb{P}$, $(\widetilde{X}_t, \tilde{k}_t, \widetilde{B}_t)_{t\geq 0}$ under probability $\widetilde{\mathbb{P}}$ satisfying $\mathscr{L}_{X_0|\mathbb{P}} = \mathscr{L}_{\widetilde{X}_0|\widetilde{\mathbb{P}}}$, then $\mathscr{L}_{(X_t,k_t)|\mathbb{P}} = \mathscr{L}_{(\widetilde{X}_t,\tilde{k}_t)|\widetilde{\mathbb{P}}}$ for $t > 0$, SDE (1.1) *is called weakly unique.*

We call (1.1) *weakly wellposed for distributions in $\hat{\mathscr{P}}$ if it has a unique weak solution for any $\mathscr{F}_0$-measurable variable ξ with $\mathscr{L}_\xi \in \hat{\mathscr{P}}$, and the distribution of X_t remains in $\hat{\mathscr{P}}$ for any $t > 0$. When $\hat{\mathscr{P}} = \mathscr{P}(\bar{\mathscr{O}})$, we simply say that* (1.1) *is weakly wellposed.*

The L^p-Wasserstein distance $\mathbb{W}_p$ for two probability measures $\mu, \nu \in \mathscr{P}(\bar{\mathscr{O}})$ is defined by

$$\mathbb{W}_p(\mu, \nu) = \inf_{\Gamma \in \mathscr{C}(\mu,\nu)} \left(\int_{\bar{\mathscr{O}}\times\bar{\mathscr{O}}} |x - y|^p \Gamma(\mathrm{d}x, \mathrm{d}y)\right)^{\frac{1}{p}}, \quad p \geq 1,$$

where $\mathscr{C}(\mu, \nu)$ stands for the collection of all couplings of μ and ν.

Let U be a compact set in $\mathbb{R}^k$ for some $k \geq 1$. Assume that the coefficients $b : [0, T] \times \bar{\mathscr{O}} \times \mathscr{P}(\bar{\mathscr{O}}) \times U \to \mathbb{R}^d$, $\sigma \in \mathbb{R}^{d\times d}$, satisfy the following:

(H$_1$) $\exists K_1 > 0$ such that for all $s, t \in [0, T]$, $x, y \in \bar{\mathscr{O}}$, $\mu, \nu \in \mathscr{P}(\bar{\mathscr{O}})$, $\alpha, \tilde{\alpha} \in U$,

$$|b(t, x, \mu, \alpha) - b(s, y, \nu, \tilde{\alpha})| \leq K_1\big(|s - t| + |x - y| + \mathbb{W}_2(\mu, \nu) + |\alpha - \tilde{\alpha}|\big).$$

(H_2) $\exists\,\lambda_0\ge 1$ such that for all $x,z\in\mathbb{R}^d$, $\mu\in\mathscr{P}(\mathbb{R}^d)$,

$$\lambda_0^{-1}|z|^2\le\langle Az,z\rangle\le\lambda_0|z|^2,$$

where $A=(a_{ij})=\sigma\sigma^*$, and σ^* denotes the transpose of the matrix σ.

Let $\widetilde{\Pi}$ be the class of functions $F:[0,T]\times\bar{\mathscr{O}}\times\mathscr{P}(\bar{\mathscr{O}})\to U$ such that there exists $C_F>0$,

$$\begin{aligned}&|F(t,x,\mu)-F(t,y,\nu)|\le C_F\big(|x-y|+\mathbb{W}_2(\mu,\nu)\big),\\ &\int_0^T|F(s,0,\delta_0)|^2\mathrm{d}s<\infty,\quad t\in[0,T],\,x,y\in\bar{\mathscr{O}},\,\mu,\nu\in\mathscr{P}(\bar{\mathscr{O}}).\end{aligned}\tag{2.2}$$

According to [1, Theorem 3.2] or [33], under the condition (H_1), for each $F\in\widetilde{\Pi}$ and $\xi\in\mathscr{F}_s$, there exists a unique solution to the reflected SDE: for $0\le s\le t\le T$,

$$X_t=\xi+\int_s^t b(r,X_r,\mathscr{L}_{X_r},F(r,X_r,\mathscr{L}_{X_r}))\mathrm{d}r+\sigma(B_t-B_s)-\int_s^t\vec{\mathbf{n}}(X_r)\mathrm{d}k_r.\tag{2.3}$$

Definition 2.2. *For $s\in[0,T)$ and $\mu\in\mathscr{P}(\bar{\mathscr{O}})$, a control policy $\boldsymbol{\alpha}=(\alpha_t)_{t\in[s,T]}$ is said to be in the class of admissible feedback controls $\Pi_{s,\mu}$ if there exists a function $F\in\widetilde{\Pi}$ such that*

$$\alpha_t=F(t,X_t,\mathscr{L}_{X_t}),$$

where $(X_t)_{t\in[s,T]}$ is the solution to the reflected SDE (2.3) with $X_s\in\mathscr{F}_s$ satisfying $\mathscr{L}_{X_s}=\mu$.

We use $(X_t^{s,\mu})_{t\in[s,T]}$ to denote the solution of (2.3) with initial value $X_s=\xi$ and $\mathscr{L}_\xi=\mu$ associated with $\boldsymbol{\alpha}$. It follows from the weak uniqueness of (2.3) that the distribution of $X_t^{s,\mu}$ for $t\in[s,T]$ depends on ξ only through its law μ. Given two measurable functions $\vartheta:[0,T]\times\bar{\mathscr{O}}\times\mathscr{P}(\bar{\mathscr{O}})\times U\to[0,\infty)$ and $g:\bar{\mathscr{O}}\times\mathscr{P}(\bar{\mathscr{O}})\to[0,\infty)$, our aim is to minimize the objective function

$$J(s,\mu;\boldsymbol{\alpha}):=\mathbb{E}\left[\int_s^T\vartheta(r,X_r^{s,\mu},\mathscr{L}_{X_r^{s,\mu}},\alpha_r)\mathrm{d}r+g(X_T^{s,\mu},\mathscr{L}_{X_T^{s,\mu}})\right].\tag{2.4}$$

We should notice that $J(s,\mu;\boldsymbol{\alpha})$ is well defined, that is, it depends only on the initial law μ no matter which random variable ξ or $\tilde{\xi}$ with $\mathscr{L}_\xi=\mathscr{L}_{\tilde{\xi}}=\mu$ has been used as the initial value of SDE (2.3). Indeed, for $\boldsymbol{\alpha}\in\Pi_{s,\mu}$ in the form $\alpha_t=F(r,X_t,\mathscr{L}_{X_t})$, we have

$$\mathbb{E}\left[\int_s^T\vartheta(r,X_r^{s,\xi},\mathscr{L}_{X_r^{s,\xi}},\alpha_r)\mathrm{d}r\right]=\int_s^T\int_{\bar{\mathscr{O}}}\vartheta\big(r,x,\mathscr{L}_{X_r^{s,\xi}},F(r,x,\mathscr{L}_{X_r^{s,\xi}})\big)\mathscr{L}_{X_r^{s,\xi}}(\mathrm{d}x)\mathrm{d}r.$$

A similar deduction yields that the term $\mathbb{E}\big[g(X_T^{s,\xi},\mathscr{L}_{X_T^{s,\xi}})\big]$ also depends on ξ through its law.

The value function is defined by

$$V(s,\mu)=\inf_{\boldsymbol{\alpha}\in\Pi_{s,\mu}}J(s,\mu;\boldsymbol{\alpha}).\tag{2.5}$$

Next, we present some properties of the value function. In particular, the value function satisfies the dynamic programming principle, which is based on the flow property of the solution to (1.1).

LEMMA 2.3. *Assume* (H_1), (H_2) *hold. For any* U*-valued* $\mathscr{F}_t$*-adapted process* $(\alpha_t)_{t\in[s,T]}$ *and* $\bar{\mathscr{O}}$*-valued random variables* $\xi, \tilde{\xi} \in \mathscr{F}_s$*, consider the solutions* $(X_t, k_t)_{t\in[s,T]}$*,* $(\widetilde{X}_t, \tilde{k}_t)_{t\in[s,T]}$ *to* (1.1) *with initial values* $X_s = \xi$ *and* $\widetilde{X}_s = \tilde{\xi}$*, respectively. Then,*

$$\mathbb{W}_2(\mathscr{L}_{X_t}, \mathscr{L}_{\widetilde{X}_t})^2 \le \mathbb{E}|X_t - \widetilde{X}_t|^2 \le \big(\mathbb{E}|\xi - \tilde{\xi}|^2\big)\mathrm{e}^{2(K_1+K_1^2)(t-s)}, \quad t\in[s,T]. \tag{2.6}$$

Proof. Since $\mathscr{O}$ is convex and $\vec{\mathbf{n}}(x)$ is unit outward normal of $\mathscr{O}$, it holds that

$$\langle \vec{\mathbf{n}}(x), y - x\rangle \le 0 \quad \forall\, x\in\partial\mathscr{O},\ y\in\mathscr{O}.$$

By Itô's formula,

$$\begin{aligned} \mathrm{d}|X_t - \widetilde{X}_t|^2 &= 2\langle X_t - \widetilde{X}_t, \mathrm{d}(X_t - \widetilde{X}_t)\rangle + \mathrm{d}(X_t - \widetilde{X}_t)\cdot \mathrm{d}(X_t - \widetilde{X}_t)\\ &\quad - 2\langle X_t - \widetilde{X}_t, \vec{\mathbf{n}}(X_t)\rangle \mathrm{d}k_t + 2\langle X_t - \widetilde{X}_t, \vec{\mathbf{n}}(\widetilde{X}_t)\rangle \mathrm{d}\tilde{k}_t. \end{aligned}$$

Since k_t increases only when $X_t\in\partial\mathscr{O}$ and $\tilde{k}_t$ increases only when $\widetilde{X}_t\in\partial\mathscr{O}$, we have

$$\langle X_t - \widetilde{X}_t, \vec{\mathbf{n}}(X_t)\rangle \mathrm{d}k_t \ge 0, \quad \langle X_t - \widetilde{X}_t, \vec{\mathbf{n}}(\widetilde{X}_t)\rangle \mathrm{d}\tilde{k}_t \le 0.$$

Thus, by (H_1),

$$\mathbb{E}|X_t - \widetilde{X}_t|^2 \le \mathbb{E}|\xi - \tilde{\xi}|^2 + (K_1 + K_1^2)\int_s^t \mathbb{E}\Big[\big(|X_r - \widetilde{X}_r| + \mathbb{W}_2(\mathscr{L}_{X_r}, \mathscr{L}_{\widetilde{X}_r})\big)^2\Big]\mathrm{d}r.$$

As $\mathbb{W}_2(\mathscr{L}_{X_r}, \mathscr{L}_{\widetilde{X}_r})^2 \le \mathbb{E}|X_r - \widetilde{X}_r|^2$, it follows from Gronwall's inequality that

$$\begin{aligned} &\mathbb{E}|X_t - \widetilde{X}_t|^2 \le \mathbb{E}|\xi - \tilde{\xi}|^2 + 2(K_1 + K_1^2)\int_s^t \mathbb{E}|X_r - \widetilde{X}_r|^2\mathrm{d}r,\\ &\mathbb{W}_2(\mathscr{L}_{X_r}, \mathscr{L}_{\widetilde{X}_r})^2 \le \mathbb{E}|X_t - \widetilde{X}_t|^2 \le \big(\mathbb{E}|\xi - \tilde{\xi}|^2\big)\mathrm{e}^{2(K_1+K_1^2)(t-s)}. \end{aligned}$$

Therefore, we arrive at the desired estimate (2.6). ∎

Let us introduce the regular condition on the cost functions ϑ and g as follows.

(H_3) There exist K_2, $K_3 > 0$ such that

$$\begin{aligned} &|\vartheta(s,x,\mu,\alpha)| \le K_2 \quad \forall\, s\in[0,T], x\in\bar{\mathscr{O}}, \mu\in\mathscr{P}(\bar{\mathscr{O}}), \alpha\in U;\\ &|\vartheta(s,x,\mu,\alpha) - \vartheta(t,y,\nu,\alpha)| + |g(x,\mu) - g(y,\nu)| \le K_3\big(|s-t| + |x-y| + \mathbb{W}_2(\mu,\nu)\big) \end{aligned}$$

for all $s,t\in[0,T]$, $x,y\in\bar{\mathscr{O}}$, μ, $\nu\in\mathscr{P}(\bar{\mathscr{O}})$, $\alpha\in U$.

PROPOSITION 2.4. *Suppose* (H_1)–(H_3) *hold. Then the value function satisfies*

$$|V(s,\mu) - V(s',\mu')| \le C\Big(\sqrt{|s-s'|} + \mathbb{W}_2(\mu,\mu')\Big), \quad s,s'\in[0,T],\ \mu,\mu'\in\mathscr{P}(\bar{\mathscr{O}}), \tag{2.7}$$

for some constant $C > 0$.

Proof. Let $0\le s < s' \le T$. By the definition of $V(s',\mu')$, for any $\varepsilon > 0$ there exists a control $\boldsymbol{\alpha}^\varepsilon \in \Pi_{s',\mu'}$ such that

$$J(s',\mu') \le V(s',\mu') + \varepsilon.$$

Let $(X_t^\varepsilon, k_t^\varepsilon)$ be the associated controlled process to $\boldsymbol{\alpha}^\varepsilon$ with $X_{s'}^\varepsilon = \xi'$ and $\mathscr{L}_\xi = \mu'$. Due to $\boldsymbol{\alpha}^\varepsilon\in\Pi_{s',\mu'}$, there exists $F^\varepsilon : [0,T]\times\bar{\mathscr{O}}\times\mathscr{P}(\bar{\mathscr{O}})\to U$ in the class $\widetilde{\Pi}$ such that $\alpha_r^\varepsilon = F^\varepsilon(r, X_r^\varepsilon, \mathscr{L}_{X_r^\varepsilon})$ for $r\in[s',T]$. Let

$$\widetilde{F}(r,x,\mu)=\begin{cases}F^{\varepsilon}(s',x,\mu), & r\in[0,s'],\\ F^{\varepsilon}(r,x,\mu), & r\in[s',T].\end{cases}$$

We can check directly that $\widetilde{F}\in\widetilde{\Pi}$. Consider the reflected SDEs: $\tilde{k}_t=\int_s^t \mathbf{1}_{\partial\mathscr{O}}(\widetilde{X}_r)\mathrm{d}\tilde{k}_r$,

$$\widetilde{X}_t=\tilde{\xi}+\int_s^t b(r,\widetilde{X}_r,\mathscr{L}_{\widetilde{X}_r},\widetilde{F}(r,\widetilde{X}_r,\mathscr{L}_{\widetilde{X}_r}))\mathrm{d}r+\int_s^t\sigma(r)\mathrm{d}B_r-\int_s^t\vec{\mathbf{n}}(\widetilde{X}_r)\mathrm{d}\tilde{k}_r.$$

Here the random variable $\tilde{\xi}\in\mathscr{F}_s$ is chosen so that $\mathscr{L}_{\tilde{\xi}}=\mu$ and $\mathbb{E}|\tilde{\xi}-\xi'|^2=\mathbb{W}_2(\mu,\mu')^2$, whose existence is a result of the existence of optimal coupling of μ and μ' (cf. [32]). Define $\tilde{\alpha}_r=\widetilde{F}(r,\widetilde{X}_r,\mathscr{L}_{\widetilde{X}_r})$, $r\in[s,T]$, then $\tilde{\boldsymbol{\alpha}}=(\tilde{\alpha}_r)_{r\in[s,T]}$ is in $\Pi_{s,\mu}$.

Due to (H_3), we have ϑ and g are bounded. Therefore, by Lemma 2.3,

$$\begin{aligned}
&V(s,\mu)-V(s',\mu')\\
&\le\mathbb{E}\left[\int_s^T\vartheta(r,\widetilde{X}_r,\mathscr{L}_{\widetilde{X}_r},\tilde{\alpha}_r)\mathrm{d}r+g(\widetilde{X}_T,\mathscr{L}_{\widetilde{X}_T})\right]\\
&\quad-\mathbb{E}\left[\int_{s'}^T\vartheta(r,X_r^{\varepsilon},\mathscr{L}_{X_r^{\varepsilon}},\alpha_r^{\varepsilon})\mathrm{d}r+g(X_T^{\varepsilon},\mathscr{L}_{X_T^{\varepsilon}})\right]+\varepsilon\\
&\le\mathbb{E}\left[\int_s^{s'}\vartheta(r,\widetilde{X}_r,\mathscr{L}_{\widetilde{X}_r},\tilde{\alpha}_r)\mathrm{d}r\right]+2K_3\int_{s'}^T\big(\mathbb{E}|\widetilde{X}_r-X_r^{\varepsilon}|^2\big)^{\frac12}\mathrm{d}r\\
&\quad+2K_3\big(\mathbb{E}|\widetilde{X}_T-X_T^{\varepsilon}|^2\big)^{\frac12}+\varepsilon\\
&\le c_1|s'-s|+c_1\big(\mathbb{E}\big[|\widetilde{X}_{s'}-\xi|^2\big]\big)^{\frac12}+\varepsilon\\
&\le c_1|s'-s|+c_1\big(\mathbb{E}[|\widetilde{X}_{s'}-\widetilde{X}_s|^2]\big)^{\frac12}+c_1\big(\mathbb{E}[|\widetilde{X}_s-\xi|^2]\big)^{\frac12}+\varepsilon\\
&\le c_2\big(|s'-s|+\sqrt{|s'-s|}+\mathbb{W}_2(\mu,\mu')\big)+\varepsilon,
\end{aligned}$$

where we have used $\mathbb{W}_2(\mathscr{L}_{\widetilde{X}_r},\mathscr{L}_{X_r^{\varepsilon}})\le\big(\mathbb{E}|\widetilde{X}_r-X_r^{\varepsilon}|^2\big)^{\frac12}$, and c_1, c_2 are constants depending only on K_3, T, and the diameter of $\mathscr{O}$. Therefore,

$$V(s,\mu)-V(s',\mu')\le c_3\left(\sqrt{|s'-s|}+\mathbb{W}_2(\mu,\mu')\right)+\varepsilon$$

for some $c_3>0$. Letting $\varepsilon\to0$, we get the desired estimate of $V(s,\mu)-V(s',\mu')$. The estimate $V(s',\mu')-V(s,\mu)$ can be proved in a similar way. The proof is completed. ◻

PROPOSITION 2.5 (dynamic programming principle). *Suppose* (H_1), (H_2) *hold. Then, for any* $0\le s\le t\le T$, $\mu\in\mathscr{P}(\bar{\mathscr{O}})$,

$$V(s,\mu)=\inf_{\boldsymbol{\alpha}\in\Pi_{s,\mu}}\left\{\mathbb{E}\left[\int_s^t\vartheta(r,X_r^{s,\mu,\alpha},\mathscr{L}_{X_r^{s,\mu,\alpha}},\alpha_r)\mathrm{d}r+V(t,\mathscr{L}_{X_t^{s,\mu,\alpha}})\right]\right\},\tag{2.8}$$

where for each $\boldsymbol{\alpha}\in\Pi_{s,\mu}$, $(X_t^{s,\mu,\alpha})_{t\in[s,T]}$ *stands for the corresponding controlled process with initial value* $X_s^{s,\mu,\alpha}$ *satisfying* $\mathscr{L}_{X_s^{s,\mu,\alpha}}=\mu$.

Proof. For each $\boldsymbol{\alpha}\in\Pi_{s,\mu}$, the wellposedness of reflected McKean–Vlasov (1.1) yields that the flow property holds:

$$X_r^{s,\xi}=X_r^{t,X_t^{s,\xi}},\quad r\in[t,T],\ s\le t.$$

This assertion can be proved in the same way as the McKean–Vlasov equations without reflection; see [7, section 3].

Denote the right-hand side of (2.8) by $\widetilde{V}(s,\mu)$. Then, according to the definition of $V(s,\mu)$, for any $\varepsilon>0$ there exists an admissible feedback control $\boldsymbol{\alpha}\in\Pi_{s,\mu}$ such that

$$\begin{aligned}
&V(s,\mu)\\
&\geq \mathbb{E}\bigg[\int_s^t \vartheta(r,X_r^{s,\mu,\alpha},\mathscr{L}_{X_r^{s,\mu,\alpha}},\alpha_r)\mathrm{d}r\\
&\quad+\int_t^T \vartheta(r,X_r^{s,\mu,\alpha},\mathscr{L}_{X_r^{s,\mu,\alpha}},\alpha_r)\mathrm{d}r+g(X_T^{s,\mu,\alpha},\mathscr{L}_{X_T^{s,\mu,\alpha}})\bigg]-\varepsilon\\
&\geq \mathbb{E}\left[\int_s^t \vartheta(r,X_r^{s,\mu,\alpha},\mathscr{L}_{X_r^{s,\mu,\alpha}},\alpha_r)\mathrm{d}r+V(t,\mathscr{L}_{X_t^{s,\mu,\alpha}})\right]-\varepsilon\geq\widetilde{V}(s,\mu)-\varepsilon,
\end{aligned}$$

where in the second inequality we have used the flow property of $(X_t^{s,\mu,\alpha})$. Letting $\varepsilon\to 0$, we obtain that $V(s,\mu)\geq\widetilde{V}(s,\mu)$.

For the inverse inequality, for all $\varepsilon>0$, by the definition of $\widetilde{V}(s,\mu)$, there is $\boldsymbol{\alpha}\in\Pi_{s,\mu}$ associated with $F\in\widetilde{\Pi}$ such that

$$\varepsilon+\widetilde{V}(s,\mu)\geq\mathbb{E}\left[\int_s^t \vartheta(r,X_r^{s,\mu,\alpha},\mathscr{L}_{X_r^{s,\mu,\alpha}},\alpha_r)\mathrm{d}r+V(t,\mathscr{L}_{X_t^{s,\mu,\alpha}})\right]. \tag{2.9}$$

By the definition of $V(t,\mathscr{L}_{X_t^{s,\mu,\alpha}})$, there exists a feedback control $\boldsymbol{\alpha}'\in\Pi_{t,\mathscr{L}_{X_t^{s,\mu,\alpha}}}$ corresponding to a function $\widetilde{F}\in\widetilde{\Pi}$ such that

$$\varepsilon+V(t,\mathscr{L}_{X_t^{s,\mu,\alpha}})\geq\mathbb{E}\left[\int_t^T \vartheta(r,X_r^{t,\nu_t},\mathscr{L}_{X_r^{t,\nu_t}},\alpha'_r)\mathrm{d}r+g(X_T^{t,\nu_t},\mathscr{L}_{X_T^{t,\nu_t}})\right], \tag{2.10}$$

where $\nu_t=\mathscr{L}_{X_t^{s,\mu,\alpha}}$. We define a new function $\hat{F}$ by

$$\hat{F}(r,x,\mu)=F(r,x,\mu)\mathbf{1}_{r\leq t}+\widetilde{F}(r,x,\mu)\mathbf{1}_{t<r\leq T},$$

and check directly that $\hat{F}\in\widetilde{\Pi}$. Then, corresponding to $\hat{F}$, consider the following SDE:

$$\mathrm{d}\hat{X}_r=b\big(r,\hat{X}_r,\mathscr{L}_{\hat{X}_r},\hat{F}(r,\hat{X}_r,\mathscr{L}_{\hat{X}_r})\big)\mathrm{d}r+\sigma(r)\mathrm{d}B_r-\vec{\mathbf{n}}(\hat{X}_r)\mathrm{d}k_r \tag{2.11}$$

with initial value $\hat{X}_s=\xi$ and $\mathscr{L}_\xi=\mu$. By the uniqueness of the solution to SDE (2.11), it holds that $\hat{X}_r=X_r^{s,\mu,\alpha}$ for $r\in[s,t]$ and $\hat{X}_r=X_r^{t,\nu_t}$ for $r\in[t,T]$. Associated with $\hat{F}$, there is an admissible feedback control $\hat{\boldsymbol{\alpha}}\in\Pi_{s,\mu}$ and $\hat{\boldsymbol{\alpha}}$ satisfies

$$\hat{\alpha}_r=\alpha_r\mathbf{1}_{s\leq r\leq t}+\alpha'_r\mathbf{1}_{t<r\leq T}.$$

Then, invoking (2.9), (2.10), by the definition of $V(s,\mu)$,

$$2\varepsilon+\widetilde{V}(s,\mu)\geq\mathbb{E}\left[\int_s^T \vartheta(r,\hat{X}_r,\mathscr{L}_{\hat{X}_r},\hat{\alpha}_r)\mathrm{d}r\ +g(\hat{X}_T,\mathscr{L}_{\hat{X}_T})\right]\geq V(s,\mu).$$

Letting $\varepsilon\to 0$, we get $\widetilde{V}(s,\mu)\geq V(s,\mu)$. In all, we have shown $V(s,\mu)=\widetilde{V}(s,\mu)$ and the proof is completed. ∎

3. Characterization of the value function: Existence of viscosity solution.

3.1. Riemannain structure of the Wasserstein space. In this subsection, we adopt the Riemannian interpretation of the Wasserstein space developed by Otto in [27] to introduce an HJB equation on the Wasserstein space and show that the value function is a viscosity solution to it. However, we defer the discussion on the uniqueness of the viscosity solution to this HJB equation to section 4.

The tangent space, geodesics, and Ricci curvature can be developed on $\mathscr{P}_2(\mathbb{R}^d) := \{\mu \in \mathscr{P}(\mathbb{R}^d); \int_{\mathbb{R}^d} |x|^2 \mathrm{d}\mu(x) < \infty\}$ endowed with the L^2-Wasserstein distance $\mathbb{W}_2$ based on the theory on optimal transport maps. See, e.g., [3] and [32]. As $\bar{\mathscr{O}}$ is bounded, it is clear that $\mathscr{P}_2(\bar{\mathscr{O}}) = \mathscr{P}(\bar{\mathscr{O}})$. As we are interested in the reflected stochastic processes on $\bar{\mathscr{O}}$, similar to $\mathscr{P}_2(\mathbb{R}^d)$, we consider the following Riemannian structure of $\mathscr{P}(\bar{\mathscr{O}})$. For each $\mu \in \mathscr{P}(\bar{\mathscr{O}})$, the tangent space at μ is defined by

$$\mathscr{T}_\mu := \left\{ v : \bar{\mathscr{O}} \to \mathbb{R}^d \text{ is measurable satisfying } \mu(|v|^2) < \infty \text{ and } \langle Av, \vec{\mathbf{n}} \rangle = 0 \text{ on } \partial\mathscr{O} \right\}, \tag{3.1}$$

where $A = \sigma\sigma^*$, and $\vec{\mathbf{n}}$ is the unit outward normal of $\mathscr{O}$. Then, $\mathscr{T}_\mu$ is a Hilbert space under the inner product

$$\langle v, v \rangle_{\mathscr{T}_\mu} = \|v\|^2_{\mathscr{T}_\mu} := \mu(|v|^2).$$

Definition 3.1. *Let $u : \mathscr{P}(\bar{\mathscr{O}}) \to \mathbb{R}$ be a continuous function, and let* Id *be the identity map on $\mathbb{R}^d$. u is said to be intrinsically differentiable at a point $\mu \in \mathscr{P}(\bar{\mathscr{O}})$ if there is a linear functional $D^L u : \mathscr{T}_\mu \to \mathbb{R}$ such that*

$$D^L_v u(\mu) = \lim_{\varepsilon \downarrow 0} \frac{u(\mu \circ (\mathrm{Id} + \varepsilon v)^{-1}) - u(\mu)}{\varepsilon}, \quad v \in \mathscr{T}_\mu, \ \mu \in \mathscr{P}(\bar{\mathscr{O}}).$$

In this situation, the unique element $D^L u(\mu) \in \mathscr{T}_\mu$ such that

$$\langle D^L u(\mu), v \rangle_{\mathscr{T}_\mu} = \int_{\bar{\mathscr{O}}} \langle D^L u(\mu)(x), v(x) \rangle \mu(\mathrm{d}x) = D^L_v u(\mu), \quad v \in \mathscr{T}_\mu,$$

is called the intrinsic derivative of u at μ.

Definition 3.2. *Let $\hat{\mathscr{P}}$ be a subset of $\mathscr{P}(\bar{\mathscr{O}})$. We write $u \in C^1_{L,b}(\hat{\mathscr{P}})$ if u is Lipschitz continuous in $(\mathscr{P}(\bar{\mathscr{O}}), \mathbb{W}_2)$ and intrinsically differentiable at any point $\mu \in \hat{\mathscr{P}}$, and its intrinsic derivative $D^L u(\mu)(x)$ satisfies that*

(i) *for each $\mu \in \hat{\mathscr{P}}$, $x \mapsto D^L u(\mu)(x)$ is continuously differentiable;*

(ii) $\sup\left\{ |D^L u(\mu)(x)| + |\nabla_x D^L u(\mu)(x)|; \mu \in \hat{\mathscr{P}}, x \in \bar{\mathscr{O}} \right\} < \infty$*;*

(iii) *$\mu \mapsto D^L u(\mu)(\cdot)$ is continuous from $\hat{\mathscr{P}}$ to $L^1(\bar{\mathscr{O}})$ in the sense that if $\mu_n, \mu \in \hat{\mathscr{P}}$ and $\mathbb{W}_2(\mu_n, \mu) \to 0$ as $n \to \infty$, then for any $\varepsilon > 0$*

$$\mu_n\left(\left\{ x \in \mathscr{O} \,\middle|\, |\nabla_x D^L \psi(t, \mu_n)(x) - \nabla_x D^L \psi(t, \mu)(x)| \geq \varepsilon \right\} \right) \longrightarrow 0, \quad \text{as } n \to \infty.$$

For a function $\psi : [0,T] \times \hat{\mathscr{P}} \to \mathbb{R}$, if for each $\mu \in \hat{\mathscr{P}}$, $\psi(\cdot, \mu)$ is continuously differential; for each $t \in [0,T]$, $\psi(t, \cdot) \in C^1_{L,b}(\hat{\mathscr{P}})$, and

$$\|D^L \psi\|_\infty := \sup\{ |D^L \psi(t, \mu)(x)|; t \in [0,T], \mu \in \hat{\mathscr{P}}, x \in \bar{\mathscr{O}} \} < \infty,$$

we say that $\psi \in C^{1,1}_{L,b}([0,T] \times \hat{\mathscr{P}})$.

DEFINITION 3.3 (absolutely continuous curves). *A curve $\mu:(a,b)\to\mathscr{P}(\bar{\mathscr{O}})$ is said to be in $AC(a,b)$ for $a,b\ge 0$ if there exists $m\in L^1(a,b)$ such that*

$$\mathbb{W}_2(\mu_s,\mu_t)\le\int_s^t m(r)\,\mathrm{d}r,\qquad a<s<t<b.$$

For an absolutely continuous curve $\mu:(a,b)\to\mathscr{P}(\bar{\mathscr{O}})$ the limit

$$|\mu'|(t):=\lim_{s\to t}\frac{\mathbb{W}_2(\mu_s,\mu_t)}{|s-t|}$$

exists for Leb-a.e. $t\in(a,b)$, which is called the metric derivative of the curve (μ_t).

Next, let us recall some results on the absolutely continuous curves in $\mathscr{P}(\bar{\mathscr{O}})$ as a subspace of $\mathscr{P}_2(\mathbb{R}^d)$, which can be proved in the same way as in $\mathscr{P}_2(\mathbb{R}^d)$.

THEOREM 3.4 (see [3, Theorem 8.3.1]). *Let $\mu:[0,T]\to\mathscr{P}(\bar{\mathscr{O}})$ be an absolutely continuous curve and let $|\mu'|\in L^1([0,T])$ be its metric derivative. Then there exists a Borel vector field $v:(t,x)\mapsto v_t(x)$ such that*

$$v_t\in L^2(\bar{\mathscr{O}}\to\mathbb{R}^d;\mu_t),\quad \|v_t\|_{L^2(\bar{\mathscr{O}};\mu_t)}\le|\mu'|(t)\quad \text{fora.e. } t\in[0,T], \tag{3.2}$$

and the continuity equation

$$\partial_t\mu_t+\nabla\cdot(v_t\mu_t)=0\quad \text{in } [0,T]\times\bar{\mathscr{O}} \tag{3.3}$$

holds in the sense of distribution, i.e.,

$$\int_0^T\int_{\mathscr{O}}\Big(\partial_t\psi(t,x)+\langle v_t(x),\nabla_x\psi(t,x)\rangle\Big)\mathrm{d}\mu_t(x)\mathrm{d}t=0\quad\forall\,\psi\in C_c^\infty((0,T)\times\mathscr{O}), \tag{3.4}$$

where $C_c^\infty((0,T)\times\mathscr{O})$ denotes the set of smooth functions on $(0,T)\times\mathscr{O}$ with compact support.

Conversely, if a continuous curve $\mu:[0,T]\to\mathscr{P}(\bar{\mathscr{O}})$ satisfies the continuity equation (3.3) *for some Borel velocity field v_t with $\|v_t\|_{L^2(\bar{\mathscr{O}};\mu_t)}\in L^1([0,T])$, then $\mu:[0,T]\to\mathscr{P}(\bar{\mathscr{O}})$ is absolutely continuous and $|\mu'|(t)\le\|v_t\|_{L^2(\bar{\mathscr{O}};\mu_t)}$ for Leb-a.e. $t\in[0,T]$.*

PROPOSITION 3.5 (see [3, Theorem 8.4.6]). *Let $\mu:[0,T]\to\mathscr{P}(\bar{\mathscr{O}})$ be absolutely continuous, and let $v_t\in\mathscr{T}_{\mu_t}$ be such that* (3.2), (3.3) *hold. Then*

$$\lim_{\varepsilon\downarrow 0}\frac{\mathbb{W}_2(\mu_{t+\varepsilon},\mu_t\circ(\mathrm{Id}+\varepsilon v_t)^{-1})}{\varepsilon}=0. \tag{3.5}$$

It is easy to show that the curve $(\mathscr{L}_{X_t})_{t\in[s,T]}$ of the controlled process $(X_t)_{t\in[s,T]}$ is an absolutely continuous curve in $\mathscr{P}(\bar{\mathscr{O}})$ under (H$_1$), (H$_2$). However, it is not easy to describe the velocity (v_t) of $(\mathscr{L}_{X_t})$ in $\mathscr{P}(\bar{\mathscr{O}})$.

Let $\mu_0\in\mathscr{P}^r(\bar{\mathscr{O}})$, and ξ is a random variable in $\mathscr{F}_0$ with $\mathscr{L}_\xi=\mu_0$. For $\boldsymbol{\alpha}\in\Pi_{0,\mu_0}$, denote $(X_t^{0,\mu_0},k_t^{0,\mu_0})_{t\in[0,T]}$ its associated controlled process satisfying (1.1). Under the nondegenerate condition (H$_2$), the law of X_t^{0,μ_0} admits a density $\rho_t(x)$, which satisfies the nonlinear Fokker–Planck equation:

$$\begin{cases}\partial_t\rho_t(x)=\mathcal{L}_\alpha^*\rho_t(x), & x\in\mathscr{O},t\in(0,T),\\ \langle A\nabla\rho_t(x),\vec{\mathbf{n}}(x)\rangle=0, & x\in\partial\mathscr{O},t\in(0,T),\end{cases} \tag{3.6}$$

where

$$\mathcal{L}_\alpha^*\rho_t(x)=\frac{1}{2}\sum_{i,j=1}^d a_{ij}\partial^2_{x_ix_j}\rho_t(x)-\sum_{i=1}^d\partial_{x_i}\big(b_i(t,x,\rho_t(x)\mathrm{d}x,\alpha_t)\rho_t(x)\big).$$

Using the decoupling method, via fixing the distribution of the process (X_t), the controlled process (X_t,k_t) can be viewed as a solution to the following SDE:

$$\begin{cases}\mathrm{d}\widetilde{X}_t=b(t,\widetilde{X}_t,\mu_t,\alpha_t)\mathrm{d}t+\sigma\mathrm{d}B_t-\vec{\mathbf{n}}(\widetilde{X}_t)\mathrm{d}\tilde{k}_t,\\ \tilde{k}_t=\displaystyle\int_0^t\mathbf{1}_{\partial\mathscr{O}}(\widetilde{X}_s)\mathrm{d}\tilde{k}_s,\end{cases}$$

where $\mu_t=\mathscr{L}_{X_t}$ is fixed by the unique solution of SDE (1.1). In particular, the law of X_t coincides with that of $\widetilde{X}_t$. Let $p(s,x;t,y)$ be the transition probability of the process $(\widetilde{X}_t)$, which is a fundamental solution of a parabolic equation with Neumann boundary condition. There is a large number of works in the literature on the estimates of fundamental solutions to parabolic equations with the Dirichlet boundary condition or Neumann boundary condition; see, for instance, [4, 5, 14, 17, 34, 35, 36] and references therein. In particular, [34] generalized the work [36] to the time-homogeneous parabolic equation with mixed boundary condition. [14] deals with time-inhomogeneous parabolic equations with Neumann boundary. Under (H_1), the drift b admits a bound M determined by K_1 and the diameters of $\mathscr{O}$ and U. Then, the Gaussian type estimates hold for $p(s,x;t,y)$. Namely, there exist constants κ_1, $\kappa_2>0$, depending on T, such that

$$\begin{aligned}&\frac{1}{\kappa_1(t-s)^{d/2}}\exp\left(-\frac{|y-x|^2}{\kappa_2(t-s)}\right)\le p(s,x;t,y)\le\frac{\kappa_1}{(t-s)^{d/2}}\exp\left(-\kappa_2\frac{|y-x|^2}{t-s}\right),\\ &|\partial_t p(s,x;t,y)|\le\frac{\kappa_1}{(t-s)^{(d+2)/2}}\exp\left(-\kappa_2\frac{|y-x|^2}{t-s}\right),\qquad x,y\in\bar{\mathscr{O}},0\le s<t\le T.\end{aligned}\tag{3.7}$$

Furthermore, the density $\rho_t(x)$ of $\mathscr{L}_{X_t}$ can be represented by

$$\rho_t(x)=\int_{\mathscr{O}}p(s,z;t,x)\mu_0(\mathrm{d}z),\quad t>s.\tag{3.8}$$

Consequently, under the nondegenerate condition (H_2), the distribution of the solution X_t to SDE (1.1) will always stay in $\mathscr{P}^r(\bar{\mathscr{O}})$.

THEOREM 3.6 (tangent vector fields: regular case). *Assume* (H_1) *and* (H_2) *hold. Let* $(X_t,k_t)_{t\in[0,T]}$ *be a solution to* (1.1) *associated with a feedback control* $\boldsymbol{\alpha}\in\Pi_{0,\mu_0}$ *in the form* $\alpha_t=F(t,X_t,\mathscr{L}_{X_t})$ *and* $\mathscr{L}_{X_0}=\mu_0\in\mathscr{P}^r(\bar{\mathscr{O}})$. *Then,*

(i) $[0,T]\ni t\mapsto\mu_t:=\mathscr{L}_{X_t}$ *is an absolutely continuous curve in* $\mathscr{P}^r(\bar{\mathscr{O}})$*; its associated velocity field* v_t *satisfying* (3.2), (3.3) *is given by*

$$v_t(x)=\sum_{i=1}^d\left(b_i(t,x,\mu_t,F(t,x,\mu_t))-\frac{1}{2}\sum_{j=1}^d\frac{a_{ij}\partial_{x_j}\rho_t(x)}{\rho_t(x)}\right)\mathbf{e}_i,\tag{3.9}$$

where $\rho_t(x)=\frac{\mathrm{d}\mu_t(x)}{\mathrm{d}x}$ *denotes the density of* μ_t*, and* $\{\mathbf{e}_1,\dots,\mathbf{e}_d\}$ *is the canonical orthonormal basis of* $\mathbb{R}^d$*;*

(ii) *let* $u \in C^1_{L,b}(\mathscr{P}^r(\bar{\mathscr{O}}))$, *then*

(3.10)
$$\begin{aligned}\frac{\mathrm{d}u(\mu_t)}{\mathrm{d}t} &= D^L_{v_t}u(\mu_t)\\ &= \int_{\bar{\mathscr{O}}}\langle b(t,x,\mu_t,\alpha_t), D^L u(\mu_t)(x)\rangle \mathrm{d}\mu_t(x) - \frac{1}{2}\int_{\bar{\mathscr{O}}}\langle D^L u(\mu_t)(x), A\nabla_x\rho_t(x)\rangle \mathrm{d}x\\ &= \int_{\bar{\mathscr{O}}}\langle b(t,x,\mu_t,\alpha_t), D^L u(\mu_t)(x)\rangle \mathrm{d}\mu_t(x) + \frac{1}{2}\int_{\bar{\mathscr{O}}}\mathrm{tr}\big(A\nabla_x D^L u(\mu_t)(x)\big)\mathrm{d}\mu_t(x).\end{aligned}$$

Proof. (i) By (H_1) and Lemma 2.3, we have that for $0 \le s < t \le T$

$$\begin{aligned}\mathbb{W}_2(\mu_t,\mu_s)^2 &\le \mathbb{E}|X_t - X_s|^2\\ &\le C\left((t-s)\mathbb{E}\int_s^t (1+|X_r|^2)\mathrm{d}r + \|\sigma\|^2(t-s)\right) \le C\big(|t-s|^2 + |t-s|\big)\end{aligned}$$

for a generic positive constant C whose value may change from line to line. Therefore, $t \mapsto \mu_t$ is absolutely continuous in $\mathscr{P}(\bar{\mathscr{O}})$. Theorem 3.4 implies that there exists a velocity v_t such that (3.2) and (3.3) hold.

Due to (3.4), for any $\psi(t,x) = \beta(t)h(x) \in C_c^\infty((0,T)\times\bar{\mathscr{O}})$,

$$\begin{aligned}&\int_0^T\int_{\mathscr{O}}\big(\beta'(t)h(x) + \langle v_t(x), \beta(t)\nabla_x h(x)\rangle\big)\mathrm{d}\mu_t(x)\mathrm{d}t\\ &\quad= \int_0^T\Big(\beta'(t)\mathbb{E}h(X_t) + \beta(t)\mathbb{E}\big[\langle v_t(X_t), \nabla_x h(X_t)\rangle\big]\Big)\mathrm{d}t = 0.\end{aligned}$$

This yields

(3.11)
$$\int_0^T \beta(t)\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}h(X_t)\,\mathrm{d}t = -\int_0^T\beta'(t)\mathbb{E}h(X_t)\,\mathrm{d}t = \int_0^T\beta(t)\mathbb{E}\big[\langle v_t(X_t), \nabla_x h(X_t)\rangle\big]\mathrm{d}t.$$

Applying Itô's formula and Green's formula, for $h \in C_c^\infty(\mathscr{O})$, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}h(X_t) &= \mathbb{E}\left[\langle b(t,X_t,\mu_t,\alpha_t), \nabla_x h(X_t)\rangle + \frac{1}{2}\mathrm{tr}\big(A\nabla_x^2 h(X_t)\big)\right]\\ &= \mathbb{E}\left[\langle b(t,X_t,\mu_t,\alpha_t), \nabla_x h(X_t)\rangle\right] - \frac{1}{2}\mathbb{E}\left[\sum_{i=1}^d \frac{\sum_{j=1}^d a_{ij}\partial_{x_j}\rho_t(X_t)}{\rho_t(X_t)}\partial_{x_i}h(X_t)\right].\end{aligned}$$

Inserting this into the left-hand side of (3.11), the arbitrariness of $\beta(t)h(x) \in C_c^\infty((0,T)\times\mathscr{O})$ can yield that $v_t(x)$ can be represented as (3.9).

(ii) Since u is Lipschitz continuous in $(\mathscr{P}(\bar{\mathscr{O}}), \mathbb{W}_2)$, there exists $C>0$ such that

$$|u(\mu_{t+\varepsilon}) - u(\mu_t\circ(\mathrm{Id}+\varepsilon v_t)^{-1})| \le C\mathbb{W}_2(\mu_{t+\varepsilon}, \mu_t\circ(\mathrm{Id}+\varepsilon v_t)^{-1}) = o(\varepsilon).$$

According to Proposition 3.5,

$$\mathbb{W}_2(\mu_{t+\varepsilon}, \mu_t\circ(\mathrm{Id}+\varepsilon v_t)^{-1}) = o(\varepsilon),$$

where v_t is given by (3.9). Thus,

$$\begin{aligned}\lim_{\varepsilon\downarrow 0}\frac{u(\mu_{t+\varepsilon}) - u(\mu_t)}{\varepsilon} &= \lim_{\varepsilon\downarrow 0}\frac{u(\mu_t\circ(\mathrm{Id}+\varepsilon v_t)^{-1}) - u(\mu_t)}{\varepsilon} = \langle D^L u(\mu_t), v_t\rangle_{\mathscr{T}_{\mu_t}}\\ &= \int_{\bar{\mathscr{O}}}\langle b(t,x,\mu_t,\alpha_t), D^L u(\mu_t)(x)\rangle\mathrm{d}\mu_t(x)\\ &\quad - \frac{1}{2}\int_{\bar{\mathscr{O}}}\langle D^L u(\mu_t)(x), A\nabla\rho_t(x)\rangle\mathrm{d}x.\end{aligned}$$

Since $\langle AD^L u(\mu), \mathbf{n}\rangle = 0$ on $\partial\mathscr{O}$, we derive from Green's formula that

$$\langle D^L u(\mu_t), v_t\rangle_{\mathscr{T}_{\mu_t}} = \int_{\bar{\mathscr{O}}} \langle b(t,x,\mu_t,\alpha_t), D^L u(\mu_t)(x)\rangle \mathrm{d}\mu_t(x) + \frac{1}{2}\int_{\bar{\mathscr{O}}} \mathrm{tr}\big(A\nabla_x D^L u(\mu_t)(x)\big)\mathrm{d}\mu_t(x).$$

We complete the proof. ∎

THEOREM 3.7 (tangent vector fields: general case). *Assume* (H_1) *and* (H_2) *hold. Let* $(X_t,k_t)_{t\in[0,T]}$ *be a solution to* (1.1) *associated with a feedback control* $\boldsymbol{\alpha}\in\Pi_{0,\mu_0}$ *in the form* $\alpha_t = F(t,X_t,\mathscr{L}_{X_t})$ *and* $\mathscr{L}_{X_0} = \mu_0 \in \mathscr{P}(\bar{\mathscr{O}})$. *Then, for every* $u \in C^1_{L,b}(\mathscr{P}(\bar{\mathscr{O}}))$ *and* $0\le t_1 < t_2 \le T$, *it holds that*

$$u(\mu_{t_2}) - \mu(\mu_{t_1}) = \int_{t_1}^{t_2}\int_{\bar{\mathscr{O}}} \Big(\langle b(t,x,\mu_t,\alpha_t), D^L u(\mu_t)(x)\rangle + \frac{1}{2}\mathrm{tr}\big(A\nabla_x D^L u(\mu_t)(x)\big)\Big)\mathrm{d}\mu_t(x)\mathrm{d}t.$$

Proof. Similar to the proof of Theorem 3.6(i), for $\mu_0\in\mathscr{P}(\bar{\mathscr{O}})$ instead of in $\mathscr{P}^r(\bar{\mathscr{O}})$, the curve (μ_t) is still an absolutely continuous curve in $\mathscr{P}(\bar{\mathscr{O}})$. The existence of the vector field v_t satisfying (3.3) in the sense of (3.4) still exists according to Theorem 3.4. Now, we cannot have the explicit expression (3.9) for v_t. Nevertheless, by (3.4), similar to the deduction in (3.11), using Itô's formula and smooth approximation, it holds that for any $\psi(t,x)\in C^{0,2}([0,T]\times\bar{\mathscr{O}})$ (i.e., $\psi(t,x)$ is continuous in t and second order continuously differentiable in x), for $0\le t_1<t_2\le T$,

$$\begin{aligned}\int_{t_1}^{t_2}\int_{\bar{\mathscr{O}}}\langle v_t(x),\nabla\psi(t,x)\rangle\mathrm{d}\mu_t(x)\mathrm{d}t &= \int_{\bar{\mathscr{O}}}\psi(t_1,x)\mathrm{d}\mu_{t_1}(x) - \int_{\bar{\mathscr{O}}}\psi(t_2,x)\mathrm{d}\mu_{t_2}(x)\\ &+ \int_{t_1}^{t_2}\int_{\bar{\mathscr{O}}}\Big(\langle b(t,x,\mu_t,\alpha_t),\nabla_x\psi(t,x)\rangle + \frac{1}{2}\mathrm{tr}\big(A\nabla_x^2\psi(t,x)\big)\Big)\mathrm{d}\mu_t(x)\mathrm{d}t.\end{aligned} \tag{3.12}$$

The relation

$$\mathbb{W}_2(\mu_{t+\varepsilon}, \mu_t\circ(\mathrm{Id}+\varepsilon v_t)^{-1}) = o(\varepsilon)$$

still holds due to Proposition 3.5, and hence for $u\in C^1_{L,b}(\mathscr{P}(\bar{\mathscr{O}}))$,

$$u(\mu_{t_2}) - u(\mu_{t_1}) = \int_{t_1}^{t_2}\frac{\mathrm{d}u(\mu_t)}{\mathrm{d}t}\mathrm{d}t = \int_{t_1}^{t_2}\int_{\bar{\mathscr{O}}}\langle D^L u(\mu_t)(x), v_t(x)\rangle\mathrm{d}\mu_t(x)\mathrm{d}t.$$

Define

$$\psi(t,x) = \int_{x_0}^{x} D^L u(\mu_t)(x)\mathrm{d}x + M \quad \text{for some } x_0\in\mathscr{O},$$

where M is a constant such that $\int_{\bar{\mathscr{O}}}\psi(0,x)\mathrm{d}\mu_0(x) = \int_{\bar{\mathscr{O}}}\psi(T,x)\mathrm{d}\mu_T(x)$. As $u\in C^1_{L,b}(\mathscr{P}(\bar{\mathscr{O}}))$, we have $\psi\in C^{0,2}([0,T]\times\bar{\mathscr{O}})$ and

$$\nabla_x\psi(t,x) = D^L u(\mu_t)(x), \quad x\in\bar{\mathscr{O}}, t\in(0,T), \tag{3.13}$$

then we derive from (3.12) the desired conclusion,

$$u(\mu_{t_2}) - \mu(\mu_{t_1}) = \int_{t_1}^{t_2}\int_{\bar{\mathscr{O}}} \Big(\langle b(t,x,\mu_t,\alpha_t), D^L u(\mu_t)(x)\rangle + \frac{1}{2}\mathrm{tr}\big(A\nabla_x D^L u(\mu_t)(x)\big)\Big)\mathrm{d}\mu_t(x)\mathrm{d}t.$$ ∎

3.2. Viscosity solutions to HJB equations. Based on Theorem 3.7 and Proposition 2.5, we shall characterize the value function as a unique viscosity solution to the following HJB equation:

$$\begin{cases} -\partial_t u(t,\mu) - \inf_{\alpha\in U} \mathcal{H}(t,\mu,u,D^L u,\alpha) = 0, & t\in[0,T), \mu\in\mathscr{P}(\bar{\mathscr{O}}),\\ u(T,\mu) = \int_{\mathscr{O}} g(x,\mu)\mathrm{d}\mu(x), & \mu\in\mathscr{P}(\bar{\mathscr{O}}), \end{cases} \tag{3.14}$$

where the Hamiltonian

$$\begin{aligned} \mathcal{H}(t,\mu,u,D^L u,\alpha) = &\int_{\mathscr{O}} \langle b(t,x,\mu,\alpha), D^L u(t,\mu)\rangle \mathrm{d}\mu(x)\\ &+ \frac{1}{2}\int_{\mathscr{O}} \mathrm{tr}\big(A\nabla_x D^L u(t,\mu)(x)\big)\mathrm{d}\mu(x)\\ &+ \int_{\mathscr{O}} \vartheta(t,x,\mu,\alpha)\mathrm{d}\mu(x). \end{aligned} \tag{3.15}$$

Let us first introduce the notation of viscosity solution for (3.14).

Definition 3.8. *Let $u:[0,T]\times\mathscr{P}(\bar{\mathscr{O}})\to\mathbb{R}$ be a continuous function.*

(i) *u is called a viscosity subsolution to* (3.14) *if $u(T,\mu)=\int_{\mathscr{O}} g(x,\mu)\mathrm{d}\mu(x)$, and*

$$-\partial_t\psi(t_0,\mu_0) - \inf_{\alpha\in U}\mathcal{H}\big(t_0,\mu_0,\psi,D^L\psi,\alpha\big) \le 0 \tag{3.16}$$

for all $\psi\in C^{1,1}_{L,b}\big([0,T)\times\mathscr{P}(\bar{\mathscr{O}})\big)$ and all $(t_0,\mu_0)\in[0,T)\times\mathscr{P}(\bar{\mathscr{O}})$ being a maximum point of $u-\psi$.

(ii) *u is called a viscosity supersolution to* (3.14) *if $u(T,\mu)=\int_{\mathscr{O}} g(x,\mu)\mathrm{d}\mu(x)$, and*

$$-\partial_t\psi(t_0,\mu_0) - \inf_{\alpha\in U}\mathcal{H}\big(t_0,\mu_0,\psi,D^L\psi,\alpha\big) \ge 0 \tag{3.17}$$

for all $\psi\in C^{1,1}_{L,b}([0,T)\times\mathscr{P}(\bar{\mathscr{O}}))$ and all $(t_0,\mu_0)\in[0,T)\times\mathscr{P}(\bar{\mathscr{O}})$ being a minimum point of $u-\psi$.

(iii) *If u is both a viscosity subsolution and a viscosity supersolution to* (3.14), *then u is called a viscosity solution to* (3.14).

Lemma 3.9. *Assume* (H_1)–(H_3) *hold. Let $\psi\in C^{1,1}_{L,b}\big([0,T)\times\mathscr{P}(\bar{\mathscr{O}})\big)$. If μ, $\mu_n\in\mathscr{P}(\bar{\mathscr{O}})$, $n\ge 1$, satisfy $\lim_{n\to\infty}\mathbb{W}_1(\mu_n,\mu)=0$, then*

$$\lim_{n\to\infty}\mathcal{H}(t,\mu_n,\psi,D^L\psi,\alpha) = \mathcal{H}(t,\mu,\psi,D^L\psi,\alpha) \quad \text{uniformlyw.r.t. } \alpha\in U. \tag{3.18}$$

Proof. We shall estimate the convergence of three terms in $\mathcal{H}(t,\mu_n,\psi,D^L\psi,\alpha)$ separately. Notice that since $\bar{\mathscr{O}}$ is compact, for every $p\ge 1$, $\lim_{n\to\infty}\mathbb{W}_p(\mu_n,\mu)=0$ is equivalent to the weak convergence of μ_n to μ (cf. [32, Chapter 6]).

First, consider the convergence of the term

$$\begin{aligned} &\int_{\mathscr{O}}\langle D^L\psi(t,\mu)(x), b(t,x,\mu,\alpha)\rangle\mathrm{d}\mu(x) - \int_{\mathscr{O}}\langle D^L\psi(t,\mu_n)(x), b(t,x,\mu_n,\alpha)\rangle\mathrm{d}\mu_n(x)\\ &\quad= \int_{\mathscr{O}}\langle D^L\psi(t,\mu)(x) - D^L\psi(t,\mu_n)(x), b(t,x,\mu,\alpha)\rangle\mathrm{d}\mu_n(x)\\ &\qquad+ \int_{\mathscr{O}}\langle D^L\psi(t,\mu_n)(x), b(t,x,\mu_n,\alpha) - b(t,x,\mu,\alpha)\rangle\mathrm{d}\mu_n(x)\\ &\qquad+ \int_{\mathscr{O}}\langle D^L\psi(t,\mu)(x), b(t,x,\mu,\alpha)\rangle\big(\mathrm{d}\mu(x) - \mathrm{d}\mu_n(x)\big)\\ &\quad=: (\mathrm{I}_1) + (\mathrm{I}_2) + (\mathrm{I}_3). \end{aligned} \tag{3.19}$$

Put

$$M_b = \sup\left\{|b(t,x,\nu,\alpha)|; (t,x,\nu,\alpha)\in[0,T]\times\bar{\mathscr{O}}\times\mathscr{P}(\bar{\mathscr{O}})\times U\right\},$$
$$M_\psi = \sup\left\{|D^L\psi(t,\nu)(x)| + |\nabla_x D^L\psi(t,\nu)(x)|; (t,x,\nu)\in[0,T]\times\bar{\mathscr{O}}\times\mathscr{P}(\bar{\mathscr{O}})\right\},$$

which are all finite due to (H_1), $\psi\in C^{1,1}_{L,b}\big([0,T)\times\mathscr{P}(\bar{\mathscr{O}})\big)$, and the compactness of $\bar{\mathscr{O}}$ and U. Then,

$$\big|(I_1)\big| \le \int_{\mathscr{O}} M_b|D^L\psi(t,\mu)(x) - D^L\psi(t,\mu_n)(x)|\mathrm{d}\mu_n(x).$$

By Definition 3.2(iii),

$$\mu_n\big(\{x\in\mathscr{O}; |D^L\psi(t,\mu_n)(x) - D^L\psi(t,\mu)(x)|\ge\varepsilon\}\big)\longrightarrow 0, \quad \text{as } n\to\infty, \tag{3.20}$$

and hence, the dominated convergence theorem yields that

$$\lim_{n\to\infty}\big|(I_1)\big| \le \lim_{n\to\infty}\int_{\mathscr{O}} M_b|D^L\psi(t,\mu)(x) - D^L\psi(t,\mu_n)(x)|\mathrm{d}\mu_n(x) = 0, \tag{3.21}$$

uniformly w.r.t. α. Next, for term (I_2), it follows from (H_1) that

$$\lim_{n\to\infty}\big|(I_2)\big| \le \lim_{n\to\infty} K_1 M_\psi \mathbb{W}_2(\mu_n,\mu) = 0, \quad \text{uniformly w.r.t. } \alpha\in U. \tag{3.22}$$

Now we proceed to estimate the term (I_3). Under the condition (H_1), one can check directly that $x\mapsto\langle D^L\psi(t,\mu)(x), b(t,x,\mu,\alpha)\rangle$ is a bounded, Lipschitz continuous function with

$$\sup_{\alpha\in U}\sup_{x\ne y}\frac{\langle D^L\psi(t,\mu)(x), b(t,x,\mu,\alpha)\rangle - \langle D^L\psi(t,\mu)(y), b(t,y,\mu,\alpha)\rangle}{|x-y|} < \infty.$$

According to the dual representation of Wasserstein distance $\mathbb{W}_1$, i.e.,

$$\mathbb{W}_1(\mu,\nu) = \sup\left\{\int_{\mathscr{O}} h(x)\mathrm{d}\mu(x) - \int_{\mathscr{O}} h(x)\mathrm{d}\nu(x);\ \sup_{x\ne y}\frac{|h(x)-h(y)|}{|x-y|}\le 1\right\}, \tag{3.23}$$

there is some constant $C>0$ such that

$$|(I_3)| \le C\mathbb{W}_1(\mu_n,\mu)\longrightarrow 0, \quad \text{as } n\to\infty, \quad \text{uniformly w.r.t. } \alpha\in U. \tag{3.24}$$

Inserting the estimates (3.21), (3.22), (3.24) into (3.19), we get

$$\lim_{n\to\infty}\int_{\mathscr{O}}\langle D^L\psi(t,\mu_n)(x), b(t,x,\mu_n,\alpha)\rangle\mathrm{d}\mu_n(x) = \int_{\mathscr{O}}\langle D^L\psi(t,\mu)(x), b(t,x,\mu,\alpha)\rangle\mathrm{d}\mu(x) \tag{3.25}$$

uniformly w.r.t. $\alpha\in U$.

Second,

$$\begin{aligned}
&\left|\int_{\mathscr{O}}\mathrm{tr}\big(A\nabla_x D^L\psi(t,\mu_n)(x)\big)\mathrm{d}\mu_n(x) - \int_{\mathscr{O}}\mathrm{tr}\big(A\nabla_x D^L\psi(t,\mu)(x)\big)\mathrm{d}\mu(x)\right|\\
&\quad\le\left|\int_{\mathscr{O}}\big[\mathrm{tr}\big(A\nabla_x D^L\psi(t,\mu_n)(x)\big) - \mathrm{tr}\big(A\nabla_x D^L\psi(t,\mu)(x)\big)\big]\mathrm{d}\mu_n(x)\right|\\
&\qquad+\left|\int_{\mathscr{O}}\mathrm{tr}\big(A\nabla_x D^L\psi(t,\mu)(x)\big)\mathrm{d}(\mu_n-\mu)(x)\right|.
\end{aligned}$$

Then, by Definition 3.2(iii) and the weak convergence of μ_n to μ, we get that

$$\lim_{n\to\infty}\Big|\int_{\mathscr{O}}\mathrm{tr}\big(A\nabla_x D^L\psi(t,\mu_n)(x)\big)\mathrm{d}\mu_n(x)-\int_{\mathscr{O}}\mathrm{tr}\big(A\nabla_x D^L\psi(t,\mu)(x)\big)\mathrm{d}\mu(x)\Big|=0. \tag{3.26}$$

At last, due to (H$_3$) and the dual representation (3.23) of $\mathbb{W}_1$,

$$\begin{aligned}&\Big|\int_{\mathscr{O}}\vartheta(t,x,\mu_n,\alpha)\mathrm{d}\mu_n(x)-\int_{\mathscr{O}}\vartheta(t,x,\mu,\alpha)\mathrm{d}\mu(x)\Big|\\ &\le\int_{\mathscr{O}}|\vartheta(t,x,\mu_n,\alpha)-\vartheta(t,x,\mu,\alpha)|\mathrm{d}\mu_n(x)+\Big|\int_{\mathscr{O}}\vartheta(t,x,\mu,\alpha)\mathrm{d}(\mu_n-\mu)(x)\Big|\\ &\le K_3\mathbb{W}_2(\mu_n,\mu)+K_3\mathbb{W}_1(\mu_n,\mu)\longrightarrow 0,\quad \text{as } n\to\infty, \text{ uniformly w.r.t. } \alpha\in U.\end{aligned} \tag{3.27}$$

Consequently, the desired conclusion (3.18) follows immediately from (3.27), (3.26), and (3.25). The proof is complete. ∎

LEMMA 3.10. *Assume* (H$_1$) *and* (H$_2$) *hold. Then for any* $\mu_{t_0}\in\mathscr{P}(\bar{\mathscr{O}})$ *and* $\Theta\in\Pi_{t_0,\mu_{t_0}}$, *the law* $\mathscr{L}_{X_t}$ *of the controlled process* X_t *satisfies that there exists* $C>0$ *such that*

$$\begin{aligned}&\mathbb{W}_1(\mu_t,\mu_{t_0})\le C\sqrt{t-t_0},\quad t\in[t_0,T],\\ &\|\mu_t-\mu_s\|_{\mathrm{var}}\le C\big(\ln(t-t_0)-\ln(s-t_0)\big),\quad t,s\in(t_0,T].\end{aligned}$$

Proof. If $s=t_0$, for any h with $|h|_{Lip}:=\sup_{x\neq y}\frac{|h(x)-h(y)|}{|x-y|}\le 1$, by (3.8) and (3.7),

$$\begin{aligned}&\Big|\int_{\bar{\mathscr{O}}}h(x)\mathrm{d}\mu_t(x)-\int_{\bar{\mathscr{O}}}h(x)\mathrm{d}\mu_{t_0}(x)\Big|\\ &=\Big|\int_{\bar{\mathscr{O}}}\int_{\bar{\mathscr{O}}}h(x)p(t_0,y;t,x)\mathrm{d}\mu_{t_0}(y)\mathrm{d}x-\int_{\bar{\mathscr{O}}}h(y)\mathrm{d}\mu_{t_0}(y)\Big|\\ &\le\int_{\bar{\mathscr{O}}}\int_{\bar{\mathscr{O}}}p(t_0,y;t,x)|h(x)-h(y)|\mathrm{d}x\mathrm{d}\mu_{t_0}(y)\Big|\\ &\le\int_{\bar{\mathscr{O}}}\int_{\mathbb{R}^d}\frac{\kappa_1}{|t-t_0|^{\frac{d}{2}}}\mathrm{e}^{-\kappa_2\frac{|x-y|^2}{t-t_0}}|h(x)-h(y)|\mathrm{d}x\mathrm{d}\mu_{t_0}(y)\\ &=\int_{\bar{\mathscr{O}}}\int_{\mathbb{R}^d}\kappa_1\mathrm{e}^{-\kappa_2|z|^2}|h(y+\sqrt{t-t_0}z)-h(y)|\mathrm{d}z\mathrm{d}\mu_{t_0}(y)\\ &\le\kappa_1\sqrt{t-t_0}\int_{\mathbb{R}^d}|z|\mathrm{e}^{-\kappa_2|z|^2}\mathrm{d}z.\end{aligned}$$

Thus, the dual representation (3.23) of $\mathbb{W}_1$ yields

$$\mathbb{W}_1(\mu_t,\mu_{t_0})\le C\sqrt{t-t_0}.$$

If $s>t_0$, for any continuous function h with $|h|_\infty:=\sup_{x\in\mathbb{R}^d}|h(x)|\le 1$,

$$\begin{aligned}&\Big|\int_{\bar{\mathscr{O}}}h(x)\mathrm{d}\mu_t(x)-\int_{\bar{\mathscr{O}}}h(x)\mathrm{d}\mu_s(x)\Big|\le\Big|\int_{\bar{\mathscr{O}}}\int_{\bar{\mathscr{O}}}|h(x)|\big|p(t_0,y;t,x)-p(t_0,y;s,x)\big|\mathrm{d}x\mathrm{d}\mu_{t_0}(y)\\ &\le\int_{\bar{\mathscr{O}}}\int_{\bar{\mathscr{O}}}\int_s^t\big|\partial_r p(t_0,y;r,x)\big|\mathrm{d}r\mathrm{d}x\mathrm{d}\mu_{t_0}(y)\le\int_{\bar{\mathscr{O}}}\int_{\mathbb{R}^d}\int_s^t\frac{\kappa_1}{(r-t_0)^{\frac{d}{2}+1}}\mathrm{e}^{-\kappa_2\frac{|x-y|^2}{r-t_0}}\mathrm{d}r\mathrm{d}x\mathrm{d}\mu_{t_0}(y)\\ &\le\kappa_1\ln\Big(\frac{t-t_0}{s-t_0}\Big)\int_{\mathbb{R}^d}\mathrm{e}^{-\kappa_2|z|^2}\mathrm{d}z.\end{aligned}$$

The proof is complete. ∎

We shall use the estimate of $\mathbb{W}_1(\mu_t,\mu_s)$ below, and the estimate of the total variation distance between μ_t and μ_s is presented as a supplemental property.

THEOREM 3.11. *Under the conditions* (H_1)–(H_3)*, the value function* $V(t,\mu)$ *given in* (2.5) *is a viscosity solution to the HJB equation* (3.14)*.*

Proof. Viscosity subsolution. According to Proposition 2.4, V is a continuous function. Let $(t_0,\mu_{t_0})\in[0,T)\times\mathscr{P}(\bar{\mathscr{O}})$ and $\psi\in C^{1,1}_{L,b}\big([0,T)\times\mathscr{P}(\bar{\mathscr{O}})\big)$ be a test function such that

$$0=(V-\psi)(t_0,\mu_{t_0})=\max\left\{(V-\psi)(t,\mu);\ (t,\mu)\in[0,T)\times\mathscr{P}(\bar{\mathscr{O}})\right\}.$$

Let (X_t,k_t) be the solution to SDE (1.1) associated with the control $\alpha_t\equiv\alpha\in U$ with initial value $\mathscr{L}_{X_{t_0}}=\mu_{t_0}$. Denote $\mu_t=\mathscr{L}_{X_t}$ for $t\geq t_0$. By the dynamic programming principle,

$$V(t_0,\mu_{t_0})\leq\mathbb{E}\left[\int_{t_0}^t\vartheta(r,X_r,\mu_r,\alpha)\mathrm{d}r+V(t,\mu_t)\right],$$

which yields that

$$\psi(t,\mu_t)-\psi(t_0,\mu_{t_0})+\int_{t_0}^t\int_{\bar{\mathscr{O}}}\vartheta(r,x,\mu_r,\alpha)\mathrm{d}\mu_r(x)\mathrm{d}r\geq 0. \tag{3.28}$$

By Theorem 3.7, we get

$$\begin{aligned}\int_{t_0}^t\Bigg[&\partial_r\psi(r,\mu_r)+\int_{\bar{\mathscr{O}}}\Big(\langle b(r,x,\mu_r,\alpha),D^L\psi(r,\mu_r)(x)\rangle\\&+\frac{1}{2}\mathrm{tr}\big(A\nabla_xD^L\psi(r,\mu_r)(x)\big)\Big)\mathrm{d}\mu_r(x)\\&+\int_{\bar{\mathscr{O}}}\vartheta(r,x,\mu_r,\alpha)\mathrm{d}\mu_r(x)\Bigg]\mathrm{d}r\geq 0.\end{aligned}$$

Using Lemmas 3.9 and 3.10, dividing both sides of the previous inequality with $t-t_0$, and letting $t\downarrow t_0$, we obtain that

$$\begin{aligned}&-\partial_t\psi(t_0,\mu_{t_0})-\int_{\bar{\mathscr{O}}}\Big(\langle b(t_0,x,\mu_{t_0},\alpha),D^L\psi(t_0,\mu_{t_0})(x)\rangle\\&\quad+\frac{1}{2}\mathrm{tr}\big(A\nabla_xD^L\psi(t_0,\mu_{t_0})(x)\big)\Big)\mathrm{d}\mu_{t_0}(x)\\&\quad-\int_{\bar{\mathscr{O}}}\vartheta(t_0,x,\mu_{t_0},\alpha)\mathrm{d}\mu_{t_0}(x)\leq 0.\end{aligned}$$

By the arbitrariness of $\alpha\in U$, we obtain that

$$-\partial_t\psi(t_0,\mu_0)-\inf_{\alpha\in U}\mathcal{H}(t_0,\mu_{t_0},\psi,D^L\psi,\alpha)\leq 0.$$

Hence, V is a viscosity subsolution to (3.14).

Viscosity supersolution. Let $(t_0,\mu_{t_0})\in[0,T)\times\mathscr{P}(\bar{\mathscr{O}})$, $\psi\in C^{1,1}_{L,b}\big([0,T)\times\mathscr{P}(\bar{\mathscr{O}})\big)$ such that

$$0=(V-\psi)(t_0,\mu_{t_0})=\min\left\{(V-\psi)(t,\mu);\ (t,\mu)\in[0,T)\times\mathscr{P}(\bar{\mathscr{O}})\right\}. \tag{3.29}$$

We shall prove

$$-\partial_t\psi(t_0,\mu_{t_0}) - \inf_{\alpha\in U}\mathcal{H}(t_0,\mu_{t_0},\psi,D^L\psi,\alpha)\geq 0 \tag{3.30}$$

by contradiction. Suppose

$$-\partial_t\psi(t_0,\mu_{t_0}) - \inf_{\alpha\in U}\mathcal{H}(t_0,\mu_{t_0},\psi,D^L\psi,\alpha)< 0. \tag{3.31}$$

For any $\boldsymbol{\alpha}\in\Pi_{t_0,\mu_{t_0}}$, the associated controlled process $(X_t^{t_0,\mu_{t_0}},k_t^{t_0,\mu_{t_0}})_{t\in[t_0,T]}$ is given in (1.1). Under the nondegenerate condition (H_2), the law of $X_t^{t_0,\mu_{t_0}}$ admits a density $\rho_t(x)$. Due to (3.8), ρ_t admits a representation

$$\rho_t(x)=\int_{\bar{\mathscr{O}}} p(t_0,z;t,x)\mathrm{d}\mu_{t_0}(z).$$

Therefore, by Lemmas 3.9 and 3.10, there exist ε, $\zeta_1>0$ such that for any $|t-t_0|<\zeta_1$ and any $\boldsymbol{\alpha}\in\Pi_{t_0,\mu_{t_0}}$,

(3.32)

$$\begin{aligned}&-\partial_t\psi(t,\mu_t)-\int_{\bar{\mathscr{O}}}\Big(\langle b(t,x,\mu_t,\alpha_t),D^L\psi(t,\mu_t)(x)\rangle+\frac12\mathrm{tr}\big(A\nabla_x D^L\psi(t,\mu_t)(x)\big)\Big)\mathrm{d}\mu_t(x)\\ &\quad-\int_{\bar{\mathscr{O}}}\vartheta(t,x,\mu_t,\alpha_t)\mathrm{d}\mu_t(x)\leq-\varepsilon.\end{aligned}$$

Take two sequences δ_n, $\gamma_n>0$, $n\geq 1$, satisfying

$$\delta_n<\zeta_1,\quad \lim_{n\to\infty}\gamma_n/\delta_n=0.$$

By the dynamic programming principle, there exists a sequence of admissible feedback controls $\boldsymbol{\alpha}_n\in\Pi_{t_0,\mu_{t_0}}$ such that

$$V(t_0,\mu_0)\geq\mathbb{E}\bigg[\int_{t_0}^{t_0+\delta_n}\vartheta(r,X_r^n,\mu_r^n,\alpha_r^n)\mathrm{d}r+V(t_0+\delta_n,\mu_{t_0+\delta_n}^n)\bigg]-\gamma_n,$$

where (X_t^n) denotes the controlled process associated with $\boldsymbol{\alpha}_n$, and μ_t^n denotes the law of X_t^n. Due to (3.29),

$$\psi(t_0,\mu_0)\geq\mathbb{E}\bigg[\int_{t_0}^{t_0+\delta_n}\vartheta(r,X_r^n,\mu_r^n,\alpha_r^n)\mathrm{d}r+\psi(t_0+\delta_n,\mu_{t_0+\delta_n}^n)\bigg]-\gamma_n.$$

Hence,

$$\frac{\gamma_n}{\delta_n}\geq\frac{1}{\delta_n}\mathbb{E}\bigg[\int_{t_0}^{t_0+\delta_n}\vartheta(r,X_r^n,\mu_r^n,\alpha_r^n)\mathrm{d}r+\int_{t_0}^{t_0+\delta_n}\frac{\mathrm{d}}{\mathrm{d}r}\big(\psi(r,\mu_r^n)\big)\mathrm{d}r\bigg].$$

Since $\psi\in C_{L,b}^{1,1}\big([0,T)\times\mathscr{P}(\bar{\mathscr{O}})\big)$, by Theorem 3.7 and (3.32),

$$\begin{aligned}\frac{\gamma_n}{\delta_n}&\geq\frac{1}{\delta_n}\int_{t_0}^{t_0+\delta_n}\bigg[\partial_r\psi(r,\mu_r^n)+\int_{\bar{\mathscr{O}}}\Big(\langle b(r,x,\mu_r^n,\alpha_r^n),D^L\psi(r,\mu_r^n)(x)\rangle\\ &\qquad+\frac12\mathrm{tr}\big(A\nabla_x D^L\psi(r,\mu_r^n)(x)\big)+\vartheta(r,x,\mu_r^n,\alpha_r^n)\Big)\mathrm{d}\mu_r^n(x)\bigg]\mathrm{d}r\\ &\geq\frac{1}{\delta_n}\int_{t_0}^{t_0+\delta_n}\varepsilon\mathrm{d}r=\varepsilon>0.\end{aligned}$$

Letting $n \to \infty$, this contradicts $\lim_{n\to\infty} \frac{\gamma_n}{\delta_n} = 0$. Consequently, the assertion (3.32) is false, and $V(t,\mu)$ is a viscosity supersolution to the HJB equation (3.14). In all, according to Definition 3.8, V is a viscosity solution to (3.14). ∎

4. Comparison principle for HJB equations. In this part we proceed to study the uniqueness of the viscosity solution to the HJB equation (3.14) associated with the intrinsic derivative. To this aim, the crucial point is to find suitable test functions to approximate the viscosity solution. In the study of HJB equations on $\mathbb{R}^d$, the Euclidean distance $|x-y|^2$ plays an important role in the argument of the comparison principle. On $\mathscr{P}(\bar{\mathscr{O}})$, although $\mathbb{W}_2$ is intrinsically differentiable (cf. Proposition 4.2 below or [3, Theorem 8.4.7]), the L^2-Wasserstein distance $\mathbb{W}_2$ is not smooth enough to establish the comparison principle for the HJB equation (3.14) on $\mathscr{P}(\bar{\mathscr{O}})$, which will be clarified in the study below.

The regularity of $\mathbb{W}_2$ w.r.t. the intrinsic derivative depends heavily on the theory of optimal transport maps between probability measures, which essentially depends on the study of the Monge–Ampère equation. A large number of works have been devoted to the study of the Monge–Ampère equation. We refer to the works of Trudinger and Wang [31], Caffarelli and McCann [9], and Chen, Liu, and Wang [13], among others.

Let us recall a result in [13] to be used later.

THEOREM 4.1 (see [13, Theorem 1.1]). *Suppose $\mathcal{O},\mathcal{O}^*$ are bounded convex domains in $\mathbb{R}^d$ with $C^{1,1}$ boundary. Suppose u is a convex solution to the Monge–Ampère equation*

$$\begin{cases} \det\big(D^2u(x)\big) = \frac{\rho(x)}{\tilde\rho(Du(x))}, & x \in \mathcal{O},\\ Du(\mathcal{O}) = \mathcal{O}^*, \end{cases} \tag{4.1}$$

where $\det(B)$ stands for the determinant of matrix B. The following assertion holds: if $\rho \in C^\beta(\bar{\mathcal{O}})$, $\tilde\rho \in C^\beta(\bar{\mathcal{O}}^*)$ for some $\beta \in (0,1)$, then

$$\|u\|_{C^{2,\beta}(\bar{\mathcal{O}})} \le C,$$

where C is a constant depending on $d,\beta,\rho,\tilde\rho,\mathcal{O}$, and $\mathcal{O}^*$.

Applying the theory on optimal transport maps between probability measures (cf., for example, [32]), for two probability measures $\mu = \rho(x)\mathrm{d}x$ and $\nu = \tilde\rho(x)\mathrm{d}x$ on $\mathbb{R}^d$, there exists a convex function $u: \mathbb{R}^d \to \mathbb{R}^d$ such that the mapping $\mathcal{T}_\nu^\mu(x) := Du(x)$ satisfies

$$\nu = (\mathcal{T}_\nu^\mu)_\# \mu := \mu \circ (\mathcal{T}_\nu^\mu)^{-1}, \text{ i.e., } \int h(x)\mathrm{d}\nu(x) = \int h(\mathcal{T}_\nu^\mu(x))\mathrm{d}\mu(x), \quad \forall\, h \in \mathscr{B}_b(\mathbb{R}^d),$$

$$\mathbb{W}_2^2(\mu,\nu) = \int_{\mathbb{R}^d} |x - \mathcal{T}_\nu^\mu(x)|^2 \mathrm{d}\mu(x).$$

Thus, u is a solution to the Monge–Ampère equation:

$$\det\big(D^2u(x)\big) = \frac{\rho(x)}{\tilde\rho(Du(x))}.$$

Moreover, although u is not unique, its gradient and hence the mapping $\mathcal{T}_\nu^\mu$ are unique and invertible. Also, the convexity of u yields that $D^2u \ge 0$. Here we present a result for the cost function $|x-y|^2$, and much effort has been devoted to the study on the general cost functions and on general spaces (cf. [3, 31, 32]).

PROPOSITION 4.2 (derivative of Wasserstein distance). *For each $\zeta\in\mathscr{P}^r(\bar{\mathscr{O}})$, the associated functional $\mu\mapsto W_2^2(\mu,\zeta)$ belongs to $C^1_{L,b}(\mathscr{P}^r(\bar{\mathscr{O}}))$ and*

$$D^L\mathbb{W}_2^2(\mu,\zeta)(x)=2\big(x-\mathcal{T}_\zeta^\mu(x)\big),\quad x\in\bar{\mathscr{O}},\,\mu\in\mathscr{P}^r(\bar{\mathscr{O}}),\tag{4.2}$$

where $\mathcal{T}_\zeta^\mu:\bar{\mathscr{O}}\to\bar{\mathscr{O}}$ denotes the unique optimal transport map such that

$$(\mathcal{T}_\zeta^\mu)_\#\mu=\zeta \text{ and } \mathbb{W}_2^2(\mu,\zeta)=\int_{\bar{\mathscr{O}}}|x-\mathcal{T}_\zeta^\mu(x)|^2\mathrm{d}\mu(x).$$

Proof. For $\mu\in\mathscr{P}^r(\bar{\mathscr{O}})$, and for any tangent vector $v\in\mathscr{T}_\mu$, the curve $\mu_\varepsilon:=\mu\circ(\mathrm{Id}+\varepsilon v)^{-1}$ for $\varepsilon\in[0,1]$ is an absolutely continuous curve in $\mathscr{P}(\bar{\mathscr{O}})$. According to [3, Theorem 8.4.7],

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Big|_{\varepsilon=0}\mathbb{W}_2^2(\mu_\varepsilon,\zeta)=\int_{\bar{\mathscr{O}}^2}2\langle x_1-x_2,v(x_1)\rangle\mathrm{d}\gamma(x_1,x_2),\tag{4.3}$$

where γ is a probability measure on $\bar{\mathscr{O}}\times\bar{\mathscr{O}}$ satisfying

$$\int_{\bar{\mathscr{O}}^2}f(x_1)+g(x_2)\mathrm{d}\gamma(x_1,x_2)=\int_{\bar{\mathscr{O}}}f(x_1)\mathrm{d}\mu(x_1)+\int_{\bar{\mathscr{O}}}g(x_2)\mathrm{d}\zeta(x_2),\quad f,g\in\mathcal{B}_b(\bar{\mathscr{O}}),$$
$$\int_{\bar{\mathscr{O}}^2}|x_1-x_2|^2\mathrm{d}\gamma(x_1,x_2)=\mathbb{W}_2^2(\mu,\zeta).$$

By virtue of the results on optimal transport maps (cf., e.g., [3, Chapter 6]), since $\mu\in\mathscr{P}^r(\bar{\mathscr{O}})$ admits a density w.r.t. the Lebesgue measure, the previous optimal plan γ is uniquely determined by $\gamma=(\mathrm{Id}\times\mathcal{T}_\zeta^\mu)_\#\mu$. Moreover, there exists a function $u:\bar{\mathscr{O}}\to\mathbb{R}$ satisfying the Monge–Ampère equation (4.1) with $\bar{\mathscr{O}}^*=\bar{\mathscr{O}}$, $\rho=\mathrm{d}\mu/\mathrm{d}x$, and $\tilde\rho=\mathrm{d}\zeta/\mathrm{d}x$ such that

$$\mathcal{T}_\zeta^\mu(x)=Du(x).$$

By Theorem 4.1, as $\rho,\tilde\rho\in C^1(\bar{\mathscr{O}})$, for each $\beta\in(0,1)$, there exists a constant $C>0$ such that

$$\|u\|_{C^{2,\beta}(\bar{\mathscr{O}})}\le C.$$

This yields that $\mathcal{T}_\zeta^\mu=Du$ is in $C^{1,\beta}(\bar{\mathscr{O}})$. Consequently, we can rewrite (4.3) to

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Big|_{\varepsilon=0}\mathbb{W}_2^2(\mu_\varepsilon,\zeta)=\int_{\bar{\mathscr{O}}}2\langle x_1-\mathcal{T}_\zeta^\mu(x_1),v(x_1)\rangle\mathrm{d}\mu(x_1),\quad v\in\mathscr{T}_\mu.$$

This yields that $\mathbb{W}_2^2(\cdot,\zeta)$ is intrinsically differentiable at μ with

$$D^L\mathbb{W}_2^2(\mu,\zeta)(x)=2(x-\mathcal{T}_\zeta^\mu(x)).$$

Moreover, as $u\in C^{2,\beta}(\bar{\mathscr{O}})$, $x\mapsto D^L\mathbb{W}_2^2(\mu,\zeta)(x)=2(x-Du(x))$ is continuous in $x\in\bar{\mathscr{O}}$. Obviously, $D^L\mathbb{W}_2^2(\mu,\zeta)$ is also bounded as $\bar{\mathscr{O}}$ is compact. ∎

Remark 4.3. From Proposition 4.2 we can see that $\mu\mapsto\mathbb{W}_2^2(\mu,\zeta)$ is intrinsically differentiable on the subset $\mathscr{P}^r(\bar{\mathscr{O}})$. To ensure the existence of second order differentiability of $x\mapsto D^L\mathbb{W}_2^2(\mu,\zeta)(x)$, a further smoothness condition on the densities of μ and ζ is needed, to act as a smooth approximation function to the HJB equation (3.14). However, the completion of $\mathscr{P}^r(\bar{\mathscr{O}})$ under the metric $\mathbb{W}_2$ will be $\mathscr{P}(\bar{\mathscr{O}})$, which cannot guarantee the desired smoothness of densities. Thus, $\mathbb{W}_2^2$ is not an appropriate smooth approximation function to study the uniqueness of the viscosity solution to the HJB equation (3.14).

Recently, Burzoni et al. [8] studied the optimal control problem for McKean–Vlasov jump-diffusion processes and the viscosity solution to HJB equations on the Wasserstein space in terms of the linear functional derivative. To establish the comparison principle, they constructed a distance-like function by

$$d(\mu,\nu)=\sum_{j=1}^{\infty}c_j\langle\mu-\nu,f_j\rangle^2,$$

where the countable set $\{f_j\}_{j\in\mathbb{N}}$ is carefully constructed so that the linear functional derivative of d can be estimated by itself. The construction of $\{f_j\}_{j\in\mathbb{N}}$ is very subtle especially in the presence of jumps in the controlled process. [8] only constructed $d(\mu,\nu)$ on $\mathscr{P}(\mathbb{R})$.

In this work we shall generalize the construction of $d(\mu,\nu)$ in [8] to the Wasserstein space over $\mathscr{O}\subset\mathbb{R}^d$. Moreover, we shall show that such a distance-like function is also useful to establish the comparison principle for HJB equations on the Wasserstein space with intrinsically differential structure. Unfortunately, an additional assumption on the drift b is needed like in [8], that is, the drift $b(t,x,\mu,\alpha)$ cannot depend on variable x and depends on μ via its moments.

Before establishing the comparison principle, we introduce the generalization of the distance-like function $d(\mu,\nu)$ on $\mathscr{P}(\bar{\mathscr{O}})$. Let us begin with the 1-dimensional case by recalling the construction in [8]. Since our controlled processes are diffusion processes without jumps, we can simplify the expression $\{f_j\}_{j\in\mathbb{N}}$ in [8].

A set of polynomials χ is said to have an $(*)$-property if it satisfies that

$$\text{for any } f\in\chi, f^{(i)}\in\chi, \forall\, i\geq 0,$$

where $f^{(i)}$ denotes ith order derivative of f with $f^{(0)}=f$. For any given polynomial f, let $\chi(f)$ be the smallest set of polynomials with the $(*)$-property that includes f. So, $\chi(f)=\{f^{(i)};i\geq 0\}$. Put

$$\widetilde{\Theta}=\bigcup_{j=1}^{\infty}\chi(x^j).$$

Then $\widetilde{\Theta}$ contains all monomials $\{x^j\}_{j=1}^{\infty}$, it is countable, and $\chi(f)\subset\widetilde{\Theta}$ for every $f\in\widetilde{\Theta}$. Let $\{f_j\}_{j=1}^{\infty}$ be an enumeration of $\widetilde{\Theta}$, which is fixed in what follows. We refer to [8] for more discussion on χ, $\chi(f)$, and the $(*)$-property.

Now we generalize the previous notions to the multidimensional situation. Denote $\mathbb{Z}_+=\{0,1,2,\ldots\}$, and $\boldsymbol{x}^{\boldsymbol{n}}=x_1^{n_1}x_2^{n_2}\cdots x_d^{n_d}$ for $\boldsymbol{n}=(n_1,\ldots,n_d)\in\mathbb{Z}_+^d$, $\boldsymbol{x}=(x_1,\ldots,x_d)\in\mathbb{R}^d$. Let $|\boldsymbol{n}|=n_1+\cdots+n_d$. For $\mu\in\mathscr{P}(\bar{\mathscr{O}})$ and $f:\bar{\mathscr{O}}\to\mathbb{R}^k$, denote by $\langle\mu,f\rangle=\int_{\bar{\mathscr{O}}}f(x)\mathrm{d}\mu(x)$. Based on the above fixed enumeration $\{f_j\}_{j=1}^{\infty}$ of $\widetilde{\Theta}$, we define $f_{\boldsymbol{n}}(\boldsymbol{x})=f_{n_1}(x_1)f_{n_2}(x_2)\cdots f_{n_d}(x_d)$, for $\boldsymbol{n}=(n_1,\ldots,n_d)$, and

$$s_{\boldsymbol{n}}=1+\sup\nolimits_{\mu\in\mathscr{P}(\bar{\mathscr{O}})}\langle\mu,f_{\boldsymbol{n}}\rangle^2.$$

Put

$$\chi(f_{\boldsymbol{n}})=\{g_1(x_1)g_2(x_2)\cdots g_d(x_d);g_i\in\chi(f_{n_i}),1\leq i\leq d\},$$

which contains all the partial derivatives of $f_{\boldsymbol{n}}$. Since $\chi(f_{\boldsymbol{n}})$ contains a finite number of polynomials, there exists a finite index set $\mathcal{I}_{\boldsymbol{n}}$ satisfying

$$\chi(f_{\boldsymbol{n}})=\{f_{\boldsymbol{m}};\ \boldsymbol{m}\in\mathcal{I}_{\boldsymbol{n}}\}.$$

Let

$$c_{\boldsymbol{n}}=\left(\sum_{\boldsymbol{m}\in\mathcal{I}_{\boldsymbol{n}}}2^{|\boldsymbol{m}|}C_{d-1}^{|\boldsymbol{m}|+d-1}\right)^{-1}\left(\sum_{\boldsymbol{m}\in\mathcal{I}_{\boldsymbol{n}}}s_{\boldsymbol{m}}\right)^{-1},\ \boldsymbol{n}\in\mathbb{Z}_+^d,$$

where $C_k^m=\frac{m!}{k!(m-k)!}$. As $s_{\boldsymbol{m}}\ge 1$ and $f_{\boldsymbol{n}}\in\chi(f_{\boldsymbol{n}})$, it holds that

$$c_{\boldsymbol{n}}\le\frac{2^{-|\boldsymbol{n}|}}{C_{d-1}^{|\boldsymbol{n}|+d-1}}.$$

If $\boldsymbol{l}\in\mathcal{I}_{\boldsymbol{n}}$, then $f_{\boldsymbol{l}}\in\chi(f_{\boldsymbol{n}})$, $\chi(f_{\boldsymbol{l}})\subset\chi(f_{\boldsymbol{n}})$, and hence $\mathcal{I}_{\boldsymbol{l}}\subset\mathcal{I}_{\boldsymbol{n}}$. By the definition of $c_{\boldsymbol{n}}$, this implies that

$$c_{\boldsymbol{l}}\ge c_{\boldsymbol{n}}. \tag{4.4}$$

LEMMA 4.4. *Define a function* $S:\mathscr{P}(\bar{\mathscr{O}})\to\mathbb{R}$ *by*

$$S(\mu)=\sum_{k=1}^{\infty}\sum_{|\boldsymbol{n}|=k}c_{\boldsymbol{n}}\langle\mu,f_{\boldsymbol{n}}\rangle^2. \tag{4.5}$$

Then S *satisfies* $S(\mu)\le 1$ *for every* $\mu\in\mathscr{P}(\bar{\mathscr{O}})$ *and is intrinsically differentiable with*

$$D^LS(\mu)(x)=\sum_{k=1}^{\infty}\sum_{|\boldsymbol{n}|=k}2c_{\boldsymbol{n}}\langle\mu,f_{\boldsymbol{n}}\rangle\nabla f_{\boldsymbol{n}}(x). \tag{4.6}$$

Proof. Noting that $\boldsymbol{n}\in\mathcal{I}_{\boldsymbol{n}}$, we have

$$S(\mu)\le\sum_{k=1}^{\infty}\sum_{|\boldsymbol{n}|=k}2^{-k}\big(C_{d-1}^{k+d-1}\big)^{-1}\frac{\langle\mu,f_{\boldsymbol{n}}\rangle^2}{s_{\boldsymbol{n}}}\le\sum_{k=1}^{\infty}2^{-k}=1.$$

For any $v\in\mathscr{T}_\mu$,

$$\begin{aligned}D_v^LS(\mu)&=\lim_{\varepsilon\to0}\frac{S(\mu\circ(\mathrm{Id}+\varepsilon v)^{-1})-S(\mu)}{\varepsilon}\\&=\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Big|_{\varepsilon=0}\sum_{k=1}^{\infty}\sum_{|\boldsymbol{n}|=k}c_{\boldsymbol{n}}\left(\int_{\mathscr{O}}f_{\boldsymbol{n}}(x+\varepsilon v(x))\mathrm{d}\mu(x)\right)^2\\&=\sum_{k=1}^{\infty}\sum_{|\boldsymbol{n}|=k}2c_{\boldsymbol{n}}\langle\mu,f_{\boldsymbol{n}}\rangle\int_{\mathscr{O}}\langle\nabla f_{\boldsymbol{n}}(x),v(x)\rangle\mathrm{d}\mu(x).\end{aligned}$$

Therefore, S is intrinsically differentiable with $D^LS(\mu)$ given by (4.6). ◻

We need to modify the condition of the drift b to establish the comparison principle.

(H$_1'$) The drift $b(t,x,\mu,\alpha)$ does not depend on x. There are $K_4>0$ and a finite set $\mathcal{I}\subset\mathbb{N}^d$ such that for any μ, $\nu\in\mathscr{P}(\bar{\mathscr{O}})$, $\alpha\in U$, t, $s\in[0,T]$,

$$|b(t,\mu,\alpha)|^2\le K_4,\quad |b(t,\mu,\alpha)-b(s,\nu,\alpha)|^2\le K_4\left(|t-s|^2+\textstyle\sum_{\boldsymbol{i}\in\mathcal{I}}\langle\mu-\nu,\boldsymbol{x}^{\boldsymbol{i}}\rangle^2\right).$$

This condition is especially suitable to deal with SDEs with drifts depending only on the moments of (X_t), for example,

$$\mathrm{d}X_t=b(t,\mathbb{E}[X_t],\mathbb{E}[|X_t|^2],\alpha_t)\mathrm{d}t+\sigma\mathrm{d}B_t-\mathbf{n}(X_t)\mathrm{d}k_t.$$

THEOREM 4.5 (comparison principle). *Suppose that* (H_1'), (H_2), *and* (H_3) *hold. Let* W *and* V *be, respectively, a viscosity subsolution and a viscosity supersolution to the HJB equation* (3.14) *satisfying the continuity property* (2.7). *Then*

$$W(t,\mu)\le V(t,\mu),\qquad t\in[0,T),\ \mu\in\mathscr{P}(\bar{\mathscr{O}}). \tag{4.7}$$

Proof. We prove (4.7) by contradiction. Assume there is $(\tilde{t},\bar{\mu})\in[0,T)\times\mathscr{P}(\bar{\mathscr{O}})$ such that

$$W(\tilde{t},\bar{\mu})>V(\tilde{t},\bar{\mu}). \tag{4.8}$$

Then, by the continuity of W and V, there exist some $(\bar{t},\bar{\mu})\in(0,T)\times\mathscr{P}(\bar{\mathscr{O}})$ such that

$$W(\bar{t},\bar{\mu})>V(\bar{t},\bar{\mu}). \tag{4.9}$$

Consider the auxiliary function

$$\Phi(t,s,\mu,\nu)=W(t,\mu)-V(s,\nu)-\varphi(t,s,\mu,\nu),\quad t,s\in(0,T],\ \mu,\nu\in\mathscr{P}(\bar{\mathscr{O}}),$$

with

$$\varphi(t,s,\mu,\nu)=\frac{1}{2\delta}\big(|t-s|^2+S(\mu-\nu)\big)+\beta(2T-t-s)+\frac{\lambda}{t}+\frac{\lambda}{s} \tag{4.10}$$

for parameters $\beta,\lambda,\delta\in(0,1)$, where the functional S is defined in Lemma 4.4. Due to the compactness of $[0,T]\times[0,T]\times\mathscr{P}(\bar{\mathscr{O}})\times\mathscr{P}(\bar{\mathscr{O}})$ when $\mathscr{P}(\bar{\mathscr{O}})$ is endowed with weak convergence topology, there exists a point $(t_0,s_0,\mu_0,\nu_0)\in(0,T]\times(0,T]\times\mathscr{P}(\bar{\mathscr{O}})\times\mathscr{P}(\bar{\mathscr{O}})$ such that

$$\Phi(t_0^\tau,s_0^\tau,\mu_0^\tau,\nu_0^\tau)=\sup\big\{\Phi(t,s,\mu,\nu);t,s\in[0,T],\mu,\nu\in\mathscr{P}(\bar{\mathscr{O}})\big\}. \tag{4.11}$$

Notice that $(t_0^\tau,s_0^\tau,\mu_0^\tau,\nu_0^\tau)$ depend on the parameters $\tau=(\beta,\lambda,\delta)$. By (4.11),

$$2\Phi(t_0^\tau,s_0^\tau,\mu_0^\tau,\nu_0^\tau)\ge\Phi(t_0^\tau,t_0^\tau,\mu_0^\tau,\mu_0^\tau)+\Phi(s_0^\tau,s_0^\tau,\nu_0^\tau,\nu_0^\tau),$$

which yields that

$$W(t_0^\tau,\mu_0^\tau)-W(s_0^\tau,\nu_0^\tau)+V(t_0^\tau,\mu_0^\tau)-V(s_0^\tau,\nu_0^\tau)\ge\frac{1}{\delta}\big(|t_0^\tau-s_0^\tau|^2+S(\mu_0^\tau-\nu_0^\tau)\big). \tag{4.12}$$

Since W and V are bounded, which follows from the boundedness of ϑ and g, this implies

$$\lim_{\delta\downarrow 0}|t_0^\tau-s_0^\tau|^2+S(\mu_0^\tau-\nu_0^\tau)=0. \tag{4.13}$$

Due to the compactness of $[0,T]\times[0,T]\times\mathscr{P}(\bar{\mathscr{O}})\times\mathscr{P}(\bar{\mathscr{O}})$, there is a subsequence $(t_0^{\tau_k},s_0^{\tau_k},\mu_0^{\tau_k},\nu_0^{\tau_k})$ of $(t_0^\tau,s_0^\tau,\mu_0^\tau,\nu_0^\tau)$ satisfying $\tau_k=(\beta,\lambda_k,\delta_k)\longrightarrow(\beta,0,0)$ as $k\to\infty$, and

$$(t_0^{\tau_k},s_0^{\tau_k},\mu_0^{\tau_k},\nu_0^{\tau_k})\text{ converges to some }(\bar{t}_0,\bar{s}_0,\bar{\mu}_0,\bar{\nu}_0). \tag{4.14}$$

Due to (4.13), it holds that

$$\bar{t}_0=\bar{s}_0,\quad S(\bar{\mu}_0-\bar{\nu}_0)=\lim_{k\to\infty}S(\mu_0^{\tau_k}-\nu_0^{\tau_k})=0.$$

Since the class of polynomials $\{\boldsymbol{x}^{\boldsymbol{n}}\}_{\boldsymbol{n}\in\mathbb{Z}_+^d} \subset \{f_{\boldsymbol{n}}\}_{\boldsymbol{n}\in\mathbb{Z}_+^d}$ is a measure determining class of functions, then we get from $S(\bar\mu_0-\bar\nu_0)=0$ that $\bar\mu_0=\bar\nu_0$. Combining this with (4.14), (2.7), and (4.12), we obtain that

$$\lim_{k\to\infty}\frac{1}{\delta_k}\big(|t_0^{\tau_k}-s_0^{\tau_k}|^2+S(\mu_0^{\tau_k}-\nu_0^{\tau_k})\big)=0. \tag{4.15}$$

Case 1: If for some sequence $\tau_k=(\beta,\lambda_k,\delta_k)$, the corresponding maximum points $(t_0^{\tau_k},s_0^{\tau_k},\mu_0^{\tau_k},\nu_0^{\tau_k})$ satisfy $t_0^{\tau_k}\vee s_0^{\tau_k}=T$. By (4.11),

$$\Phi(\bar t,\bar t,\bar\mu,\bar\mu)\le\Phi(t_0^{\tau_k},s_0^{\tau_k},\mu_0^{\tau_k},\nu_0^{\tau_k}),$$

which yields

$$\begin{aligned}&W(\bar t,\bar\mu)-V(\bar t,\bar\mu)-2\beta(T-\bar t)-\frac{2\lambda_k}{\bar t}\\&\quad\le W(t_0^{\tau_k},\mu_0^{\tau_k})-V(s_0^{\tau_k},\nu_0^{\tau_k})-\frac{1}{2\delta_k}\big(|t_0^{\tau_k}-s_0^{\tau_k}|^2+S(\mu_0^{\tau_k}-\nu_0^{\tau_k})\big).\end{aligned}$$

Letting $k\to\infty$, due to (4.15) and $W(T,\mu)=V(T,\mu)$ for any $\mu\in\mathscr{P}(\bar{\mathscr{O}})$,

$$W(\bar t,\bar\mu)-V(\bar t,\bar\mu)-\beta(T-\bar t)\le 0.$$

Then, letting $\beta\to0$, we get $W(\bar t,\bar\mu)-V(\bar t,\bar\mu)\le 0$, which contradicts (4.9).

Case 2: For any $\tau_k=(\beta,\lambda_k,\delta_k)$, the corresponding maximum points satisfy $t_0^{\tau_k}\vee s_0^{\tau_k}<T$. Let

$$\psi(t,\mu)=V(s_0^{\tau_k},\nu_0^{\tau_k})+\frac{1}{2\delta_k}\big(|t-s_0^{\tau_k}|^2+S(\mu-\nu_0^{\tau_k})\big)+\beta(2T-t-s_0^{\tau_k})+\frac{\lambda_k}{t}+\frac{\lambda_k}{s_0^{\tau_k}}.$$

According to Lemma 4.4, $\psi\in C_{L,b}^{1,1}\big((0,T]\times\mathscr{P}(\bar{\mathscr{O}})\big)$. Consider the function

$$(t,\mu)\mapsto W(t,\mu)-\psi(t,\mu)=\Phi(t,s_0^{\tau_k},\mu,\nu_0^{\tau_k})$$

which attains its maximum at $(t_0^{\tau_k},\mu_0^{\tau_k})$ by (4.11). Because W is a viscosity subsolution to (3.14), it holds that

$$-\partial_t\psi(t_0^{\tau_k},\mu_0^{\tau_k})-\inf_{\alpha\in U}\mathcal{H}\big(t_0^{\tau_k},\mu_0^{\tau_k},\psi,D^L\psi,\alpha\big)\le 0. \tag{4.16}$$

Analogously, let

$$\tilde\psi(s,\nu)=W(t_0^{\tau_k},\mu_0^{\tau_k})-\frac{1}{2\delta_k}\big(|t_0^{\tau_k}-s|^2+S(\mu_0^{\tau_k}-\nu)-\beta(2T-t_0^{\tau_k}-s)-\frac{\lambda_k}{t_0^{\tau_k}}-\frac{\lambda_k}{s}.$$

Then, $(s,\nu)\mapsto V(s,\nu)-\tilde\psi(s,\nu)$ attains its minimum at $(s_0^{\tau_k},\nu_0^{\tau_k})$. As V is a viscosity supersolution to (3.14), it holds that

$$-\partial_s\tilde\psi(s_0^{\tau_k},\nu_0^{\tau_k})-\inf_{\alpha\in U}\mathcal{H}\big(s_0^{\tau_k},\nu_0^{\tau_k},\tilde\psi,D^L\tilde\psi,\alpha\big)\ge 0. \tag{4.17}$$

Combining (4.17) with (4.16), we obtain

$$\begin{aligned}&\partial_t\psi(t_0^{\tau_k},\mu_0^{\tau_k})-\partial_s\tilde\psi(s_0^{\tau_k},\nu_0^{\tau_k})\\&\quad\ge\inf_{\alpha\in U}\mathcal{H}\big(s_0^{\tau_k},\nu_0^{\tau_k},\tilde\psi,D^L\tilde\psi,\alpha\big)-\inf_{\alpha\in U}\mathcal{H}\big(t_0^{\tau_k},\mu_0^{\tau_k},\psi,D^L\psi,\alpha\big)\\&\quad\ge\inf_{\alpha\in U}\Big\{\mathcal{H}\big(s_0^{\tau_k},\nu_0^{\tau_k},\tilde\psi,D^L\tilde\psi,\alpha\big)-\mathcal{H}\big(t_0^{\tau_k},\mu_0^{\tau_k},\psi,D^L\psi,\alpha\big)\Big\}.\end{aligned} \tag{4.18}$$

According to Lemma 4.4,

$$D^L\psi(t_0^{\tau_k},\mu_0^{\tau_k})(x)=D^L\tilde\psi(s_0^{\tau_k},\nu_0^{\tau_k})=\frac{1}{\delta_k}\sum_{m=1}^{\infty}\sum_{|\boldsymbol{n}|=m}c_{\boldsymbol{n}}\langle\mu_0^{\tau_k}-\nu_0^{\tau_k},f_{\boldsymbol{n}}\rangle\nabla f_{\boldsymbol{n}}(x).$$

By direct calculation, we get from (4.18) that

(4.19)

$$\begin{aligned}-2\beta\geq\inf_{\alpha\in U}&\Bigg\{\int_{\bar{\mathscr{O}}}\langle b(s_0^{\tau_k},\nu_0^{\tau_k},\alpha),D^L\tilde\psi(s_0^{\tau_k},\nu_0^{\tau_k})\rangle\mathrm{d}\nu_0^{\tau_k}-\int_{\bar{\mathscr{O}}}\langle b(t_0^{\tau_k},\mu_0^{\tau_k},\alpha),D^L\psi(t_0^{\tau_k},\mu_0^{\tau_k})\rangle\mathrm{d}\mu_0^{\tau_k}\\&+\frac12\left(\int_{\bar{\mathscr{O}}}\mathrm{tr}\big(A\nabla_xD^L\tilde\psi(s_0^{\tau_k},\nu_0^{\tau_k})\big)\mathrm{d}\nu_0^{\tau_k}-\int_{\bar{\mathscr{O}}}\mathrm{tr}\big(A\nabla_xD^L\psi(t_0^{\tau_k},\mu_0^{\tau_k})\big)\mathrm{d}\mu_0^{\tau_k}\right)\\&+\int_{\bar{\mathscr{O}}}\vartheta(s_0^{\tau_k},x,\nu_0^{\tau_k},\alpha)\mathrm{d}\nu_0^{\tau_k}-\int_{\bar{\mathscr{O}}}\vartheta(t_0^{\tau_k},x,\mu_0^{\tau_k},\alpha)\mathrm{d}\mu_0^{\tau_k}\Bigg\}\\=:\inf_{\alpha\in U}&\big\{(\mathrm{I})+(\mathrm{II})+(\mathrm{III})\big\}.\end{aligned}$$

We shall estimate these three terms one by one.

First, let us estimate term (I). By (H_1'), there exists $K>0$ such that

$$|b(t,\mu,\alpha)-b(s,\nu,\alpha)|^2\leq K\big(|t-s|^2+S(\mu-\nu)\big),\quad t,s\in[0,T],\ \mu,\nu\in\mathscr{P}(\mathscr{O}),$$

where the functional S is given in Lemma 4.4.

$$\begin{aligned}(\mathrm{I})=&\frac{1}{\delta_k}\sum_{m=1}^{\infty}\sum_{|\boldsymbol{n}|=m}c_{\boldsymbol{n}}\langle\mu_0^{\tau_k}-\nu_0^{\tau_k},f_{\boldsymbol{n}}\rangle\Bigg(\int_{\bar{\mathscr{O}}}\langle b(s_0^{\tau_k},\nu_0^{\tau_k},\alpha)-b(t_0^{\tau_k},\mu_0^{\tau_k},\alpha),\nabla f_{\boldsymbol{n}}(x)\rangle\mathrm{d}\nu_0^{\tau_k}(x)\\&+\int_{\bar{\mathscr{O}}}\langle b(t_0^{\tau_k},\mu_0^{\tau_k},\alpha),\nabla f_{\boldsymbol{n}}(x)\rangle\mathrm{d}(\nu_0^{\tau_k}-\mu_0^{\tau_k})(x)\Bigg)\\\geq&-\frac{1}{2\delta_k}\sum_{m=1}^{\infty}\sum_{|\boldsymbol{n}|=m}c_{\boldsymbol{n}}\Bigg[\langle\mu_0^{\tau_k}-\nu_0^{\tau_k},f_{\boldsymbol{n}}\rangle^2+2\left(\langle b(s_0^{\tau_k},\nu_0^{\tau_k},\alpha)-b(t_0^{\tau_k},\mu_0^{\tau_k},\alpha),\int_{\bar{\mathscr{O}}}\nabla f_{\boldsymbol{n}}(x)\mathrm{d}\nu_0^{\tau_k}(x)\rangle\right)^2\\&+2\left(\langle b(t_0^{\tau_k},\mu_0^{\tau_k},\alpha),\int_{\bar{\mathscr{O}}}\nabla f_{\boldsymbol{n}}(x)\mathrm{d}(\nu_0^{\tau_k}-\mu_0^{\tau_k})(x)\right)^2\Bigg]\\\geq&-\frac{1}{2\delta_k}\sum_{m=1}^{\infty}\sum_{|\boldsymbol{n}|=m}c_{\boldsymbol{n}}\Bigg[\langle\mu_0^{\tau_k}-\nu_0^{\tau_k},f_{\boldsymbol{n}}\rangle^2+2|b(s_0^{\tau_k},\nu_0^{\tau_k},\alpha)-b(t_0^{\tau_k},\mu_0^{\tau_k},\alpha)|^2\left|\int_{\bar{\mathscr{O}}}\nabla f_{\boldsymbol{n}}(x)\mathrm{d}\nu_0^{\tau_k}(x)\right|^2\\&+2|b(t_0^{\tau_k},\mu_0^{\tau_k},\alpha)|^2\left|\int_{\bar{\mathscr{O}}}\nabla f_{\boldsymbol{n}}(x)\mathrm{d}(\nu_0^{\tau_k}-\mu_0^{\tau_k})(x)\right|^2\Bigg]\\\geq&-\frac{1}{2\delta_k}\sum_{m=1}^{\infty}\sum_{|\boldsymbol{n}|=m}c_{\boldsymbol{n}}\Bigg[\langle\mu_0^{\tau_k}-\nu_0^{\tau_k},f_{\boldsymbol{n}}\rangle^2+2K\Big(|t_0^{\tau_k}-s_0^{\tau_k}|^2+S(\nu_0^{\tau_k}-\mu_0^{\tau_k})\Big)\left|\int_{\bar{\mathscr{O}}}\nabla f_{\boldsymbol{n}}(x)\mathrm{d}\nu_0^{\tau_k}(x)\right|^2\\&+2K_4\left|\int_{\bar{\mathscr{O}}}\nabla f_{\boldsymbol{n}}(x)\mathrm{d}(\nu_0^{\tau_k}-\mu_0^{\tau_k})(x)\right|^2\Bigg].\end{aligned}$$

Notice that $\partial_{x_i}f_{\boldsymbol{n}}(x)\in\chi(f_{\boldsymbol{n}})$ for each $i\in\{1,\dots,d\}$, and so there exists $\boldsymbol{l}_i(\boldsymbol{n})\in\mathcal{I}_{\boldsymbol{n}}$ such that $\partial_{x_i}f_{\boldsymbol{n}}(x)=f_{\boldsymbol{l}_i(\boldsymbol{n})}(x)$. By (4.4), $c_{\boldsymbol{l}_i(\boldsymbol{n})}\geq c_{\boldsymbol{n}}$, and

$$c_{\boldsymbol{n}}\left|\int_{\bar{\mathscr{O}}}\partial_{x_i}f_{\boldsymbol{n}}(x)\mathrm{d}\nu_0^{\tau_k}(x)\right|^2\leq c_{\boldsymbol{l}_i(\boldsymbol{n})}\left|\int_{\bar{\mathscr{O}}}f_{\boldsymbol{l}_i(\boldsymbol{n})}(x)\mathrm{d}\nu_0^{\tau_k}(x)\right|^2.$$

Besides, Lemma 4.4 tells us that $S(\nu_0^{\tau_k}) \le 1$. This yields that

$$\sum_{m=1}\sum_{|\boldsymbol{n}|=m} c_{\boldsymbol{n}} \left| \int_{\bar{\mathscr{O}}} \nabla f_{\boldsymbol{n}}(x) \mathrm{d}\nu_0^{\tau_k}(x)\right|^2 \le \sum_{m=1}^{\infty}\sum_{|\boldsymbol{n}|=m}\sum_{i=1}^{d} c_{\boldsymbol{n}} \left|\int_{\bar{\mathscr{O}}} \partial_{x_i} f_{\boldsymbol{n}}(x)\mathrm{d}\nu_0^{\tau_k}(x)\right|^2$$
$$\le \sum_{m=1}^{\infty}\sum_{|\boldsymbol{n}|=m}\sum_{i=1}^{d} c_{\boldsymbol{l}_i(\boldsymbol{n})} \langle \nu_0^{\tau_k}, f_{\boldsymbol{l}_i(\boldsymbol{n})}\rangle^2 \le d,$$

and

(4.20)

$$\sum_{m=1}\sum_{|\boldsymbol{n}|=m} c_{\boldsymbol{n}} \left|\int_{\bar{\mathscr{O}}} \nabla f_{\boldsymbol{n}}(x) \mathrm{d}(\nu_0^{\tau_k}-\mu_0^{\tau_k})(x)\right|^2 \le d \sum_{m=1}\sum_{|\boldsymbol{n}|=m} c_{\boldsymbol{n}} \langle \nu_0^{\tau_k}-\mu_0^{\tau_k}, f_{\boldsymbol{n}}\rangle^2.$$

Invoking the previous estimates, we finally obtain that

$$\text{(I)} \ge -\frac{1}{2\delta_k} S(\mu_0^{\tau_k}-\nu_0^{\tau_k}) - \frac{C}{2\delta_k}\big(|t_0^{\tau_k}-s_0^{\tau_k}|^2 + S(\mu_0^{\tau_k}-\nu_0^{\tau_k})\big) \tag{4.21}$$

for some constant $C>0$ independent of k.

Now we deal with term (II). Similar to the estimate of (4.20), there exists $C>0$ independent of k such that

(4.22)

$$\begin{aligned}
\text{(II)} &= \frac{1}{\delta_k}\sum_{m=1}^{\infty}\sum_{|\boldsymbol{n}|=m} c_{\boldsymbol{n}} \langle \mu_0^{\tau_k}-\nu_0^{\tau_k}, f_{\boldsymbol{n}}\rangle \int_{\bar{\mathscr{O}}} \mathrm{tr}(A\nabla^2 f_{\boldsymbol{n}}(x))\mathrm{d}(\nu_0^{\tau_k}-\mu_0^{\tau_k})(x)\\
&\ge -\frac{1}{2\delta_k}\sum_{m=1}^{\infty}\sum_{|\boldsymbol{n}|=m} c_{\boldsymbol{n}} \left[\langle \mu_0^{\tau_k}-\nu_0^{\tau_k}, f_{\boldsymbol{n}}\rangle^2 + \left(\int_{\bar{\mathscr{O}}} \mathrm{tr}\big(A\nabla^2 f_{\boldsymbol{n}}(x)\big)\mathrm{d}(\nu_0^{\tau_k}-\mu_0^{\tau_k})(x)\right)^2\right]\\
&\ge -\frac{C}{\delta_k}\sum_{m=1}^{\infty}\sum_{|\boldsymbol{n}|=m} c_{\boldsymbol{n}} \langle \mu_0^{\tau_k}-\nu_0^{\tau_k}, f_{\boldsymbol{n}}\rangle^2 = -\frac{C}{\delta_k} S(\mu_0^{\tau_k}-\nu_0^{\tau_k}).
\end{aligned}$$

At last, we estimate term (III). By virtue of (4.14), $t_0^{\tau_k}$ converges to $\bar{t}_0$, $\mu_0^{\tau_k}$, $\nu_0^{\tau_k}$ converges and weakly to $\bar{\mu}_0$ as $k\to\infty$. As $\mathscr{O}$ is bounded, this also implies that $\mathbb{W}_2(\mu_0^{\tau_k},\bar{\mu}_0)\to 0$ as $k\to\infty$. By (H_3),

(4.23)

$$\begin{aligned}
\lim_{k\to\infty}\text{(III)} &= \lim_{k\to\infty}\Big\{\int_{\bar{\mathscr{O}}}\big(\vartheta(s_0^{\tau_k},x,\nu_0^{\tau_k},\alpha)-\vartheta(t_0^{\tau_k},x,\mu_0^{\tau_k},\alpha)\big)\mathrm{d}\nu_0^{\tau_k}(x)\\
&\quad + \int_{\bar{\mathscr{O}}}\big(\vartheta(t_0^{\tau_k},x,\mu_0^{\tau_k},\alpha)-\vartheta(\bar{t}_0,x,\bar{\mu}_0,\alpha)\big)\mathrm{d}(\nu_0^{\tau_k}-\mu_0^{\tau_k})(x)\\
&\quad + \int_{\bar{\mathscr{O}}}\vartheta(\bar{t}_0,x,\bar{\mu}_0,\alpha)\mathrm{d}(\nu_0^{\tau_k}-\mu_0^{\tau_k})(x)\Big\}\\
&\ge \lim_{k\to\infty}\Big\{-K_3\big(|s_0^{\tau_k}-t_0^{\tau_k}|+\mathbb{W}_2(\nu_0^{\tau_k},\mu_0^{\tau_k})\big)-2K_3\big(|t_0^{\tau_k}-\bar{t}_0|+\mathbb{W}_2(\mu_0^{\tau_k},\bar{\mu}_0)\big)\\
&\quad + \int_{\bar{\mathscr{O}}}\vartheta(\bar{t}_0,x,\bar{\mu}_0,\alpha)\mathrm{d}\nu_0^{\tau_k}(x) - \int_{\bar{\mathscr{O}}}\vartheta(\bar{t}_0,x,\bar{\mu}_0,\alpha)\mathrm{d}\mu_0^{\tau_k}(x)\Big\}\\
&= 0.
\end{aligned}$$

Finally, inserting the estimates (4.23), (4.22), (4.21) into (4.19), due to (4.15), we get

$$\lim_{k\to\infty} -2\beta \geq \lim_{k\to\infty} (\mathrm{I}) + (\mathrm{II}) + (\mathrm{III}) = 0,$$

which contradicts the fact $\beta > 0$.

Consequently, we have shown that the existence of $(\bar{t}, \bar{\mu})$ satisfying (4.9) is false, and so is the existence of $(\tilde{t}, \tilde{\mu})$. Thus, we conclude that $W(t,\mu) \leq V(t,\mu)$ for all $(t,\mu) \in [0,T) \times \mathscr{P}(\bar{\mathscr{O}})$ as desired. ∎

At last, we propose two possible applications of the conclusions of this work. One is to develop the viscosity solution theory on the Wasserstein space over Riemannian manifolds associated with the optimal control problem for stochastic processes on Riemannian manifolds. Another is the study of the control problem for N-particle systems or mean-field games. There are many works in the literature dedicated to the study of convergence of mean field games and N-particles systems, for instance, [11, 12, 16, 22, 23] and references therein. There are very few explicit solvable mean field game models; see, for example, linear quadratic models [6, 21], and the optimal investment model [24]. The idea of using the HJB equation or the Master equation to prove the limit theorem of an N-particles system has proven to be powerful; see, e.g., [10] and references therein. However, a key point of the approach of [10] is that it works under the sole assumption that the Master equation admits a classical solution. How to generalize the approach of [10] to the setting of viscosity solutions is a question worth investigation in the future.

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