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Obstructions to reversing Lagrangian surgery in Lagrangian fillings

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Given an immersed, Maslov-0, exact Lagrangian filling of a Legendrian knot, if the filling has a vanishing index and action double point, then through Lagrangian surgery it is possible to obtain a new immersed, Maslov-0, exact Lagrangian filling with one less double point and with genus increased by one. We show that it is *not* always possible to reverse the Lagrangian surgery: not every immersed, Maslov-0, exact Lagrangian filling with genus $g \geq 1$ and p double points can be obtained from such a Lagrangian surgery on a filling of genus $g-1$ with $p+1$ double points. To show this, we establish the connection between the existence of an immersed, Maslov-0, exact Lagrangian filling of a Legendrian Λ that has p double points with action 0 and the existence of an embedded, Maslov-0, exact Lagrangian cobordism from p copies of a Hopf link to Λ. We then prove that a count of augmentations provides an obstruction to the existence of embedded, Maslov-0, exact Lagrangian cobordisms between Legendrian links.

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1. Introduction

An important problem in smooth topology is to understand the 4-ball genus and the 4-ball crossing number of a smooth knot. Through a variety of techniques, including Heegaard Floer homology, gauge theory, and instanton homology [OS16, KM21], the 4-ball genus and crossing numbers have been calculated for all prime knots with crossing number 10 or less. Less is known about these invariants for connect sums; see, for example, [LVC18]. In general, the 4-ball genus and crossing numbers give information about what combinations of genus and double points can be realized by surfaces in the 4-ball with a fixed knot as their boundary: a transverse double point can be resolved at the cost of increasing the genus of the surfaces, and sometimes a disk that intersects the surface transversely along its boundary allows one to reduce the genus at the cost of increasing the number of double points.

One can study analogous problems when the knot and surface satisfy additional geometric conditions imposed by symplectic geometry. The development of symplectic field theory [EGH00] motivated the study of *Lagrangian* cobordisms between *Legendrian* submanifolds; these are embedded Lagrangian submanifolds in the symplectization of a contact manifold that have cylindrical ends over the Legendrians, see Figure 3 for a schematic picture. Lagrangian fillings occur when the bottom Legendrian is the empty set.

For a fixed Legendrian knot, obstructions to the existence of embedded, exact Lagrangian fillings arise from classical and non-classical invariants of the Legendrian; see, for example, [Cha10, Ekh12, DR16, ST13]. Legendrians that admit embedded, Lagrangian fillings are relatively rare and Lagrangian fillings that do exist are known to be more topologically rigid than their smooth counterparts: an embedded, oriented, exact Lagrangian filling will always realize the smooth 4-ball genus of the knot [Cha10].

Immersed Lagrangian fillings are more plentiful: any Legendrian with rotation number 0 will admit an immersed Lagrangian filling, see, for example, [Cha10, Remark 4.2]. Currently, there are fewer known obstructions for immersed Lagrangian fillings. Classical invariants, linearized contact homology, and generating family homology can give some insight into the possible combinations of genus and double points that can be realized in an immersed, Maslov-0, exact Lagrangian filling of a Legendrian knot, [Cha10, Pez18, PT22, PR22]. Sometimes the existence of one such immersed filling will lead to the existence of another: if Λ admits an immersed, Maslov-0, exact Lagrangian filling of genus g with $p \geq 1$ double points such that one of the double points has "index and action equal to 0" (see Section 2 for definitions), then through Lagrangian surgery it is possible to construct a new immersed, Maslov-0, exact Lagrangian filling of genus $g+1$ with $p-1$ double points. In this paper we address the following question: is it always possible to "reverse" the surgery process? Namely, can every immersed, Maslov-0, exact Lagrangian filling with genus $g \geq 1$ and p double points be obtained by Lagrangian surgery on an action-0 and index-0 double point of an immersed, Maslov-0, exact Lagrangian filling of genus $g-1$ with $p+1$ double points? See Figure 1 for a schematic of this question.

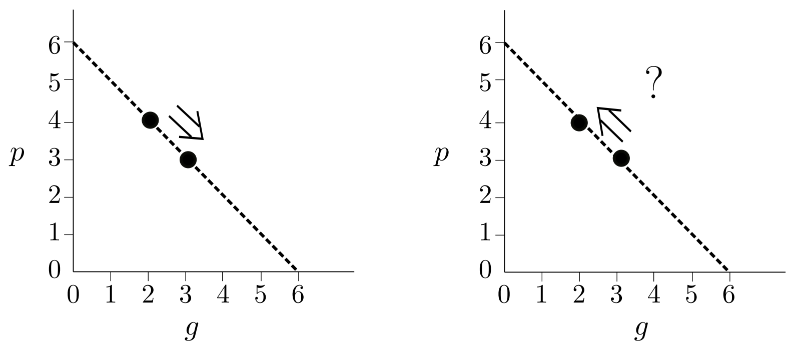


Figure 1. Asking if a filling arises from Lagrangian surgery is asking if it is possible to decrease g at the expense of increasing p.

We answer this question by first translating the existence of an immersed, Maslov-0, exact Lagrangian filling with action-0 double points to the existence of an embedded, Maslov-0, exact Lagrangian cobordism from a disjoint union of Hopf links to Λ. We then construct new obstructions to the existence of *embedded*, Maslov-0, exact Lagrangian cobordisms between Legendrian links in $\mathbb{R}^3_{\mathrm{std}}$ through the theory of augmentations. Finally, we apply our obstruction techniques to find families of Legendrian knots admitting immersed, Maslov-0, exact Lagrangian fillings that do not arise from Lagrangian surgery as defined in Definition 3.3.

1.1. Immersed to embedded Lagrangian cobordisms

In [Cha15, Theorem 1.3] Chantraine showed that the existence of an immersed, exact Lagrangian filling of Λ with a single action-0 double point implies the existence of an embedded, exact Lagrangian cobordism from a Hopf link to Λ. We give an extension of this result to more general cobordisms, more double points, and higher dimensions; Definition 2.5 defines Λ_{H}^k, the Hopf link with Maslov potential induced by the integer k.

Theorem 1.1. *Suppose $\Lambda_\pm$ are Legendrian links in $\mathbb{R}^{2n-1}_{std}$, $n \geq 2$. If there exists an immersed, Maslov-0, exact Lagrangian cobordism $L^\times$ from Λ_- to Λ_+ with genus g and p double points, m of which, $x_1, \ldots, x_m$, have action 0, then there exists an immersed, Maslov-0, exact Lagrangian cobordism L of genus g with $(p-m)$ double points from $\bigsqcup_{k=1}^m \Lambda_{\mathrm{H}}^{i_k} \cup \Lambda_-$ to Λ_+, where the Maslov potential on the Hopf links are induced by the indices i_k of x_{i_k}.*

As a corollary, we see that if each of the p double points of $L^\times$ has action 0, then we can conclude the existence of an embedded, Maslov-0, exact Lagrangian cobordism L of genus g from $\sqcup_{k=1}^p \Lambda_{\mathrm{H}}^{i_k} \cup \Lambda_-$ to Λ_+.

Remark 1.2. The hypothesis that all the double points of the immersed exact Lagrangian cobordism have action 0 is not generic. Indeed, it corresponds to the assumption that all Reeb chords in the Legendrian lift $\widetilde{L}$ of the Lagrangian cobordism have length 0. One can instead generalize to consider a *contractible double point*, which is a double point X whose corresponding Reeb chord c_X is contractible, i.e., its length can be shrunk to 0 without the front projection of $\widetilde{L}$ needing to undergo any moves; see [EHK16, Definition 6.13] for a precise description of a contractible Reeb chord. The notion of multiple action-0 double points can be generalized to multiple "simultaneously contractible" double points. The Legendrian Hopf link from Figure 9 illustrates that two individually contractible Reeb chords need not be simultaneously contractible: here, the two interstrand Reeb chords b_1 and b_2 are not simultaneously contractible since they cobound a disk. For any immersed, exact Lagrangian filling we can apply a Legendrian isotopy so that all Reeb chords in the Legendrian lift have *nonzero* length without any births or deaths of pairs of Reeb chords; such a Legendrian isotopy on the Legendrian lift can be realized by a *safe Hamiltonian isotopy* of the Lagrangian filling, see [CDRGG]. However, in general there are obstructions in going from a set of contractible Reeb chords to a set of action-0 double points.

Remark 1.3. Theorem 1.1 can be extended beyond transverse double points to more general singularities of exact Lagrangians. In particular, we can consider any Lagrangian singularity f such that the boundary of a Darboux ball centered at the singularity, or a real morsification of the singularity, intersects the exact Lagrangian as a Legendrian and the primitive is constant on the Legendrian. See [Cas22] for some examples of such singularities.

1.2. Obstructions to embedded exact Lagrangian cobordisms

For a Legendrian link Λ in the standard contact manifold $\mathbb{R}^3_{\mathrm{std}}$, the Chekanov-Eliashberg DGA [Che02, Eli98] $(\mathcal{A}(\Lambda), \partial)$ is a powerful invariant that arises from symplectic field theory [EGH00]. An augmentation ϵ of $\mathcal{A}(\Lambda)$ to a unital, commutative ring $\mathbb{F}$ is a DGA map $\epsilon : (\mathcal{A}(\Lambda), \partial) \to (\mathbb{F}, 0)$, where $(\mathbb{F}, 0)$ is a DGA with $\mathbb{F}$ in degree 0 and differential identically 0. Let $Aug(\Lambda; \mathbb{F})$ denote the set of augmentations of $\mathcal{A}(\Lambda)$ to $\mathbb{F}$. An embedded, Maslov-0, exact Lagrangian cobordism L from Λ_- to Λ_+ induces a DGA map from $\mathcal{A}(\Lambda_+)$ to $\mathcal{A}(\Lambda_-)$ [EHK16] that by composition with an augmentation of $\mathcal{A}(\Lambda_-)$ induces a map

$$\mathcal{F}_L : Aug(\Lambda_-; \mathbb{F}) \to Aug(\Lambda_+; \mathbb{F}). \tag{1.1}$$

Let $Aug(\Lambda; \mathbb{F})/\sim_{Aug_+}$ denote the set of augmentations up to the equivalence relation $\sim_{Aug_+}$ given by the natural equivalence given in the augmentation category $\mathcal{A}ug_+(\Lambda)$, see Definition 5.1 , or equivalently with respect to split-DGA homotopy, see Definition 5.3 and Proposition 5.5. We will use $|Aug(\Lambda; \mathbb{F})/\sim_{Aug_+}|$ to denote the cardinality of the set $Aug(\Lambda; \mathbb{F})/\sim_{Aug_+}$.

Theorem 1.4. *Let $\Lambda_\pm$ be Legendrian links in $\mathbb{R}^3_{\mathrm{std}}$ such that there exists an embedded, Maslov-0, exact Lagrangian cobordism L from Λ_- to Λ_+. Suppose $\mathbb{F}$ is a commutative ring; if $\mathbb{F}$ does not have characteristic 2 we further assume that L is spin. Given augmentations $\epsilon_1, \epsilon_2 \in Aug(\Lambda_-, \mathbb{F})$, if $\mathcal{F}_L(\epsilon_1), \mathcal{F}_L(\epsilon_2)$ are equivalent with respect to $\sim_{Aug_+}$, then ϵ_1, ϵ_2 are equivalent with respect to $\sim_{Aug_+}$. In particular,*

$$|Aug(\Lambda_-; \mathbb{F})/\sim_{Aug_+}| \leq |Aug(\Lambda_+; \mathbb{F})/\sim_{Aug_+}|. \tag{1.2}$$

If $\Lambda_\pm$ are single component Legendrian knots or $\mathbb{F} = \mathbb{Z}_2$, the map

$$\mathcal{F}_L : Aug(\Lambda_-; \mathbb{F})/\sim_{Aug_+} \to Aug(\Lambda_+; \mathbb{F})/\sim_{Aug_+} \tag{1.3}$$

exists and is injective.

Although the map $\mathcal{F}_L$ on the set of augmentations (see Equation 1.1) always exists, the map $\mathcal{F}_L$ on the set of equivalence classes of augmentations (see Equation 1.3) does not exist for multi-component links or when $\mathbb{F} \neq \mathbb{Z}_2$. See Remark 5.7. The fifth author [Pan17] proved a result that implies Theorem 1.4 when $\Lambda_\pm$ are Legendrian knots. We will see that equation (1.2) provides a practical way to obstruct the existence of embedded cobordisms when $\mathbb{F} = \mathbb{Z}_2$. When $\mathbb{F}$ is not of characteristic 2, then, as in [CDRGG20, EES05b, Kar20, Sei08], rigid holomorphic disks in the moduli spaces that arise in the proof of Theorem 1.4 are counted with signs.

Fillings induce augmentations, and so one of the many reasons to consider augmentations to a more general $\mathbb{F}$ is that they can give information on the number of fillings of a Legendrian link. It is known that Hamiltonian isotopic, embedded, Maslov-0, exact Lagrangian fillings induce $\sim_{Aug_+}$ equivalent augmentations to $\mathbb{Z}$, [EHK16, Kar20]. Examples of Legendrian links that have an infinite number of distinct fillings up to Hamiltonian isotopy were first given in [CG22] and later also in [CZ22, GSW20b, GSW20a**?**]. From the existence of a Legendrian with an infinite number of distinct fillings distinguished by augmentations to $\mathbb{Z}$, we can apply Theorem 1.4 to deduce the existence of more such Legendrians.

Corollary 1.5. *(c.f. [CN21, Proposition 7.5, Remark 7.6]) Let $N \in \mathbb{N} \cup \{\infty\}$. Suppose Λ_- is a Legendrian link that has N augmentations to $\mathbb{Z}$ up to $\sim_{Aug_+}$ equivalence that are induced by embedded, Maslov-0, exact Lagrangian fillings, and there exists an embedded, Maslov-0, exact Lagrangian cobordism from Λ_- to Λ_+, then Λ_+ admits N embedded, Maslov-0, exact Lagrangian fillings that are distinct up to Hamiltonian isotopy.*

Proof. Consider two embedded, Maslov-0, exact Lagrangian fillings of Λ_- that induce augmentations $\epsilon_1, \epsilon_2 \in Aug(\Lambda_-, \mathbb{Z})$ that are not equivalent with respect to $\sim_{Aug_+}$. Concatenating these fillings with the cobordism L from Λ_- to Λ_+ produces two embedded, Maslov-0, exact Lagrangian fillings of Λ_+; the augmentations induced by these fillings agree with $\mathcal{F}_L(\epsilon_1), \mathcal{F}_L(\epsilon_2) \in Aug(\Lambda_+, \mathbb{Z})$. By Theorem 1.4, $\mathcal{F}_L(\epsilon_1), \mathcal{F}_L(\epsilon_2)$ are not equivalent with respect to $\sim_{Aug_+}$, and thus the fillings of Λ_+ are not Hamiltonian isotopic. □

In the case when $\Lambda_\pm$ are knots, Theorem 1.4 was derived in [Pan17] from studying the *augmentation category* $\mathcal{A}ug_+(\Lambda)$, which is an A_∞-category associated to a Legendrian Λ, see [NRS$^+$20]. The objects of $\mathcal{A}ug_+(\Lambda)$ are augmentations $\epsilon : \mathcal{A}(\Lambda) \to \mathbb{F}$ and morphisms $Hom_+(\epsilon^1, \epsilon^2)$ are modules over Reeb chords between Λ and its "push-off". When $\Lambda_\pm$ are knots, the functoriality of the DGA under cobordisms naturally extends the map $\mathcal{F}_L$ from Equation (1.1) to a functor

$$(\mathcal{F}_L)_+ : \mathcal{A}ug_+(\Lambda_-) \to \mathcal{A}ug_+(\Lambda_+)$$

between the augmentation categories. In [Pan17] it is proved that $\mathcal{F}_L$ is injective on equivalence classes of objects when $\Lambda_\pm$ are knots by showing that the functor $(\mathcal{F}_L)_+$ induces an isomorphism on the degree 0 cohomology of morphism spaces; i.e. $H^0\mathrm{Hom}_+(\epsilon^1, \epsilon^2) \cong H^0\mathrm{Hom}_+(\mathcal{F}_L(\epsilon^1), \mathcal{F}_L(\epsilon^2))$. However, this latter statement fails for links. Moreover, the functor $(\mathcal{F}_L)_+$ is not even well-defined for cobordisms between links. Instead, we employ the machinery of wrapped Floer theory for Lagrangian cobordisms developed in [CDRGG20] (see Section 6), to argue that if $\mathcal{F}_L(\epsilon^1), \mathcal{F}_L(\epsilon^2)$ are equivalent, then ϵ^1, ϵ^2 are equivalent, where equivalence is with respect to $\sim_{\mathcal{A}ug_+}$. To do this, we construct "wrong-way" maps, namely maps in direction opposite to those induced by $(\mathcal{F}_L)_+$,

$$\iota : H^*\mathrm{Hom}_+(\mathcal{F}_L(\epsilon^1), \mathcal{F}_L(\epsilon^2)) \to H^*\mathrm{Hom}_+(\epsilon^1, \epsilon^2).$$

Combining the work of the second author [Leg20] and wrapped Floer theory, we show that ι is unital and preserves the product structure on $H^*\mathrm{Hom}_+$

In Section 7.4, we build two additional obstructions to the existence of embedded, Maslov-0, exact Lagrangian cobordisms in terms of linearized contact homology $LCH^\epsilon_*(\Lambda)$ (see Section 4.2) and the ruling polynomial $R_\Lambda(z)$ (see Equation (8.1), which are Legendrian invariants that are associated to augmentations. These results are extensions of parallel results in [Pan17].

Proposition 1.6 (see Proposition 7.5). *Assume $\mathbb{F}$ is a field, $\Lambda_\pm$ are Legendrian links in $\mathbb{R}^3_{\mathrm{std}}$, ϵ is an augmentation of Λ_-, and L is an embedded, Maslov-0, exact Lagrangian cobordism from Λ_- to Λ_+, which we further assume to be spin if $\mathbb{F}$ does not have characteristic 2. Then,*

$$LCH^{\mathcal{F}_L(\epsilon)}_k(\Lambda_+) \cong LCH^\epsilon_k(\Lambda_-) \tag{1.4}$$

for $k < 0$ and $k > 1$.

Proposition 1.7 (see Corollary 7.8). *Let L be a spin, embedded, Maslov-0, exact Lagrangian cobordism from Λ_- to Λ_+. Then,*

$$R_{\Lambda_-}(q^{1/2} - q^{-1/2}) \leq q^{-\chi(L)/2} R_{\Lambda_+}(q^{1/2} - q^{-1/2})$$

for any q that is a power of a prime number.

1.3. Obstructions to reversing Lagrangian surgery

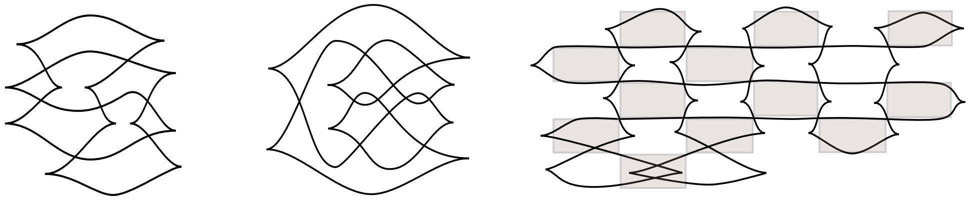
We apply Theorem 1.1 and Theorem 1.4 to find examples of Legendrian knots in $\mathbb{R}^3_{std}$ admitting immersed, Maslov-0, exact Lagrangian fillings that do not arise from Lagrangian surgery. We say that an immersed, Maslov-0, exact Lagrangian filling F_g^p of a Legendrian Λ with genus g and p double points *does not arise from Lagrangian surgery* if there does not exist an immersed, Maslov-0, exact Lagrangian filling F_{g-1}^{p+1} with genus $g-1$ and $p+1$ double points where the indices and actions of p of the double points agree with those of F_g^p and there is an additional double point of action and index 0 that could be surgered to produce F_g^p; see Definition 3.3.

As a simple illustration of our techniques, consider the Legendrian knot Λ_{7_4} in Figure 2(a), which is the maximal-tb representative of the knot 7_4. Using known construction techniques, described in Section 8, we know that Λ_{7_4} admits an embedded, Maslov-0, exact Lagrangian filling of genus 1; we prove this filling cannot be obtained by applying Lagrangian surgery on an immersed, Maslov-0, exact Lagrangian disk filling with one double point. Indeed, if it was the case, Λ_{7_4} would admit an immersed, Maslov-0, exact Lagrangian disk filling with a double point of action 0 and index 0. By Theorem 1.1 the existence of such an immersed filling is equivalent to the existence of an embedded, Maslov-0, exact Lagrangian cobordism from the Hopf link Λ_{H}^0 to Λ_{7_4}. However, since we can compute

$$|Aug(\Lambda_{\mathrm{H}}^0;\mathbb{Z}_2)/\sim_{Aug_+}| = 3, \text{ and } |Aug(\Lambda_{7_4};\mathbb{Z}_2)/\sim_{Aug_+}| = 1,$$

by Theorem 1.4 such an embedded cobordism does not exist. In fact, for this specific example, there is an underlying smooth reason that such an immersed Lagrangian disk filling does not exist for Λ_{7_4}: it has been shown in [OS16] using Heegaard Floer homology that the smooth knot 7_4 does not have any smooth, immersed disk filling with 1 double point. The following theorem gives examples of Legendrian knots with obstructed immersed Lagrangian fillings, where there is no smooth obstruction. The Legendrian knot shown in Figure 2(b) is an example of a Legendrian in Theorem 1.8

(1), and the Legendrian shown in Figure 2(c) is an example of a Legendrian in Theorem 1.8 (2).



(a) (b) (c)

Figure 2. Legendrian knots admitting fillings that do not arise from Lagrangian surgery. (a) Λ_{7_4}; (b) $\Lambda_k|_{k=1} = \Lambda_{9_{48}}$; (c) the clasped checkerboard Λ^1_2.

Theorem 1.8. 1) *For all $k \geq 1$, there exists a Legendrian knot Λ_k, with Λ_1 being a Legendrian 9_{48} knot, that admits an immersed, Maslov-0, exact Lagrangian filling F^k_k, which has genus k and k double points, that does not arise from Lagrangian surgery, even though Λ_k admits a smooth filling of genus $(k-1)$ with $(k+1)$ double points.*

2) *Given $g \in \mathbb{Z}^+$, and $p \in \mathbb{Z}^{\geq 0}$, there is a Legendrian knot Λ^p_g that has an immersed, Maslov-0, exact Lagrangian filling F^p_g, which has genus g and p double points, that does not arise from Lagrangian surgery.*

The family Λ^p_g in Theorem 1.8(2) generalizes Λ_{7_4}: $\Lambda^0_1 = \Lambda_{7_4}$. Other than Λ^0_1, the knots in this family have crossing numbers that are at least 11 and can be arbitrarily large: a SnapPy calculation shows that Λ^1_1 is the smooth knot 11_{495}, and, to the best of our knowledge, this and the others in the family do not have smooth obstructions.

Remark 1.9. The Poincaré polynomial for the Legendrian contact homology of $\Lambda_{9_{48}}$ is $t^{-1} + 2 + 2t$, [CN13]. Using the techniques of generating families, this implies that any immersed, gf-compatible (and thus Maslov-0 and exact) Lagrangian disk filling of $\Lambda_{9_{48}}$ must have at least two double points, of indices 0 and 1 [Pez18, PT22]. With the techniques of this paper, we obstruct the case where both the double points must satisfy the additional action-0 hypothesis, or the equivalent "contractible" formulation described in Remark 1.2.

We end this introduction with the following observation. The fact that the immersed Lagrangian fillings in Theorem 1.8 are not obtained from Lagrangian surgery on other fillings tells us about the non-existence of particular Lagrangian disks. As explained in Section 3, after a change of coordinates and the removal of a cylindrical end, an exact Lagrangian filling L of a Legendrian Λ in the symplectization of $\mathbb{R}^3_{std}$ becomes a compact, exact Lagrangian filling $\overline{L}$ in $(\mathbb{B}^4, \omega_{std})$ of $\Lambda \subset S^3$.

We will call an essential, embedded curve $\gamma \subset L$ is a **pre-singularity loop** if it is obtained by the transversal intersection of a Lagrangian disk $D \subset (\mathbb{B}^4, \omega_{std})$ with the interior of $\overline{L}$. As shown in [Yau13], given a pre-singularity loop, it is always possible to reverse Lagrangian surgery. Thus, we obtain the following corollary to Theorem 1.8.

Corollary 1.10. *Let Λ be one of the Legendrian knots from Theorem 1.8 that admits an immersed, Maslov-0, exact Lagrangian filling F_g^p with genus g and p double points that cannot be obtained by Lagrangian surgery. Then the filling F_g^p does not admit a pre-singularity loop.*

Remark 1.11. Given an embedded, orientable, exact Lagrangian filling L with a pre-singularity loop $\gamma \subset L$ that bounds a Lagrangian disk with interior disjoint from L, one can shrink the Lagrangian disk to a point and perform Lagrangian surgery in one of the two ways, as explained in Section 3, to obtain two embedded exact Lagrangian fillings L_1 and L_2. Note that L_1 and L_2 are smoothly isotopic but not Hamiltonian isotopic. This has been employed to great effect in the construction of infinitely many orientable embedded exact Lagrangian fillings for certain Legendrian links up to Hamiltonian isotopy by [CZ22, Theorem 4.21]. Obstructing the existence of pre-singularity loops allows one to understand when such constructions are not possible. The obstruction tools that we construct however do not determine which curves in L are pre-singularity loops. They also only provide an upper bound on the number of pre-singularity loops γ in L.

Outline: In Section 2, we define immersed, Maslov-0, exact Lagrangian cobordisms and the action and index of double points. In Section 3, we review Lagrangian surgery and prove Theorem 1.1 by employing the theory of Liouville and Weinstein structures. We then review concepts that are used in proving Theorem 1.4 including the Chekanov-Eliashberg DGA, the augmentation category, and the wrapped Floer theory for cobordisms, in Sections 4, 5, and 6, respectively. In Section 5, the equivalence relation $\sim_{Aug_+}$ is reviewed and the new definition of split-DGA homotopy is introduced. In

Section 7, we integrate everything together and prove Theorem 1.4 as well as the other obstructions provided by Propositions 1.6 and 1.7. Finally, in Section 8, we apply Theorem 1.1 and Theorem 1.4 to prove Theorem 1.8: for one of the families we count augmentations directly while for the other family we apply the theory of rulings to count augmentations.

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2. Actions and indices of double points

In the first subsection, we define immersed, exact Lagrangian cobordisms between Legendrian links and the *action* of a double point. In the second subsection, we define the *index* of a double point.

2.1. Immersed Lagrangian cobordisms and the action of a double point

Let Λ be a Legendrian knot or link in the standard contact manifold $\mathbb{R}^{2n+1}_{\mathrm{std}} = (\mathbb{R}^{2n+1}, \ker\alpha)$, where $\alpha = dz - \sum_{i=1}^{n} y_i\, dx_i$ and $(x_1, \ldots, x_n, y_1, \ldots, y_n, z)$ are the coordinates of $\mathbb{R}^{2n+1}$. There are two useful projections of Λ: the **Lagrangian projection** $\pi_{xy}(\Lambda)$ where $\pi_{xy} : \mathbb{R}^{2n+1} \to \mathbb{R}^{2n}, (\mathbf{x}, \mathbf{y}, z) \to (\mathbf{x}, \mathbf{y})$, and the **front projection** $\pi_{xz}(\Lambda)$ where $\pi_{xz} : \mathbb{R}^{2n+1} \to \mathbb{R}^{n+1}, (\mathbf{x}, \mathbf{y}, z) \to (\mathbf{x}, z)$, where $\mathbf{x}$ and $\mathbf{y}$ are $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$. We will always assume that Λ is **chord generic**, meaning that the self-intersection points of $\pi_{xy}(\Lambda)$ consists of a finite number of transverse double points.

Now we define immersed, exact Lagrangian cobordisms between Legendrian links, which are immersed manifolds with "cylindrical ends" over

Legendrian links; see Figure 3. This extends the definition of embedded, exact Lagrangian cobordisms of [EHK16, Definition 1.1].

Definition 2.1. Let $\Lambda_\pm$ be Legendrian links in $\mathbb{R}^{2n-1}_{\text{std}}$. An **immersed, exact Lagrangian cobordism L from Λ_- to Λ_+** is an immersed, Lagrangian submanifold in the symplectization, $L = i(\Sigma)$ for a Lagrangian immersion $i : \Sigma \to (\mathbb{R}_t \times \mathbb{R}^{2n-1}, d(e^t\alpha))$, such that for some $N > 0$,

1) $L \cap ([-N, N] \times \mathbb{R}^{2n-1})$ is compact,
2) $L \cap ([N, \infty) \times \mathbb{R}^{2n-1}) = [N, \infty) \times \Lambda_+$,
3) $L \cap ((-\infty, -N] \times \mathbb{R}^{2n-1}) = (-\infty, -N] \times \Lambda_-$, and
4) there exists a function $f : \Sigma \to \mathbb{R}$ and constants $\mathfrak{c}_\pm$ such that $i^*\left(e^t\alpha\right) = df$, where $f|_{i^{-1}((-\infty,-N]\times\Lambda_-)} = \mathfrak{c}_-$, and $f|_{i^{-1}([N,\infty)\times\Lambda_+)} = \mathfrak{c}_+$.

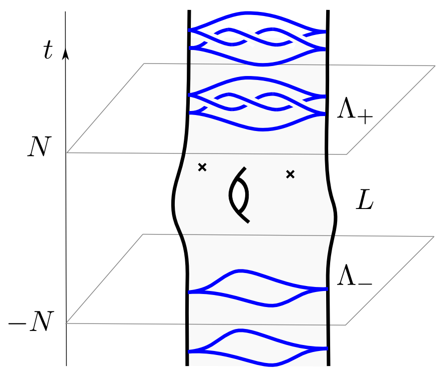


Figure 3. A schematic picture of an immersed, exact Lagrangian cobordism L from Λ_- to Λ_+ with genus 1 and two double points.

Remark 2.2.

1) The function f in condition (4) in Definition 2.1 is a **primitive** of L. Since $\Lambda_\pm$ are Legendrian, it follows that on the ends of L, the primitive f is locally constant. The condition (4) enforces that when Λ_- (or Λ_+) is not connected, the constant $\mathfrak{c}_-$ (or $\mathfrak{c}_+$) does not vary from component to component. By the addition of a constant, we can always assume that $\mathfrak{c}_- = 0$; this will be the convention that we use in Section 6.

2) Generically all immersion points of L are isolated, transverse double points. In this paper, when we write L for an immersed exact Lagrangian cobordism we implicitly assume that it comes as the image of an immersion $i : \Sigma \to \mathbb{R} \times \mathbb{R}^{2n-1}$ satisfying the conditions in Definition 2.1 and that all the immersion points are isolated and transverse double points.

Given an immersed, exact Lagrangian cobordism $L \subset \mathbb{R} \times \mathbb{R}^{2n-1}$ from Λ_- to Λ_+, the primitive f guaranteed by Definition 2.1(4) allows one to construct the **Legendrian lift of** L, defined as $\widetilde{L} = \{(i(q), -f(q)) | q \in \Sigma\}$ in the contactization of $\big(\mathbb{R}_t \times \mathbb{R}^{2n-1}, d(e^t\alpha)\big)$, which is the contact manifold $\big((\mathbb{R}_t \times \mathbb{R}^{2n-1}) \times \mathbb{R}_u, du + e^t\alpha\big)$. Double points of L are in one-to-one correspondence with **Reeb chords** of $\widetilde{L}$, which are trajectories of the Reeb vector field $\frac{\partial}{\partial u}$ that begin and end on $\widetilde{L}$.

The **action of a double point** X of L is defined to be the **length of the corresponding Reeb chord** c_X of $\widetilde{L}$ starting at $c^- \in \widetilde{L}$ and ending at $c^+ \in \widetilde{L}$, which is given by $u(c^+) - u(c^-) \geq 0$. From our construction of $\widetilde{L}$, if X is the image of $p_1, p_2 \in \Sigma$ the action of a double point X is the absolute value of the difference of the primitives at p_1, and p_2: $|f(p_1) - f(p_2)|$.

Remark 2.3. For an immersed, exact Lagrangian cobordism $L = i(\Sigma)$, the primitive, as defined in Remark 2.2, is defined on Σ, $f : \Sigma \to \mathbb{R}$. When all the double points of L have action 0, the primitive is a well-defined function $f : L \to \mathbb{R}$.

2.2. Maslov class and index of a double point

We now clarify what we mean by the *index* of a double point in an immersed, Maslov-0, exact Lagrangian cobordism. Briefly, the index of a non-zero action double point will be defined in a standard way using the Conley-Zehnder index of the corresponding Reeb chord (of strictly positive length) in the Legendrian lift. We then define the index of an action-0 double point of an immersed Lagrangian.

2.2.1. Maslov index of a loop of Lagrangians and Maslov class of a Lagrangian. First, notice that our Lagrangian cobordisms live in $\big(\mathbb{R} \times \mathbb{R}^{2n-1}, d(e^t\alpha)\big)$ which is equivalent via an exact symplectic diffeomorphism to $\big(\mathbb{R}^{2n}, \sum dq_i \wedge dp_i\big)$. Then, there is a standard way of associating an integer, known as the **Maslov index**, to a smooth loop on an immersed, Lagrangian submanifold in $\mathbb{R}^{2n}$; see, for example, [EES05a, Section 2.2].

All examples of Lagrangian cobordisms that we consider in this paper have *Maslov class* 0 (denoted **Maslov-0**), meaning that all loops have Maslov index 0. In particular, this implies that the Lagrangians are orientable since the Maslov class modulo 2 is the first Stiefel–Whitney class. In general, Maslov-n ensures a well-defined $\mathbb{Z}_n$-grading for generators of the Chekanov-Eliashberg DGA (Section 4.1) and generators of the Cthulhu complex (Section 6.2); all augmentations and chain maps are also $\mathbb{Z}_n$-graded.

2.2.2. Index of a double point. Consider an embedded, connected Legendrian $\Lambda \subset \mathbb{R}^{2n+1}$ and its Lagrangian projection $\pi_{xy}(\Lambda) \subset \mathbb{R}^{2n}$. Given a Reeb chord c of Λ, a *capping path* γ *along* Λ from the point corresponding to the end of the Reeb chord c^+ to the start of the Reeb chord c^- together with a standard closure, as defined in [EES05a], gives rise to a smooth loop of Lagrangian subspaces. The Maslov index of this loop defines the **Conley-Zehnder index** of the Reeb chord c, denoted $CZ_\gamma(c)$. When the Maslov class of the Lagrangian $\pi_{xy}(\Lambda)$ is 0, the Conley-Zehnder index does not depend on the choice of the capping path along Λ, and so we denote it $CZ(c)$. Given this, if L is an immersed, Maslov-0, exact Lagrangian with *embedded*, Maslov-0, Legendrian lift $\widetilde{L}$, a double point X of L lifts to a Reeb chord c_X, and we define the **index of** X as

$$\mathrm{ind}(X) = CZ(c_X) - 1. \tag{2.1}$$

For low-dimensional Legendrians, there is a combinatorial way to compute the Conley-Zehnder index of a Reeb chord of Λ using a *Maslov potential* on the front projection, $\pi_{xz}(\Lambda)$. Let Λ denote an embedded Legendrian knot in $\mathbb{R}^3_{std}$ (resp. $\mathbb{R}^5_{std}$) with generic front projection, and let Λ_{sing} be the subset of Λ where the front projection is not an immersion, i.e. the preimage by π_{xz} of the set of cusp points (resp. cusp edges and swallow tails). If the Lagrangian $\pi_{xy}(\Lambda)$ has Maslov class 0, a **Maslov potential** is a locally constant map

$$\mu : \Lambda/\Lambda_{sing} \to \mathbb{Z},$$

such that near a cusp point, or cusp edge, the Maslov potential of the upper sheet is 1 more than that of the lower sheet. The Maslov potential is well defined up to a global shift by an integer. Now let c be a Reeb chord of Λ from c^- to c^+. In a neighborhood of c^+ (resp. c^-), Λ is the 1-jet of a Morse function f_u (resp. f_l) defined on a neighborhood of $\pi_x(c)$, and $\pi_x(c)$ is a critical point of the function $f_{ul} := f_u - f_l$. Given a Maslov potential μ on

Λ, we have

$$CZ(c) = \mu(u) - \mu(l) + \operatorname{ind}_{f_{ul}}(\pi_x(c)), \tag{2.2}$$

where u and l are the sheets of Λ containing c^+ and c^- respectively, see [EES05a, Lemma 3.4].

In the case when Λ is not connected, there is no capping path for Reeb chords between two different components, so we need to make additional choices, as explained in, for example, [EHK16, Section 3.1]. In particular, the capping paths involve the choice of points in each component of Λ as well as paths between the corresponding Lagrangian tangent spaces at these points. The Conley-Zehnder index of a particular Reeb chord between components depends on these choices, but for two such Reeb chords, the difference is independent of the choices. One can again compute the index of a Reeb chord combinatorially using Equation 2.2; the paths determine "the jump" of Maslov potential between the two components Λ_i and Λ_j.

The above definition of the index of a Reeb chord applies to the case where the Legendrian Λ is embedded, and so $c^\pm$ are distinct points of Λ for each Reeb chord c. In other words, the double points of the Lagrangian projection $\pi_{xy}(\Lambda)$ are all of strictly positive action. When Λ is immersed and c is a Reeb chord of length 0, meaning $c^+ = c^-$ (by assumption this Reeb chord still corresponds to a transverse double point in the Lagrangian projection), the Conley-Zehnder index may depend on the choice of capping path even if $\pi_{xy}(\Lambda)$ has Maslov class 0. Indeed, for any non-trivial path $\gamma : [0,1] \to \Lambda$ from $c = c^\pm$ to itself starting on one sheet of Λ and coming back to c along the other sheet, both γ and its reverse $-\gamma$ are capping paths for the Reeb chord c. Since in a neighborhood of c, Λ consists of two sheets meeting tangentially at c, using Equation 2.2, we find that

$$CZ_{-\gamma}(c) = n - CZ_\gamma(c),$$

where n is the dimension of the Legendrian. Thus if X is an action-0 double point of an n-dimensional, exact Lagrangian L, and c_X denotes the associated length 0 Reeb chord in the Legendrian lift, then comparing a capping path γ and its reverse, we have that

$$\begin{aligned} \operatorname{ind}_\gamma(X) = CZ_\gamma(c_X) - 1 &= (n - CZ_{-\gamma}(c_X)) - 1 \\ &= n - 1 - CZ_{-\gamma}(c_X) = n - 2 - \operatorname{ind}_{-\gamma}(X). \end{aligned}$$

In particular, when $n = 2$, the index of X using a capping path γ or its reverse differs by a sign:

$$\mathrm{ind}_\gamma(X) = -\mathrm{ind}_{-\gamma}(X).$$

Definition 2.4. Suppose X is an action-0 double point in an n-dimensional, immersed, Maslov-0, exact Lagrangian. The index of X is defined to be the greater of $\mathrm{ind}_\gamma(X)$ and $\mathrm{ind}_{-\gamma}(X)$, for any capping path γ for c_X. When $n = 2$, we have that $\mathrm{ind}(X) = |\mathrm{ind}_\gamma(X)|$.

The index of a double point arises when considering Legendrian Hopf links.

Definition 2.5. The $(n-1)$-dimensional **Legendrian Hopf link** Λ_{H}^k is given by the intersection of the standard local model of an index-k double point of an n-dimensional Lagrangian submanifold (namely, $\mathbb{R}^n \cup i\mathbb{R}^n \subset \mathbb{C}^n$) and the unit sphere S^{2n-1} with its standard contact structure.

For $n = 2$, we can give a more specific description of the 1-dimensional Legendrian Hopf link Λ_{H}^k.

Example 2.6 (Hopf links). When $n = 2$, consider the Hopf link Λ_{H}^k given by the intersection of the local model for an index-k double point of a Lagrangian surface ($\mathbb{R}^2 \cup i\mathbb{R}^2 \subset \mathbb{C}^2$) and S^3. We claim that, potentially after a Legendrian isotopy, there is a front projection of Λ_{H}^k as shown in the leftmost diagram in Figure 9, where the Maslov potential, near the right cusps, from bottom to top, on the four strands is given by $0, 1, k+1$ and $k+2$ (up to a global addition of an integer). To see this correspondence for Λ_{H}^0, we will observe in Lemma 3.2 that in order to get a *Maslov-0* exact Lagrangian cobordism from another *Maslov-0*, immersed, exact Lagrangian cobordism on which we perform Lagrangian surgery, the index of the double point we surgered must be 0. The Hopf link corresponding to this double point (link of the singularity) will thus admit an embedded, Malsov-0, exact Lagrangian filling. From consideration on augmentations and using the Seidel's isomorphism, see Example 4.3, one can check that Λ_{H}^0 is the only Hopf link that bounds an embedded, Maslov-0, exact Lagrangian filling. Then, if the double point is of index k, the difference in Maslov potential of the two components of $\mathbb{R}^2 \cup i\mathbb{R}^2$ must be k. Therefore, the boundary Λ_{H}^k inherits the required Maslov potential from that of the surface $\mathbb{R}^2 \cup i\mathbb{R}^2$. An explicit Legendrian isotopy via Legendrian Reidemeister moves shows that Λ_{H}^k and $\Lambda_{\mathrm{H}}^{-k}$ are Legendrian isotopic.

3. Lagrangian surgery

We start this section by reviewing the *Lagrangian surgery* operation on immersed Lagrangian submanifolds, which was first defined for Lagrangian surfaces by Lalonde and Sikorav in [LS91] and then generalized to higher dimensions by Polterovich [Pol91]. We then prove Theorem 1.1, which translates the existence of immersed fillings into the existence of embedded cobordisms with the double points of action 0 being replaced by Hopf links.

3.1. Lagrangian surgery construction

In this subsection, our goal is to prove the following:

Proposition 3.1. *If a Legendrian link $\Lambda \subset \mathbb{R}^3_{std}$ admits an immersed, Maslov-*0*, exact Lagrangian filling L of genus g with p double points such that one of the double points has index* 0 *and action* 0*, then Λ also admits an immersed, Maslov-*0*, exact Lagrangian filling L' of genus $g+1$ with $p-1$ double points.*

To resolve a double point X of a Lagrangian, we remove a small neighborhood of X and glue back in a Lagrangian handle. In the setting where the Lagrangian L is exact, we can understand Lagrangian surgery in terms of the Legendrian lift $\widetilde{L}$ of L. This is the approach taken in [CMP19, Section 6.2] where Casals Murphy Presas give explicit parametrizations of two Lagrangian handles that can be constructed to replace an action-0 double point. The Legendrian lift of one of these handles can be seen as a "cusp-sum", and the Legendrian lift of the other can be seen as a "cone-sum"; see Figure 4. These two Lagrangian surgeries are smoothly the same [Pol91, Proposition 2]. Observe that L' obtained from either of these surgeries is necessarily exact since it is constructed through its Legendrian lift. The proof of Proposition 3.1 then follows immediately from the next lemma that tells us that if the double point has index 0, the Maslov-0 condition is preserved under surgery.

Lemma 3.2. *(cf. [Pol91, CMP19]) Suppose L is an immersed, Maslov-*0*, exact Lagrangian surface that contains an action-*0 *double point X; let L' denote an exact Lagrangian obtained from one of the two Lagrangian surgeries that correspond to the Legendrian "cusp-sum" or "cone-sum" resolutions of the lift described above. If the index of X is* 0*, then L' has Maslov class* 0*.*

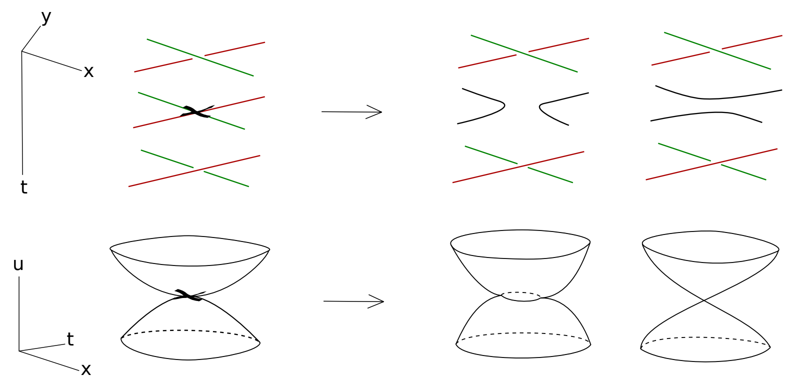


Figure 4. On the top row, left side, are schematized π_{xy} slices of the Lagrangian $L \subset \mathbb{R}_t \times \mathbb{R}^3$ in a neighborhood of a double point, and on the top right are slices of the Lagrangian obtained after the two possible handle attachments. The bottom row, left side, schematizes the Legendrian lift $\widetilde{L} \subset \mathbb{R}_t \times \mathbb{R}^3 \times \mathbb{R}_u$ of L and on the right the Legendrian lifts of each handle attachment.

Proof. The Maslov class of L' is 0 if and only if its Legendrian lift $\widetilde{L}'$ admits a ($\mathbb{Z}$-valued) Maslov potential. Before surgery, L has Maslov class 0 so its lift $\widetilde{L}$ admits a Maslov potential μ. In the lower left model shown in Figure 4, denote the upper and lower sheets of $\widetilde{L}$ by u and ℓ respectively. For both the cusp edge and the cone singularity cases, the Maslov potential μ can be "extended" after surgery to $\widetilde{L}'$ if and only if $\mu(u) - \mu(\ell) = 1$, (see also [DR11, Figure 3] for the cusp edges arising after perturbing the cone). The condition $\mu(u) - \mu(\ell) = 1$ is equivalent to the condition $\mathrm{ind}(X) = 0$ according to Definition 2.4 and Formulas (2.2) and (2.1). □

Definition 3.3. Let F_g^p denote an immersed, Maslov-0, exact Lagrangian filling F_g^p of a Legendrian Λ with genus g and p double points of indices $i_1, \ldots, i_p$ and actions $a_1, \ldots, a_p$. We say that F_g^p **arises from Lagrangian surgery** if there exists an immersed, Maslov-0, exact Lagrangian filling F_{g-1}^{p+1} of Λ with genus $g-1$ and $p+1$ double points such that

1) p of the double points have indices $i_1, \ldots, i_p$ and actions $a_1, \ldots, a_p$,
2) there exists a double point x_0 of index 0 and action 0, and
3) the Lagrangian surgery corresponding to the Legendrian cusp-sum or cone-sum at x_0 produces F_g^p.

If there is no such Lagrangian filling F^{p+1}_{g-1}, then we say that F^p_g **does not arise from Lagrangian surgery**.

3.2. Proof of Theorem 1.1

In [Cha15, Theorem 1.3] Chantraine showed that the existence of an immersed, exact Lagrangian filling of Λ with a single action-0 double point implies the existence of an embedded exact Lagrangian cobordism from a Hopf link to Λ. In this section, we prove Theorem 1.1, which generalizes this result to more general cobordisms, more double points, and higher dimensions.

The proof of Theorem 1.1 will use the theory of Liouville structures. Below we briefly describe some of the key terms. See, for example, [CE12, Chapters 11 and 12] for more details. A 1-form λ on a manifold M such that $\omega = d\lambda$ is symplectic is called a **Liouville form**; the associated ω-dual vector field V, defined by $i_V\omega = \lambda$, is the **Liouville vector field** of λ. A **Liouville domain**, (W, ω, V), is a compact manifold with boundary, W, equipped with an exact symplectic structure $\omega = d\lambda$ such that the associated Liouville vector field V points outward along ∂W. The boundary ∂W is a contact manifold with contact form $\alpha := \lambda_{|\partial W}$. A **Liouville manifold** is a manifold M together with a Liouville form λ, equivalently a triple $(M, \omega = d\lambda, V)$, such that V is complete and M admits an exhaustion $M = \cup_{k=1}^{\infty} W^k$ where (W^k, ω, V) are Liouville domains. The **skeleton** of a Liouville manifold $(M, \omega = d\lambda, V)$ is the isotropic set of points that do not escape to infinity under the Liouville flow. More concretely, $Skel(M, \omega, V) = \cup_{k=1}^{\infty} \cap_{t>0} \phi^{-1}(W^k)$, where $\cup_{k=1}^{\infty} W^k$ is an exhaustion of M, and $\phi^t : M \to M$ is the flow along V for time t. A Liouville manifold is obtained from a Liouville domain W by attaching the semi-infinite cylinder $([0, \infty) \times \partial W)$ to W and extending the Liouville form by $e^t\alpha$. For example,

$$\left(\mathbb{R}^{2n}, \omega_{std} = \sum dq_i \wedge dp_i, V_{rad} = \frac{1}{2}\sum_{i=1}^{n}\left(q_i\frac{\partial}{\partial q_i} + p_i\frac{\partial}{\partial p_i}\right)\right) \tag{3.1}$$

is a Liouville manifold. In a Liouville manifold (M, ω, V), any hypersurface $\Sigma \overset{i}{\hookrightarrow} M$ transverse to V is a contact manifold, with contact form given by $\alpha = i^*\lambda$. For any Legendrian $\Lambda \subset \Sigma$, flowing Λ along V defines a **Lagrangian** that is cylindrical over Λ. Weinstein domains are Liouville domains with a compatible Morse handlebody decomposition. For $k \leq n$, a $2n$-dimensional **Weinstein handle of index** k has underlying Liouville domain given as $\left(\mathbb{B}^k \times \mathbb{B}^{2n-k}, \omega_{std}, V_k\right)$, where

$$\omega_{std} = \sum_{i=1}^{n} dq_i \wedge dp_i,$$
$$V_k = \sum_{i=1}^{k} \left(-q_i \frac{\partial}{\partial q_i} + 2p_i \frac{\partial}{\partial p_i}\right) + \frac{1}{2} \sum_{i=k+1}^{n} \left(q_i \frac{\partial}{\partial q_i} + p_i \frac{\partial}{\partial p_i}\right).$$

The **core** (respectively, **cocore)** of the k-handle is $\mathbb{B}^k \times \{0\}$ (respectively, $\{0\} \times \mathbb{B}^{2n-k}$) and the handle has **attaching sphere** given by the boundary of the core, $S^{k-1} \times \{0\}$. It is possible to build Weinstein cobordisms via attaching handles by gluing the isotropic attaching sphere to isotropic spheres in the contact level sets, [CE12, Proposition 11.13].

Proof of Theorem 1.1. Let $L^\times$ be an immersed, Maslov-0, exact Lagrangian cobordism from Λ_- to Λ_+ with p double points, m of which, $x_1, \ldots, x_m$, have action 0. By Definition 2.1, we know that the value of the primitive is constant along all components of Λ_-. For the reader's convenience, we outline the argument.

1) Map $(\mathbb{R}_t \times \mathbb{R}^{2n-1}, d(e^t\alpha))$ to $(\mathbb{R}^{2n} - \{\text{ray}\}, \omega_{std} = \sum dq_i \wedge dp_i) \subset (\mathbb{R}^{2n}, \omega_{std})$ with an exact symplectomorphism so that $L^\times$ is sent to an exact Lagrangian $\widetilde{L}^\times$ that is cylindrical outside of $\mathbb{B}_0(\rho_+)$ and inside $\mathbb{B}_0(\rho_-)$, where $\mathbb{B}_0(\rho)$ is the standard Euclidean ball centered at 0 of radius ρ.
2) Change the Liouville structure on $\mathbb{R}^{2n}$ from (ω_{std}, V_{rad}) to a Liouville structure $(\omega_{std}, V_{\#}^{rad})$ so that a "*multi-dumbbell region*" $\mathbb{D}_{\#} \subset \mathbb{B}_0(\rho_-)$ has a Liouville structure obtained from attaching m "exterior" Weinstein 0-handles to a "center" Weinstein 0-handle via m Weinstein 1-handles.
3) Apply a Hamiltonian isotopy to drag the double points of $\widetilde{L}^\times$ to the center of the exterior 0-handles of $\mathbb{D}_{\#}$ and move $\widetilde{L}^\times$ to agree with standard intersecting Lagrangian disks near each double point. Now $\widetilde{L}^\times \cap \partial\mathbb{D}_{\#} = \widetilde{\Lambda}_-$ consists of the disjoint union of m *Legendrian* Hopf links and the Legendrian link corresponding to Λ_-.
4) By modifying $V_{\#}^{rad}$ inside $\mathbb{D}_{\#}$, we change the Liouville structure from $(\omega_{std}, V_{\#}^{rad})$ to (ω_{std}, V_0) so that V_0 only vanishes at the origin. Furthermore, we also ensure that on a small ball $\mathbb{B}_0(\epsilon) \subset \text{Int}(\mathbb{D}_{\#})$, V_0 agrees with the radial Liouville structure. The flow of the Liouville vector

field V_0 over the Legendrian $\widetilde{\Lambda}_-$ defines an exact Lagrangian cylinder L_{V_0}. We construct a new, immersed, Maslov-0, exact Lagrangian cobordism $\widehat{L}$ with only $(p-m)$ double points by replacing $\widetilde{L}^\times \cap \operatorname{Int} \mathbb{D}_\#$ with the Lagrangian $L_{V_0} \cap (\mathbb{D}_\# \setminus \{0\})$. Since $\partial(\mathbb{B}_0(\epsilon))$ is transverse to V_0, $\widehat{\Lambda}_- = \widehat{L} \cap \partial(\psi_1(B_0(\epsilon))$ is Legendrian consists of $\kappa(\Lambda_-)$ and m copy of Hopf link. The Legendrian $\widehat{\Lambda}_-$ is the new negative end of $\widehat{L}$ even though the primitives on $\kappa(\Lambda_-)$ and the Hopf links may not agree with each other.

5) Sard's Theorem guarantees the existence of a trajectory of V_0 that avoids $\widehat{L}$. This allows us to map the Lagrangian cobordism $\widehat{L}$ back to an immersed, Maslov-0, exact Lagrangian cobordism $L \subset (\mathbb{R}_t \times \mathbb{R}^{2n-1}, d(e^t\alpha))$ with only $(p-m)$ double points. By applying another Hamiltonian isotopy, we can guarantee that the primitive agrees with the same constant on all components of the negative end.

We now give more details for these steps.

Step 1: As shown in, for example, [Gei08, Proposition 2.1.8] there is a contactomorphism

$$\kappa : (\mathbb{R}^{2n-1}, \ker\alpha) \to \left(S^{2n-1} - \{pt\}, \ker\left(\frac{1}{2}\left(\sum q_i dp_i - p_i dq_i\right)\right)\right).$$

This contactomorphism lifts to an exact symplectomorphism between the symplectizations:

$$\begin{aligned} \widetilde{\kappa} : (\mathbb{R}_t \times \mathbb{R}^{2n-1}, d(e^t\alpha)) &\to \left(\mathbb{R}^{2n} - \{ray\}, \omega_{std} = \sum dq_i \wedge dp_i\right) \\ \widetilde{\kappa}(t,p) &\mapsto e^t\kappa(p). \end{aligned}$$

We can view the image of $\widetilde{\kappa}$ as a subset of the Liouville manifold $(\mathbb{R}^{2n}, \omega_{std}, V_{rad})$, as defined in Equation (3.1). Then, $\widetilde{L}^\times := \widetilde{\kappa}(L^\times)$ is an immersed, Maslov-0, exact Lagrangian surface that is cylindrical over the Legendrians $\kappa(\Lambda_\pm)$ with respect to the radial Liouville vector field V_{rad}. In particular, if $L^\times$ is cylindrical outside $t_\pm$, there exist $\rho_\pm$ such that $\widetilde{L}^\times$ is cylindrical outside $\mathbb{B}_0(\rho_+)$ and inside $\mathbb{B}_0(\rho_-)$, which are balls with respect to the standard Euclidean metric of radius $\rho_\pm$ centered at the origin.

Step 2: Choose $y_1, \ldots, y_m \in \mathbb{B}_0(\rho_-)$, and consider balls $\mathbb{B}_{y_1}, \ldots, \mathbb{B}_{y_m} \subset \mathbb{B}_0(\rho_-)$ centered at $y_1, \ldots, y_m$, and attach each of these balls via radial paths $\delta_1, \ldots, \delta_m$ to a disjoint center ball $\mathbb{B}_0 \subset \mathbb{B}_0(\rho_-)$ centered at the origin. View the balls $\mathbb{B}_0$ and $\mathbb{B}_{y_k}$, $k = 1, \ldots, m$, as Weinstein 0-handles and construct m Weinstein 1-handles with core δ_k. Thus, it is possible to glue these Weinstein structures together to obtain a Weinstein structure on a neighborhood of a dumbbell region $\mathbb{D}_{\#}$, [CE12, Proposition 11.13]; see Figure 5 for a schematic picture. Let $(\mathbb{D}_{\#}, \omega_{std}, V_{\#})$ denote the resulting Liouville domain. Now we define a new Liouville structure $(\mathbb{R}^{2n}, \omega_{std}, V_{\#}^{rad})$ that agrees with (ω_{std}, V_{rad}) outside a neighborhood of $\mathbb{D}_{\#}$ and with the Liouville structure $(\omega_{std}, V_{\#})$ on $\mathbb{D}_{\#}$. Let $N(\mathbb{D}_{\#})$ denote a contractible neighborhood of $\mathbb{D}_{\#}$ where $V_{\#}$ is defined. Let λ_{rad} and $\lambda_{\#}$ denote the Liouville 1-forms for V_{rad} and $V_{\#}$ in $(\mathbb{R}^{2n}, \omega_{std})$. Since $d(\lambda_{\#} - \lambda_{rad}) = \omega_{std} - \omega_{std} = 0$, and all closed 1-forms on $N(\mathbb{D}_{\#})$ are exact, we know $\lambda_{\#} - \lambda_{rad} = dH$ for some function $H : N(\mathbb{D}_{\#}) \to \mathbb{R}$. Let σ be a smooth bump function for $\mathbb{D}_{\#}$ supported on $N(\mathbb{D}_{\#})$: $\sigma(p) = 1$ for all $p \in \mathbb{D}_{\#}$, and $\operatorname{supp} \sigma \subset N(\mathbb{D}_{\#})$. Then consider $\lambda_{\#}^{rad} = \lambda_{rad} + d(\sigma H)$. On $\mathbb{D}_{\#}$, $\lambda_{\#}^{rad} = \lambda_{\#}$, while on the complement of $N(\mathbb{D}_{\#})$, $\lambda_{\#}^{rad} = \lambda_{rad}$. By construction, $\lambda_{\#}^{rad}$ is a Liouville 1-form of $(\mathbb{R}^{2n}, \omega_{std})$, so it provides a uniquely defined Liouville vector field $V_{\#}^{rad}$ on $(\mathbb{R}^{2n}, \omega_{std})$. By construction of $\lambda_{\#}^{rad}$, $\widetilde{L}$ is still exact in the new Liouville manifold $(\mathbb{R}^{2n}, \omega_{std}, V_{\#}^{rad})$.

Step 3: By the n-transitivity of Hamiltonian isotopies, see for example [Boo69, Theorem A], we can assume that after applying a compactly supported Hamiltonian isotopy the double points x_k are at the point y_k for $k = 1, \ldots, m$. By Moser's arguments (as in, for example, [MS95, Section 3.3]), we can further assume that, after applying a Hamiltonian isotopy, the immersed $\widetilde{L}^{\times}$ agrees with standard intersecting Lagrangian disks passing through y_k parallel to $\mathbb{R}^n$ and $i\mathbb{R}^n$. Then $\widetilde{\Lambda}_{-} := \widetilde{L}^{\times} \cap \partial \mathbb{D}_{\#}$ consists of m Legendrian Hopf link and the Legendrian $\kappa(\Lambda_{-})$, and the immersed $\widetilde{L}^{\times}$ is cylindrical over the Legendrians $\kappa(\Lambda_{\pm})$. By exactness of $\widetilde{L}^{\times}$, $\lambda_{\#}^{std}|_{\widetilde{L}^{\times}} = d\widetilde{f}$, for $\widetilde{f} : \Sigma \to \mathbb{R}$, where $\widetilde{L}^{\times}$ is the immersed image of Σ. Observe that on the intersecting Lagrangian disks at y_k, $\lambda_{\#}^{std} = 0$. Thus $\widetilde{f}$ is constant on each of these disks, and this constant must agree with $\widetilde{f}(y_k)$. Letting $\widetilde{f}(y_k) = c_k$, $k = 1, \ldots, m$, we then know that the primitive restricts to the constant c_k on the k-th Hopf link in $\widetilde{\Lambda}_{-}$. By hypothesis, $\widetilde{f}$ is constant on the Legendrian $\kappa(\Lambda_{-}) \subset \partial \mathbb{B}_0 \cap \partial \mathbb{D}_{\#}$; we denote this constant by c_0.

Step 4: First, we construct a new Liouville vector filed V_0. By construction, the skeleton of $V_{\#}^{rad}$, $\operatorname{Skel}(V_{\#}^{rad})$, consists of the origin, the points y_k, and the paths δ_k between the origin and y_k for $k = 1, \ldots, m$. Choose $\epsilon > 0$ such that $\mathbb{B}_0(\epsilon) \subset \operatorname{Int} \mathbb{D}_{\#}$, and fix an open neighborhood $N_0 \subset \operatorname{Int} \mathbb{D}_{\#}$ containing $\operatorname{Skel}(V_{\#}^{rad}) \cup \mathbb{B}_0(\epsilon)$. We will change the Liouville structure from $(\mathbb{R}^{2n}, \omega_{std} = d\lambda_{std}, V_{\#}^{rad})$ to $(\mathbb{R}^{2n}, \omega_{std} = d\lambda_0, V_0)$ where V_0 agrees with the radial Liouville vector field V_{rad} (colored in blue) on N_0, and V_0 agrees with $V_{\#}^{rad}$ (colored in black) on $\mathbb{R}^{2n} \setminus \operatorname{Int} \mathbb{D}_{\#}$ as shown in Figure 6.

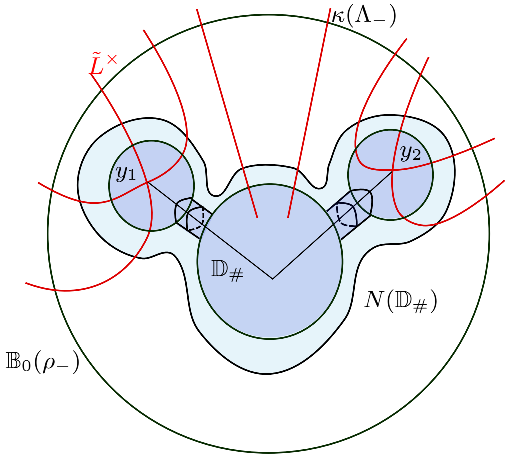


Figure 5. A schematic of the dumbbell region $\mathbb{D}_\# \subset \operatorname{Int} \mathbb{B}_0(\rho_-)$.

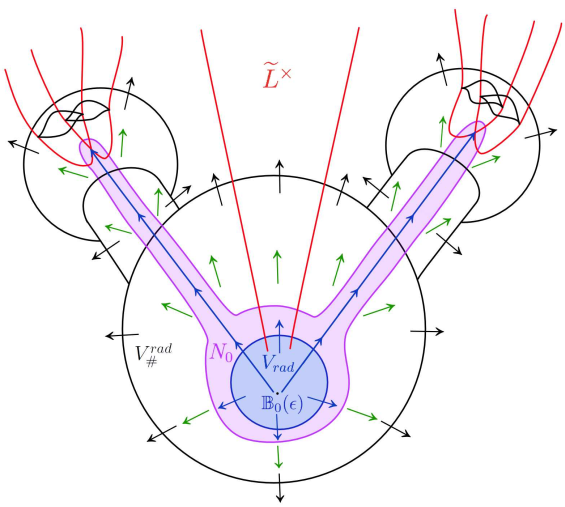


Figure 6. The Liouville vector field V_0 consists of three parts that are colored in blue, green and black respectively.

Since both λ_{rad} and $\lambda_\#^{rad}$ are Liouville 1-forms for $(\mathbb{R}^{2n}, \omega_{std})$, then, as argued in Step 3, $\lambda_{rad} - \lambda_\#^{rad} = dH_0$ for some function $H_0 : \mathbb{R}^{2n} \to \mathbb{R}$. An important observation is that H_0 is only determined up to a global shift. Let $\sigma_0 : \mathbb{R}^{2n} \to [0,1]$ be a smooth bump function for N_0 supported in $\operatorname{Int} \mathbb{D}_\#$: $\sigma_0(p) \equiv 1$ for all $p \in N_0$, $\operatorname{supp} \sigma_0 \subset \operatorname{Int} \mathbb{D}_\#$. Now consider $\lambda_0 =$

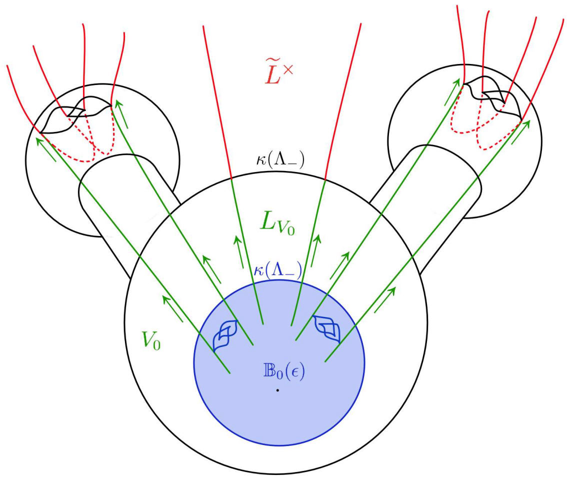


Figure 7. The Lagrangian $\widehat{L}$ is constructed in two parts: part of $\widetilde{L}^{\times}$ colored in red and L_{V_0} colored in green.

$\lambda_{\#}^{rad} + d(\sigma_0 H_0)$. On N_0, $\lambda_0 = \lambda_{rad}$, while on the complement of $\operatorname{Int}\mathbb{D}_{\#}$, we have that $\lambda_0 = \lambda_{\#}^{rad}$. By construction λ_0 is a Liouville 1-form of $(\mathbb{R}^{2n}, \omega_{std})$ so it provides a uniquely defined Liouville vector field V_0 on $(\mathbb{R}^{2n}, \omega_{std})$.

We now show that on $\mathbb{D}_{\#}$, we can choose λ_0 and H_0 well enough so that V_0 vanishes only at the origin. First observe that on N_0, $V_0 = V_{rad}$ and thus V_0 only vanishes at the origin within this subset. All that remains to be shown is that, with a good choice of λ_0 and H_0, we can ensure $V_0 \neq 0$ on $\mathbb{D}_{\#} \setminus N_0$. Observe that, with respect to the standard almost complex structure J, $\iota_{-J\nabla(\sigma_0 H_0)}\omega_{std} = d(\sigma_0 H_0)$. Also note that $V_{\#}^{rad} = V_{\#}$ on $\mathbb{D}_{\#} \setminus N_0$. Hence, to show $V_0 = V_{\#} - J\nabla(\sigma_0 H_0)$ does not vanish, we only need to show $V_{\#} \neq J\nabla(\sigma_0 H_0)$ on $\mathbb{D}_{\#} \setminus N_0$. A calculation for the right side shows that

$$\begin{aligned} -J\nabla(\sigma_0 H_0) &= -\sigma_0 J\nabla H_0 - H_0 J\nabla\sigma_0 \\ &= \sigma_0(V_{rad} - V_{\#}) + H_0(-J\nabla\sigma_0), \text{ on } \mathbb{D}_{\#} \setminus N_0 \end{aligned}$$

where the last equation follows from the fact that $\iota_{-J\nabla H_0}\omega_{std} = dH_0 = \lambda_{rad} - \lambda_{\#}$ on $\mathbb{D}_{\#} \setminus N_0$. We can choose σ_0 such that $\nabla\sigma_0$ is parallel to $-V_{\#}$, and thus $-J\nabla\sigma_0$ is parallel to $-JV_{\#}$. Our goal is to show that no point p in $\mathbb{D}_{\#}\setminus N_0$ satisfies $\sigma_0(V_{rad} - V_{\#}) + H_0(-J\nabla\sigma_0) = -V_{\#}$, which implies that at p the following three properties hold at the same time.

1) $V_{rad} - V_\#$ is in the 2-plane spanned by $V_\#$ and $JV_\#$, and thus V_{rad} is contained in the 2-plane spanned by $V_\#$ and $JV_\#$,
2) $\langle \sigma_0(V_{rad} - V_\#), V_\# \rangle = -\|V_\#\|^2$, and
3) $\langle \sigma_0(V_{rad} - V_\#), JV_\# \rangle JV_\# + H_0(-J\nabla\sigma_0) = 0$.

Note that conditions (1) and (2) are closed conditions and thus the set of points in $\mathbb{D}_\# \setminus N_0$ that satisfy the first two conditions is a bounded closed set. By globally shifting H_0 by some constant, we can ensure that all points in $\mathbb{D}_\# \setminus N_0$ that satisfy the first two conditions can not satisfy the third one. Thus, we finish proving that V_0 only vanish at the origin in $\mathbb{D}_\#$.

Next, we construct a new, immersed, Maslov-0, exact Lagrangian cobordism $\widehat{L}$ in $(\mathbb{R}^{2n}, \omega_{std} = d\lambda_0)$ by replacing $\widetilde{L}^\times \cap \operatorname{Int} \mathbb{D}_\#$ with a Lagrangian L_{V_0} formed by the trajectories of $-V_0$ in $\mathbb{D}_\# - \{0\}$ through $\widetilde{\Lambda}_- \subset \partial\mathbb{D}_\#$, see Figure 7. The fact that L_{V_0} is Lagrangian implies that $\lambda_0 = 0$ on TL_{V_0} since $\lambda_0(w) = \omega_{std}(V_0, w) = 0$ for any vector $w \in TL_{V_0}$. Thus L_{V_0} is exact and so is $\widehat{L}$. Since $\partial(\mathbb{B}_0(\epsilon))$ is transverse to V_0, $\widehat{\Lambda}_- = \widehat{L} \cap \partial(\mathbb{B}_0(\epsilon))$ is Legendrian. The fact that $\lambda_0|_{TL_{V_0}} = 0$ implies the primitive of L_{V_0} is constant on each connected component and thus evaluates to the same constants $c_0, c_1, \ldots, c_m$ on the components of $\widehat{\Lambda}_-$ (as was the case for the evaluation of the primitive $\widetilde{f}$ on all components of the Legendrian $\widetilde{\Lambda}_-$ of $\widetilde{L}^\times$). Moreover, since the Maslov potential of the k-th Hopf link $\Lambda_{\mathrm{H}}^{i_k}$ is inherited from the Maslov potential of $\widetilde{L}^\times$, replacing part of the surface does not affect the Maslov-0 condition.

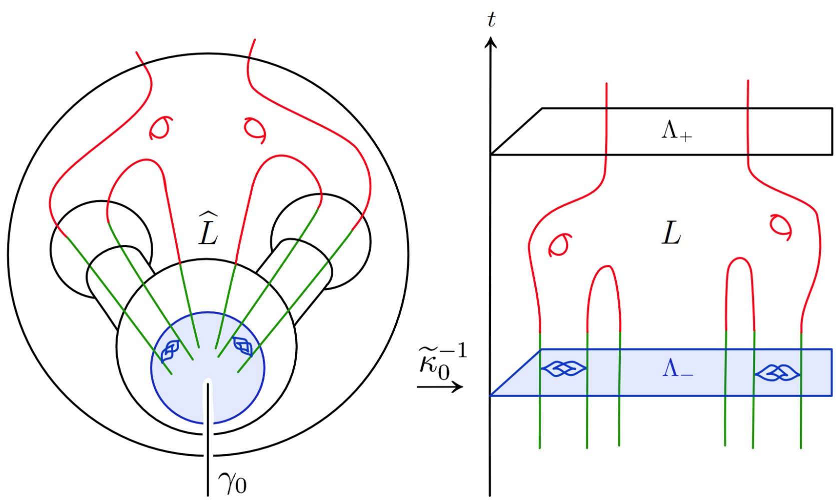


Figure 8. A schematic picture for the map $\widetilde{\kappa}_0^{-1}$.

Step 5: We now send $\widehat{L}$ from in $(\mathbb{R}^{2n}, \omega_{std} = d\lambda_0)$ back to $(\mathbb{R}_t \times \mathbb{R}^{2n-1}, d(e^t\alpha))$. Similar to the map κ in Step 1, we have a contactomorphism

$$\kappa_0 : (\mathbb{R}^{2n-1}, \ker\alpha) \to (\partial(B_0(\epsilon)) - \{pt\}, \ker\lambda_0)\,.$$

This contactomorphism lifts to an exact symplectomorphism between the symplectizations via the flow lines $\psi_t^{V_0}$ of V_0:

$$\widetilde{\kappa}_0 : (\mathbb{R}_t \times \mathbb{R}^{2n-1}, d(e^t\alpha)) \to \big(\mathbb{R}^{2n} - \{\gamma_0\}, \omega_{std} = d\lambda_0\big)$$
$$\widetilde{\kappa}_0(t,p) \mapsto \psi_t^{V_0}(\kappa(p)),$$

where γ_0 is a trajectory of V_0 over the point taken from $\partial(B_0(\epsilon))$, see Figure 8. Thus to send $\widehat{L}$ back through $\widetilde{\kappa}_0^{-1}$, all we need to do is to find a trajectory γ_0 of V_0 that does not intersect $\widehat{L}$.

We can ensure the existence of γ_0 for the following reason. Note that $\widehat{L}$ is a Lagrangian immersion $i(\Sigma)$ for $i : \Sigma \to \mathbb{R}^{2n} - \{0\}$, where Σ is an n-dimensional embedded surface. We can project $\widehat{L}$ to $\partial(B_0(\epsilon))$ though the flow line of V_0 and get a smooth map from Σ to S^{2n-1}. By Sard's Theorem, this map cannot be surjective for $n > 1$, and therefore we can always find a point q on $\partial(B_0(\epsilon))$ that is not on the image of Σ and thus the preimage γ_0 of q does not intersect $\widehat{L}$. Once back in $(\mathbb{R}_t \times \mathbb{R}^{2n-1}, d(e^t\alpha))$, by a Hamiltonian isotopy we can adjust the primitives to be the same constant on all components at the negative end (see, for example, [CDRGG20, Section 10.1]). Thus we get an exact, Maslov 0, Lagrangian cobordism L with genus g and $p - m$ double points from $\bigsqcup_{k=1}^{m} \Lambda_{\mathrm{H}}^{i_k} \cup \Lambda_- $ to Λ_+. □

4. Legendrian contact homology

In this section we recall the definition of Legendrian contact homology, which was originally formulated by Chekanov [Che02] and Eliashberg [Eli98]. We recall also the definition of augmentations and of linearized and bilinearized Legendrian contact homology. Throughout this section, we follow notations and conventions of [CDRGG20] and refer to this paper for more details. More details about the situation when coefficients are taken in a field can be found, for example, in [EES05b] or [EN22].

4.1. Chekanov-Eliashberg DGA

Here we give the key definitions and set the notation that we will use. A careful description of the Chekanov-Eliashberg DGA can be found, for example, in [EN22, CDRGG20].

The **Chekanov-Eliashberg differential graded algebra (DGA)** of Λ, $(\mathcal{A}(\Lambda), \partial)$ is the unital, graded algebra over a commutative ring $\mathbb{F}$ generated by Reeb chords of Λ. Let $R(\Lambda)$ denote the set of Reeb chords of Λ. The grading on $\mathcal{A}(\Lambda)$ is defined on the Reeb chord generators by

$$|c| = CZ(c) - 1, \tag{4.1}$$

where $CZ(c)$ is as described in Section 2.2.2. The differential ∂ on $\mathcal{A}(\Lambda)$ is defined by a count of rigid pseudo-holomorphic disks in the symplectization $(\mathbb{R}_t \times \mathbb{R}^3, d(e^t\alpha))$, with boundary on $\mathbb{R} \times \Lambda$. For any Reeb chords $a, b_1, \ldots, b_m \in R(\Lambda)$, and any almost complex structure J which is a cylindrical lift of an admissible almost complex structure on $\mathbb{R}^2$ (see [CDRGG20, Section 2.2]), define the **LCH moduli space** $\widetilde{\mathcal{M}}_J^{\mathbb{R}\times\Lambda}(a; b_1, \ldots, b_m)$ to be the space of J-holomorphic maps $u : (D^2_{m+1}, \partial D^2_{m+1}) \to (\mathbb{R} \times \mathbb{R}^3, \mathbb{R} \times \Lambda)$, with a *positive* asymptotic to the Reeb chord a and *negative* asymptotics to the Reeb chords $b_1, \ldots, b_m$, up to conformal reparametrization of the domain; see [CDRGG20, §3.2.3]. This moduli space admits an $\mathbb{R}$-action by translation along the symplectization direction; we let

$$\mathcal{M}_J^{\mathbb{R}\times\Lambda}(a; b_1, \ldots, b_m)$$

denote the quotient of $\widetilde{\mathcal{M}}_J^{\mathbb{R}\times\Lambda}(a; b_1, \ldots, b_m)$ by $\mathbb{R}$. A disk $u \in \mathcal{M}_J^{\mathbb{R}\times\Lambda}(a; b_1, \ldots, b_m)$ is called *rigid* if $\dim \mathcal{M}_J^{\mathbb{R}\times\Lambda}(a; b_1, \ldots, b_m) = 0$. Compactness results ensure that there are finitely many rigid holomorphic disks, which are used to define the differential ∂:

$$\partial(a) = \sum_{\dim\left(\mathcal{M}_J^{\mathbb{R}\times\Lambda}(a;b_1,\ldots,b_m)\right)=0} |\mathcal{M}_J^{\mathbb{R}\times\Lambda}(a; b_1, \ldots, b_m)| b_1 \ldots b_m.$$

The **Legendrian contact homology of** Λ, denoted $LCH_*(\Lambda)$, is the homology of $(\mathcal{A}(\Lambda), \partial)$.

Example 4.1 (DGA of Hopf links). Consider the Hopf link Λ_{H}^k whose front and Lagrangian projections as well as Maslov potential are depicted in Figure 9. The algebra $\mathcal{A}(\Lambda_{\mathrm{H}}^k)$ is generated by four Reeb chords a_1, a_2, b_1 and b_2 with $|a_i| = 1$ and $|b_1| = -|b_2| = k$. Using results of [DR16], the

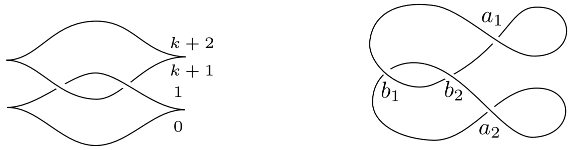


Figure 9. The front and Lagrangian projections of Λ^k_{H}.

differential as described above can be computed in a combinatorial way (see for example in [Che02, EN22]), and the non-trivial part of the differential is given by $\partial a_1 = b_1 b_2$ and $\partial a_2 = b_2 b_1$.

4.2. Augmentations

In this section, we review how augmentations, first used in [Che02], can be used to construct a variety of "linearizations" of Legendrian contact homology.

First observe that a commutative ring $\mathbb{F}$ can be considered as a DGA, where all elements of $\mathbb{F}$ have degree 0 and the differential is identically 0. Then an **augmentation** of $\mathcal{A}(\Lambda)$ to $\mathbb{F}$ is a DGA-morphism, which is a graded algebra homomorphism that preserves the differential. In particular, $\epsilon : (\mathcal{A}(\Lambda), \partial) \to (\mathbb{F}, 0)$ is a chain map such that $\epsilon(1) = 1$, and for any element a of nonzero degree, $\epsilon(a) = 0$.

Definition 4.2. $Aug(\Lambda; \mathbb{F})$ will denote the set of augmentations of $\mathcal{A}(\Lambda)$ to $\mathbb{F}$. As shown in [EHK16], an embedded, Maslov-0, exact Lagrangian cobordism L from Λ_- to Λ_+ induces a DGA map $\Phi_L : \mathcal{A}(\Lambda_+) \to \mathcal{A}(\Lambda_-)$ and thus a map:

$$\mathcal{F}_L : Aug(\Lambda_-; \mathbb{F}) \to Aug(\Lambda_+; \mathbb{F}),$$
$$\epsilon_- \mapsto \epsilon_- \circ \Phi_L.$$

As above, let $R(\Lambda)$ denote the set of Reeb chords of Λ, and then let $C(\Lambda)$ denote the graded $\mathbb{F}$-module generated by elements in $R(\Lambda)$, where the grading is as in Equation (4.1). Given an augmentation ϵ of $\mathcal{A}(\Lambda)$, the **linearized Legendrian contact homology** of Λ, denoted $LCH^\epsilon_*(\Lambda)$, is the homology of the chain complex $(C(\Lambda), \partial^\epsilon)$ with

$$\partial^\epsilon(a) = \sum_{\dim\left(\mathcal{M}^{\mathbb{R}\times\Lambda}_J(a;\mathbf{p}b\mathbf{q})\right)=0} |\mathcal{M}^{\mathbb{R}\times\Lambda}_J(a; \mathbf{p}b\mathbf{q})| \epsilon(\mathbf{p})\epsilon(\mathbf{q}) b,$$

where $a, b \in R(\Lambda)$, and $\mathbf{p}, \mathbf{q}$ are words of Reeb chords.

Example 4.3 (Augmentations of Hopf Links). Continuing with Example 4.1, one computes that the Hopf link Λ_{H}^0 admits three augmentations to $\mathbb{Z}_2$ defined by sending the pair of chords (b_1, b_2) to $(0,0)$, $(1,0)$ and $(0,1)$, while the Hopf links Λ_{H}^k for $k \neq 0$ admit only the augmentation sending all chords to 0. One can now complete the explanation of the claim in Example 2.6, namely that Λ_{H}^0 is the only Hopf link admitting an embedded, Maslov-0, exact Lagrangian filling. If a Hopf link Λ_{H} bounds a connected, embedded, Maslov-0, exact Lagrangian filling L, then by Seidel's isomorphism [Ekh12, DR16] the Poincaré polynomial of the Legendrian contact homology linearized by the augmentation induced by L must be of the form $t + 2g(L) + 1$, where $g(L)$ is the genus of L. The LCH polynomial of Λ_{H}^k is $2t + t^k + t^{-k}$ when $k \neq 0$ and is $2t + 2$ or $t + 1$ when $k = 0$ (depending on the choice of augmentation). Thus, when $k \neq 0$, Seidel's isomorphism obstructs the existence of a connected, embedded, exact, Maslov-0 Lagrangian filling of Λ_{H}^k.

In fact, one can use two augmentations to linearize: given augmentations ϵ^1, ϵ^2 of Λ, the **bilinearized Legendrian contact homology** $LCH_*^{\epsilon^1,\epsilon^2}(\Lambda)$, defined first in [BC14], is the homology of $(C(\Lambda), \partial^{\epsilon^1,\epsilon^2})$, where

$$\partial^{\epsilon^1,\epsilon^2}(a) = \sum_{\dim\left(\mathcal{M}_J^{\mathbb{R}\times\Lambda}(a;\mathbf{p}b\mathbf{q})\right)=0} |\mathcal{M}_J^{\mathbb{R}\times\Lambda}(a;\mathbf{p}b\mathbf{q})|\epsilon^1(\mathbf{p})\epsilon^2(\mathbf{q})b.$$

In Section 5 and 6, we will be using moduli spaces that are defined using a partition of a Legendrian link into components. In the case $\Lambda = \Lambda^1 \cup \Lambda^2$, where Λ^1, Λ^2 are Legendrian links, denote $R(\Lambda^i, \Lambda^j)$ the set of Reeb chords from Λ^j to Λ^i. If $c \in R(\Lambda^i, \Lambda^j)$ with $i \neq j$, we call c a **mixed Reeb chord**, otherwise we call c a **pure Reeb chord**. Denote $C(\Lambda^1, \Lambda^2)$ the graded $\mathbb{F}$-module generated by elements in $R(\Lambda^1, \Lambda^2)$. Augmentations ϵ^1 of Λ^1 and ϵ^2 of Λ^2 induce an augmentation $\epsilon = (\epsilon^1, \epsilon^2)$ of $\Lambda^1 \cup \Lambda^2$ that agrees with ϵ^i on pure chords of Λ^i, for $i = 1, 2$, and vanishes on mixed Reeb chords. Then, the differential of the Legendrian contact homology of Λ linearized by ϵ, and restricted to mixed Reeb chords in $R(\Lambda^1, \Lambda^2)$ is defined via a count of J-holomorphic disks in **mixed LCH moduli spaces**:

$$\partial^{\epsilon}(a^{12}) = \sum_{\dim\left(\mathcal{M}_J^{\mathbb{R}\times(\Lambda^1\cup\Lambda^2)}(a^{12};\mathbf{p}^{11}b^{12}\mathbf{q}^{22})\right)=0} |\mathcal{M}_J^{\mathbb{R}\times(\Lambda^1\cup\Lambda^2)}(a^{12};\mathbf{p}^{11}b^{12}\mathbf{q}^{22})|\epsilon^1(\mathbf{p}^{11})\epsilon^2(\mathbf{q}^{22})b^{12},$$

where $a^{12}, b^{12} \in R(\Lambda^1, \Lambda^2)$, $\mathbf{p}^{11}$ is a word of Reeb chords of Λ^1, and $\mathbf{q}^{22}$ is a word of Reeb chords of Λ^2. Note that $\left(C(\Lambda^1, \Lambda^2), \partial^{\epsilon}_{|C(\Lambda^1,\Lambda^2)}\right)$ is a subcomplex of $(C(\Lambda), \partial^{\epsilon})$.

5. The augmentation category

In this section, we give a brief summary of the augmentation category mainly following [NRS+20]. We then define a new notion of *split-DGA homotopy* for augmentations of multicomponent links. This gives rise to a simple criterion, Corollary 5.6, to determine when two augmentations are not equivalent in $\mathcal{A}ug_+$, which will be used frequently in Section 8 when applying Theorem 1.4.

5.1. Definitions

Let Λ be a Legendrian knot or link in $\mathbb{R}^3_{std}$. Assume that the Lagrangian projection $\pi_{xy}(\Lambda)$ has Maslov class 0 and that each connected component of Λ is decorated with a base point.

The augmentation category $\mathcal{A}ug_+(\Lambda)$ is an A_∞-category whose *objects* are elements of $Aug(\Lambda;\mathbb{F})$, namely augmentations of the Chekanov-Eliashberg DGA $\mathcal{A}(\Lambda)$ to $\mathbb{F}$. In order to define the morphisms in $\mathcal{A}ug_+(\Lambda)$, we use the DGA of a 2-copy of Λ, denoted by $2\Lambda = \Lambda^1 \cup \Lambda^2$. The copy Λ^1 is a perturbed push-off of Λ^2 in the z-direction, perturbed via a positive Morse function $f : \Lambda \to \mathbb{R}^+$ having one maximum and one minimum on each component of Λ, located near its base point as in Figure 10. Both Λ^1 and Λ^2 have the same Maslov potential. The Lagrangian projection of 2Λ for the max *tb* right-handed (positive) trefoil is shown in Figure 11.

For any two objects $\epsilon^1, \epsilon^2 \in Aug(\Lambda;\mathbb{F})$ of $\mathcal{A}ug_+(\Lambda)$, the *morphism space from* ϵ^1 *to* ϵ^2, denoted $Hom_+(\epsilon^1,\epsilon^2)$, is the graded $\mathbb{F}$-module generated by the Reeb chords in $R(\Lambda^1,\Lambda^2)$ of 2Λ, with the grading of generators shifted up by 1, commonly denoted as

$$Hom_+(\epsilon^1,\epsilon^2) := C(\Lambda^1,\Lambda^2)[1].$$

We use $|\cdot|$ to denote the gradings in the $\mathbb{F}$-module $C(\Lambda)$, as given in Equation (4.1), and $|\cdot|_+$ to denote the shifted gradings in $Hom_+(\epsilon^1,\epsilon^2)$.

Taking a closer look at the generator set of $Hom_+(\epsilon^1,\epsilon^2)$, we note that for each Reeb chord a of Λ, there is a corresponding mixed Reeb chord $a^{12} \in R(\Lambda^1,\Lambda^2)$ of 2Λ with grading given by

$$|a^{12}|_+ = |a| + 1.$$

The other generators of $Hom_+(\epsilon^1,\epsilon^2)$ are the Morse Reeb chords, corresponding to the critical points of the Morse function f. Assume Λ has m components, and denote the Morse Reeb chords corresponding to the maxima of f by x_i^{12} and the ones corresponding to the minima of f by y_i^{12}, for

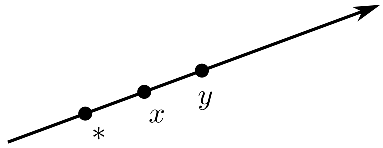


Figure 10. The local model near the base point $*$ of each component of Λ, with the arrow representing the orientation of Λ, and x (y) denoting the maximum (minimum) of the function f.

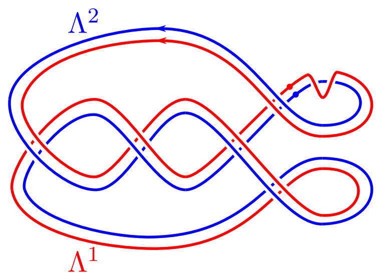


Figure 11. The Lagrangian projection of the 2-copy 2Λ, where Λ is a max tb positive trefoil.

$i = 1, \ldots, m$. By Equations (4.1) and (2.2), we find

$$|x_i^{12}|_+ = 1, \text{ and } \quad |y_i^{12}|_+ = 0. \tag{5.1}$$

As a graded module, the morphism space $Hom_+(\epsilon^1, \epsilon^2)$ does not depend on ϵ^1 or ϵ^2, but the A_∞ operators, called *compositions*,

$$m_n : Hom_+(\epsilon^n, \epsilon^{n+1}) \otimes Hom_+(\epsilon^{n-1}, \epsilon^n) \otimes \cdots \otimes Hom_+(\epsilon^1, \epsilon^2) \\ \to Hom_+(\epsilon^1, \epsilon^{n+1})$$

do depend on the choice of augmentations $\epsilon^1, \ldots, \epsilon^{n+1}$. These A_∞ operators m_n can be defined using the DGA of an $(n+1)$-copy of Λ. The $(n+1)$-copy is perturbed in such a way that every Reeb chord generator of $Hom_+(\epsilon^1, \epsilon^2)$ has corresponding versions on consecutive pairs of the $(n+1)$-copy; see [NRS$^+$20, Figure 6]. We recall below the definitions of the operators m_1 and m_2 that will be used in this paper; see [NRS$^+$20, Section 4] for details of this construction.

- The operator $m_1 : Hom_+(\epsilon^1,\epsilon^2) \to Hom_+(\epsilon^1,\epsilon^2)$ is defined by a count of rigid holomorphic disks with boundary on $\mathbb{R} \times 2\Lambda = \mathbb{R} \times (\Lambda^1 \cup \Lambda^2)$ with one positive asymptotic and one negative asymptotic to Reeb chords in $R(\Lambda^1,\Lambda^2)$ and possibly some other negative asymptotics to *pure* Reeb chords. Indeed,

$$m_1(b^{12}) = \sum_{\dim(\mathcal{M}(a^{12};\mathbf{p}^{11}b^{12}\mathbf{q}^{22}))=0} |\mathcal{M}^{\mathbb{R}\times(\Lambda^1\cup\Lambda^2)}(a^{12};\mathbf{p}^{11}b^{12}\mathbf{q}^{22})|\epsilon^1(\mathbf{p}^{11})\epsilon^2(\mathbf{q}^{22})a^{12},$$

where $b^{12}, a^{12} \in R(\Lambda^1,\Lambda^2)$, and $\mathbf{p}^{11}$, $\mathbf{q}^{22}$ are words of pure Reeb chords in $R(\Lambda^1)$ and $R(\Lambda^2)$, respectively. The operator m_1 is a degree 1 map that satisfies $m_1^2 = 0$, and we denote $H^*Hom_+(\epsilon^1,\epsilon^2)$ the cohomology of the complex $(Hom_+(\epsilon^1,\epsilon^2), m_1)$. In addition, one has the following isomorphism from [NRS$^+$20, Corollary 5.6]:

$$H^*Hom_+(\epsilon^1,\epsilon^2) \cong LCH^{\epsilon^1,\epsilon^2}_{1-*}(\Lambda).$$

- To define the operator $m_2 : Hom_+(\epsilon^2,\epsilon^3) \otimes Hom_+(\epsilon^1,\epsilon^2) \to Hom_+(\epsilon^1,\epsilon^3)$, we first consider the 3-copy $3\Lambda = \Lambda^1 \cup \Lambda^2 \cup \Lambda^3$, where 3Λ is constructed such that for any $i < j$, the DGA of $\Lambda^i \cup \Lambda^j$ is canonically identified with the DGA of 2Λ; see [NRS$^+$20, Figure 6]. The operator m_2 counts rigid holomorphic disks with boundary on $\mathbb{R} \times 3\Lambda$, with a positive asymptotic to a Reeb chord in $R(\Lambda^1,\Lambda^3)$, two negative asymptotics to Reeb chords in $R(\Lambda^1,\Lambda^2)$ and $R(\Lambda^2,\Lambda^3)$ respectively, and possibly additional negative asymptotics to pure Reeb chords. More precisely,

$$m_2(c^{23}, b^{12}) = \sum_{\dim(\mathcal{M}(a^{13};\mathbf{p}^{11}b^{12}\mathbf{q}^{22}c^{23}\mathbf{r}^{33}))=0} |\mathcal{M}^{\mathbb{R}\times 3\Lambda}(a^{13};\mathbf{p}^{11}b^{12}\mathbf{q}^{22}c^{23}\mathbf{r}^{33})|\epsilon^1(\mathbf{p}^{11})\epsilon^2(\mathbf{q}^{22})\epsilon^3(\mathbf{r}^{33})\ a^{13},$$

where $a^{13} \in R(\Lambda^1,\Lambda^3), b^{12} \in R(\Lambda^1,\Lambda^2), c^{23} \in R(\Lambda^2,\Lambda^3)$, and $\mathbf{p}^{11},\mathbf{q}^{22},\mathbf{r}^{33}$ are words of pure chords. The operator m_2 is of degree 0 and induces a product structure on the cohomology H^*Hom_+:

$$m_2 : H^iHom_+(\epsilon^2,\epsilon^3) \otimes H^jHom_+(\epsilon^1,\epsilon^2) \to H^{i+j}Hom_+(\epsilon^1,\epsilon^3).$$

5.2. Unital A_∞ category

A key property of $\mathcal{A}ug_+(\Lambda)$ is that it is a **strictly unital A_∞ category**: for any ϵ, there is an element $e_\epsilon \in Hom_+(\epsilon,\epsilon)$ with $|e_\epsilon|_+ = 0$ such that

- $m_1(e_\epsilon) = 0$;

- for all $a \in Hom_+(\epsilon, \epsilon')$ and $b \in Hom_+(\epsilon', \epsilon)$,

$$m_2(e_\epsilon, b) = b, \quad m_2(a, e_\epsilon) = a; \qquad \text{and}$$

- any higher order composition, m_n for $n \geq 3$ vanishes when e_ϵ is one of the inputs.

In fact, if Λ has m components, the unit is given by

$$e_\epsilon = -\sum_{i=1}^{m} y_i^{12} \in Hom_+(\epsilon, \epsilon),$$

for y_i^{12} as defined in Equation (5.1). It follows that the induced cohomology category $H^*\mathcal{A}ug_+(\Lambda)$ is a unital category. This allows us to define a notion of equivalence of two objects.

Definition 5.1. Two augmentations ϵ^1 and ϵ^2 of $\mathcal{A}(\Lambda)$ are **equivalent** in the augmentation category $\mathcal{A}ug_+(\Lambda)$, denoted by $\epsilon^1 \sim_{\mathcal{A}ug_+} \epsilon^2$, if they are isomorphic in $H^*\mathcal{A}ug_+(\Lambda)$, that is, if there exist $[\alpha] \in H^0 Hom_+(\epsilon^1, \epsilon^2)$ and $[\beta] \in H^0 Hom_+(\epsilon^2, \epsilon^1)$ such that

$$\begin{aligned} &m_2([\alpha], [\beta]) = [e_{\epsilon^2}] \in H^0 Hom_+(\epsilon^2, \epsilon^2), \\ \text{and} \quad &m_2([\beta], [\alpha]) = [e_{\epsilon^1}] \in H^0 Hom_+(\epsilon^1, \epsilon^1), \end{aligned}$$

where $[e_{\epsilon^i}]$ is the unit in $H^0 Hom_+(\epsilon^i, \epsilon^i)$ for $i = 1, 2$.

It can be difficult to show that two augmentations of Λ are not equivalent using Definition 5.1. However, by relating this definition of equivalence to the notion of DGA-homotopic augmentations, there is an easier criterion for distinguishing non-equivalent augmentations; see Corollary 5.6.

An augmentation is a DGA morphism, and there is an established notion of a homotopy between DGA morphisms; see, for example, [Kál05, Section 2.3] and [NRS$^+$20, Definition 5.15]. It is proved in [NRS$^+$20, Proposition 5.19] that if Λ is a Legendrian *knot*, then two augmentations are DGA-homotopic if and only if they are equivalent in $\mathcal{A}ug_+(\Lambda)$. In order to obtain a similar result in the case where Λ is a Legendrian *link*, we use the fact that the DGA of a Legendrian link has a "homotopy splitting," which was first defined by Mishachev [Mis03]. Before we explain this splitting of the DGA for a Legendrian link, we give the general definition of a split DGA and morphisms of split DGAs.

Definition 5.2. A (unitary) **split DGA** $(\mathcal{A}_{**}, \partial_{**})$ over $\mathbb{F}$ is an algebra $\mathcal{A}_{**}$ over $\mathbb{F}$ such that $\mathcal{A}_{**} = \oplus_{j_1,j_2=1}^{n} \mathcal{A}_{j_1 j_2}$, where

1) each $\mathcal{A}_{j_1 j_2}$ is a module over $\mathbb{F}$,
2) there are bilinear multiplication maps $\mathcal{A}_{j_1 j_2} \times \mathcal{A}_{j_3 j_4} \to \mathcal{A}_{j_1 j_4}$ that are 0 unless $j_2 = j_3$,
3) for all j, $\mathcal{A}_{jj}$ contains an element e_j that acts as the identity under multiplication, and
4) ∂_{**} respects the splitting, namely $\partial_{**} : \mathcal{A}_{j_1 j_2} \to \mathcal{A}_{j_1 j_2}$, for all $1 \leq j_1, j_2 \leq n$.

Given two split DGAs, $\left(\oplus_{i,j=1}^{n} \mathcal{A}_{ij}, \partial\right)$ and $\left(\oplus_{i,j=1}^{m} \mathcal{A}'_{ij}, \partial'\right)$, a **split-DGA morphism** $f : (\oplus_{i,j=1}^{n} \mathcal{A}_{ij}, \partial) \to (\oplus_{i,j=1}^{m} \mathcal{A}'_{ij}, \partial')$ is a DGA morphism such that for all i, j, there exist i', j' such that $f(\mathcal{A}_{ij}) \subset \mathcal{A}'_{i'j'}$. Observe that $(\mathbb{F}, 0)$ can be viewed as a split DGA with no splitting.

The following is a new definition, which extends the definition of DGA homotopy given, for example, in [NRS⁺20, Definition 5.15].

Definition 5.3. Given a unital, commutative ring $\mathbb{F}$, let $\mathbb{F}^*$ denote the set of units. Two split-DGA morphisms $f_1, f_2 : (\oplus_{i,j=1}^{n} \mathcal{A}_{ij}, \partial) \to (\oplus_{i,j=1}^{m} \mathcal{A}'_{ij}, \partial')$ are **split-DGA homotopic** if there exists $K : \oplus_{i,j=1}^{n} \mathcal{A}_{ij}, \to \oplus_{i,j=1}^{m} \mathcal{A}'_{ij}$ such that:

1) K is split, $\mathbb{F}$-linear, and degree 1,
2) for all i, j there exists $\alpha_i, \alpha_j \in \mathbb{F}^*$ such that for all $a \in \mathcal{A}_{ij}$,
$$\alpha_i f_1(a) - \alpha_j f_2(a) = \partial' K(a) + K\partial(a), \quad \text{and}$$
3) $K(x \cdot y) = K(x) \cdot f_2(y) + (-1)^{|x|} f_1(x) \cdot K(y)$, for all $x, y \in \oplus_{i,j=1}^{n} \mathcal{A}_{ij}$.

Remark 5.4. 1) If $\alpha_i = 1$ for all i or $\mathbb{F} = \mathbb{Z}_2$, then Definition 5.3 agrees the usual definition of DGA homotopy.

2) If Λ has a single component, and ϵ^1, ϵ^2 are two augmentations of $\mathcal{A}(\Lambda)$, then the existence of a DGA homotopy between ϵ^1, ϵ^2 is equivalent to the existence of a split-DGA homotopy between ϵ^1, ϵ^2. It is immediate to see that a DGA homotopy implies the existence of a split DGA homotopy. In the other direction, a split DGA homotopy implies the

existence of K and $\alpha \in \mathbb{F}^*$ satisfying

$$\alpha f_1(a) - \alpha f_2(a) = \partial' K + K\partial, \quad \text{for all } a \in \mathcal{A}.$$

Then $K' = \alpha^{-1}K$ is the desired DGA homotopy.

Given a Legendrian link $\Lambda = (\Lambda^1, \ldots, \Lambda^m)$, we can split what is essentially a submodule of the Chekanov-Eliashberg DGA into m^2 pieces that are invariant under Legendrian isotopy as has been shown in, for example, [Ng03, Definition 2.18] and [NT04, Section 2.4]). Let $\mathcal{A}_{ij}$ be the module generated by words of Reeb chords that begin on Λ^i and end on Λ^j, i.e. Reeb chords in $R(\Lambda^j, \Lambda^i)$. If $i = j$ we also add in an indeterminate e_j. The differential ∂_{**} is defined on the generators a as follows: if the Reeb chord a begins and ends on distinct components of Λ, then $\partial_{**}(a) = \partial(a)$; if a is a Reeb chord that begins and ends on the same component Λ^j of Λ, then replace any occurrence of 1 in $\partial(a)$ by e_j, that is, every holomorphic disk with boundary on Λ_j with positive asymptotic to a and no negative asymptotics contributes e_j to $\partial_{**}(a)$. Then ∂_{**} extends to $\mathcal{A}_{**}$ by applying the Leibniz rule and setting $\partial_{**}(e_j) = 0$, for all j. Augmentations $\epsilon : (\mathcal{A}, \partial) \to (\mathbb{F}, 0)$ are in bijective correspondence with **split augmentations** $\epsilon_{**} : (\mathcal{A}_{**}, \partial) \to (\mathbb{F}, 0)$: on any Reeb chord generator a, $\epsilon(a) = \epsilon_{**}(a)$ and $\epsilon(1) = 1 = \epsilon_{**}(e_j)$, for all j.

Using Definition 5.3, a slight modification of the proof of [NRS$^+$20, Proposition 5.19] gives the following proposition, whose proof is given in Appendix A.

Proposition 5.5. *Given a Legendrian link $\Lambda \subset \mathbb{R}^3_{std}$, two augmentations $\epsilon^1, \epsilon^2 : \mathcal{A}(\Lambda) \to \mathbb{F}$ are equivalent in $\mathcal{A}ug_+(\Lambda)$ if and only if the corresponding split augmentations ϵ^1_{**} and ϵ^2_{**} are split-DGA homotopic.*

Proposition 5.5 gives us a simple way to determine if two augmentations are not equivalent in $\mathcal{A}ug_+(\Lambda)$.

Corollary 5.6. *Suppose that the Legendrian link $\Lambda = (\Lambda^1, \ldots, \Lambda^n)$ does not have any degree -1 Reeb chords. Then any two augmentations ϵ^1 and ϵ^2 of Λ are equivalent in $\mathcal{A}ug_+(\Lambda)$ if and only if for all $i, j \in \{1, \ldots, n\}$ there exist $\alpha_i, \alpha_j \in \mathbb{F}^*$ such that $\alpha_i \epsilon^1(a) = \alpha_j \epsilon^2(a)$, for all degree 0 Reeb chords $a \in R(\Lambda^j, \Lambda^i)$. If $\mathbb{F} = \mathbb{Z}_2$, then two augmentations are equivalent in $\mathcal{A}ug_+(\Lambda)$ if and only if they are identically the same.*

Proof. Recall that the support of an augmentation is contained in the degree 0 portion of $\mathcal{A}(\Lambda)$. By Proposition 5.5, it suffices to show that when a Legendrian Λ does not have any degree -1 Reeb chords, $\epsilon^1, \epsilon^2 : (\mathcal{A}(\Lambda), \partial) \to (\mathbb{F}, 0)$

are split-DGA homotopic if and only if for all i, j there exist $\alpha_i, \alpha_j \in \mathbb{F}^*$ such that $\alpha_i \epsilon^1(a) - \alpha_j \epsilon^2(a) = 0$, for all degree 0 Reeb chords $a \in R(\Lambda^j, \Lambda^i)$. Suppose ϵ^1, ϵ^2 are split-DGA homotopic via $K : (\mathcal{A}(\Lambda), \partial) \to (\mathbb{F}, 0)$. Since K is degree 1 and $\mathbb{F}$ is in degree 0, K is supported in the degree -1 portion of $\mathcal{A}(\Lambda)$, which since there are no -1 degree Reeb chords is spanned by monomials of words length at least 2. Then an induction argument using the condition (3) of Definition 5.3 tells us that $K = 0$. It follows that for an arbitrary degree 0 Reeb chord $a \in R(\Lambda^j, \Lambda^i)$, $\alpha_i \epsilon^1(a) - \alpha_j \epsilon^2(a) = 0$. For the other direction, if for all $i, j \in \{1, \dots, n\}$ there exist $\alpha_i, \alpha_j \in \mathbb{F}^*$ such that $\alpha_i \epsilon^1(a) - \alpha_j \epsilon^2(a) = 0$, for all degree 0 Reeb chords $a \in R(\Lambda^j, \Lambda^i)$, by setting $K = 0$, we get the desired split-DGA homotopy. □

Remark 5.7. Recall the map $\mathcal{F}_L : Aug(\Lambda_-; \mathbb{F}) \to Aug(\Lambda_+; \mathbb{F})$ in Definition 4.2, induced by an embedded, Maslov-0, exact Lagrangian cobordism L from Λ_- to Λ_+.

1) It is known that this map descends to

$$\mathcal{F}_L : Aug(\Lambda_-; \mathbb{F}) / \sim_{DGA\ hom} \to Aug(\Lambda_+; \mathbb{F}) / \sim_{DGA\ hom},$$

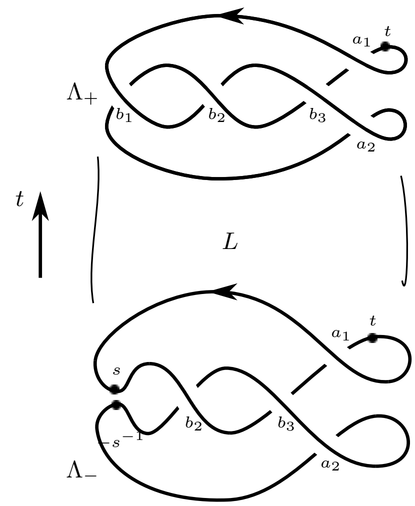
where $\sim_{DGA\ hom}$ denotes the equivalence relation defined by DGA homotopy: if K_- is a DGA-homotopy between two augmentations ϵ^1 and ϵ^2 of Λ_-, then $K_- \circ \Phi_L$ is a DGA-homotopy between $\mathcal{F}_L(\epsilon^1)$ and $\mathcal{F}_L(\epsilon^2)$. Thus, as observed in Remark 5.4 if $\Lambda_\pm$ are Legendrian knots or if $\Lambda_\pm$ are Legendrian links and $\mathbb{F} = \mathbb{Z}_2$, $\sim_{Aug_+}$ is the same as $\sim_{DGA\ hom}$, and the map

$$\mathcal{F}_L : Aug(\Lambda_-; \mathbb{F}) / \sim_{Aug_+} \to Aug(\Lambda_+; \mathbb{F}) / \sim_{Aug_+}$$

exists.

2) In general the map $\mathcal{F}_L$ does not descend to augmentations defined up to equivalence by split-DGA homotopy. See the following example for details.

Example 5.8. In this example, we show that there exists an embedded, Maslov-0, exact Lagrangian cobordism L from a Hopf link to the trefoil such that the Hopf link has two augmentations over $\mathbb{Z}$ that are split-DGA homotopic, while their images under $\mathcal{F}_L$ are not (split-)DGA homotopic augmentations of the trefoil. See [ENS02] for a combinatorial definition of the DGA over $\mathbb{Z}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$, and Section 4.2 of [CN21] for a combinatorial definition of the DGA maps induced exact by pinch moves.



$$\partial_+ a_1 = t^{-1} + b_1 + b_3 + b_1 b_2 b_3$$
$$\partial_+ a_2 = 1 - b_1 - b_3 - b_3 b_2 b_1$$
$$\partial_+ b_i = 0$$

$\Phi_L \downarrow$
$$\Phi_L(b_1) = s$$
$$\Phi_L(b_2) = b_2 - s^{-1}$$
$$\Phi_L(x) = x \text{ for } x \neq b_1, b_2$$

$$\partial_- a_1 = t^{-1} + s(1 + b_2 b_3)$$
$$\partial_- a_2 = 1 - (b_3 b_2 + 1)s$$
$$\partial_+ b_i = 0$$

Figure 12. Example of a cobordism from a Hopf link to a trefoil and the corresponding DGA differentials and DGA map.

Let L be the exact Lagrangian cobordism from the Hopf link Λ_- to the max-tb trefoil Λ_+ given by pinching the Reeb chord b_1 as shown in Figure 12. The DGA $\mathcal{A}(\Lambda_+)$ is a $\mathbb{Z}[t^{\pm 1}]$ tensor algebra generated by the Reeb chords a_1, a_2, b_1, b_2, b_3 and the DGA $\mathcal{A}(\Lambda_-)$ is a $\mathbb{Z}[t^{\pm 1}, s^{\pm 1}]$ tensor algebra over a_1, a_2, b_2, b_3 as labeled in Figure 12. The differentials $\partial_\pm$ and the DGA map Φ_L induced by the cobordism are described in Figure 12. Note that augmentations ϵ_1, ϵ_2 of $\mathcal{A}(\Lambda_-)$ that send (t, s, b_2, b_3) to $(-1, 1, 1, 0)$ and $(-1, 1, -1, 0)$, respectively, are split-DGA homotopic. Indeed, since b_2 is a mixed Reeb chord, and $\partial_- b_2 = 0$, then we can choose units $\alpha_1 = 1$ and $\alpha_2 = -1$ such that

$$\alpha_1 \epsilon_1(b_2) - \alpha_2 \epsilon_2(b_2) = 1 - 1 = 0,$$

so ϵ_1 and ϵ_2 are split-DGA homotopic. On the other hand, their induced augmentations $\mathcal{F}_L(\epsilon_1), \mathcal{F}_L(\epsilon_2)$ are augmentations of $\mathcal{A}(\Lambda_+)$ that send (t, b_1, b_2, b_3) to $(-1, 1, 0, 0)$ and $(-1, 1, -2, 0)$. Since b_2 is a pure Reeb chord of Λ_+ and $\partial_+ b_2 = 0$, then $\mathcal{F}_L(\epsilon_1)$ and $\mathcal{F}_L(\epsilon_2)$ are DGA-homotopic only if $\mathcal{F}_L(\epsilon_1)(b_2) - \mathcal{F}_L(\epsilon_2)(b_2) = 0$, which is not the case.

6. Wrapped Floer theory

In this section we review the setup and some properties of Floer theory for Lagrangian cobordisms as developed in [CDRGG20] for our setting of interest. Namely, we consider the Cthulhu complex $Cth(L^1, L^2)$ over a unital, commutative ring $\mathbb{F}$ associated to a pair of transverse, embedded, Maslov-0, exact Lagrangian cobordisms L^1, L^2. If $\mathbb{F}$ is not characteristic 2, we further assume the cobordisms L^1 and L^2 are spin. Without loss of generality, we assume that the constant value of the primitive of any cobordism we consider vanishes on the negative end, i.e. $\mathfrak{c}_- = 0$; see Remark 2.2. We review the result established in [Pan17] that we can construct an isomorphism ϕ_* between the cohomology of a quotient complex of the Cthulhu complex, $H^*(C_{-\infty}, d_{-\infty})$, and $H^*Hom_+(\epsilon^1_+, \epsilon^2_+)$, see Equation (6.3). In fact, the cohomology groups on both sides of this isomorphism possess a product structure, $m_2^{-\infty}, m_2^+$, and we review the fact, from [Leg20], that ϕ_* preserves the product structure, see Proposition 6.3. Understanding the definition of $m_2^{-\infty}$ will be important in Section 7 where we will establish in Proposition 7.2, the key result needed to prove Theorem 1.4.

6.1. A special pair

Let L be an embedded, Maslov-0, exact Lagrangian cobordism in the symplectization of $\mathbb{R}^3_{std}$ from Λ_- to Λ_+. Consider a perturbed 2-copy of L, $2L = L^1 \cup L^2$, where L^1 is a push-off of $L^2 := L$ in the positive z-direction via a Morse function $F : L \to \mathbb{R}^+$ such that $L^1 \cup L^2$ on the two cylindrical ends agrees with a cylinder over the 2-copies $\Lambda^1_\pm \cup \Lambda^2_\pm$ in the corresponding $\mathcal{A}ug_+$ categories; for details see [Pan17]. In particular, the Morse function F on $[N, \infty) \times \Lambda_+$ and $(-\infty, -N] \times \Lambda_-$ agrees with $e^t f_\pm$, where $f_\pm$ are Morse functions on $\Lambda_\pm$ that have the same critical points as the ones used in the construction of $2\Lambda_\pm$ in $\mathcal{A}ug_+(\Lambda_\pm)$; see Section 5.1. Moreover, we assume that on $\big([-N, N] \times \mathbb{R}^3\big) \cap L$ the value of the Morse function F on each point is less than the *cobordism action* of any pure Reeb chord γ of Λ_-, given by $e^{-N} \int_\gamma \alpha$. Such an assumption is necessary in order to get the identifications of complexes in Proposition 6.2 below.

Remark 6.1. We refer to [CDRGG20, Section 3.4.2] for more details on the relation between the energy of pseudo-holomorphic disks with boundary on $L^1 \cup L^2$ and the action of intersection points and Reeb chords. In our special pair case, intersection points in $L^1 \cap L^2$ are in one-to-one correspondence with critical points of the Morse function F, and the action of $p \in L^1 \cap L^2$ is given by the value of F at $p \in L$.

The particular type of perturbation used on the cylindrical ends implies that the algebras $\mathcal{A}(\Lambda^1_\pm)$ and $\mathcal{A}(\Lambda^2_\pm)$ are canonically isomorphic: there are canonical identifications of Reeb chords and the differentials agree under this identification. An augmentation ϵ_- of $\mathcal{A}(\Lambda^2_-)$ gives under this identification an augmentation of $\mathcal{A}(\Lambda^1_-)$. Moreover, if the cobordisms L^1 and L^2 are sufficiently C^1-close, then they induce the same augmentation of $\mathcal{A}(\Lambda^1_+)$ and $\mathcal{A}(\Lambda^2_+)$, i.e. $\epsilon_- \circ \Phi_{L^1} = \epsilon_- \circ \Phi_{L^2}$, under the canonical identification of generators, see [CDRGG15, Theorem 2.15].

6.2. The Cthulhu complex $Cth(L^1, L^2)$

Given the special pair of cobordisms L^1, L^2 as above, for $i = 1, 2$ suppose that ϵ^i_- is an augmentation for $\mathcal{A}(\Lambda^i_-)$ and $\epsilon^i_+ = \mathcal{F}_{L^i}(\epsilon^i_-)$ is the augmentation of $\mathcal{A}(\Lambda^i_+)$ induced by ϵ^i_- through L^i. The **Cthulhu complex** $Cth(L^1, L^2)$ can be described as follows. It is a graded $\mathbb{F}$-module generated by three types of generators:

$$Cth(L^1, L^2) = C_+(L^1, L^2) \oplus C_0(L^1, L^2) \oplus C_-(L^1, L^2),$$

where

- $C_+(L^1, L^2) = C(\Lambda^1_+, \Lambda^2_+)[2]$ is the $\mathbb{F}$-module generated by Reeb chords from Λ^2_+ to Λ^1_+ with a grading shift, i.e. a Reeb chord $a \in C_+(L^1, L^2)$ has grading $|a|_{Cth} = |a| + 2$, for $|a|$ as in Equation (4.1).
- $C_-(L^1, L^2) = C(\Lambda^1_-, \Lambda^2_-)[1]$.
- $C_0(L^1, L^2)$ is the $\mathbb{F}$-module generated by intersection points in $L^1 \cap L^2$. The grading of intersection points is given by the Conley-Zehnder index of the corresponding Reeb chords in the Legendrian lift, which is the same as the grading in Lagrangian intersection homology.

We use the shortened notation $Cth(L^1, L^2) = C_+ \oplus C_0 \oplus C_-$. The fact that the Morse function F is positive implies, by energy restrictions, that the differential d on $Cth(L^1, L^2)$ is upper triangular [CDRGG20, Lemma 7.2]:

$$d = \begin{pmatrix} d_{++} & d_{+0} & d_{+-} \\ 0 & d_{00} & d_{0-} \\ 0 & 0 & d_{--} \end{pmatrix},$$

where each component is defined by a count of rigid pseudo-holomorphic disks with boundary on $L^1 \cup L^2$ that we now describe. Let $\mathcal{J}^{cyl}$, $\mathcal{J}^{adm}$ be respectively the sets of cylindrical and admissible almost complex structures on $(\mathbb{R} \times \mathbb{R}^3, d(e^t\alpha))$, defined as in [CDRGG20, Section 2.2].

1) The maps $d_{\pm\pm}$ are the bilinearized codifferentials with respect to $(\epsilon^1_\pm, \epsilon^2_\pm)$, as reviewed in Section 4.2, and therefore count rigid holomorphic disks with boundary on $\mathbb{R} \times (\Lambda^1_\pm \cup \Lambda^2_\pm)$ with one puncture positively asymptotic and one puncture negatively asymptotic to mixed Reeb chords in $R(\Lambda^1_\pm, \Lambda^2_\pm)$. More explicitly,

$$d_{\pm\pm}(b^{12}_\pm) = \sum_{\dim\left(\mathcal{M}^{\mathbb{R}\times(\Lambda^1_\pm\cup\Lambda^2_\pm)}_{J^\pm}(a^{12}_\pm;\mathbf{p}^{11}_\pm b^{12}_\pm \mathbf{q}^{22}_\pm)\right)=0} |\mathcal{M}^{\mathbb{R}\times(\Lambda^1_\pm\cup\Lambda^2_\pm)}_{J^\pm}(a^{12}_\pm;\mathbf{p}^{11}_\pm b^{12}_\pm \mathbf{q}^{22}_\pm)|\epsilon^1_\pm(\mathbf{p}^{11}_\pm)\epsilon^2_\pm(\mathbf{q}^{22}_\pm)\, a^{12}_\pm,$$

where $J^\pm \in \mathcal{J}^{cyl}$, $b^{12}_\pm$, $a^{12}_\pm$ are generators in $C_\pm$, and $\mathbf{p}^{11}_\pm$, and $\mathbf{q}^{22}_\pm$ are words of pure Reeb chords of $\Lambda^1_\pm$ and $\Lambda^2_\pm$, respectively.

2) The maps d_{ij} from C_j to C_i, for $(i,j) = (+-), (+,0), (0,0)$ or $(0,-)$, are defined by a count of rigid holomorphic disks with boundary on $L^1 \cup L^2$ with a puncture positively asymptotic to a generator c_+ of C_i, a puncture negatively asymptotic to a generator c_- of C_j, and possibly other punctures negatively asymptotic to pure Reeb chords of $\Lambda^1_- \cup \Lambda^2_-$, as shown in Figure 13. We use $(\epsilon^1_-, \epsilon^2_-)$ to augment the Reeb chords at the negative pure punctures. The definition of such moduli spaces is similar to the definition of mixed LCH moduli spaces in Section 4.2 except that the Lagrangian boundary condition is not cylindrical anymore. This means that there is no $\mathbb{R}$-action, and so we need a path $\mathbf{J}_s$ of almost complex structures in $\mathcal{J}^{adm}$ to ensure transversality (see [CDRGG20, Section 3] for more details):

$$d_{i,j}(c_-) = \sum_{\dim\left(\mathcal{M}^{L^1\cup L^2}_{\mathbf{J}_s}(c_+;\mathbf{p}^{11}_- c_- \mathbf{q}^{22}_-)\right)=0} |\mathcal{M}^{L^1\cup L^2}_{\mathbf{J}_s}(c_+;\mathbf{p}^{11}_- c_- \mathbf{q}^{22}_-)|\epsilon^1_-(\mathbf{p}^{11}_-)\epsilon^2_-(\mathbf{q}^{22}_-)\, c_+,$$

We can identify some subcomplex and quotient complex of $(Cth(L^1, L^2), d)$ with cochain complexes defined in Section 5.

Proposition 6.2 ([Pan17, Theorem 5.1]). *The top and bottom cochain complexes admit the following identifications*

$$(C_+, d_{++}) = (Hom_+(\epsilon^1_+, \epsilon^2_+)[1], m^+_1), \quad (C_-, d_{--}) = (Hom_+(\epsilon^1_-, \epsilon^2_-), m^-_1),$$

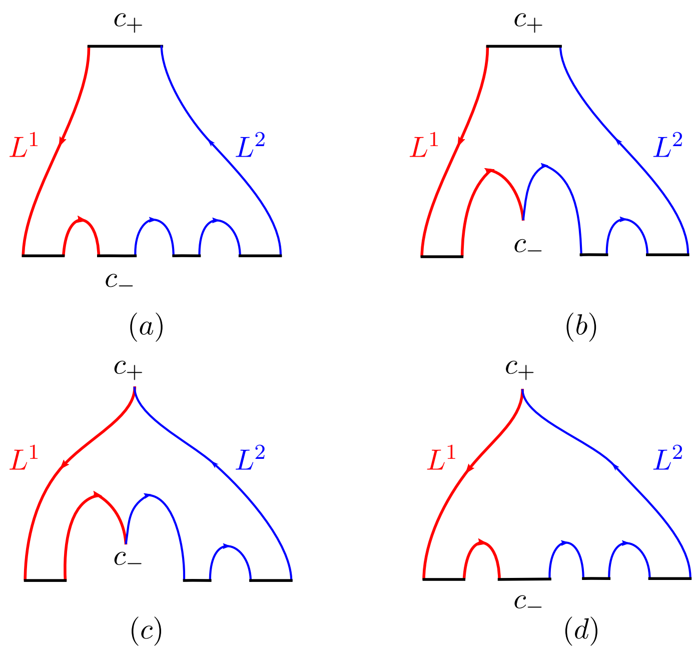


Figure 13. $(a)-(d)$ are the types of holomorphic disks counted by $d_{i,j}$ for $(i,j) = (+-), (+,0), (0,0)$ and $(0,-)$, respectively, where c_+ is a generator of C_i and c_- a generator of C_j.

and (C_0, d_{00}) *is the cochain complex of the Morse cohomology for* F *with differential counting Morse flow lines of* F*.*

Observe that the Cthulhu complex $(Cth(L^1, L^2), d)$ is the cone of $\phi := d_{+0} + d_{+-}$. The long exact sequence induced by the cone together with the fact that the complex is acyclic [CDRGG20, Theorem 6.6] implies that ϕ induces an isomorphism

$$\phi_* : H^*(C_{-\infty}, d_{-\infty}) \to H^{*+1}(C_+, d_{++}), \tag{6.1}$$

where

$$C^*_{-\infty} = C^*_0 \oplus C^*_-, \quad \text{and} \quad d_{-\infty} = \begin{pmatrix} d_{00} & d_{0-} \\ 0 & d_{--} \end{pmatrix}.$$

Note that ϕ_* may depend on the perturbation F. We have also that $C_{-\infty}$ is the cone of d_{0-}. The long exact sequence induced by a cone together with the isomorphism ϕ_* and the identifications in Proposition 6.2, give the following long exact sequence:

$$\cdots \to H^k(L,\Lambda_-) \to H^k Hom_+(\epsilon_+^1,\epsilon_+^2) \to H^k Hom_+(\epsilon_-^1,\epsilon_-^2) \to H^{k+1}(L,\Lambda_-) \to \cdots . \tag{6.2}$$

6.3. Product structure

The isomorphism ϕ_* from Equation (6.1) together with the identification of $H^{*+1}(C_+, d_{++}) = H^* Hom_+(\epsilon_+^1,\epsilon_+^2)$ given by Proposition 6.2 gives an isomorphism

$$\phi_* : H^*(C_{-\infty}, d_{-\infty}) \to H^* Hom_+(\epsilon_+^1,\epsilon_+^2). \tag{6.3}$$

As recalled in Section 5.1, there is a product structure on $H^* Hom_+$ given by the map m_2^+ in the category $\mathcal{A}ug_+(\Lambda_+)$. There is also a product structure $m_2^{-\infty}$ on $H^*(C_{-\infty}, d_{-\infty})$ defined by the second author of this paper [Leg20]. In fact, the isomorphism ϕ_* preserves the product structures; see Proposition 6.3.

Let us give a more detailed overview of the construction of $m_2^{-\infty}$ in the case of a 3-copy $3L = L^1 \cup L^2 \cup L^3$ such that any pair (L^i, L^j) for $i < j$ has the same Cthulhu complex as $2L$; that is, for $i < j$, the cobordism L^i is a push-off of L^j using a positive Morse function F^{ij} satisfying the same conditions as the Morse function F. In particular, the Morse functions F^{ij} are chosen so that the action of any intersection point in the Cthulhu complex $Cth(L^i, L^j)$ is less than the action of any pure Reeb chord of Λ_- and that the top and bottom cylinders of $3L$ agree with the cylinder over $3\Lambda_\pm$ in $\mathcal{A}ug_+(\Lambda_\pm)$.

For $i = 1, 2, 3$, given augmentations ϵ_-^i of $\mathcal{A}(\Lambda_-)$, the induced augmentations ϵ_+^i of $\mathcal{A}(\Lambda_+)$, and a *domain dependent almost complex structure* $\mathbf{J}_z$ with values in $\mathcal{J}^{adm}$, the operator

$$m_2^{-\infty} : C_{-\infty}(L^2, L^3) \otimes C_{-\infty}(L^1, L^2) \to C_{-\infty}(L^1, L^3)$$

counts rigid holomorphic disks with a puncture positively asymptotic to a generator of $C_{-\infty}(L^1, L^3)$, two punctures negatively asymptotic to a generator of $C_{-\infty}(L^1, L^2)$ and a generator of $C_{-\infty}(L^2, L^3)$, and punctures negatively asymptotic to pure Reeb chords of $3\Lambda_-$ that are augmented by ϵ_-^i. The operator $m_2^{-\infty}$ can be decomposed as the sum of the maps

$$\mu_{i,j}^k : C_i(L^2, L^3) \otimes C_j(L^1, L^2) \to C_k(L^1, L^3)$$

for $i, j, k \in \{0, -\}$, satisfying the following:

1) In the special setting we consider here, namely the 3-copy $3L$, energy restrictions guarantee that if one of the inputs of $m_2^{-\infty}$ is in C_0, then the output is in C_0, i.e. $\mu_{0,j}^- = \mu_{i,0}^- = 0$, for $i, j = 0, -$.
2) The map $\mu_{-,-}^-$ agrees with the usual m_2^- of $\mathcal{A}ug_+(\Lambda_-)$.
3) For the rest of the cases, $(i, j, k) = (-, 0, 0), (0, -, 0), (0, 0, 0), (-, -, 0)$, the map $\mu_{i,j}^k$ counts rigid holomorphic disks with boundary on $3L$:

$$\mu_{i,j}^k(c^{23}, b^{12}) = \sum_{\dim \mathcal{M}_{\mathbf{J}_z}(a^{13};\mathbf{p}_-^{11}b^{12}\mathbf{q}_-^{22}c^{23}\mathbf{r}_-^{33})=0} |\mathcal{M}_{\mathbf{J}_z}^{L^1\cup L^2\cup L^3}(a^{13};\mathbf{p}_-^{11}b^{12}\mathbf{q}_-^{22}c^{23}\mathbf{r}_-^{33})|\epsilon_-^1(\mathbf{p}_-^{11})\epsilon_-^2(\mathbf{q}_-^{22})\epsilon_-^3(\mathbf{r}_-^{33})\ a^{13},$$

where a^{13} is a generator of $C_k(L^1, L^3)$, b^{12} is a generator of $C_j(L^1, L^2)$, c^{23} is a generator of $C_i(L^2, L^3)$, and $\mathbf{p}_-^{11}, \mathbf{q}_-^{22}, \mathbf{r}_-^{33}$ are words of pure Reeb chords of Λ_-^1, Λ_-^2, Λ_-^3, respectively.

In [Leg20, Section 5.2], it is shown that $m_2^{-\infty}$ commutes with the differentials $d_{-\infty}$ and thus induces a product map on cohomology

$$m_2^{-\infty} : H^m C_{-\infty}(L^2, L^3) \otimes H^n C_{-\infty}(L^1, L^2) \to H^{m+n} C_{-\infty}(L^1, L^3).$$

Moreover, we have the following proposition:

Proposition 6.3 ([Leg20, Theorem 2]). *The map ϕ_* from Equation (6.3) preserves the product structures, i.e.*

$$\phi_* \circ m_2^{-\infty}([a], [b]) = m_2^+(\phi_*[a], \phi_*[b])$$

for $[a] \in H^* C_{-\infty}(L^2, L^3)$ and $[b] \in H^* C_{-\infty}(L^1, L^2)$.

Remark 6.4. Note that we abuse the notation of ϕ_* here, for simplicity of notation. This isomorphism ϕ_* is defined on each pair of cobordisms $2L$ in $3L$ and could be different for the different 2-copies. A more rigorous way of writing the identity in the proposition above would be

$$\phi_*^{(\epsilon_-^1, \epsilon_-^3)} \circ m_2^{-\infty}([a], [b]) = m_2^+\left(\phi_*^{(\epsilon_-^2, \epsilon_-^3)}[a], \phi_*^{(\epsilon_-^1, \epsilon_-^2)}[b]\right).$$

7. Obstructions to exact Lagrangian cobordisms between links

In this section, we give an obstruction to the existence of embedded, Maslov-0, exact Lagrangian cobordisms through a count of augmentations of the bottom and top Legendrian links. We will count augmentations up to $\sim_{Aug_+}$, the equivalence in $\mathcal{A}ug_+(\Lambda_\pm)$ (Definition 5.1), which by Proposition 5.5 is the same as the split-DGA homotopy equivalence (Definition 5.3). The obstruction through a count of augmentations is proven in Section 7.1, with the proofs of key propositions provided in 7.2 and 7.3. Section 7.4 provides other obstructions in terms of linearized contact homology and ruling polynomials.

7.1. Proof of Theorem 1.4

Throughout this subsection, we suppose that ϵ_- is an augmentation for $\mathcal{A}(\Lambda_-)$ and $\epsilon_+ = \mathcal{F}_L(\epsilon_-)$ is the augmentation of $\mathcal{A}(\Lambda_+)$ induced by ϵ_- through L. Since we will be counting augmentations up to equivalence in $\mathcal{A}ug_+(\Lambda_\pm)$, we first define maps $\iota : H^0Hom_+(\epsilon_+^i, \epsilon_+^j) \to H^0Hom_+(\epsilon_-^i, \epsilon_-^j)$, for $i, j \in \{1, 2\}$. Consider the special pair of cobordisms $2L$ as described in Section 6.1 and the isomorphism $\phi_* : H^*(C_{-\infty}, d_{-\infty}) \to H^*Hom_+(\epsilon_+^i, \epsilon_+^j)$ in Equation (6.3). Note that (C_0, d_{00}) is a subchain complex of $(C_{-\infty}, d_{-\infty})$. Combining this fact with Proposition 6.2, it follows that the quotient map $\pi : C_{-\infty} \to C_-$ induces a map on cohomology:

$$\pi_* : H^*(C_{-\infty}, d_{-\infty}) \to H^*Hom_+(\epsilon_-^i, \epsilon_-^j).$$

Precomposing with ϕ_*^{-1} gives a map

$$\iota = \pi_* \circ \phi_*^{-1} : H^*Hom_+(\epsilon_+^i, \epsilon_+^j) \to H^*Hom_+(\epsilon_-^i, \epsilon_-^j).$$

The next proposition shows that ι is “natural”: although ϕ_* may depend on the Morse perturbation function F used to construct $2L$, ι does not. The proof of this proposition can be found in Section 7.2.

Proposition 7.1. *The maps $\iota : H^*Hom_+(\epsilon_+^i, \epsilon_+^j) \to H^*Hom_+(\epsilon_-^i, \epsilon_-^j)$ are independent of the choice of the Morse perturbation function F, up to compactly supported homotopy.*

The following properties of the ι map are used in the proof of Theorem 1.4, and are proved in Section 7.3.

Proposition 7.2. *The map* $\iota : H^k Hom_+(\epsilon^i_+, \epsilon^j_+) \to H^k Hom_+(\epsilon^i_-, \epsilon^j_-)$ *satisfies the following properties:*

1) ι *preserves the product structures, i.e.*

$$m_2^-(\iota[a], \iota[b]) = \iota(m_2^+([a],[b]))$$

for $[a] \in H^* Hom_+(\epsilon^2_+, \epsilon^3_+)$ *and* $[b] \in H^* Hom_+(\epsilon^1_+, \epsilon^2_+)$*, where* $m_2^\pm$ *are the products in the augmentation categories* $\mathcal{A}ug_+(\Lambda_\pm)$*,*

2) ι *is unital, meaning that when* $\epsilon^1_\pm = \epsilon^2_\pm = \epsilon_\pm$*, we have* $\iota([e_{\epsilon_+}]) = [e_{\epsilon_-}]$*.*

Proof of Theorem 1.4. Let L be an embedded, Maslov-0, exact Lagrangian cobordism from Λ_- to Λ_+, and $\epsilon^1_-, \epsilon^2_-$ be two augmentations of $\mathcal{A}(\Lambda_-)$. To show that

$$|Aug(\Lambda_-; \mathbb{F})/\sim_{Aug_+}| \le |Aug(\Lambda_+; \mathbb{F})/\sim_{Aug_+}|$$

we show that if the induced augmentations $\epsilon^1_+ = \mathcal{F}(\epsilon^1_-)$ and $\epsilon^2_+ = \mathcal{F}(\epsilon^2_-)$ are equivalent then ϵ^1_- and ϵ^2_- are also equivalent. Since $\epsilon^1_+, \epsilon^2_+$ are equivalent, there exist $[\alpha] \in H^0 Hom_+(\epsilon^1_+, \epsilon^2_+)$ and $[\beta] \in H^0 Hom_+(\epsilon^2_+, \epsilon^1_+)$ such that

$$\begin{aligned} & m_2^+([\alpha],[\beta]) = [e_{\epsilon^2_+}] \in H^0 Hom_+(\epsilon^2_+, \epsilon^2_+), \\ \text{and} \quad & m_2^+([\beta],[\alpha]) = [e_{\epsilon^1_+}] \in H^0 Hom_+(\epsilon^1_+, \epsilon^1_+), \end{aligned}$$

where $[e_{\epsilon^i_+}]$ is the unit in $H^0 Hom_+(\epsilon^i_+, \epsilon^i_+)$, for $i = 1, 2$. By Proposition 7.2,

$$m_2^-(\iota[\alpha], \iota[\beta]) = \iota(m_2^+([\alpha],[\beta])) = \iota([e_{\epsilon^2_+}]) = [e_{\epsilon^2_-}].$$

Analogously, one can prove that $m_2^-(\iota[\beta], \iota[\alpha]) = [e_{\epsilon^1_-}]$. It follows that ϵ^1_- and ϵ^2_- are equivalent, as desired. □

If $\Lambda_\pm$ are Legendrian knots or if $\Lambda_\pm$ are Legendrian links and $\mathbb{F} = \mathbb{Z}_2$, as mentioned in Remark 5.7(1), the map

$$\mathcal{F}_L : Aug(\Lambda_-; \mathbb{F})/\sim_{Aug_+} \to Aug(\Lambda_+; \mathbb{F})/\sim_{Aug_+}$$

exists; the above argument shows that $\mathcal{F}_L$ is injective.

7.2. Proof of Proposition 7.1

Proof of Proposition 7.1. Following the construction in Section 6.1, suppose that F and F' are two Morse functions on L, homotopic through a homotopy with compact support, and let $2L = L^1 \cup L^2$ and $2L' = L^{1'} \cup L^2$ denote the corresponding 2-copies. The homotopy between F and F' induces a compactly supported Lagrangian isotopy between $2L$ and $2L'$; note that the isotopy keeps the two cylindrical ends fixed. According to [CDRGG20, Proposition 6.4], the isotopy induces a chain map

$$\varphi : \; Cth(L^1, L^2) \to Cth(L^{1'}, L^2).$$

Following [Ekh12], we will show that the map φ is the identity map on $C_+(L^1, L^2) \to C_+(L^{1'}, L^2)$. Along a generic isotopy $\{L^1_s\}_{s\in[0,1]}$ from $L^1_0 := L^1$ to $L^1_1 := L^{1'}$, one can assume that except for a finite number of distinct points $0 < s_0 < s_1 < \cdots < s_r < 1$, the cobordisms L^1_s and L^2 are transverse and the moduli spaces contributing to the differential of $Cth(L^1_s, L^2)$ are transversely cut out. At the points s_j, two different situations can occur:

1) The birth/death of a pair of intersection points, $c_1, c_2 \in C_0$ with $|c_1| = |c_2| + 1$;
2) The appearance of a (-1)-disk $u \in \mathcal{M}(c_1; \mathbf{p}c_2\mathbf{q})$ with boundary on the non-cylindrical parts of the cobordisms.

Moreover, one can assume that these two cases do not occur simultaneously. Hence, from now on, let us assume that $s_0 \in (0, 1)$ is the only point in the isotopy when situations (1) or (2) can occur. Suppose first that case (1) occurs, and denote the Cthulhu chain complex with (resp. without) the pair of intersection points by $(C[+], d[+])$ (resp. $(C[-], d[-])$). We have $d[+](c_2) = c_1 + v$ where v does not contain c_1. The induced chain map $C[+] \to C[-]$ corresponding to the death of the pair of intersection points c_1, c_2 maps $c_2 \to 0$, $c_1 \to -v$ and other elements to themselves. The induced chain map $C[-] \to C[+]$ corresponding to the birth of c_1, c_2 sends an element c to $c - c_1^*(d[+]c)c_2$, where c_1^* is the dual element for c_1. Note that both c_1 and c_2 are intersection points, thus the induced chain maps are identity maps on C_+. In the second case, a (-1)-disk $u \in \mathcal{M}(c_1; \mathbf{p}c_2\mathbf{q})$ appears. The induced map φ sends c_2 to $c_2 + \lambda c_1$, for some number λ, and all other elements to themselves. Since the negative puncture c_2 is not in C_+, the induced chain map is the identity on C_+.

Denote $\varphi_{-\infty}$ the component

$$\varphi_{-\infty}:\ (C_{-\infty}(L^1,L^2),d_{-\infty})\to(C_{-\infty}(L^{1'},L^2),d'_{-\infty})$$

The fact that φ is a chain map and fixes C_+ implies that $\varphi_{-\infty}$ is a chain map, i.e. $\varphi_{-\infty}\circ d_{-\infty}=d'_{-\infty}\circ\varphi_{-\infty}$. Let us then denote φ_+ the component

$$\varphi_+: Cth(L^1,L^2)\to C_+(L^{1'},L^2)$$

The fact that φ is a chain map implies that for any cycle $c\in C_{-\infty}(L^1,L^2)$ one has

$$\varphi_+\circ(\phi+d_{-\infty})(c)=d'_{++}\circ\varphi_+(c)+\phi'\circ\varphi_{-\infty}(c)$$

Using the fact that φ is the identity map on C_+ and $d_{-\infty}(c)=0$, this equation becomes,

$$\phi(c)=d'_{++}\circ\varphi_+(c)+\phi'\circ\varphi_{-\infty}(c).$$

It follows that

$$[\phi(c)]=[\phi'\circ\varphi_{-\infty}(c)]\in H^*(C_+(L^{1'},L^2))=H^*Hom_+(\epsilon^i_+,\epsilon^j_+).$$

In order to show that $\iota:=\pi_*\circ\phi_*^{-1}=\pi'_*\circ(\phi'_*)^{-1}=:\iota'$, where π and π' are the projection maps from $C_{-\infty}\to C_-$ for the two cases, respectively, we will prove that

$$\text{if } c\in C_{-\infty}(L^1,L^2),\ \text{then } \pi(c)=\pi'\circ\varphi_{-\infty}(c). \tag{7.1}$$

Again, it suffices to understand how $\varphi_{-\infty}$ behaves when either case (1) or (2) occurs in the isotopy. If a (-1)-disk $u\in\mathcal{M}(c_1;\mathbf{p}c_2\mathbf{q})$ occurs, the positive puncture c_1 can be an element in C_+ or C_0, and by definition of $\varphi_{-\infty}$ we only need to consider disks u with $c_1\in C_0$. Then, the induced chain map $\varphi_{-\infty}$ sends any element c to $c+m$ for $m=0$ or $m\in C_0$. If case (1) occurs, and we have a birth/death of intersection points c_1,c_2 in C_0, denote the chain complex with (resp. without) the pair of intersection points by $(C_{-\infty}[+],d_{-\infty}[+])$ (resp. $(C_{-\infty}[-],d_{-\infty}[-])$). Suppose that $d_{-\infty}[+](c_2)=c_1+v$. Since the differential of $C_{-\infty}$ is upper triangular, we know that v is in C_0. Thus, the map from $C_{-\infty}[+]$ to $C_{-\infty}[-]$ maps c_1 to C_0 and c_2 to 0. If we have a birth of intersection points, the map from $C_{-\infty}[-]$ to $C_{-\infty}[+]$ sends an element c to $c-c_1^*(d_{-\infty}[+]c)c_2$, which is also in C_0. In both cases we have shown (7.1) is true and can conclude that the map ι does not depend on the choice of Morse function F. □

7.3. Proof of Proposition 7.2

To prove the first statement of Proposition 7.2, first recall that $\iota = \pi_* \circ \phi_*^{-1}$ and that ϕ_*^{-1} preserves the product structures; see Proposition 6.3. Thus Proposition 7.2 (1) follows immediately from

Lemma 7.3. *The map $\pi_* : H^*(C_{-\infty}, d_{-\infty}) \to H^*(C_-, d_{--})$ preserves the products.*

Proof. Recall that $m_2^{-\infty}(a, b) \in C_0$ if a or b is in C_0. Thus the component of $m_2^{-\infty}(a, b)$ with values in C_- only comes from $m_2^-(\pi(a), \pi(b))$, i.e.

$$\pi \circ m_2^{-\infty}(a, b) = m_2^-(\pi(a), \pi(b)).$$

$\square$

In order to prove Proposition 7.2 (2), we need that for any augmentation ϵ_- of $\mathcal{A}(\Lambda_-)$ and its induced augmentation ϵ_+ of $\mathcal{A}(\Lambda_+)$, the map

$$\iota : H^0 Hom_+(\epsilon_+, \epsilon_+) \to H^0 Hom_+(\epsilon_-, \epsilon_-)$$

preserves the unit. Note that ϕ_*^{-1} is an isomorphism that preserves the product structures and thus sends the unit $[e_+]$ of $H^0 Hom_+(\epsilon_+, \epsilon_+)$ to a unit $[e_{-\infty}]$ of $H^0(C_{-\infty})$. In order to show $\pi_*([e_{-\infty}]) \in H^0 Hom_+(\epsilon_-, \epsilon_-)$ is the unit $[e_-]$ of $H^0 Hom_+(\epsilon_-, \epsilon_-)$, we only need to prove the following lemma.

Lemma 7.4. *There is an element $e = e_- + e_0 \in C_{-\infty}$, where e_0 is an element in C_0, such that $d_{-\infty}(e) = 0$.*

Proof of Proposition 7.2. With Lemma 7.4 in hand, the fact that π_* preserves the product structure, and the fact that $[e_{-\infty}]$ and $[e_-]$ are the units of $H^*(C_{-\infty})$ and $H^0 Hom_+(\epsilon_-, \epsilon_-)$ respectively, we have that

$$\begin{aligned} \pi_*([e_{-\infty}]) = m_2^-([e_-], \pi_*([e_{-\infty}])) &= m_2^-(\pi_*[e], \pi_*([e_{-\infty}])) \\ &= \pi_* \circ m_2^{-\infty}([e], [e_{-\infty}]) = \pi_*[e] = [e_-]. \end{aligned}$$

Thus, $\iota([e_+]) = \pi_* \circ \phi_*^{-1}([e_+]) = \pi_*([e_{-\infty}]) = [e_-]$. $\square$

Proof of Lemma 7.4. Recall that the unit e_- of $Hom_+^0(\epsilon_-, \epsilon_-)$ is given by $e_- = -\sum y_i^{12}$, where y_i^{12} are the Reeb chord of $2\Lambda_-$ corresponding to the Morse minima of the Morse function f_- used to define $2\Lambda_-$. Let e_0 be negative of the sum of all the intersections that corresponds to the minima of the Morse function F, and then let $e = e_0 + e_-$. We have that

$$d_{-\infty}(e) = d_{00}(e_0) + d_{0-}(e_-) + d_{--}(e_-).$$

The fact that e_- is closed in $Hom_+(\epsilon_-, \epsilon_-)$ implies $d_{--}(e_-) = 0$. It follows from Proposition 6.2 that d_{00} counts negative Morse flow lines of the Morse function F. We need to interpret the holomorphic disks counted by $d_{0-}(e_-)$ in terms of Morse flow lines of a Morse function $\widetilde{F}$ that agrees with F in the main part but also encodes the Morse function f_- on the bottom cylinder. This can be done by concatenating a cobordism from the bottom and comparing the Cthulhu complexes of the two pairs of cobordisms using a *transfer map* defined in [CDRGG20]. The remainder of the proof is dedicated to describing $d_{00}(e_0)$ and $d_{0-}(e-)$ in detail.

Recall that Λ^1_- is a push off of Λ^2_- using a very small positive Morse function f_-. Let $A \in \mathbb{R}^+$ be twice the maximum value of f_-. Consider the cylinder $\mathbb{R} \times \Lambda^1_-$ and push the negative end of the cylinder in the $-z$ direction by A. Denote this new Legendrian in the negative end by $\Lambda^1_- - A$. Thus, we get a cobordism W^1 from $\Lambda^1_- - A$ to Λ^1_- as shown in Figure 14. Concretely, consider a non-increasing Morse function $\delta(t) : \mathbb{R} \to \mathbb{R}$ which is 0 when $t > -N - 1$ and is equal to the constant A when $t < N'$, for some $N' < -N - 1$. Note that $X_H = -\delta(t)\partial/\partial z$ is a Hamiltonian vector field, and denote its time 1 flow by Φ_H. It follows that $W^1 := \Phi_H(\mathbb{R} \times \Lambda^1_-)$ is an exact Lagrangian cobordism. Denote by W^2 the cylinder $\mathbb{R} \times \Lambda^2_-$.

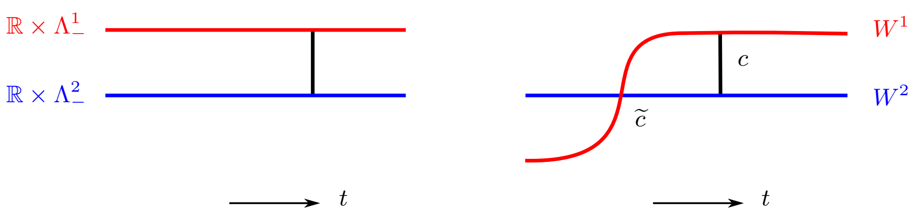


Figure 14. A schematic picture of wrapping the negative end of a cobordism.

Observe that there is a natural bijection between $C_0(W^1, W^2)$ and the Morse Reeb chords in $C_+(W^1, W^2)$ with degree shifted up by 1. Moreover, we can show that $d_{+0}^{W^1,W^2}$ sends an intersection point to the corresponding Morse Reeb chord, as follows. First, the projection map $\pi_{xy} : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}_{xy}$ sends $W^1 \cup W^2$ to $\pi_{xy}(\Lambda^1_- \cup \Lambda^2_-)$. Then, according to [DR16, Proposition 5.11], the projection map also sends holomorphic disks with boundary on $W^1 \cup W^2$ to holomorphic disks with boundary on $\pi_{xy}(\Lambda^1_- \cup \Lambda^2_-)$. Suppose that a disk

$u \in \mathcal{M}^{W^1 \cup W^2}(a, \mathbf{p}_-^{11} b \mathbf{q}_-^{22})$ contributes to $d_{+0}^{W^1,W^2}$, i.e. a is a mixed Reeb chord of $\Lambda_-^1 \cup \Lambda_-^2$, b is an intersection point in $W^1 \cap W^2$ and $\mathbf{p}_-^{11}, \mathbf{q}_-^{22}$ are words of pure degree 0 Reeb chords of $\Lambda_-^1 - A$ and Λ_-^2, respectively. The rigidity of u implies that $|a| - |b| = 1$ using the grading in the Cthulhu complex. Projecting down to the xy-plane, we have that $\pi_{xy}(u) \in \mathcal{M}(\pi_{xy}(a); \mathbf{p}_-^{11} \pi_{xy}(b) \mathbf{q}_-^{22})$ is a holomorphic disk with boundary on $\pi_{xy}(\Lambda_-^1 \cup \Lambda_-^2)$. Comparing the grading in the Cthulhu complex and the grading in $\mathcal{A}(\Lambda_-^1 \cup \Lambda_-^2)$, we have

$$|a| = |\pi_{xy}(a)|_{LCH} + 2 \text{ and } |b| = |\pi_{xy}(b)|_{LCH} + 1.$$

It follows that $|\pi_{xy}(a)|_{LCH} - |\pi_{xy}(b)|_{LCH} = 0$, or in other words, the expected dimension of $\mathcal{M}(\pi_{xy}(a); \mathbf{p}_-^{11} \pi_{xy}(b) \mathbf{q}_-^{22})$ is -1, which implies that $\pi_{xy}(u)$ is constant and thus $|\pi_{xy}(a)| = |\pi_{xy}(b)|$. Therefore, we have proved that $d_{+0}^{W^1,W^2}$ sends an intersection point in $W^1 \cap W^2$ to the corresponding Morse Reeb chord of $\Lambda_-^1 \cup \Lambda_-^2$.

Consider now the Cthulhu complex of the pair of concatenated cobordisms $(W^1 \odot L^1, W^2 \odot L^2)$. Its generators can be decomposed into four types.

$$\begin{aligned} Cth(W^1 \odot L^1, W^2 \odot L^2) = C_-(W^1, W^2) &\oplus C_0(W^1, W^2) \\ &\oplus C_0(L^1, L^2) \oplus C_+(L^1, L^2) \end{aligned}$$

According to [CDRGG20], there is a chain map

$$\Psi^W : Cth(W^1 \odot L^1, W^2 \odot L^2) \to Cth(L^1, L^2)$$

which is $d_{+0}^{W^1,W^2}$ on $C_0(W^1, W^2)$, is $d_{+-}^{W^1,W^2}$ on $C_-(W^1, W^2)$ and is the identity on $C_0(L^1, L^2) \oplus C_+(L^1, L^2)$ (in the case of the special pair of cobordisms we are considering in this paper). Due to action restrictions, Morse Reeb chords do not show up in the image of $d_{+-}^{W^1,W^2}$ but only in the image of $d_{+0}^{W^1,W^2}$.

Denote the intersection point in $W^1 \cap W^2$ corresponding to a Morse Reeb chord c of $\Lambda_-^1 \cup \Lambda_-^2$ by $\widetilde{c}$, as shown in Figure 14. Due to the description of $d_{+0}^{W^1,W^2}$, the chain map Ψ^W identifies the holomorphic disks counted by $d_{00}^{W^1 \odot L^1, W^2 \odot L^2}(\widetilde{c})$ such that the positive puncture is in $C_0(L^1, L^2)$, with the holomorphic disks counted by $d_{0-}^{L^1,L^2}(c)$. Thus, we can describe $d_{0-}^{L^1,L^2}(e_-)$ through $d_{00}^{W^1 \odot L^1, W^2 \odot L^2}(\widetilde{e}_-)$, where $\widetilde{e}_- = -\sum \widetilde{y}_i$ and $\widetilde{y}_i$ are the intersection points corresponding to the Morse Reeb chords y_i of $\Lambda_-^1 \cup \Lambda_-^2$.

Observe that $W^1 \odot L^1$ happens on a small neighborhood of $W^2 \odot L^2 = L^2$ and thus can be described as a push-off of L^2 along a Morse function $\widetilde{F}$.

Note that the Morse function $\widetilde{F}$ agrees with F on $([-N,N]\times\mathbb{R}^3)\cap L^2$ but has also minima at $\widetilde{y}_i$ and saddle points at $\widetilde{x}_i$. Since $W^1\odot L^1$ and $W^2\odot L^2$ are close enough, the differential $d_{00}^{W^1\odot L^1, W^2\odot L^2}$ counts the negative Morse flow lines of $\widetilde{F}$. Let $\widetilde{e}=\widetilde{e}_-+e_0$ be the negative sum of all the minima of $\widetilde{F}$. Observe that $d_{00}^{W^1\odot L^1, W^2\odot L^2}(\widetilde{e})=0$ since each saddle point of $\widetilde{F}$ has two Morse trajectories flowing down with the opposite sign and they have to approach some minima. It follows from Ψ^W being a chain map that

$$d_{00}(e_0)+d_{0-}(e_-)=\pi_0\circ d^{L^1,L^2}\circ\Psi^W(\widetilde{e})=\pi_0\circ\Psi^W\circ d^{W^1\odot L^1,W^2\odot L^2}(\widetilde{e})=0,$$

where π_0 is the projection map: $Cth(L^1,L^2)\to C_0(L^1,L^2)$. □

7.4. Other obstructions

In this section, we give two additional obstructions to the existence of exact Lagrangian cobordisms in terms of linearized contact homology and ruling polynomials, which generalize the results in [Pan17].

Proposition 7.5. *Assume $\mathbb{F}$ is a field and let L be an exact Lagrangian cobordism from Λ_- to Λ_+ with Maslov-0. Suppose that ϵ_- is an augmentation of Λ_- and ϵ_+ is the induced augmentation of Λ_+. Then we have that*

$$LCH_k^{\epsilon_+}(\Lambda_+)\cong LCH_k^{\epsilon_-}(\Lambda_-) \tag{7.2}$$

for $k<0$ and $k>1$.

Proof. From Equation (6.2), we have a long exact sequence

$$\cdots\to H^k(L,\Lambda_-)\to H^kHom_+(\epsilon_+,\epsilon_+)\\ \to H^kHom_+(\epsilon_-,\epsilon_-)\to H^{k+1}(L,\Lambda_-)\to\cdots.$$

Note that $H^k(L,\Lambda_-)=0$ when $k<0$ and $k>2$. For $k=2$, we know that $H^2(L,\Lambda_-)$ is 0 because any two components of Λ_- cannot bound a closed surface in L, i.e. a Lagrangian cap of two components of Λ_-. Otherwise we get a cobordism from a subset of Λ_- (that admits an augmentation restricted from ϵ_-) to the empty set, which is a contradiction by [DR15, Corollary 1.9].

The long exact sequence implies that

$$H^k Hom_+(\epsilon_-, \epsilon_-) \cong H^k Hom_+(\epsilon_+, \epsilon_+),$$

for $k < -1$ and $k > 1$. Recall that $H^k Hom_+(\epsilon, \epsilon) \cong LCH^{\epsilon}_{1-k}(\Lambda)$, so we get

$$LCH^{\epsilon_+}_k(\Lambda_+) \cong LCH^{\epsilon_-}_k(\Lambda_-),$$

for $k > 2$ and $k < 0$.

The isomorphism for $k = 2$ comes from the Sabloff duality [EES09], which gives a long exact sequence:

$$\cdots \to H^k(\Lambda) \to LCH^k_{\epsilon}(\Lambda) \to LCH^{\epsilon}_{-k}(\Lambda) \to H^{k+1}(\Lambda) \to \cdots .$$

The fact that $H^k(\Lambda)$ vanishes unless $k = 0$ or 1 implies that $LCH^{\epsilon}_{-k}(\Lambda) \cong LCH^k_{\epsilon}(\Lambda)$ for $k > 1$ and $k < -1$. Note that $LCH^{\epsilon}_k(\Lambda)$ are vector spaces over a field $\mathbb{F}$. It follows from the universal coefficient theorem that $\dim LCH^k_{\epsilon}(\Lambda) = \dim LCH^{\epsilon}_k(\Lambda)$. Thus, we have that $\dim LCH^{\epsilon}_{-k}(\Lambda) \cong \dim LCH^{\epsilon}_k(\Lambda)$ for $k > 1$. Since the isomorphism (7.2) holds for $k = -2$, the dimension of the LCH homologies are the same for $k = 2$, which implies the isomorphism for $k = 2$ as they are vector spaces over $\mathbb{F}$. □

We do not get the relation between the LCH's on degree 0 and 1 as Pan did for cobordisms between knots in [Pan17, Corollary 1.4].

Example 7.6. Take $\mathbb{F} = \mathbb{Z}_2$ and consider two exact Lagrangian cobordisms L^1, L^2 from the Hopf link Λ^0_{H} to the trefoil obtained by pinching the chords b_1 and b_2 of the trefoil, respectively, as shown in Figure 15. Let ϵ^1_-, resp. ϵ^2_-, be the augmentation of Λ^0_{H} which sends the two Reeb chords (c_1, c_2) to $(0, 0)$, resp. $(0, 1)$. Both augmentations ϵ^i_-, induce through L^i for $i = 1, 2$ the augmentation of the trefoil ϵ_+, which sends the three Reeb chords (b_1, b_2, b_3) to $(1, 1, 0)$. However, the Legendrian contact homology of Λ^0_{H} linearized by ϵ^1_- has rank one in degrees 0 and 1, while linearized by ϵ^2_- it has rank 2 in degrees 0 and 1. Thus, the data $(L, \Lambda_+, \Lambda_-, \epsilon_+)$ cannot determine $LCH^{\epsilon_-}(\Lambda_-)$.

Another way to count the number of augmentations in the augmentation category is the **homotopy cardinality** [NRSS17], which is defined by

$$\pi_{\geq 0}\mathcal{A}ug_+(\Lambda; \mathbb{F}_q)^* = \sum_{[\epsilon] \in \mathcal{A}ug_+(\Lambda;\mathbb{F}_q)/\sim} \frac{1}{|Aut(\epsilon)|} \cdot \frac{|H^{-1}Hom_+(\epsilon,\epsilon)| \cdot |H^{-3}Hom_+(\epsilon,\epsilon)| \cdots}{|H^{-2}Hom_+(\epsilon,\epsilon)| \cdot |H^{-4}Hom_+(\epsilon,\epsilon)| \cdots},$$

where $[\epsilon]$ is the equivalence class of ϵ in the augmentation category and $|Aut(\epsilon)|$ is the number of invertible elements in $H^0 Hom_+(\epsilon, \epsilon)$.

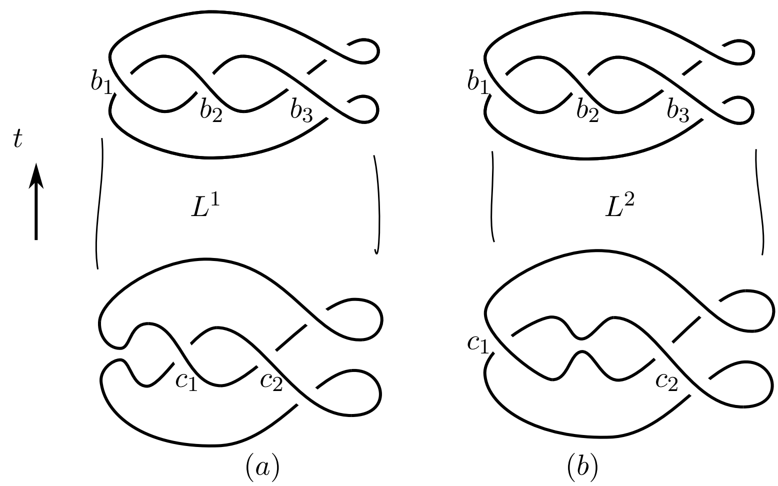


Figure 15. Part (a) and (b) shows two cobordisms obtained by doing pinch move on b_1 and b_2, respectively.

Proposition 7.7. *Let L be a spin exact Lagrangian cobordism from Λ_- to Λ_+ with Maslov number 0. Then for any finite field $\mathbb{F}_q$, we have that*

$$\pi_{\geq 0}\mathcal{A}ug_+(\Lambda_+;\mathbb{F}_q)^* \geq \pi_{\geq 0}\mathcal{A}ug_+(\Lambda_-;\mathbb{F}_q)^*.$$

Proof. For each equivalence class in $Aug_+(\Lambda_-;\mathbb{F}_q)$, we take a representative ϵ_- and compare the term of $[\epsilon_-]$ in the sum with the term of the induced augmentation ϵ_+ for $Aug_+(\Lambda_+;\mathbb{F}_q)$. It follows from Proposition 7.5 that the $H^k Hom_+$ spaces are isomorphic between ϵ_- and ϵ_+ for $k<0$. Moreover, it follows from Theorem 1.4 that if an element $[\alpha_+] \in H^0 Hom_+(\epsilon_+,\epsilon_+)$ is invertible, then $\iota[\alpha_+] \in H^0 Hom_+(\epsilon_-,\epsilon_-)$ is invertible. Thus $H^0 Hom_+(\epsilon_-,\epsilon_-)$ may have more invertible elements than $H^0 Hom_+(\epsilon_+,\epsilon_+)$. It follows that for each equivalent class represented by ϵ_-, the term in the summand for ϵ_+ is bigger than or equal to the term for ϵ_-. Moreover, there may be more equivalence classes in $\mathcal{A}ug_+(\Lambda_+)$ than in $\mathcal{A}ug_+(\Lambda_-)$. Thus the proposition follows. □

The homotopy cardinality is related to the *ruling polynomial* $R_\Lambda(z)$, a combinatorial invariant of Legendrian knots that is easily computed, in the following way:

$$\pi_{\geq 0}\mathcal{A}ug_+(\Lambda;\mathbb{F}_q)^* = q^{tb(\Lambda)/2}R_\Lambda(q^{1/2}-q^{-1/2}).$$

See Section 8.3 for more details on the ruling polynomial. Thus we have the following corollary.

Corollary 7.8. *Let L is a spin exact Lagrangian cobordism from Λ_- to Λ_+ with Maslov number* 0*. Then, we have that*

$$R_{\Lambda_-}(q^{1/2} - q^{-1/2}) \leq q^{-\chi(L)/2} R_{\Lambda_+}(q^{1/2} - q^{-1/2})$$

for any q that is a power of a prime number.

8. Examples of obstructed fillings

In this section, we will prove Theorem 1.8. To prove that certain immersed Lagrangian fillings of a Legendrian knot Λ do exist, we will use the "decomposable" moves described below to prove the existence of embedded Lagrangian cobordisms from a disjoint union of Legendrian Hopf links to Λ. Recall that, by definition, the Legendrian Hopf link Λ_{H}^k admits an immersed, Maslov-0, exact Lagrangian filling with one action-0 double point of index k. We will prove that certain types of Lagrangian fillings of Λ cannot exist by applying Theorems 1.1 and 1.4. Throughout this section, we consider DGAs over $\mathbb{Z}_2$ and augmentations to $\mathbb{Z}_2$. For the family Λ_k in Theorem 1.8(1), we will count augmentations directly, while for the family Λ_g^p in Theorem 1.8(2), we will employ the theory of rulings to count augmentations.

All of the embedded, Maslov-0, exact Lagrangian fillings and cobordisms that we construct in this section are *decomposable* in the following sense. It is known that there exists an embedded, Maslov-0, exact Lagrangian cobordisms between two Legendrian links $\Lambda_\pm$ if Λ_+ differs from Λ_- by Legendrian isotopy, pinch moves, and the death of a max *tb* unknotted component. Figure 16 illustrates the local front projections of an orientable downward in time pinch move and the downward in time death of a max *tb* unknot. In order to produce an orientable surface, the pinch move can only be performed on strands with opposite orientations, and in order for the Lagrangian to be Maslov-0, pinch moves can only be performed on strands whose upper branch has a Maslov potential 1 greater than that of the lower branch, as shown in Figure 16. A Lagrangian cobordism L from Λ_- to Λ_+ is called **elementary** if it arises from isotopy, a single pinch move, or a single disk filling. A Lagrangian cobordism is **decomposable** it is obtained by stacking elementary cobordisms. The elementary moves that make up decomposable cobordisms were introduced by Ekholm, Honda, and Kálmán in [EHK16, Section 6].

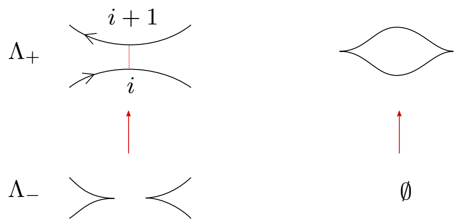


Figure 16. Local front projections of a merge/pinch move (topologically a saddle cobordism/1-handle) and the birth/death of a max *tb* unknot (topologically a disk/0-handle). The red arrows represent the positive t direction and the labels on the strands indicate the Maslov potential.

As we apply Theorem 1.4, it will be useful to have the following augmentation count.

Lemma 8.1.

$$|Aug(\Lambda_{\mathrm{H}}^k;\mathbb{Z}_2)/\sim_{Aug_+}| = \begin{cases} 3, & k=0 \\ 0, & k\neq 0. \end{cases}$$

Moreover,

$$|Aug(\bigsqcup_{i=1,\dots,m} \Lambda_{\mathrm{H}}^{k_i};\mathbb{Z}_2)/\sim_{Aug_+}| = 3^Z,$$

where $Z = |\{i : k_i = 0\}|$.

Proof. As explained in Example 4.3, the Hopf link Λ_{H}^k has 3 augmentations when $k=0$ and no augmentations otherwise. When $k=0$, there are no degree -1 chords, and thus, by Corollary 5.6, the count of augmentations up to the equivalence relation $\sim_{Aug_+}$ is the same as the count of augmentations. □

8.1. Proof of Theorem 1.8(1)

We construct the family of Legendrian knots Λ_k such that $\Lambda_1 = \Lambda_{9_{48}}$ as follows. Consider the tangle T in Figure 17. Arrange k copies $T_1,\dots,T_k$ of T in a row and connect them by a tangle sum; then perform the standard rainbow tangle closure after introducing 1 more crossing, as shown in Figure 17. The resulting Legendrian Λ_k admits a Maslov potential whose values on each strand is also indicated in the figure. When $k=1$, the Legendrian knot obtained this way is a 9_{48} knot; its front projection is shown in Figure 18 and its Lagrangian projection in Figure 19.

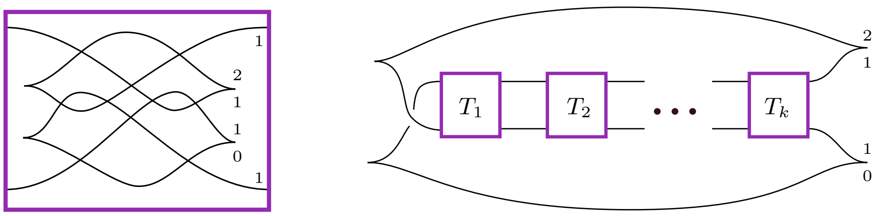


Figure 17. Left: front projection of the tangle T; right: front projection of Λ_k. The numbers indicate the Maslov potential.

Proposition 8.2. *Λ_k admits an immersed, Maslov-0, exact Lagrangian filling F_k^k of genus k with k double points, each of which has action 0 and index 1.*

Proof. When $k = 1$, by performing a sequence of pinch moves as indicated by the red lines in the Figure 18 and Reidemeister moves, we obtain an embedded, Maslov-0, exact Lagrangian cobordism from the Hopf link Λ_{H}^1 to $\Lambda_{9_{48}}$. For $k \geq 2$, by performing pinch moves on each copy of the tangle T as in the case of $\Lambda_{9_{48}}$, we obtain an embedded, Maslov-0, exact Lagrangian cobordism of genus k from $\sqcup_k \Lambda_{\mathrm{H}}^1$ to Λ_k. Each Λ_{H}^1 has an immersed, Maslov-0, exact Lagrangian filling with a double point of action 0 and index 1. Stacking these Lagrangian cobordisms produces the desired filling F_k^k of Λ_k. □

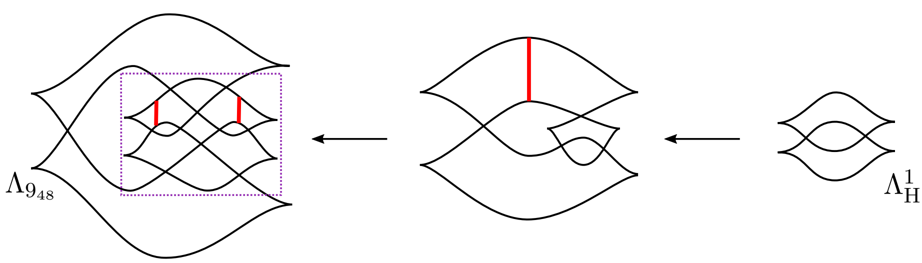


Figure 18. Sequence of three pinch moves that prove the existence of an embedded, Maslov-0, exact Lagrangian cobordism of genus 1 from the Hopf link Λ_{H}^1 to $\Lambda_{9_{48}}$.

Proposition 8.3. *Λ_k does not admit an immersed, Maslov-0, exact Lagrangian disk filling F_{k-1}^{k+1} with $k+1$ double points, all of action 0 and k of index 1 and one of index 0.*

The proof of this proposition will follow easily once we prove the following count of augmentations.

Lemma 8.4. *For all $k \geq 1$, $|Aug(\Lambda_k;\mathbb{Z}_2)/\sim_{Aug_+}| = 1$.*

Proof. When $k = 1$, the DGA $\mathcal{A}(\Lambda_{9_{48}})$ is generated by $a_i, i = 1, \cdots 6$, $b_i, i = 1, \cdots, 7$, $c_i, i = 1, 2$ with grading $|a_i| = 1, |b_i| = 0, |c_i| = -1$ as shown in Figure 19. The differential is given by

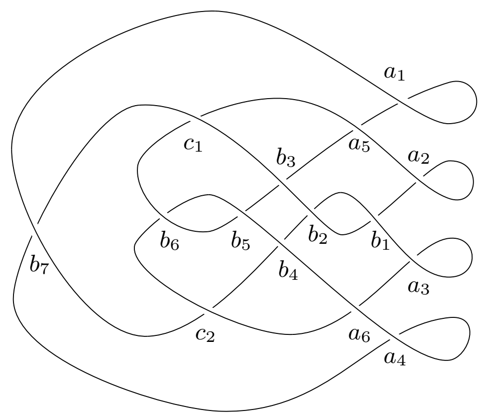


Figure 19. A Lagrangian projection for $\Lambda_{9_{48}}$.

$$\begin{aligned}
\partial a_1 &= 1 + b_7(b_3 + c_1 a_5) & \partial b_2 &= c_1 b_6 b_4 + b_3 b_6 c_2\\
\partial a_2 &= 1 + a_5 b_6 c_2 b_1 + b_6 b_4 b_1 & \partial b_3 &= c_1(1 + b_6 b_5)\\
\partial a_3 &= 1 + b_1 c_1 b_6 a_6 + b_1 b_3 b_6 & \partial b_4 &= (1 + b_5 b_6)c_2\\
\partial a_4 &= 1 + (b_4 + a_6 c_2)b_7 & \partial b_i &= 0, \text{ for } i \neq 2, 3, 4\\
\partial a_5 &= 1 + b_6 b_5 & \partial c_i &= 0, \text{ for } i = 1, 2.\\
\partial a_6 &= 1 + b_5 b_6 & &
\end{aligned}$$

There are two augmentations ϵ_0 and ϵ_1 of $\mathcal{A}(\Lambda_{9_{48}})$ to $\mathbb{Z}_2$ with $\epsilon_i(b_2) = i$, and $\epsilon_i(b_j) = 1$ for $j \neq 2$, $i = 0, 1$. These two augmentations are DGA homotopic since $\epsilon_0 - \epsilon_1 = K \circ \partial$, where K sends c_1 to 1 and the other Reeb chords to 0. Since $\mathbb{F} = \mathbb{Z}_2$, By Proposition 5.5 and Remark 5.4, equivalence with respect to DGA homotopy is the same as equivalent with respect to $\sim_{Aug_+}$, and thus we have that $|Aug(\Lambda_{9_{48}};\mathbb{Z}_2)/\sim_{Aug_+}| = 1$.

The calculation for $k \geq 2$ is similar. Label the Reeb chords in the jth tangle of Λ_k by b_{i_j}, a_{i_j} and c_{i_j} following a similar labeling scheme as for $\Lambda_{9_{48}}$, see Figure 20. Let $\tilde{b}_7$, $\tilde{a}_1$ and $\tilde{a}_4$ denote the Reeb chords of Λ_k not contained in any of the k tangles, and such that $|\tilde{b}_7| = 0$ and $|\tilde{a}_1| = |\tilde{a}_4| = 1$.

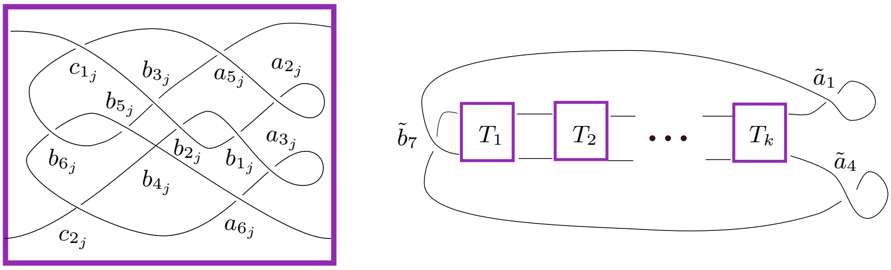


Figure 20. Lagrangian projections of the tangle T and the Legendrian Λ_k.

Then, one can find that any augmentation ϵ of Λ_k to $\mathbb{Z}_2$ takes the following values: $\epsilon(\tilde{b}_7) = 1$, $\epsilon(b_{i_j}) = 1$ for any $i_j \neq 2_j$, and $\epsilon(b_{2_j}) \in \{0, 1\}$. Therefore, for any Λ_k we have 2^k augmentations to $\mathbb{Z}_2$. Suppose that ϵ_1 and ϵ_2 are two augmentations of Λ_k such that $\epsilon_1(b_{2_j}) - \epsilon_2(b_{2_j}) = 1$ for j contained in some subset $J \subset \{1, \ldots, k\}$. Then, there exists a DGA homotopy K from ϵ_1 to ϵ_2 where $K(c_{1_j}) = 1$ for $j \in J$, and which maps all other Reeb chords to 0. Therefore, the Legendrians Λ_k have a unique augmentation to $\mathbb{Z}_2$ up to DGA homotopy and thus up to $\sim_{Aug_+}$. □

Proof of Proposition 8.3. By Theorem 1.1, the existence of the filling F_{k-1}^{k+1} is equivalent to the existence of an embedded, Maslov-0, exact Lagrangian cobordism from $\sqcup_k \Lambda_{\mathrm{H}}^1 \sqcup \Lambda_{\mathrm{H}}^0$ to Λ_k. By Lemma 8.1 and Lemma 8.4,

$$|Aug(\sqcup_k \Lambda_{\mathrm{H}}^1 \sqcup \Lambda_{\mathrm{H}}^0; \mathbb{Z}_2)/\sim_{Aug_+}| = 3, \quad \text{and} \quad |Aug(\Lambda_k; \mathbb{Z}_2)/\sim_{Aug_+}| = 1,$$

and thus by Theorem 1.4 such an embedded cobordism from $\sqcup_k \Lambda_{\mathrm{H}}^1 \sqcup \Lambda_{\mathrm{H}}^0$ to Λ_k does not exist. □

We now have all the ingredients to prove our first part of Theorem 1.8.

Proof of Theorem 1.8(1). Fix Λ_k. Proposition 8.2 shows the existence of the immersed, Maslov-0, exact Lagrangian filling F_k^k with genus k that has k double points, each with action 0 and index 1. Proposition 8.3 shows there does not exist an immersed, Maslov-0, exact Lagrangian disk filling F_{k-1}^{k+1} with $(k+1)$ double points, all of action 0, k of index 1, and one of index 0. Thus, by Definition 3.3, F_k^k does not arise from Lagrangian surgery.

For the smooth comparison, $\Lambda_1 = \Lambda_{9_{48}}$ admits a smooth disk filling with one immersed point [OS16, Section 4.6], and thus it also admits a disk filling with p immersed points, for any $p \geq 1$. One can more easily see that by two "unclasping" moves, 9_{48} has an unknotting number of 2: it follows that there exists a smooth disk filling of $\Lambda_{9_{48}}$ with 2 double points. Similarly, when $k \geq 2$, by performing unclasping moves in each of the k tangles, we see that Λ_k admits a smooth disk filling with $2k$ double points, and thus by smooth surgery a smooth genus j filling with $2k - j$ double points for all $0 \leq j \leq k$. □

8.2. Proof of Theorem 1.8(2)

For all $g \geq 1$ and $p \geq 0$, we will show the existence of a Legendrian knot Λ_g^p that has an immersed, Maslov-0, exact Lagrangian filling F_g^p, which has genus g and p double points of action and index 0, that does not arise from Lagrangian surgery. The construction of Λ_g^p is an example of the Mondrian diagrams of [Ng05].

To construct the Legendrian **checkerboard knot** Λ_g^0, $g \geq 1$, begin with a $(2g+2) \times 4$ shaded checkerboard, with the lower left square shaded. For every shaded square, replace the right (resp. left) edge with a right (resp. left) cusp. If two shaded squares share a vertex, replace the vertex with a crossing, and otherwise replace the vertex with a smoothing of the vertex. An example is given in Figure 21. We can directly check that Λ_g^0 has a single component, for all $g \geq 1$.

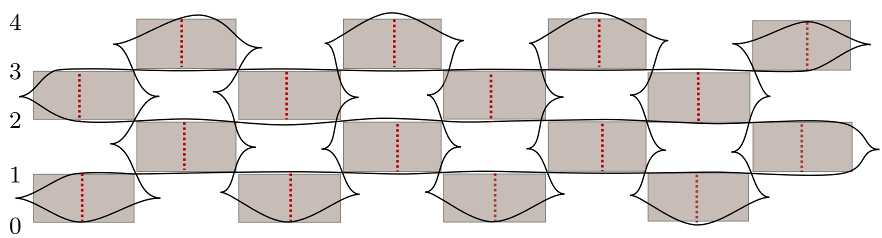


Figure 21. The Legendrian $\Lambda_{\mathbf{3}}^0$ constructed by starting with a $(2(\mathbf{3})+2) \times 4$ shaded checkerboard; the red lines denote the pinches used in the construction of F_3^0.

For $p \geq 1$, the Legendrian knot Λ_g^p will be constructed by applying Legendrian Reidemeister I moves and adding p *clasps* to Λ_g^0, as shown in Figure 22. To form Λ_g^1, for $g \geq 1$, start with the two shaded regions corresponding to the bottom row, first and third columns in the shaded $(2g+2) \times 4$-checkerboard used to construct Λ_g^0. Perform one downward Reidemeister I move on each portion of Λ_g^0 corresponding to these two shaded regions, and clasp the pair of cusps facing each other as schematized on Figure 22. We form Λ_g^2 by again starting with the two bottom left shaded regions of Λ_g^0, performing 6 Reidemeister I moves, and then forming 2 clasps in the shaded tiles of the plane.

Similarly, for all $p \geq 1$, we can form the **clasped checkerboard** Legendrian Λ_g^p, by starting with Λ_g^0, performing $4p-2$ Reidemeister moves, and adding p clasps, as shown in Figure 22. Observe that Λ_g^p has a single component.

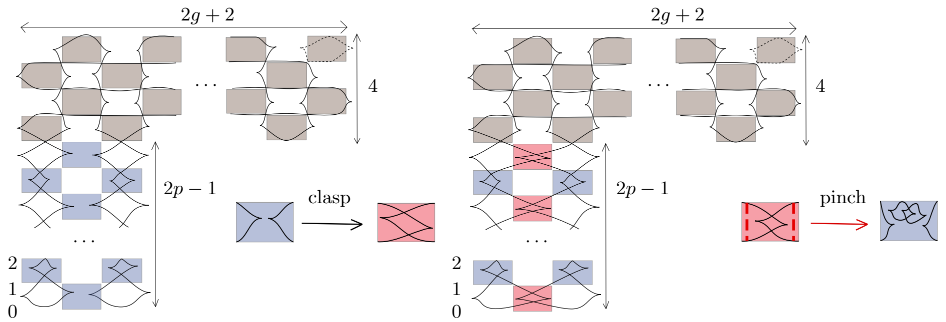


Figure 22. Construction of the clasped checkerboard Legendrian Λ_g^p, $p \geq 1$, and a schematization of pinch moves around each clasp that shows the existence of a cobordism from $\sqcup_p \Lambda_{\mathrm{H}}^0 \cup \Lambda_g^0$ to Λ_g^p.

Proposition 8.5. *For all $g \geq 1$ and $p \geq 0$, the Legendrian knot Λ_g^p admits an immersed, Maslov-*0*, exact Lagrangian filling F_g^p of genus g with p double points, each of which has action* 0 *and index* 0*.*

Proof. First fix Λ_g^0, for some $g \geq 1$. By performing pinch moves on each pair of strands that correspond to the top and bottom edges of each shaded square in the $(2g+2) \times 4$ shaded checkerboard that was used to constuct Λ_g^0, we obtain an embedded, exact, Lagrangian cobordism from a disjoint union of max *tb* Legendrian unknots to Λ_g^0; see an illustration in Figure 21. The Maslov potential on the strands on which we perform the pinch moves ensures that this cobordism has Maslov class 0. Each Legendrian unknot can be filled with a disk to obtain F_g^0, an embedded, Maslov-0, exact Lagrangian filling of Λ_g^0. As we perform $\frac{1}{2}(2g+2)4$ pinch moves and obtain $4 + (2g+1)$ unknots, we see that this filling does indeed have genus g, as desired.

Now fix Λ_g^p, for $p \geq 1$. By performing pinch moves along the red dash lines besides the clasps as schematized in Figure 22, we build a genus 0 embedded, Maslov-0, exact Lagrangian cobordism from $\sqcup_p \Lambda_{\mathrm{H}}^0 \cup \Lambda_g^0$ to Λ_g^p. The Λ_g^0 has an embedded, Maslov-0, exact Lagrangian filling of genus g, while each Hopf link Λ_{H}^0 can be filled by an immersed, Maslov-0, exact Lagrangian filling with one double point of action 0 and index 0. By stacking this Lagrangian cobordism and these fillings, we obtain the desired F_g^p. $\square$

Proposition 8.6. *The Legendrian knot Λ_g^p does not admit an immersed, Maslov-*0*, exact Lagrangian disk filling F_{g-1}^{p+1} with $p+1$ double points, all of action* 0 *and index* 0*.*

The proof follows easily from the following calculation, which will be proved in Section 8.3.

Lemma 8.7. $|Aug(\Lambda_g^p;\mathbb{Z}_2)/\sim_{Aug_+}| = 3^p$.

Proof of Proposition 8.6. By Theorem 1.1, the existence of the filling F_{g-1}^{p+1} is equivalent to the existence of an embedded, Maslov-0, exact Lagrangian cobordism of genus g from $\sqcup_{p+1}\Lambda_{\mathrm{H}}^0$ to Λ_g^p. By Lemma 8.1 and Lemma 8.7,

$$|Aug(\sqcup_{p+1}\Lambda_{\mathrm{H}}^0;\mathbb{Z}_2)/\sim_{Aug_+}| = 3^{p+1}, \quad \text{and} \quad |Aug(\Lambda_g^p;\mathbb{Z}_2)/\sim_{Aug_+}| = 3^p,$$

and thus by Theorem 1.4 such an embedded cobordism from $\sqcup_{p+1}\Lambda_{\mathrm{H}}^0$ to Λ_k does not exist. ☐

Proof of Theorem 1.8(2). Proposition 8.5 shows the existence of the immersed, Maslov-0, exact Lagrangian genus g filling F_g^p of Λ_g^p that has p double points of action 0 and index 0. Proposition 8.6 shows there does not exist an immersed, Maslov-0, exact Lagrangian genus $(g-1)$ filling F_{g-1}^{p+1} of Λ_g^p with $(p+1)$ double points, all of action 0 and index 0. Thus, by Definition 3.3, F_g^p does not arise from Lagrangian surgery. ☐

It remains to prove Lemma 8.7, which we do in the next subsection.

8.3. Proof of Lemma 8.7

As opposed to the more direct counting strategy we employed in Lemma 8.4, here we count augmentations of these arbitrarily high crossing knots Λ_g^p using the theory of rulings. So we begin with some background on rulings, first defined in [PC05, Fuc03], and review the definition of the ruling polynomial.

Following [Sab20], a (graded, normal) **ruling** of a Legendrian knot Λ is a set of crossings (called **switches**) such that resolving the switches yields a link of unknots $\Lambda_1, \ldots, \Lambda_m$ such that

1) At each switch, the two strands have the same Maslov potential;
2) Each Λ_i, $i = 1, \ldots, m$, is a Legendrian unknot with 0 crossings and 2 cusps that bounds a ruling disk D_i;
3) Exactly two components are incident to any switch;

4) Near each switch, the incident ruling disks D_i bounded by Λ_i are either nested or disjoint as shown in Figure 23.

For each ruling R of a Legendrian Λ, denote the number of switches and disks by $s(R)$ and $d(R)$, respectively. For a Legendrian Λ, the **ruling polynomial** $R_\Lambda(z)$ is the polynomial

$$R_\Lambda(z) = \sum_R z^{s(R)-d(R)}. \tag{8.1}$$

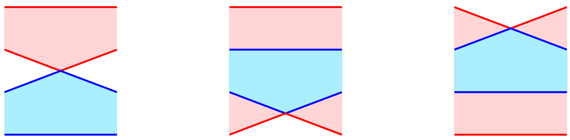


Figure 23. Around a switch, the possible nested or disjoint configurations of the incident disks.

Rulings and augmentations are related: the existence of one implies the existence of the other [Fuc03, FI04, Sab05]. The following lemma shows how we can use the ruling polynomial to find our desired count of augmentations.

Lemma 8.8. *Let Λ be a Legendrian knot with no negative degree Reeb chords. Then*

$$|Aug(\Lambda;\mathbb{Z}_2)/\sim_{Aug_+}| = 2^{\chi(\Lambda)/2} R_\Lambda(2^{-1/2}), \tag{8.2}$$

for $\chi(\Lambda) = \sum_{k\geq 0}(-1)^k a_k$, where a_k is the number of Reeb chords of degree k.

Proof. Under the assumption of the lemma, following [HR15, Remark 3.3(ii)] the number of augmentations $|Aug(\Lambda;\mathbb{Z}_2)|$ is related to the ruling polynomial $R_\Lambda(z)$ in the following way:

$$2^{-\chi(\Lambda)/2}|Aug(\Lambda;\mathbb{Z}_2)| = R_\Lambda(2^{1/2} - 2^{-1/2}),$$

where $\chi(\Lambda)$ is the Euler characteristic of $(\mathcal{A}(\Lambda), \partial)$, defined as $\chi(\Lambda) = \sum_{k\geq 0}(-1)^k a_k$, where a_k is the number of generators of $\mathcal{A}(\Lambda)$ of degree k. By Corollary 5.6, we know that $|Aug(\Lambda;\mathbb{Z}_2)| = |Aug(\Lambda;\mathbb{Z}_2)/\sim_{Aug_+}|$, and our result follows. □

Proof of Lemma 8.7. First consider the checkerboard Legendrian knot Λ_g^0. Observe that Λ_g^0 has a unique ruling that switches at every crossing: one can check this by considering the shaded regions in the top row of Λ_g^0: the left cusp of each shaded square has to match with the right cusp of that same shaded square, thus it forces the crossings at the bottom vertices of these shaded squares to be switches. Similarly for the second topmost row, the right cusp of each shaded square has to match with the left cusp of the same shaded square forcing the crossings at the bottom vertices of these shaded squares to be switches. Thus, by considering Λ_g^0 from top to bottom we can conclude that every crossing in Λ_0 is a switch. In this unique ruling R, using Equations (2.2) and (4.1), we see that all Reeb chords in $\mathcal{A}(\Lambda_g^0)$ have degree 0 or 1. Furthermore, $s(R) - d(R) = \chi(\Lambda_g^0)$ since there is a switch at each degree 0 chord and a one-to-one correspondence between disks and right cusps (which correspond to Reeb chords of degree 1). Then applying Lemma 8.8, we find

$$|Aug(\Lambda_g^0;\mathbb{Z}_2)/\sim_{Aug_+}| = 2^{\chi(\Lambda_g^0)/2} R_{\Lambda_g^0}(2^{-1/2}) = 2^{\chi(\Lambda_g^0)/2}(2^{-1/2})^{\chi(\Lambda_g^0)} = 1.$$

Now consider Λ_g^p, for $p \geq 1$. As shown in Figure 22, the first clasp in the construction Λ_g^p introduces four new degree 0 Reeb chords (two from the Reidemeister moves, two in the clasp region), and two rulings. With just one clasp, the ruling polynomial changes from $R_{\Lambda_g^0}(z) = z^{\chi\left(\Lambda_g^0\right)}$ to

$$R_{\Lambda_g^1}(z) = z^{\chi\left(\Lambda_g^1\right)}(z^{-2}+1).$$

Each additional clasp introduces 6 new degree 0 Reeb chords (4 from Reidemeister moves, 2 in the clasp region). Considering rulings, each new chord coming from a Reidemeister move must be a switch and then one can either switch at both or neither of the two crossings in the clasp region. Thus, the ruling polynomial becomes

$$R_{\Lambda_g^p}(z) = z^{\chi\left(\Lambda_g^p\right)}(z^{-2}+1)^p.$$

Using Equation (8.2), we find that the number of augmentations of Λ_g^p to $\mathbb{Z}_2$ is 3^p:

$$|Aug(\Lambda_g^p;\mathbb{Z}_2)/\sim_{Aug_+}| = 2^{\chi\left(\Lambda_g^p\right)/2}(2^{-1/2})^{\chi\left(\Lambda_g^p\right)}((2^{-1/2})^{-2}+1)^p = 3^p.$$

□

Appendix A. Equivalence in $Aug_+(\Lambda)$ for Legendrian links

In this appendix we will provide the proof of Proposition 5.5 following the proof for the case of single component knots in [NRS+20, Proposition 5.19]. We start by setting some basic notation. Let $\Lambda = \cup_{k=1}^m \Lambda_k$ be a Legendrian link with m link components. For a mixed Reeb chord a that starts on an ith link component Λ_i and ends on the jth link component Λ_j, that is $a \in R(\Lambda_j, \Lambda_i)$, we let $c(a) = i$ and $r(a) = j$.

Let Λ_f^n denote the n-copy of Λ that has been perturbed by a Morse function f with a single maximum and minimum as in [NRS+20]. Note that if $\Lambda = \cup_{k=1}^m \Lambda_k$ is a link with m link components, then $\Lambda_f^n = \cup_{i=1}^n \cup_{k=1}^m \Lambda_k^i$ is a link with mn link components. Given a Legendrian link Λ, and its perturbed two copy $\Lambda_f^2 = \Lambda^1 \cup \Lambda^2$, for any Reeb chord $a \in R(\Lambda^1, \Lambda^2)$, there is a corresponding element $\check{a} \in Hom_+(\epsilon^1, \epsilon^2)$ with degree $|\check{a}|_+ = |a| + 1$. Observe that this is a different notation convention than what we use in Section 5.

Let $(\epsilon^1, \dots, \epsilon^{n+1})$ be a tuple of augmentations of $(\mathcal{A}(\Lambda), \partial)$. Define $((\mathcal{A}(\Lambda_f^{n+1}))^\epsilon, \partial_\epsilon^{n+1})$ as follows. Let $(\mathcal{A}(\Lambda_f^{n+1}))^\epsilon := (\mathcal{A}(\Lambda_f^{n+1}) \otimes \mathbb{F})/(t_k = \epsilon(t_k))$ and set $\partial_\epsilon^{n+1} = \phi_\epsilon \circ \partial^{n+1} \circ \phi_\epsilon^{-1}$, where $\phi_\epsilon(a) = a + \epsilon(a)$. Then, the composition maps

$$m_n : Hom_+(\epsilon^n, \epsilon^{n+1}) \otimes \cdots \otimes Hom_+(\epsilon^2, \epsilon^3) \otimes Hom_+(\epsilon^1, \epsilon^2) \to Hom_+(\epsilon^1, \epsilon^{n+1}),$$

are given by

$$m_n(\check{\alpha}_n, \dots, \check{\alpha}_1) = (-1)^\sigma \sum_{a \in \mathcal{R} \cup x_k \cup y_k} \check{a} \cdot \mathrm{Coeff}_{\alpha_1^{12} \alpha_2^{23} \cdots \alpha_n^{n,n+1}}(\partial_\epsilon^{n+1} a^{1,n+1})$$

where $\alpha_i \in \{a_1, \dots, a_r, x_1, \dots, x_m, y_1, \dots, y_m\}$ for each i, and $\sigma = n(n-1)/2 + \sum_{p<q} |\check{\alpha}_p|_+ |\check{\alpha}_q|_+ + |\check{\alpha}_{n-1}|_+ + |\check{\alpha}_{n-3}|_+ + \cdots$.

Proposition A.1 ([NRS+20, Proposition 4.14]). *Let $\Lambda \subset \mathbb{R}^3_{std}$ be a Legendrian link with m link components, one basepoint t_k per link component and Reeb chords $\mathcal{R} = \{a_1, \dots, a_r\}$. The DGA of the perturbed n-copy of Λ, Λ_f^n, is generated by*

1) $(t_k^i)^{\pm 1}$ *for* $1 \le i \le n, 1 \le k \le m$, *with* $|t_k^i| = 0$;
2) a_h^{ij} *for* $1 \le i, j \le n$, *and* $1 \le h \le r$ *with* $|a_h^{ij}| = |a_h|$;
3) x_k^{ij} *for* $1 \le i, j \le n$, *and* $1 \le k \le m$, *with* $|x_k^{ij}| = 0$;

4) y_k^{ij} *for* $1 \leq i, j \leq n$, $1 \leq k \leq m$, *with* $|y^{ij}| = -1$,

and satisfies the relations $t_k^i(t_k^i)^{-1} = (t_k^i)^{-1}t_k^i = 1$ *for each* i *and* k. *The differential of* $\mathcal{A}(\Lambda_f^n, \partial^n)$ *can be described as follows. Assemble the generators of* $\mathcal{A}(\Lambda_f^n, \partial^n)$ *into* $n \times n$ *matrices:* $A_h = (a_h^{ij})$, $\Delta_k = Diag(t_k^1, \ldots, t_k^n)$,

$$X_k = \begin{pmatrix} 1 & x_k^{12} & \cdots & x_k^{1n} \\ 0 & 1 & \cdots & x_k^{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}, \text{ and } Y_k = \begin{pmatrix} 0 & y_k^{12} & \cdots & y_k^{1n} \\ 0 & 0 & \cdots & y_k^{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$

where $1 \leq k \leq m$, *and* $1 \leq h \leq r$. *Then, applying* ∂^n *to matrices entry-by-entry, we have*

$$\begin{aligned} \partial^n(A_h) &= \Phi(\partial(a_h)) + Y_{r(a_h)}A_h + (-1)^{|a_h|+1}A_hY_{c(a_h)} \\ \partial^n(X_k) &= \Delta_k^{-1}Y_k\Delta_kX_k - X_kY_k \\ \partial^n(Y_k) &= Y_k^2 \end{aligned}$$

where $\Phi : \mathcal{A}(\Lambda) \to Mat(M, \mathcal{A}^n)$ *is a ring of homomorphism such that* $\Phi(a_h) = A_h, \Phi(t_k) = \Delta_kX_k, \Phi(t_k^{-1}) = X_k^{-1}\Delta_k$.

The following Lemma A.2, and Proposition A.3 are generalizations of Lemma 5.16, Proposition 5.17 and Proposition 5.18 in [NRS+20]. Lemma A.2 is an immediate consequence of Proposition A.1 which allows us to compute m_1 and m_2 from $(\mathcal{A}(\Lambda_f^2), \partial^2)$ and $(\mathcal{A}(\Lambda_f^3), \partial^3)$.

Lemma A.2. *Let* $\Lambda \subset \mathbb{R}^3_{std}$ *be an* m *component Legendrian link with one basepoint* t_k *per link component, Reeb chords* $\mathcal{R} = \{a_1, \ldots, a_r\}$ *and augmentations* ϵ^1, ϵ^2. *In* $Hom_+(\epsilon^1, \epsilon^2)$, *we have that*

$$m_1(\check{a}_h) = \sum_{1 \leq l \leq n} \delta_{b_l,a_h}\sigma_u|(\overline{\mathcal{M}}_J^{\mathbb{R}\times\Lambda}(a_{h'}; b_1, \ldots, b_n)|\epsilon^1(b_1 \cdots b_{l-1})\epsilon^2(b_{l+1} \cdots b_n)\check{a}_{h'}$$

$$\begin{aligned} m_1(\check{y}_k) &= (\epsilon^1(t_k)^{-1}\epsilon^2(t_k) - 1)\check{x}_k + \sum_{a_h \in \{a \in \mathcal{R} | c(a) = k\}} \epsilon^2(a_h)\check{a}_h \\ &\quad + \sum_{a \in \{a \in \mathcal{R} | r(a) = k\}} (-1)^{|a_h|+1}\epsilon^1(a_h)\check{a}_h \end{aligned}$$

$m_1(\check{x}_k) \in span_{\mathbb{F}}\{\check{a}_1 \ldots, \check{a}_r\}$ *for* $k \in \{1, \ldots, N\}$.

where $b_i \in span_{\mathbb{F}}\{a_1, \ldots, a_r, t_1, \ldots, t_m\}$, *and* $\sigma_u \in \{\pm 1\}$ *denotes the product of all orientation signs at the corners of the disk* u. *We also have that for*

$i, j \in \{1, \ldots, m\}$, and $1 \leq h, h' \leq r$,

$$m_2(\check{y}_i, \check{y}_j) = \begin{cases} -\check{y}_i & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \quad m_2(\check{x}_i, \check{y}_j) = \begin{cases} -\epsilon^1(t_i)^{-1}\epsilon^2(t_i)\check{x}_i & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

$$m_2(\check{y}_i, \check{x}_j) = \begin{cases} -\check{x}_i & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \quad m_2(\check{y}_i, \check{a}_h) = \begin{cases} -\check{a}_h & \text{if } c(a_h) = i \\ 0 & \text{if } c(a_h) \neq i \end{cases}$$

$$m_2(\check{a}_h, \check{y}_i) = \begin{cases} -\check{a}_h & \text{if } r(a_h) = i \\ 0 & \text{if } r(a_h) \neq i \end{cases}$$

$m_2(\check{a}_h, \check{a}_{h'}), m_2(\check{x}_i, \check{x}_j), m_2(\check{x}_i, \check{a}_h), m_2(\check{a}_h, \check{x}_j) \in span_{\mathbb{F}}\{\check{a}_1, \ldots, \check{a}_r\}$.
Moreover, If we assume that the Reeb chords of Λ are labeled by increasing height, $h(a_1) \leq h(a_2) \leq \cdots \leq h(a_2)$, then $m_2(\check{a}_h, \check{a}_{h'}) \in span_{\mathbb{F}}\{\check{a}_l \mid l \geq max(h, h'), 1 \leq h, h' \leq r\}$.

Proposition A.3. *Consider an element $\alpha \in Hom^0_+(\epsilon^1, \epsilon^2)$ of the form $\alpha = -\sum_{k=1}^m c_k \check{y}_k - \sum_{h=1}^r K(a_h)\check{a}_h$*
where $K : (\mathcal{A}(\Lambda), \partial) \to (\mathbb{F}, 0)$ is an $\mathbb{F}$ linear map. Then, $m_1(\alpha) = 0$ if and only if K is a split DGA homotopy from ϵ^1 to ϵ^2.

Proof. First, observe that

$$\begin{aligned}
&\sum_{h=1}^r K(a_h) m_1(\check{a}_h) \\
&= \sum_{h=1}^r K(a_h)[\sum_{1\leq l\leq n} \delta_{b_l,a_h} |(\overline{\mathcal{M}}_J^{\mathbb{R}\times\Lambda}(a_{h'}; b_1, \ldots, b_n)|\epsilon^1(b_1\cdots b_{l-1})\epsilon^2(b_{l+1}\cdots b_n)\check{a}_{h'}] \\
&= \sum_{h=1}^r [\sum_{1\leq l\leq n} \delta_{b_l,a_h}(-1)^{|b_1\cdots b_{l-1}|}|(\overline{\mathcal{M}}_J^{\mathbb{R}\times\Lambda}(a_{h'}; b_1, \ldots, b_n)|\epsilon^1(b_1\cdots b_{l-1})K(b_l)\epsilon^2(b_{l+1}\cdots b_n)]\check{a}_{h'} \\
&= \sum_{h=1}^r K \circ \partial(a_h)\check{a}_h
\end{aligned}$$

where the last equality follows from the fact that $K(t_k^{\pm}) = 0$ for all $k \in \{1, \dots, m\}$. Therefore, using Lemma A.2, we know that

$$\begin{aligned}
-m_1(\alpha) &= m_1(\sum_{k=1}^{m} c_k \check{y}_k + \sum_{h=1}^{r} K(a_h)\check{a}_h) \\
&= \sum_{k=1}^{m} c_k m_1(\check{y}_k) + \sum_{h=1}^{r} (K \circ \partial(a_h))\check{a}_h \\
&= \sum_{k=1}^{m} c_k (\epsilon^1(t_k)^{-1}\epsilon^2(t_k) - 1)\check{x}_k \\
&\quad + \sum_{h=1}^{r} [c_{c(a_h)}\epsilon^2(a_h) + (-1)^{|a_h|+1} c_{r(a_h)}\epsilon^1(a_h)]\check{a}_h \\
&+ \sum_{h=1}^{r} (K \circ \partial(a_h))\check{a}_h
\end{aligned}$$

Thus, $m_1(\alpha) = 0$ if and only if $K \circ \partial(a_h) = -c_{c(a_h)}\epsilon^2(a_h) - (-1)^{|a_h|+1} c_{r(a_h)}\epsilon^1(a_h)$ for all $h \in \{1, \dots, r\}$, and $(\epsilon^1(t_k)^{-1}\epsilon^2(t_k) - 1) = 0$ for all $k \in \{1, \dots, m\}$. Note that $\mathbb{F}$ is supported in grading 0, and therefore $\epsilon^1(a_h) = (-1)^{|a_h|}\epsilon^1(a_h)$ for all h since ϵ^1 is supported in grading 0. If $\mathcal{A}(\Lambda)$ has a $\mathbb{Z}_n$ grading, and ϵ^1 is an n-graded augmentation, recall that the grading is defined mod n. Therefore, $K \circ \partial(a_h) = c_{r(a_h)}\epsilon^1(a_h) - c_{c(a_h)}\epsilon^2(a_h)$, and ϵ^1 and ϵ^2 are split DGA homotopic via the operator K. □

Proof of Proposition 5.5. Suppose that ϵ^1 and ϵ^2 are equivalent in $\mathcal{A}ug_+(\Lambda)$. Then, as stated in Definition 5.1, there exist cocycles $\alpha \in Hom_+(\epsilon^1, \epsilon^2)$, and $\beta \in Hom_+(\epsilon^2, \epsilon^1)$ such that $[m_2(\alpha, \beta)] = -\sum_{k=1}^{m} [\check{y}_k] \in H^0 Hom_+(\epsilon^1, \epsilon^2)$.That is, $m_2(\alpha, \beta) + \sum_{k=1}^{m} \check{y}_k = m_1(\gamma)$ for some $\gamma \in Hom_+(\epsilon^2, \epsilon^2)$. By Lemma A.2 and the fact that $\gamma \in Hom_+(\epsilon^2, \epsilon^2)$, we know that $\langle m_1(\gamma), \check{y}_k\rangle = 0$ and $\langle m_1(\gamma), \check{x}_k\rangle = 0$. Therefore, $m_1(\gamma) = \sum_{h=1}^{r} K(a_h)\check{a}_h$ for some $\mathbb{F}$ linear map $K : (\mathcal{A}(\Lambda), \partial) \to (\mathbb{F}, 0)$ which is naturally split. We can now write $m_2(\alpha, \beta) = -\sum_{k=1}^{m} \check{y}_k + \sum_{h=1}^{r} K(a_h)\check{a}_h$. Again by Lemma A.2 and the fact that $|\alpha|_+ = |\beta|_+ = 0$, while $|\check{x}_k|_+ = 1$, we know that

$$\begin{aligned}
\alpha &= \sum_{k=1}^{m} (c_\alpha)_k \check{y}_k + \sum_{h=1}^{r} K_\alpha(a_h)\check{a}_h \\
\beta &= \sum_{k=1}^{m} (c_\beta)_k \check{y}_k + \sum_{h=1}^{r} K_\beta(a_h)\check{a}_h
\end{aligned}$$

such that $(c_\alpha)_k, (c_\beta)_k \in \mathbb{F}^*$ and $(c_\alpha)_k(c_\beta)_k = 1$ for each k, and for some $\mathbb{F}$ linear maps $K_\alpha, K_\beta : (\mathcal{A}(\Lambda), \partial) \to (\mathbb{F}, 0)$. Both α and β are cocycles so by Proposition A.3, K_α and K_β are DGA homotopies between ϵ^1 and ϵ^2.

Suppose that ϵ^1 and ϵ^2 are split DGA homotopic, such that for any Reeb chord a,

$$c_{c(a)}\epsilon^1(a) - c_{r(a)}\epsilon^2(a) = K \circ \partial(a)$$

for constants $c_i \in \mathbb{F}^*$ and some split DGA homotopy $K : (\mathcal{A}(\Lambda), \partial) \to (\mathbb{F}, 0)$. We know that $K(a) = 0$ for any Reeb chord a such that $|a| \neq -1$. By Lemma A.3, $\alpha = \sum_{k=1}^m (c_\alpha)_k \check{y}_k + \sum_{h=1}^r K(a_h)\check{a}_h$ is a cocycle in $H^0 Hom_+(\epsilon^1, \epsilon^2)$. We now construct cocycles $\beta, \gamma \in Hom_+(\epsilon^1, \epsilon^2)$ such that $|\beta|_+ = |\gamma|_+ = 0$, $m_2(\beta, \alpha) = m_2(\alpha, \gamma) = -\sum_{k=1}^m \check{y}_k$. This implies that $[\beta] = [\gamma] \in H^0 Hom_+(\epsilon^1, \epsilon^2)$ is the multiplicative inverse of $[\alpha]$ in $\mathcal{A}ug_+(\Lambda)$. The construction of γ is similar to the construction of β which we now provide.

Suppose that the Reeb chords $\{\check{a}_1, \dots, \check{a}_r\}$ are ordered by height. Then we can write $\alpha = \sum_{k=1}^m (c_\alpha)_k \check{y}_k + A$ where $A \in span_{\mathbb{F}}\{\check{a}_1, \dots, \check{a}_r\}$. Let $\beta = \sum_{k=1}^m (c_\beta)_k \check{y}_k + B$ where $(c_\alpha)_k(c_\beta)_k = 1$ for $1 \leq k \leq m$, $B \in span_{\mathbb{F}}\{\check{a}_1, \dots, \check{a}_r\}$ and is defined inductively to satisfy $B = A + m_2(B, A)$. Then, $m_2(\beta, \alpha) = -\sum_k \check{y}_k$. To verify that β is a cocycle note that the A_∞ relations on $\mathcal{A}ug_+(\Lambda)$ imply that

$$m_1(-\sum_{k=1}^m \check{y}_k) = m_1(m_2(\beta, \alpha) = m_2(m_1(\beta), \alpha) + m_2(\beta, m_1(\alpha)).$$

We know that $m_1(\check{y}_k) = 0$ for all $1 \leq k \leq m$ and that $m_1(\alpha) = 0$ so $m_2(\beta, m_1(\alpha)) = 0$. Therefore, $m_2(m_1(\beta), \alpha) = 0$.

We will show that if $X \in span_{\mathbb{F}}\{\check{a}_1, \dots, \check{a}_r, \check{x}_1, \dots, \check{x}_N, \check{y}_1, \dots, \check{y}_N\}$, then $m_2(X, \alpha) = 0$ implies that $X = 0$. Note that $m_2(X, A) \in span_{\mathbb{F}}\{\check{a}_1, \dots, \check{a}_r, \check{x}_1, \dots, \check{x}_m, \check{y}_1, \dots, \check{y}_m\}$ by Lemma A.2. Then,

$$0 = m_2(X, \alpha) = m_2(X, \sum_{k=1}^m (c_\alpha)_k \check{y}_k + A) = m_2(X, \sum_{k=1}^m (c_\alpha)_k \check{y}_k) + m_2(X, A)$$

Thus, $m_2(X, \sum_{k=1}^m (c_\alpha)_k \check{y}_k) = m_2(X, A)$. Note that $m_2(X, A) \in span_{\mathbb{F}}\{\check{a}_1, \dots, \check{a}_r\}$ because $A \in span_{\mathbb{F}}\{\check{a}_1, \dots, \check{a}_r\}$ by Lemma A.2. Therefore, we know that $\langle X, \check{x}_k \rangle = \langle X, \check{y}_k \rangle = 0$ for all $1 \leq k \leq m$, and so $X \in span_{\mathbb{F}}\{\check{a}_1, \dots, \check{a}_r\}$. Moreover, by induction on the height of Reeb chords, and Lemma A.2, we know that $\langle X, \check{a}_h \rangle = 0$ for all $1 \leq h \leq r$. Thus, for $X = m_1(\beta) \in span\{\check{a}_1, \dots, \check{a}_r, \check{x}_1, \dots, \check{x}_m, \check{y}_1, \dots, \check{y}_m\}$, since $m_2(m_1(\beta), \alpha) = 0$ as shown above, we can conclude that $m_1(\beta) = 0$. □

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