ON THE EXISTENCE OF ZERO-SUM SUBSEQUENCES WITH LENGTH NOT DIVIDED BY A GIVEN NUMBER

WEIDONG GAO*, XIAO JIANG, YUANLIN LI, AND HUIJUAN QI

Abstract. Let G be a finite abelian group and k be an integer not dividing the exponent of G. We denote by $\mathsf{E}_k(G)$ the smallest positive integer l such that every sequence over G of length no less than l has a zero-sum subsequence of length not divisible by k. In this paper, we focus on determining $\mathsf{E}_k(G)$ for $G = C_n$, a cyclic group of order n. Specifically, we prove that

$$\mathsf{E}_k(C_n) = \left\lfloor \frac{k}{k-1}(n-1) \right\rfloor + 1$$

for $k \in \{3\} \cup (\lceil \frac{n}{2} \rceil, n)$.

1. Introduction and Main Results

For a positive integer n, we denote by C_n a cyclic group of order n. It is well known that any non-trivial finite abelian group G can be written as

$$G = C_{n_1} \oplus \cdots \oplus C_{n_r}$$

with $1 < n_1 \mid n_2 \mid \cdots \mid n_r$. We refer to r as the *rank* of G and to $n_r = \exp(G)$ as the *exponent* of G. Throughout this paper, let G be a finite abelian group, written additively, and 0 be its identity element. A sequence S over G (unordered and repetition is allowed) is called a zero-sum sequence if the sum of all elements of S is 0.

A typical direct zero-sum problem studies the condition which ensures that given sequences have non-empty zero-sum subsequences with prescribed properties. The investigations on the direct zero-sum problems were initiated by P. Erdős, A. Ginzburg and A. Ziv. In 1961 [6], they proved that $\mathsf{s}(C_n) = 2n - 1$ (see also [2] for some other proofs), where $\mathsf{s}(G)$ is defined as the smallest integer l such that every sequence of length l has a zero-sum subsequence of length $\exp(G)$. The result is well known as Erdős-Ginzburg-Ziv Theorem, which is regarded as a cornerstone of the zero-sum theory, and the invariant $\mathsf{s}(G)$ is called Erdős-Ginzburg-Ziv constant.

A few years later, another zero-sum invariant was introduced by H. Davenport in a famous western conference (cf. [5] and [14] – [15]). He raised the

2020 *Mathematics Subject Classification.* 11B30, 11P75, 20K01.
Key words and phrases. Abelian group, Zero-sum subsequence.
*Corresponding author: Weidong Gao, Email: wdgao@nankai.edu.cn.

question of determining the maximal number of distinct prime ideals that may appear in the prime ideal decomposition of an irreducible element in an algebraic number field F. Let $\mathsf{D}(G)$ denote the smallest positive integer l such that every sequence S over G of length $|S| \geq l$ contains a nonempty zero-sum subsequence. Davenport pointed out that $\mathsf{D}(G)$ is exactly the maximal number that he asked if the ideal class group of F is isomorphic to G. The invariant $\mathsf{D}(G)$ is so called the Davenport constant, and it played a crucial role in the proof of the famous result that there are infinitely many Carmichael numbers in 1994 [1].

The Erdős-Ginzburg-Ziv constant $\mathsf{s}(G)$ and the Davenport constant $\mathsf{D}(G)$ are classical invariants in combinatorial number theory and have received a great deal of attention in recent years (cf. [3]–[4], [7]–[11] and [16]). To further study $\mathsf{s}(G)$, the first author of this paper introduced the following invariant in 2000 [9].

Definition 1.1. *Let G be a finite abelian group and k be a positive integer with $k \nmid \exp(G)$. We denote by $\mathsf{E}_k(G)$ the smallest positive integer l such that every sequence S over G of length $|S| \geq l$ has a zero-sum subsequence of length not divided by k.*

It was proved in [9] that $\mathsf{E}_2(G) = 2\mathsf{D}(G) - 1$, where $G = C_n$ or G is a p-group for some prime p. $\mathsf{E}_k(G)$ for all abelian p-groups was completely determined by W. Schmid in 2001 [18] and the main result is summarized in the following lemma.

Lemma 1.2. [18, Theorem 1.2, Lemmas 2.1 and 2.4] *Let $G = C_{n_1} \oplus C_{n_2} \oplus \ldots \oplus C_{n_r}$ with $1 < n_1 | \ldots | n_r$ and k be a positive integer not dividing* $\exp(G)$.

(1)

$$\mathsf{D}(G) \leq \mathsf{E}_k(G) \leq \mathsf{s}(G).$$

Specifically, the equality on the left-hand side is valid if $k > \mathsf{D}(G)$.

(2)

$$\mathsf{E}_k(G) \geq \left\lfloor \frac{k}{k-1} \sum_{i=1}^{r} (n_i - 1) \right\rfloor + 1.$$

Moreover, the equality holds if G is a p-group and $\gcd(p,k) = 1$.

Based on Lemma 1.2 (2), it is natural to give the following conjecture.

Conjecture 1.3. *Let n and k be positive integers satisfying that $n \geq 3$, $k > 1$ and $\gcd(k,n) = 1$. Then for any cyclic group C_n, we have*

$$\mathsf{E}_k(C_n) = \left\lfloor \frac{k}{k-1}(n-1) \right\rfloor + 1.$$

Clearly, $\mathsf{E}_k(C_n) = n$ for $k > n$ by Lemma 1.2 (1). Therefore, Conjecture 1.3 remains open when $k \le n$.

The first main result of this paper verifies this conjecture for $k = 3$.

Theorem 1.4. *Let n be a positive integer with $3 \nmid n$. Then*

$$\mathsf{E}_3(C_n) = \left\lfloor \frac{3}{2}(n-1) \right\rfloor + 1.$$

We also prove the following related result.

Theorem 1.5. *Let n and k be positive integers with $k \nmid n$. If $\lceil \frac{n}{2} \rceil < k < n$, then*

$$\mathsf{E}_k(C_n) = n + 1.$$

The paper is organized in the following way. In Section 2, we gather some notations and some preliminary results. Section 3 provides some essential lemmas. Our main results are proved in Section 4.

2. Notation and Preliminaries

2.1. **Notation.** We recall some standard notation and terminology. We denote by $\mathbb{N}$ the set of positive integers, and by $\mathbb{N}_0$ the set $\mathbb{N} \cup \{0\}$. For any two integers $a, b \in \mathbb{N}_0$, we set $[a, b] = \{x \in \mathbb{N}_0 \mid a \le x \le b\}$. For a real number x, we denote by $\lfloor x \rfloor$ the largest integer that is less than or equals to x, which is called the floor of x. Similarly, we denote by $\lceil x \rceil$ the smallest integer that is greater than or equals to x, which is called the ceiling of x.

We denote by $\mathcal{F}(G)$ the free abelian monoid with basis G under multiplication. The elements of $\mathcal{F}(G)$ are called *sequences* over G. The identity element $\emptyset \in \mathcal{F}(G)$ is called the *empty* sequence. Every sequence S over G can be written in the form

$$S = g_1 \cdot \ldots \cdot g_\ell = \prod_{g \in G} g^{\mathrm{v}_g(S)},$$

where $\mathrm{v}_g(S) \in \mathbb{N}_0$ denotes the *multiplicity* of g in S. Moreover, we say that S contains g if $\mathrm{v}_g(S) > 0$. A sequence T is called a *subsequence* of S if $\mathrm{v}_g(T) \le \mathrm{v}_g(S)$ for all $g \in G$, and is denoted by $T \mid S$. If T is a subsequence of S, let ST^{-1} denote the subsequence with T deleted from S. If S_1 and S_2 are two sequences over G, let S_1S_2 denote the sequence over G satisfying that $\mathrm{v}_g(S_1S_2) = \mathrm{v}_g(S_1) + \mathrm{v}_g(S_2)$ for all $g \in G$. Two subsequences S_1 and S_2 of S are called *disjoint* if $S_1 \mid SS_2^{-1}$. We define $g + S = (g + g_1) \cdot \ldots \cdot (g + g_\ell)$ (and $g - S = (g - g_1) \cdot \ldots \cdot (g - g_\ell)$) for all $g \in G$.

We call $|S| = \ell = \sum_{g\in G} \mathrm{v}_g(S) \in \mathbb{N}_0$ the *length* of S, $\sigma(S) = \sum_{i=1}^{\ell} g_i = \sum_{g\in G} \mathsf{v}_g(S)g \in G$ the *sum* of S, $\mathrm{supp}(S) = \{g \in G \mid \mathrm{v}_g(S) > 0\} \subset G$ the *support* of S and $\mathrm{h}(S) = \max\{\mathrm{v}_g(S) \mid g \in G\}$ the maximum of the multiplicity in S. We denoted by $\mathrm{h}_2(S)$ the second largest multiplicity in S. We call that S is a *zero-sum sequence* if $\sigma(S) = 0 \in G$, S is a *zero-sum free sequence* if there is no non-empty zero-sum subsequence of S, and S is a *b-zero-sum free sequence* if there is no nonempty zero-sum subsequence T of S with length $|T| = b$.

Each map $f : G \to G'$ between finite abelian groups can extend uniquely to a monoid homomorphism $\mathcal{F}(G) \to \mathcal{F}(G')$, which is denoted by f as well. If f is a group homomorphism, then $\sigma(f(S)) = f(\sigma(S))$ for each $S \in \mathcal{F}(G)$.

2.2. **Some known results.** We collect some known results which will be needed later. First, we introduce a new invariant and state a related result.

Definition 2.1. *For each positive integer ℓ, we define $\mathrm{disc}_\ell(G)$ to be the smallest positive integer t such that every sequence S over G of length $|S| \ge t$ has a nonempty zero-sum subsequence T with $|T| \ne \ell$.*

Lemma 2.2. [12, Theorem 1.3] *Let $\ell \in [2, \mathsf{D}(G)-1]$ and m, n be positive integers. Then $\mathrm{disc}_\ell(C_n) = n + 1$ for $n \ge 3$.*

For the convenience of the reader, we give the following notations.

Definition 2.3. *Let g be a generator of C_n.*

(1) *For an element $h \in C_n$, we define $\mathrm{ind}_g(h)$ to be the least positive integer t such that $h = tg$.*

(2) *For a sequence $S = g_1 \cdot \ldots \cdot g_\ell$ over C_n, we define*

$$\|S\|_g = \frac{\mathrm{ind}_g(g_1) + \cdots + \mathrm{ind}_g(g_\ell)}{\mathrm{ord}(g)}$$

to be the g-norm of S.

In 2008, S. Savchev and F. Chen provided a characterization of the n-zero-sum free sequences over C_n of length greater than $\frac{3n}{2} - 1$, which plays an important role in the proof of Theorem 1.4.

Lemma 2.4. [17, Theorem 5, Proposition 4] *Let $S \in \mathcal{F}(C_n)$ be a sequence of length greater than $\frac{3n}{2} - 1$ over C_n and $\mathrm{v}_0(S) = \mathrm{h}(S)$. Then S is n-zero-sum free if and only if there exists a generator g of C_n such that $S = S_1 \cdot S_2$ with $\|S_1\|_g < 1$ and $\|g - S_2\|_g < 1$. In this case, we have*

$$\mathrm{v}_g(S) + \mathrm{v}_0(S) \ge 2(|S| - n + 1).$$

The following two lemmas give some additional properties of the zero-sum free sequence over C_n.

Lemma 2.5. [12, Lemma 3.2] *Let $G = C_n$ with $n \geq 3$ and S a zero-sum free sequence over C_n of length $|S| = \ell \geq (n+1)/2$. Then there is a generator $g \in G$ such that $g \mid S$, $\|S\|_g < 1$ and $\sum(S) = \{g, 2g, 3g, \ldots, n\|S\|_g g\}$.*

Lemma 2.6. [13, Theorem 5.4.5(3)] *Let $G = C_n$ with $n \geq 3$ and $S \in \mathcal{F}(G)$ be a zero-sum free sequence of length $|S| \geq \frac{n+1}{2}$. Then S contains some element g of order n. Moreover, if n is odd, then $\mathrm{v}_g(S) \geq \frac{n+5}{6}$.*

The following useful lemma was provided by Geroldinger et al. in 2006.

Lemma 2.7. [13, Lemma 5.7.10] *Let $\varphi : G \to \overline{G}$ be a group epimorphism and $k \in \mathbb{N}$. If $S \in \mathcal{F}(G)$ and $|S| \geq (k-1)\exp(\overline{G}) + \mathsf{s}(\overline{G})$, then S admits a product decomposition $S = S_1 \cdot \ldots \cdot S_k \cdot S'$, where $S_1, \ldots, S_k, S' \in \mathcal{F}(G)$ and, for every $i \in [1, k]$, $\varphi(S_i)$ has sum zero and length $|S_i| = \exp(\overline{G})$.*

3. Some Essential Lemmas

In this section, we provide several essential lemmas that will be used to prove our main results. For the convenience of the reader, we first introduce another new concept.

Definition 3.1. *A sequence $S \in \mathcal{F}(C_n)$ is called* E-trivial *if any one of the following statements holds.*

(1) $g^{\mathrm{ord}(g)} \mid S$ *for some* $g \in C_n$.
(2) S *has a zero-sum subsequence of length* n.
(3) S *has a zero-sum subsequence of length* 2 *if* n *is odd.*

We remark that clearly, if $S \in \mathcal{F}(C_n)$ is E-trivial, then for any integer $k \geq 2$ with $\gcd(n, k) = 1$, S has a zero-sum subsequence of length not divisible by k. So we do not need to consider the E-trivial sequences when discussing $\mathsf{E}_k(C_n)$. Furthermore, if $|S| = n + \frac{n-1}{2}$, then the sum of the two highest multiplicities of S must be greater than n.

Lemma 3.2. *Let $S \in \mathcal{F}(C_n)$ with $|S| = n + \frac{n-1}{2}$. If S is not E-trivial, then $\mathrm{h}(S) + \mathrm{h}_2(S) \geq n + 1$. Moreover, if $\mathrm{v}_g(S) = \mathrm{h}(S)$, then g must be a generator of C_n.*

Proof. Let $C_n = \langle g_1 \rangle$ and $\mathrm{v}_x(S) = \mathrm{h}(S)$ for some $x \in \langle g_1 \rangle \setminus \{0\}$. Since S is n-zero-sum free, $-x + S$ is also n-zero-sum free with $\mathrm{v}_0(-x + S) = \mathrm{h}(S)$.

By Lemma 2.4, there exists a generator g_2 of $\langle g_1\rangle$ such that $-x+S=S_1S_2$, $\|S_1\|_{g_2}<1$ and $\|g_2-S_2\|_{g_2}<1$, and thus in this case,

$$\mathrm{h}(S)+\mathrm{h}_2(S)\geq \mathrm{v}_0(S_1S_2)+\mathrm{v}_{g_2}(S_1S_2)\geq 2\cdot\frac{n+1}{2}=n+1.$$

If $\mathrm{v}_g(S)=\mathrm{h}(S)$, then $\mathrm{ord}(g)>\mathrm{h}(S)\geq\frac{n+1}{2}$. It follows from $\mathrm{ord}(g)\mid n$ that $\mathrm{ord}(g)=n$. □

The following lemmas establish some useful results related to sequences that contain exactly two different elements.

Lemma 3.3. *Let $n,k\geq 2$ be integers with $\gcd(n,k)=1$. Let $C_n=\langle g\rangle$ and let $T^*=g^{x_1}\cdot(bg)^{y_1}$ and $T^{**}=g^{x_2}\cdot(bg)^{y_2}$, where $b\in[1,n]$ and $x_1,x_2,y_1,y_2\in[0,n-1]$. Suppose that $\sigma(T^*)=\sigma(T^{**})=0$ and $|T^*|\equiv|T^{**}|\equiv 0 \pmod k$.*

(1) If $\|T^*\|_g$ or $\|T^{**}\|_g$ is coprime to k, then $\gcd((b-1),k)=1$.

(2) If $\|T^*\|_g\equiv\|T^{**}\|_g\equiv m \pmod k$ for some integer m coprime to k, then $x_1\equiv x_2\equiv -mn(b-1)^{-1} \pmod k$ and $y_1\equiv y_2\equiv mn(b-1)^{-1} \pmod k$.

Proof. (1) Without loss of generality, we may assume that $\|T^*\|_g\equiv m \pmod k$ for some integer m coprime to k. Then by $\gcd(n,k)=1$, we have

$$x_1+by_1=n\|T^*\|_g\equiv mn \pmod k.$$

Using the above result together with the hypothesis $x_1+y_1=|T^*|\equiv 0 \pmod k$, we have

$$y_1(b-1)\equiv x_1+y_1+(b-1)y_1\equiv mn \pmod k.$$

Hence $\gcd((b-1),k)=1$.

(2) Since $\gcd(mn,k)=1$ and $\|T^*\|_g\equiv\|T^{**}\|_g\equiv m \pmod k$, we have

$$x_1+y_1b\equiv x_2+y_2b\equiv mn \pmod k.$$

Now $|T^*|\equiv|T^{**}|\equiv 0 \pmod k$ implies that

$$x_1+y_1\equiv x_2+y_2\equiv 0 \pmod k.$$

It follows from the above two congruences that

$$y_1(b-1)\equiv y_2(b-1)\equiv mn \pmod k.$$

Using (1), we obtain

$$-x_1\equiv -x_2\equiv y_1\equiv y_2\equiv mn(b-1)^{-1} \pmod k.$$

□

Lemma 3.4. *Let $n, k \geq 2$ be integers with $\gcd(n, k) = 1$ and $C_n = \langle g \rangle$. Let*

$$S = g^{h_1} \cdot (bg)^{h_2} \in \mathcal{F}(C_n),$$

where $b \in [2, n-2]$ and $2 \leq h_2 \leq h_1 \leq n-1$. Then S has a zero-sum subsequence of length not divisible by k if any one of the following conditions holds.

(1) *$b \in [2, \frac{n}{3}]$ and $h_1 + h_2 \geq n + 1$.*
(2) *$b \in (\frac{n}{3}, n - h_1 - 1]$ and $h_1 + h_2 \geq n + \frac{n-1}{2}$.*
(3) *$b \in [n - h_1, n - \frac{n}{h_2}]$ and $h_1 + h_2 \geq n + 1$.*
(4) *$b \in (n - \frac{n}{h_2}, n - 2]$ and $h_1 + h_2 \geq n + \frac{n-1}{2}$.*

Proof. If $h_1 + h_2 \geq n + 1$, then by $2 \leq h_2 \leq h_1 \leq n - 1$, we have

$$\frac{n+1}{2} \leq h_1 \leq n - 1. \tag{3.1}$$

We prove by contradiction. Assume to the contrary that every zero-sum subsequence T of S has length $|T|$ divisible by k.

(1) If $b \in [2, \frac{n}{h_2}]$, then by the hypothesis $2 \leq h_2 \leq h_1 \leq n - 1$ and $n + 1 \leq h_1 + h_2$, we have

$$0 \leq n - h_2 b < n - (h_2 - 1)b \leq n - h_2 \leq h_1 - 1 < h_1.$$

Then

$$T_1 = g^{n - h_2 b} \cdot (bg)^{h_2} \text{ and } T_2 = g^{n-(h_2-1)b} \cdot (bg)^{h_2 - 1}$$

are zero-sum subsequences of S with $\|T_1\|_g = \|T_2\|_g = 1$. By the assumption, we have $|T_1| \equiv |T_2| \equiv 0 \pmod{k}$. Using Lemma 3.3 (2), we obtain $h_2 \equiv h_2 - 1 \pmod{k}$, yielding a contradiction.

If $b \in (\frac{n}{h_2}, \frac{n}{3}]$, then $3 \leq \lfloor \frac{n}{b} \rfloor < h_2$, which infers that

$$0 \leq n - \left\lfloor \frac{n}{b} \right\rfloor b < n - \left\lfloor \frac{n}{b} \right\rfloor b + b \leq \min\{n - 2b,\ 2b\} \leq \frac{(n-2b) + 2b}{2} = \frac{n}{2}.$$

Then by (3.1),

$$T_3 = g^{n - \lfloor \frac{n}{b} \rfloor b + b} \cdot (bg)^{\lfloor \frac{n}{b} \rfloor - 1} \text{ and } T_4 = g^{n - \lfloor \frac{n}{b} \rfloor b} \cdot (bg)^{\lfloor \frac{n}{b} \rfloor}$$

are zero-sum subsequences of S with $\|T_3\|_g = \|T_4\|_g = 1$. By the assumption, we have $|T_3| \equiv |T_4| \equiv 0 \pmod{k}$. It follows from Lemma 3.3 (2) that $\lfloor \frac{n}{b} \rfloor - 1 \equiv \lfloor \frac{n}{b} \rfloor \pmod{k}$, a contradiction.

(2) By the hypothesis of (2), we get

$$n - b - 1 \geq h_1 \geq h_2 \geq n + \frac{n-1}{2} - h_1 \geq \frac{n+1}{2} + b.$$

Thus $b \leq \frac{n-3}{4}$, which contradicts to $b \in (\frac{n}{3}, n - h_1 - 1]$.

(3) We claim that $b \neq n - \frac{n}{h_2}$. Otherwise, $h_2(n-b) = n$ implies $h_2 b \equiv 0 \pmod n$. It follows that $\mathrm{ord}(bg) \leq h_2$ and so S is E-trivial, yielding a contradiction.

If $b \in [n - h_1, \frac{n}{2}]$, then $0 \leq n - 2b < n - b \leq h_1$. Thus, by $h_2 \geq 2$,

$$T_5 = g^{n-b} \cdot (bg) \text{ and } T_6 = g^{n-2b} \cdot (bg)^2$$

are zero-sum subsequences of S with $\|T_5\|_g = \|T_6\|_g = 1$. By the assumption, we have $|T_5| \equiv |T_6| \equiv 0 \pmod k$. By Lemma (3.3) (2), we obtain $1 \equiv 2 \pmod k$, a contradiction.

If $b \in (\frac{n}{2}, n - \frac{n}{h_2})$, then set $t = n - b \in (\frac{n}{h_2}, \frac{n}{2})$ and $\ell = \lfloor \frac{n}{t} \rfloor + 1 \in [3, h_2]$. By (3.1),

$$0 < \ell t - n = \left\lfloor \frac{n}{t} \right\rfloor t - n + t \leq t < \frac{n}{2} < h_1.$$

Thus,

$$T_7 = g^t \cdot (bg) \text{ and } T_8 = g^{\ell t - n} \cdot (bg)^\ell$$

are zero-sum subsequences of S with $\|T_7\|_g = 1$ and $\|T_8\|_g = \ell - 1$. By the assumption, we have

$$t + 1 = |T_7| \equiv 0 \equiv |T_8| = \ell(t+1) - n \pmod k$$

and so $n \equiv 0 \pmod k$, yielding a contradiction to $\gcd(n, k) = 1$.

(4) By (3.1) and the hypothesis of (4), $h_2 + h_1 \geq n + \frac{n-1}{2}$, we get

$$h_2 \geq n + \frac{n-1}{2} - h_1 \geq \frac{n+1}{2}.$$

Since $b \leq n-2$, we obtain $h_2(n-b) \geq 2h_2 > n$, a contradiction to $b > n - \frac{n}{h_2}$.

In all cases, we have found contradictions. Therefore, S has a zero-sum subsequence of length not divisible by k, and we are done. □

Next, we discuss the situation that the sequences may contain more than two different elements. By Lemmas 3.2 and 3.4, it remains to discuss the situation when $b \in (\frac{n}{3}, n - h_1 - 1]$ or $b \in (n - \frac{n}{h_2}, n-2]$. The following lemma will be used frequently.

Lemma 3.5. *Let $n, k \geq 2$ be integers with $\gcd(n,k) = 1$ and $C_n = \langle g \rangle$. Let $S \in \mathcal{F}(C_n)$ with $0 \nmid S$. If every zero-sum subsequence of S has length divisible by k, then we have the following statements for every $x \mid S$.*

(1) If $\mathrm{v}_g(S) \geq n - \mathrm{ind}_g(x)$, then $n - \mathrm{ind}_g(x) \equiv -1 \pmod k$.

(2) If $\mathrm{v}_g(S) \geq \mathrm{ind}_g(x)$ and x occurs in a zero-sum subsequence of $g^{-\mathrm{v}_g(S)}S$, then $\mathrm{ind}_g(x) \equiv 1 \pmod k$.

Proof. (1) If $\mathrm{ind}_g(x) \geq n - \mathrm{v}_g(S)$, then $T = g^{n - \mathrm{ind}_g(x)} \cdot x$ is a zero-sum subsequence of S. Thus by the hypothesis, we obtain

$$n - \mathrm{ind}_g(x) + 1 = |T| \equiv 0 \pmod k.$$

(2) If $\mathrm{ind}_g(x) \le \mathrm{v}_g(S)$ and x occurs in a zero-sum subsequence W of $g^{-\mathrm{v}_g(S)}S$, then

$$T = g^{\mathrm{ind}_g(x)} \cdot x^{-1}W$$

is a zero-sum subsequence of S. Thus by the hypothesis, we get $|W| \equiv 0 \pmod k$, and hence

$$\mathrm{ind}_g(x) - 1 \equiv \mathrm{ind}_g(x) + |W| - 1 = |T| \equiv 0 \pmod k.$$

This completes the proof. □

We now give a result in the situation when the sequences contain more than two different elements and $b \in (\frac{n}{3}, n - h_1 - 1]$.

Lemma 3.6. *Let $C_n = \langle g_1 \rangle$ with an odd integer n not divisible by k. Set*

$$S = (g_1)^{h_1} \cdot (bg_1)^{h_2} \cdot U \in \mathcal{F}(C_n),$$

where $1 \le \mathrm{h}(U) \le h_2 \le h_1 \le n-1$, $b \in (\frac{n}{3}, n - h_1 - 1]$ and $|S| = n + \frac{n-1}{2}$. Then S has a zero-sum subsequence of length not divisible by k.

Proof. If S is E-trivial, then we are done. So we only need to deal with the situation that S is not E-trivial. By Lemma 3.2, we have $h_1 + h_2 \ge n+1$ and so

$$\frac{n+1}{2} \le h_1 \le n - 1. \tag{3.2}$$

It follows from $b \in (\frac{n}{3}, n - h_1 - 1]$ that

$$\frac{n}{3} < b \le n - h_1 - 1 \le \min\left\{h_2 - 2, \frac{n-3}{2}\right\} < h_1. \tag{3.3}$$

Assume to the contrary that each zero-sum subsequence of S has length divisible by k. By (3.3), we have

$$0 < n - 2b < b < h_1.$$

Thus $T_1 = g_1^{n-2b} \cdot (bg_1)^2$ is a zero-sum subsequence of S. It follows from the assumption that

$$n - 2b + 2 = |T_1| \equiv 0 \pmod k. \tag{3.4}$$

Denote $g_2 = bg_1$ and $W = g_2^{h_2} \cdot U$. Since S is not E-trivial, we have $\mathrm{ord}(g_2) > h_2 > \frac{n}{3}$ (by (3.3)). Note that n is odd and $\mathrm{ord}(g_2) \mid n$. Thus $\mathrm{ord}(g_2) = n$ and so g_2 is also a generator of C_n. We divide the rest of the proof into two cases.

Case 1. W is not zero-sum free.

For any $W' \mid W$ with $\sigma(W') = 0$, if $g_2 \mid W'$, then $T_2 = g_1^b \cdot g_2^{-1}W'$ is also a zero-sum subsequence of S as $g_2 = bg_1$ and $b < h_1$. By the assumption, we have

$$|W'| \equiv 0 \equiv |T_2| = b - 1 + |W'| \equiv b - 1 \pmod k.$$

This together with (3.4) includes $n \equiv 0 \pmod{k}$, yielding a contradiction to $k \nmid n$. Therefore,

$$\mathrm{v}_{g_2}(W') = 0. \tag{3.5}$$

By the hypothesis of this case, there exists a nonempty zero-sum subsequence $T_3 \mid U$. For any $g' \mid T_3$, note that $g_2^{n-\mathrm{ind}_{g_2}(g')} \cdot g'$ and $g_2^{\mathrm{ind}_{g_2}(g')} \cdot g'^{-1}T_3$ are both zero-sum. By (3.5), neither of them is a subsequence of W. Thus $h_2 < \min\{n - \mathrm{ind}_{g_2}(g'), \mathrm{ind}_{g_2}(g')\}$, or equivalently $h_2 < \mathrm{ind}_{g_2}(g') < n - h_2$. By (3.3), we can write

$$T_3 = (y_1 g_2) \cdot (y_2 g_2) \cdot \ldots \cdot (y_q g_2),$$

where $q \geq 3$ and

$$b + 2 \leq h_2 < y_1 \leq \ldots \leq y_q < n - h_2 \leq h_1 - 1 \leq n - b - 2. \tag{3.6}$$

Since S is not E-trivial and n is odd, $(y_1 g_2) \cdot (y_2 g_2)$ is not zero-sum and so $y_1 + y_2 \neq n$.

If $y_1 + y_2 < n$, then $g_2^{n-y_1-y_2} \cdot (y_1 g_2) \cdot (y_2 g_2)$ is a zero-sum sequence. By (3.5), it should not be a subsequence of W. By (3.6), we get

$$b + 2 \leq h_2 < n - y_1 - y_2 < n - 2b,$$

which contradicts to (3.3).

If $y_1 + y_2 > n$, then $g_2^{y_1+y_2-n} \cdot (y_1 g_2)^{-1}(y_2 g_2)^{-1} T_4$ is a zero-sum sequence. Similarly, we get

$$b + 2 \leq h_2 < y_1 + y_2 - n < n - 2b - 4,$$

which also contradicts to (3.3).

So Case 1 is proved.

Case 2. W is zero-sum free.

Now (3.2) infers that $|W| = |S| - h_1 = n + \frac{n-1}{2} - h_1 \geq \frac{n+1}{2}$. By Lemma 2.5, there is a $g_3 \in C_n$ with $\mathrm{ord}(g_3) = n$ such that

$$W = g_3^{h_3} \cdot (z_1 g_3) \cdot \ldots \cdot (z_s g_3),$$

where $2 \leq z_1 \leq \cdots \leq z_s \leq n - 2$ and $h_3 + z_1 + \cdots + z_s < n$. That is,

$$h_3 + 2s \leq h_3 + z_1 + \cdots + z_s \leq n - 1.$$

Since $h_1 + h_3 + s = |S| = n + \frac{n-1}{2}$ and $b \leq n - h_1 - 1$, we get

$$s + h_3 = \frac{n-1}{2} + n - h_1 \geq \frac{n-1}{2} + b + 1.$$

From the above two inequalities, we obtain

$$2(b+1) \leq 2(h_3 + s) - (n-1) \leq h_3 \leq \mathrm{h}(W) = h_2.$$

It follows from (3.3) and $b \le n - h_1 - 1$ that

$$h_2 \ge 2b + 2 > n - b + 2 \ge h_1 + 3,$$

which contradicts $h_1 \ge h_2$.

So Case 2 is proved. □

To deal with the situation when $b \in (n - \frac{n}{h_2}, n-2]$, we need the following auxiliary lemma.

Lemma 3.7. *Let $C_n = \langle g_1 \rangle$ with an odd integer n not divisible by k. Set*

$$S = (g_1)^{h_1} \cdot (bg_1)^{h_2} \cdot U \in \mathcal{F}(C_n),$$

where $b \in (n - \frac{n}{h_2}, n - 2]$ and $1 \le \mathrm{h}(U) \le h_2 \le h_1 \le n - 1 \le h_1 + h_2 - 2$.

(1) *Suppose that every zero-sum subsequence of S has length divisible by k. For any $g \mid U$, if $\mathrm{ind}_{g_1}(g) \in [2, n - h_1 - 1]$, then $\mathrm{ind}_{g_1}(g) \equiv 1 \pmod{k}$.*

(2) *Suppose that $\mathrm{ind}_{g_1}(g) \in [2, n - h_1 - 1]$ for every $g \mid U$. If*

$$n\|U\|_{g_1} \ge n - h_1,$$

then S has a zero-sum subsequence of length not divisible by k.

(3) *Suppose that $\mathrm{ind}_{g_1}(g) \in [h_2(n - b), n - 2]$ for every $g \mid U$. If*

$$n - n\| - U\|_{g_1} \le n - h_1 - 1,$$

then S has a zero-sum subsequence of length not divisible by k.

Proof. Set $t = n - b$. It follows from $h_2 \ge 2$ and $b \in (n - \frac{n}{h_2}, n - 2]$ that

$$t \in \Big[2, \frac{n}{h_2}\Big) \subseteq \Big[2, \frac{n}{2}\Big).$$

Since $h_2 \le h_1 \le n - 1 \le h_1 + h_2 - 2$, we have

$$n - h_1 < h_2 < h_2 t \tag{3.7}$$

and

$$t < \frac{n+1}{2} \le h_1 \le n - 1 \tag{3.8}$$

as n is odd. Thus by $t \in [2, \frac{n}{h_2})$,

$$(n + 1 - h_1)t \le h_2 t < n \le n + (t - 2)\Big(\frac{n - t}{2}\Big) = \Big(\frac{n - t}{2} + 1\Big)t,$$

which implies

$$\frac{n + t}{2} < h_1 \le n - 1.$$

Thus

$$n - h_1 < n - \frac{n + t}{2} = \frac{n + t}{2} - t < h_1 - t < \lfloor \frac{h_1}{t} \rfloor t.$$

By denoting $\ell = \min\{\lfloor \frac{h_1}{t} \rfloor, h_2\}$ and using the above result together with (3.7), we have

$$n - h_1 < \min\left\{\left\lfloor \frac{h_1}{t}\right\rfloor t, h_2 t\right\} = \ell t \leq \left\lfloor \frac{h_1}{t}\right\rfloor t \leq h_1. \tag{3.9}$$

If every zero-sum subsequence of S has length divisible by k, then it follows from (3.8) and Lemma 3.5 (1) that $h_1 > n - b$ and

$$t = n - b \equiv -1 \pmod{k}. \tag{3.10}$$

Denote $W = (bg_1)^{h_2} \cdot U$ and

$$U = (x_1 g_1) \cdot \ldots \cdot (x_r g_1)$$

with

$$2 \leq x_1 \leq \ldots \leq x_r \leq n - 2. \tag{3.11}$$

(1) If $x_i \in [2, n - h_1 - 1]$, then by (3.9),

$$0 < \ell t + h_1 + 1 - n \leq \ell t - x_i < h_1.$$

It follows from $t = n - b$ that

$$T_1 = g_1^{\ell t - x_i} \cdot (bg_1)^{\ell} \cdot (x_i g_1)$$

is a zero-sum subsequence of S. By the hypothesis and (3.10), we have

$$-x_i + 1 \equiv \ell(t+1) - x_i + 1 = |T_1| \equiv 0 \pmod{k}$$

as desired.

(2) By the hypothesis of (2), there exists a longest subsequence U_1 of U such that

$$0 < n\|U_1\|_{g_1} \leq n - h_1 - 1 < n\|U\|_{g_1}.$$

Choose $g' \mid U_1^{-1}U$ and denote $U_2 = g'U_1$. By (3.8) and the maximum of the $|U_1|$, we have

$$n - h_1 \leq n\|U_2\|_{g_1} = n\|U_1\|_{g_1} + \mathrm{ind}_{g_1}(g') \leq 2(n - h_1 - 1) < n.$$

Thus

$$T_2 = g_1^{n - n\|U_2\|_{g_1}} \cdot U_2$$

is a zero-sum subsequence of S.

If every zero-sum subsequence of S has length divisible by k, then by (1), we have $n\|U_2\|_{g_1} \equiv |U_2| \pmod{k}$ and so

$$0 \equiv |T_2| = n - n\|U_2\|_{g_1} + |U_2| \equiv n \pmod{k}.$$

It is impossible since $k \nmid n$. So (2) is proved.

(3) Assume to the contrary that every zero-sum subsequence of S has length divisible by k.

Using (3.8) and $t = n - b$, we obtain that $b > n - h_1$, which together with (3.7) and the hypothesis of (3) gives

$$\mathrm{ind}_{g_1}(g) > n - h_1 \tag{3.12}$$

for any $g \mid W = (bg_1)^{h_2} \cdot U$.

Let $D = n - n\|-U\|_{g_1}$. Then by (3.9) and the hypothesis of (3), we have

$$D \le n - h_1 - 1 < \ell t \le h_2 t. \tag{3.13}$$

We divide the rest of the proof into three cases.

Case 1. $D \ge 0$.

If $h_2 t > h_1$, then $h_2 > \frac{h_1}{t} \ge \lfloor \frac{h_1}{t} \rfloor$ and so $\ell = \min\{\lfloor \frac{h_1}{t} \rfloor, h_2\} = \lfloor \frac{h_1}{t} \rfloor < h_2$. It follows from (3.13) that

$$0 < \ell t - D \le \left\lfloor \frac{h_1}{t} \right\rfloor t \le h_1.$$

Thus

$$T_3 = g_1^{\ell t - D} \cdot (bg_1)^{\ell} \cdot U$$

is a zero-sum subsequence of S as $t = n - b$ and

$$\begin{aligned}\sigma(T_3) &= (\ell t - D + \ell b + n\|U\|_{g_1})g_1 \\ &= (n\| - U\|_{g_1} + n\|U\|_{g_1})g_1 = 0.\end{aligned}$$

Therefore, (3.12) and Lemma 3.5 (1) infer that

$$n\| - U\|_{g_1} \equiv -|-U| = -|U| \pmod{k}.$$

Using the above result together with the assumption and (3.10), we have

$$\begin{aligned}-n &\equiv n\| - U\|_{g_1} - n + |U| \\ &\equiv \ell(t+1) - D + |U| = |T_3| \equiv 0 \pmod{k},\end{aligned}$$

yielding a contradiction to $k \nmid n$.

If $h_2 t \le h_1$, then by (3.13), we have

$$0 < h_2 t - D \le h_1 - D < h_1.$$

Similarly,

$$T_4 = g_1^{h_2 t - D} \cdot (bg_1)^{h_2} \cdot U$$

is a zero-sum subsequence of S and we get a contradiction to $k \nmid n$. So Case 1 is proved.

Case 2. $D < 0$ and W is not zero-sum free.

In this case, there exists a zero-sum subsequence $W_1 \mid W$ and we assert that

$$n\| - W_1\|_{g_1} = n.$$

Otherwise, $n\| - W_1\|_{g_1} \geq 2n$. Thus there exists a longest subsequence $W_2 \mid W_1$ such that

$$n\| - W_2\|_{g_1} \leq n < n\| - W_1\|_{g_1}.$$

Choose $g' \mid W_2^{-1}W_1$. By the maximum of the $|W_2|$, we have

$$n\| - W_2\|_{g_1} \leq n < n\| - (W_2 \cdot g^{'})\|_{g_1} = n\| - W_2\|_{g_1} + n - \mathrm{ind}_{g_1}(g')$$

and so

$$0 \leq n - n\| - W_2\|_{g_1} < n - \mathrm{ind}_{g_1}(g') < h_1$$

as (3.12). Hence, we have that

$$T_5 = g_1^{n-n\|-W_2\|_{g_1}} \cdot W_2^{-1}W_1$$

is also a zero-sum subsequence of S as

$$\sigma(T_5) = (n - n\| - W_2\|_{g_1} - n\|W_2\|_{g_1} + n\|W_1\|_{g_1})g_1 = 0$$

By (3.12) and Lemma 3.5(1), we have

$$n\| - W_2\|_{g_1} \equiv -| - W_2| = -|W_2| \pmod k.$$

By the assumption, we have

$$n \equiv n - n\| - W_2\|_{g_1} - |W_2| + |W_1| = |T_5| \equiv |W_1| \equiv 0 \pmod k,$$

yielding a contradiction to $k \nmid n$. So our assertion is proved.

Choose an element w in W_1. Thus by our assertion,

$$n - \mathrm{ind}_{g_1}(w) + n\| - w^{-1}W_1\|_{g_1} = n\| - W_1\|_{g_1} = n.$$

Using the above result together with (3.12), Lemma 3.5 (1) and the assumption, we have $|W_1| \equiv 0 \pmod k$ and so

$$n \equiv \mathrm{ind}_{g_1}(w) - 1 = n\| - w^{-1}W_1\|_{g_1} - 1 \equiv -(|W_1| - 1) - 1 \equiv 0 \pmod k,$$

yielding a contradiction to $k \nmid n$. And Case 2 is proved.

Case 3. $D < 0$ and W is zero-sum free.

Since $n\| - U\|_{g_1} = n - D > n$ and W is zero-sum free, we can set $U_3 = \prod_{i=1}^{d}(x_i g_1)$ for some $d \in [1, r-1]$ such that

$$n\| - U_3\|_{g_1} < n < n\| - U_3\|_{g_1} + (n - x_{d+1}) \leq n\| - U\|_{g_1}.$$

By denoting $D' = n - n\| - U_3\|_{g_1}$, we have

$$0 < D' < n - x_{d+1}.$$

If $0 < D' < h_2 t$, then we can use the same argument as in Case 1. Consider

$$T_3' = g_1^{\ell t - D'} \cdot (bg_1)^{\ell} \cdot U_3$$

when $h_2 t > h_1$ and

$$T_4' = g_1^{h_2 t - D'} \cdot (bg_1)^{h_2} \cdot U_3$$

when $h_2 t \le h_1$. They both lead to a contradiction to $k \nmid n$.

Therefore, it remains to deal with the situation $D' \ge h_2 t$. Using (3.11) and (3.12), we have

$$h_2 t \le D' \le n - x_{d+1} \le n - x_d < h_1. \tag{3.14}$$

Next, we divide the proof of Case 3 into the following three subcases.

Case 3.1. $x_{d+1} + x_d \le n$.

Since S is not E-trivial, we have $x_{d+1} + x_d \ne n$.

Using (3.11) and (3.12), we have

$$h_1 > n - x_1 > n - (x_d + x_{d+1}) > 0.$$

Therefore,

$$T_6 = g_1^{n-x_d-x_{d+1}} \cdot (x_d g_1) \cdot (x_{d+1} g_1)$$

is a zero-sum subsequence of S. By (3.14), Lemma 3.5 (1) and the assumption, we have

$$-n \equiv n - (2n+2) + 2 \equiv n - x_d - x_{d+1} + 2 = |T_6| \equiv 0 \pmod{k},$$

yielding a contradiction to $k \nmid n$.

Case 3.2. $n < x_{d+1} + x_d \le n + h_2 t$.

By (3.14) and the hypothesis of Case 3.2, we have

$$0 < h_2 t + n - x_{d+1} - x_d < h_2 t < h_1.$$

Thus

$$T_7 = g_1^{h_2 t + n - x_{d+1} - x_d} \cdot (b g_1)^{h_2} \cdot (x_d g_1) \cdot (x_{d+1} g_1)$$

is a zero-sum subsequence of S. By the assumption, (3.10), (3.14) and Lemma 3.5 (1), we get

$$-n \equiv n - (2n+2) + 2 \equiv h_2(t+1) + n - x_d - x_{d+1} + 2 = |T_7| \equiv 0 \pmod{k},$$

yielding a contradiction to $k \nmid n$.

Case 3.3. $x_{d+1} + x_d > n + h_2 t$.

Since W is zero-sum free, Lemma (2.6) (3) infers that

$$h_2 \ge \frac{n+5}{6}.$$

By (3.14) and the hypothesis of Case 3.3, we have

$$2h_2 t \le 2n - x_{d+1} - x_d < 2n - n - h_2 t,$$

which implies $h_2 < \frac{n}{3t} \le \frac{n}{6}$ as $t \ge 2$, yielding a contradiction to $h_2 \ge \frac{n+5}{6}$. So Case 3 is proved.

In all cases, we have found contradictions and thereby we complete the proof. $\square$

Finally, we deal with the situation when $b \in (n - \frac{n}{h_2}, n-2]$.

Lemma 3.8. *Let $k \ge 3$ and $C_n = \langle g_1 \rangle$ with an odd integer n not divisible by k. Let*

$$S = (g_1)^{h_1} \cdot (bg_1)^{h_2} \cdot U \in \mathcal{F}(C_n),$$

where $b \in (n - \frac{n}{h_2}, n-2]$, $1 \le \mathrm{h}(U) \le h_2 \le h_1 \le n-1$ and $|S| = n + \frac{n-1}{2}$. Then S has a zero-sum subsequence of length not divisible by k.

Proof. If S is E-trivial, then we are done. So we may assume that S is not E-trivial. By Lemma 3.2, we have

$$h_1 + h_2 \ge n+1$$

and so

$$\frac{n}{h_2} < \frac{n+1}{2} \le h_1 \le n-1. \tag{3.15}$$

Using (3.15) and $b \in (n - \frac{n}{h_2}, n-2]$, we get

$$n - h_1 < n - \frac{n}{h_2} < b < n-2. \tag{3.16}$$

Set $t = n - b$. It follows from $h_1 + h_2 \ge n+1$ and $b \in (n - \frac{n}{h_2}, n-2]$ that

$$n - h_1 < h_2 < h_2 t < n. \tag{3.17}$$

Assume to the contrary that each zero-sum subsequence of S has length divisible by k. It follows from (3.16) and Lemma 3.5 (1) that

$$t + 1 \equiv 0 \pmod{k}. \tag{3.18}$$

Let

$$U = (x_1 g_1) \cdot \ldots \cdot (x_r g_1).$$

We distinguish the four cases according to the interval where x_i is located.

CASE 1. $n - h_1 \le x_i < h_2 t$ for some $i \in [1, r]$.

Then by (3.17), we have

$$0 < n - x_i \le h_1$$

and

$$0 < h_2 t - x_i \le h_2 t - (n - h_1) < h_1.$$

Thus $T_1 = g_1^{n-x_i} \cdot (x_i g_1)$ and $T_2 = g_1^{h_2 t - x_i} \cdot (bg_1)^{h_2} \cdot (x_i g_1)$ are zero-um subsequences of S. By (3.18) and the assumption, we have

$$\begin{cases} n - x_i + 1 = |T_1| \equiv 0 & \pmod{k}, \\ h_2(t+1) - x_i + 1 = |T_2| \equiv 0 & \pmod{k}, \end{cases}$$

which implies that $n \equiv 0 \pmod{k}$, yielding a contradiction to $k \nmid n$. Thus Case 1 is proved.

CASE 2. $2 \le x_1 \le \ldots \le x_r \le n - h_1 - 1$.

If $n - h_1 \le n\|U\|_{g_1}$, then the result follows from Lemma 3.7 (2).

If $n\|U\|_{g_1} \le n - h_1 - 1$, then by $\mathrm{h}(U) \ge 1$ and the hypothesis of Case 2, we have

$$|U| < 2|U| \le n\|U\|_{g_1} \le n - h_1 - 1.$$

Since $|S| = h_1 + h_2 + |U| = n + \frac{n-1}{2}$, we have

$$h_2 = |S| - h_1 - |U| = n + \frac{n-1}{2} - h_1 - |U| \ge \frac{n-1}{2} + 1.$$

Thus $h_2 t > n$ (as $t \ge 2$), which contradicts to (3.17), and we are done.

CASE 3. $h_2 t \le x_1 \le \ldots \le x_r \le n-2$.

If $n - n\|-U\|_{g_1} \le n - h_1 - 1$, then the result follows from Lemma 3.7 (3).

If $n - h_1 \le n - n\| - U\|_{g_1} < h_2 t$, then by (3.17) and denoting $D = n - n\| - U\|_{g_1}$, we have

$$0 < h_2 t - D \le h_2 t - (n - h_1) < h_1.$$

Thus $T_3 = g_1^{h_2 t - D} \cdot (bg_1)^{h_2} \cdot U$ is a zero-sum subsequence of S since

$$\sigma(T_3) = (h_2 t - D + h_2 b + n\|U\|)g_1 = (n\|U\| - n + n\| - U\|)g_1 = 0$$

(as $t = n - b$). By (3.17), Lemma 3.5(1) and the hypothesis of Case 3, we have

$$n\| - U\|_{g_1} \equiv -| - U| = -|U| \pmod{k}.$$

Using the above result together with (3.18) and the assumption, we have

$$-n \equiv n\| - U\|_{g_1} - n + |U| \equiv h_2(t+1) - D + |U| = |T_4| \equiv 0 \pmod{k},$$

yielding a contradiction to $k \nmid n$.

If $n - n\| - U\|_{g_1} \ge h_2 t$, then by $t = n - b$, (3.16) and the hypothesis of Case 3, we have $t \ge 2$ and

$$2h_2 \le h_2 t \le n - n\| - U\|_{g_1} \le n - 2|U| = n - 2r.$$

Thus

$$r \le \frac{n}{2} - h_2.$$

By (3.15) and $h_1 + h_2 + r = |S| = n + \frac{n-1}{2}$, we have

$$h_2 + r = \frac{n-1}{2} + n - h_1 \ge \frac{n-1}{2} + 1$$

and so

$$\frac{n+1}{2} - h_2 \le r \le \frac{n}{2} - h_2,$$

which is impossible. This completes the proof of Case 3.

CASE 4. For some $m \in [1, r-1]$,

$$2 \le x_1 \le \ldots \le x_m \le n - h_1 - 1 < h_2 t \le x_{m+1} \le \ldots \le x_r \le n-2.$$

Let $U = U_1 U_2$, where $U_1 = \prod_{i=1}^{m}(x_i g_1)$ and $U_2 = \prod_{m+1}^{r}(x_i g_1)$.

If $n - h_1 \le n\|U_1\|_{g_1}$, then by $2 \le x_1 \le \ldots \le x_m \le n - h_1 - 1$ and Lemma 3.7 (2), the result is proved. If $n - h_1 \le n - n\| - U_2\|_{g_1} < h_2 t$ or $n - n\| - U_2\|_{g_1} \le n - h_1 - 1$, then using the same argument as in Case 3, the result is proved.

Therefore, it remains to deal with the situation where

$$0 < n\|U_1\|_{g_1} \le n - h_1 - 1 \tag{3.19}$$

and

$$n - n\| - U_2\|_{g_1} \ge h_2 t.$$

It follows from $t = n - b$ and (3.16) that $t \ge 2$. By the hypothesis of Case 4, we obtain

$$2h_2 \le h_2 t \le n - n\| - U_2\|_{g_1} \le n - 2|U_2|$$

and thus

$$|U_2| \le \frac{n}{2} - h_2. \tag{3.20}$$

By the hypothesis of Case 4 and Lemma 3.7 (1), we have $x_i \equiv 1 \pmod k$ and so $x_i \ge k + 1$ for every $i \in [1, m]$, which implies that $n\|U_1\|_{g_1} \ge 4|U_1| = 4m$ (as the hypothesis $k \ge 3$). Using (3.19) and $h_1 + h_2 + |U_1| + |U_2| = |S| = n + \frac{n-1}{2}$, we have

$$\begin{aligned} h_2 + |U_2| &= \frac{n-1}{2} + n - h_1 - m \\ &\ge \frac{n-1}{2} + 1 + n\|U_1\|_{g_1} - m \ge \frac{n+1}{2} + 3m. \end{aligned}$$

It follows from (3.20) that

$$\frac{n+1}{2} + 3m - h_2 \le |U_2| \le \frac{n}{2} - h_2,$$

yielding a contradiction to $|U_1| = m \ge 1$.

In all cases, we have found contradictions and thereby we complete the proof. □

4. Proof of the Main Results

We now give the proofs of our main results.

Proof of Theorem 1.4. It follows from Lemma 1.2 that $\mathsf{E}_3(C_n) \ge \lfloor \frac{3}{2}(n-1) \rfloor + 1$. So it remains to prove $\mathsf{E}_3(C_n) \le \lfloor \frac{3}{2}(n-1) \rfloor + 1$. Further, we will show the stronger result that $\mathsf{E}_k(C_n) \le \lfloor \frac{3}{2}(n-1) \rfloor + 1$ for any integer $k \ge 3$ with $k \nmid n$.

Let $C_n = \langle g \rangle$ and $S \in \mathcal{F}(C_n)$ with $|S| = \lfloor \frac{3}{2}(n-1) \rfloor + 1$. We prove that S has a zero-sum subsequence of length not divisible by k according to the following two cases.

Case 1. n is even.

Let $C_2 = \langle \frac{n}{2}g \rangle \lhd C_n$ and $\varphi : C_n \to C_n/C_2$ be the natural epimorphism. Noticing that $|S| = \frac{n}{2} + n - 1$ and $\mathsf{s}(C_n/C_2) = n - 1$, we apply Lemma 2.7 to get the decomposition

$$S = S_1 \cdot S_2 \cdot S^{'},$$

where $|S_1| = |S_2| = \frac{n}{2}$ and $\sigma(S_1), \sigma(S_2) \in \ker(\varphi) = C_2$.

Each of S_1, S_2 and S_1S_2 has length not divisible by k since $k \nmid n$. And it is obviously impossible that $0 \neq \sigma(S_1) = \sigma(S_2) = \sigma(S_1S_2) \in C_2$. Thus, at least one of S_1, S_2 or S_1S_2 is a zero-sum subsequence of S with length not divisible by k, as desired.

Case 2. n is odd.

If S is E-trivial, then we are done. So we may assume that S is not E-trivial. By $|S| = n + \frac{n-1}{2}$ and Lemma 3.2, we can write

$$S = g_1^{h_1} \cdot (bg_1)^{h_2} \cdot U,$$

where $\mathrm{ord}(g_1) = n$, $b \in [2, n-2]$, $0 \leq \mathrm{h}(U) \leq h_2 \leq h_1 \leq n-1$ and $h_1 + h_2 \geq n + 1$.

If $b \in [2, \frac{n}{3}]$, the result follows from Lemma 3.4 (1).

If $b \in (\frac{n}{3}, n - h_1 - 1]$, then by Lemmas 3.4 (2) and 3.6, we obtain this result.

If $b \in [n - h_1, n - \frac{n}{h_2}]$, then Lemma 3.4 (3) provides the desired result.

If $b \in (n - \frac{n}{h_2}, n-2]$, then the result follows immediately from Lemmas 3.4 (4) and 3.8. This completes the proof of Theorem 1.4. $\Box$

Proof of Theorem 1.5**.** It follows from Lemma 1.2 that $\mathsf{E}_k(C_n) \geq n+1$. So it remains to prove $\mathsf{E}_k(C_n) \leq n+1$.

Let S be a sequence of length $n+1$. We need to show that there exists a subsequence T of S with $\sigma(T) = 0$ such that $k \nmid |T|$. By $\lceil \frac{n}{2} \rceil < k \leq n - 1 = \mathsf{D}(C_n) - 1$ and Lemma 2.2(1), we can find a zero-sum subsequence T of S with $|T| \neq k$. If $k \mid |T|$, then $|T| \geq 2k \geq 2(\lceil \frac{n}{2} \rceil + 1) \geq n + 2 > |S|$, a contradiction. Thus $k \nmid |T|$, so we find the desired subsequence T and we are done. $\Box$

As a consequence of Theorems 1.4 and 1.5 along with earlier work by the first author and W. Schmid, we can summarize all known results of $\mathsf{E}_k(C_n)$ in the following corollary.

Corollary 4.1. *Let n and k be positive integers with $k \nmid n$. Then*

$$\left\lfloor \frac{k}{k-1}(n-1) \right\rfloor + 1 \leq \mathsf{E}_k(C_n) \leq \left\lfloor \frac{3}{2}(n-1) \right\rfloor + 1$$

for $k \geq 3$, and

$$\mathsf{E}_k(C_n) = \left\lfloor \frac{k}{k-1}(n-1) \right\rfloor + 1$$

if $k = 2,\ 3$ *or* $k > \lceil \frac{n}{2} \rceil$.

Acknowledgements. This work was supported in part by National Natural Science Foundation of China (Grant No. 12071344), and was also supported in part by a Discovery Grant from the Natural Sciences and Engineering Research Council of Canada(Grant No. RGPIN 2024-03931).

References

[1] W. Alford, A. Granville and C. Pomerance, *There are infinitely many Carmichael numbers*, Ann. of Math. 139 (1994), 703–722.

[2] N. Alon and M. Dubiner, *Zero-sum sets of prescribed size*, Bolyai Soc. Math. Stud. 1 (1993), 33–50.

[3] N. Alon and M. Dubiner, *A lattice point problem and additive number theory*, Combinatorica 15 (1995), 301–309.

[4] R. Chi, S. Ding, W. Gao, A. Geroldinger and W. Schmid, *On zero-sum subsequences of restricted size IV*, Acta Math. Hungar. 107 (2005), 337–344.

[5] P. van Emde Boas and D. Kruyswijk, *A combinatorial problem on finite Abelian groups*, Reports ZW-1967-009, Mathematical Centre, Amsterdam, 1967.

[6] P. Erdős, A. Ginzburg and A. Ziv, *Theorem in the additive number theory*, Bull. Res. Council Israel Sect. F 10F (1961), 41–43.

[7] Y. Fan, W. Gao, G. Wang, Q. Zhong and J. Zhuang, *On short zero-sum subsequences of zero-sum sequences*, Electron. J. Combin. 19 (2012), 31–23.

[8] Y. Fan, W. Gao, L. Wang and Q. Zhong, *Two zero-sum invariants on finite abelian groups*, European J. Combin. 34 (2013), 1331–1337.

[9] W. Gao, *On zero sum subsequences of restricted size III*, Ars Combin. 61 (2001), 65–72.

[10] W. Gao, *On zero-sum subsequences of restricted size II*, Discrete Math. 271 (2003), 51–59.

[11] W. Gao and A. Geroldinger, *Zero-sum problems in finite abelian groups: a survey*, Expo. Math. 24 (2006), 337–369.

[12] W. Gao, Y. Li, P. Zhao and J. Zhuang, *On sequences over a finite Abelian group with zero-sum subsequences of forbidden lengths*, Colloq. Math. 144 (2016), 31–44.

[13] A. Geroldinger and F. Halter-Koch, *Non-unique factorizations. Algebraic, Combinatorial and Analytic Theory*, Pure Appl. Math., vol. 278, Chapman & Hall/CRC, 2006.

[14] H. Mann and J. Olson, *Sums of sets in the elementary Abelian group of type* (p, p), J. Combinatorial Theory 2 (1967), 275–284.

[15] J. Olson, *A combinatorial problem on finite Abelian groups I*, J. Number Theory 1 (1969), 8–10.

[16] C. Reiher, *On Kemnitz' conjecture concerning lattice-points in the plane*, Ramanujan J. 13 (2007), 333–337.

[17] S. Savchev and F. Chen, *Long n-zero-free sequences in finite cyclic groups*, Discrete Math. 308 (2008), 1–8.

[18] W. Schmid, *On zero-sum subsequences in finite abelian groups*, Integers 1 (2001), 1–8.

Center for Applied Mathematics and KL-AAGDM, Tianjin University, Tianjin 300072, P.R. China
E-mail address: wdgao@nankai.edu.cn

School of Mathematical Sciences, Chengdu University of Technology, Chengdu 610059, P.R. China
E-mail address: jiangxiao2022@tju.edu.cn

Brock University, Department of Mathematics and Statistics, St. Catharines ON L2S 3A1, Canada
E-mail address: yli@brocku.ca

Center for Applied Mathematics and KL-AAGDM, Tianjin University, Tianjin 300072, P.R. China
E-mail address: huijaunqi175@163.com