Research Article

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L^2-exponential ergodicity of stochastic Hamiltonian systems with α-stable Lévy noises

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Abstract: Based on the hypocoercivity approach due to Villani (2009), Dolbeault, Mouhot and Schmeiser (2015) established a new and simple framework to investigate directly the L^2-exponential convergence to the equilibrium for the solution to the kinetic Fokker–Planck equation. Nowadays, the general framework advanced by Dolbeault, Mouhot and Schmeiser (2015) is named as the DMS framework for hypocoercivity. Subsequently, Grothaus and Stilgenbauer (2014) built a dual version of the DMS framework in the kinetic Fokker–Planck setting. No matter what the abstract DMS framework by Dolbeault, Mouhot and Schmeiser (2015) or the dual counterpart by Grothaus and Stilgenbauer (2014), the densely defined linear operator involved is assumed to be decomposed into two parts, where one part is symmetric and the other part is anti-symmetric. Thus, the existing DMS framework is not applicable to investigate the L^2-exponential ergodicity for stochastic Hamiltonian systems with α-stable Lévy noises, where one part of the associated infinitesimal generators is anti-symmetric whereas the other one is not symmetric at all. In this paper, we shall develop a dual version of the DMS framework in the fractional kinetic Fokker–Planck setup, where one part of the densely defined linear operator under consideration need not to be symmetric. As a direct application, we explore the L^2-exponential ergodicity of stochastic Hamiltonian systems with α-stable Lévy noises. The proof is also based on Poincaré inequalities for non-local stable-like Dirichlet forms and the potential theory for fractional Riesz potentials.

Keywords: Fractional kinetic Fokker–Planck operator, stochastic Hamiltonian system with α-stable Lévy noise, Poincaré inequality, fractional Riesz potential

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1 Introduction and main result

1.1 Background

In physics, the Hamiltonian system, as a mathematical formalism due to W.R. Hamilton, describes the evolution of particles in physical systems. From the perspective of practical applications, the deterministic Hamiltonian systems are often subject to environmental noises. Then the environmentally perturbed system, named as the stochastic Hamiltonian system in literature, is brought into being. So far, stochastic Hamiltonian systems have been applied ubiquitously (see e.g. [23]) in finance describing some risky assets, in physics portraying the synchrotron oscillations of particles in storage rings due to the impact of external fluctuating electromagnetic fields,

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and in stochastic optimal control serving as a stochastic version of the maximum principle of Pontryagin's type, to name just a few.

With regard to the mathematical formulation, the stochastic Hamiltonian system is described by the following degenerate stochastic differential equation (SDE for short) on $\mathbb{R}^{2d} := \mathbb{R}^d \times \mathbb{R}^d$:

$$\begin{cases} \mathrm{d}X_t = \nabla_v H(X_t, V_t)\,\mathrm{d}t, \\ \mathrm{d}V_t = -(\nabla_x H(X_t, V_t) + F(X_t, V_t)\nabla_v H(X_t, V_t))\,\mathrm{d}t + \mathrm{d}Z_t, \end{cases} \tag{1.1}$$

where H is the Hamiltonian function, $\nabla_x H$ and $\nabla_v H$ stand for the gradient operators with respect to the position variable x and the velocity variable v, respectively, F means the damping coefficient, and $(Z_t)_{t\geq 0}$ is a d-dimensional stochastic noise. Throughout the paper, in some occasions, we frequently use the simplified notation ∇ to denote the gradient operator in case there are no confusions evoked. In particular, when $H(x, v) = U(x) + \frac{1}{2}|v|^2$ and $(Z_t)_{t\geq 0} = (B_t)_{t\geq 0}$, a d-dimensional standard Brownian motion, (1.1) reduces to the stochastic damping Hamiltonian system:

$$\begin{cases} \mathrm{d}X_t = V_t\,\mathrm{d}t, \\ \mathrm{d}V_t = -(\nabla U(X_t) + F(X_t, V_t)V_t)\,\mathrm{d}t + \mathrm{d}B_t, \end{cases} \tag{1.2}$$

where U might incorporate the confining potentials and the interaction potentials (e.g., the Lennard–Jones potential and the Coulomb potential).

In the past few years, great progress has been made on ergodicity of the stochastic Hamiltonian system (1.2) with regular potentials. For the polynomial-like potential U, the exponential ergodicity under the total variation distance was addressed in [33, 38] with the aid of Harris' theorem. By making use of the mixture of the reflection coupling and the synchronous coupling, concerning the underdamped Langevin SDE (i.e., (1.2) with $F(x, v) \equiv 1$), the exponential contractivity under the quasi-Wasserstein distance was tackled in [17]. Recently, the kinetic Langevin dynamics with singular potentials has also received more and more attention since the interaction potentials exhibit certain singular features. Specially, the geometric ergodicity under the total variation distance of kinetic Langevin dynamics with singular potentials has been investigated in depth via Harris' theorem; see [22] concerned with the setting on the Lennard–Jones-type interactions, and [28] regarding the setup on the Coulomb interactions, respectively.

In comparison to stochastic Hamiltonian systems subject to Brownian motion noises, the long term behavior of the counterparts environmentally perturbed by pure jump Lévy processes is sparse. All the same, there has been some progress on ergodicity of stochastic Hamiltonian systems with pure jumps in recent years. In [7], concerning stochastic Hamiltonian systems with pure jumps and regular potentials, by designing a novel Markov coupling, we dealt with the exponential ergodicity under the multiplicative Wasserstein-type distance. In the meantime, based on distinctive constructions of Lyapunov functions and the Hörmander theorem for non-local operators, the exponential ergodicity under the total variation distance was explored [6] via Harris' theorem for Lévy-driven Langevin dynamics, where the singular potentials might be the Coulomb potentials or the Lennard–Jones-like potentials.

Besides the exponential ergodicity under the total variation or the Wasserstein-type distance, there is plenty of work that is devoted to the L^2-exponential ergodicity. We recall some facts related to it. Let $(X_t)_{t\geq 0}$ be a Markov process generating a Markov semigroup $(P_t)_{t\geq 0}$, and let the probability measure μ be an invariant probability measure (IPM for abbreviation) of $(P_t)_{t\geq 0}$. The Markov process $(X_t)_{t\geq 0}$ is called L^2-exponentially ergodic if there exist constants $c, \lambda > 0$ such that for all $f \in L^2(\mu)$ and $t > 0$,

$$\mathrm{Var}_\mu(P_t f) \leq c\,\mathrm{e}^{-\lambda t}\,\mathrm{Var}_\mu(f), \tag{1.3}$$

where $\mathrm{Var}_\mu(f) := \mu(f^2) - \mu(f)^2$ with $\mu(f) := \int f\,\mathrm{d}\mu$. The L^2-exponential ergodicity above has multiple applications. For instance, the explicit bounds involved in (1.3) may provide insights into effectiveness of stochastic algorithms. Particularly, the explicit constants $c, \lambda > 0$ in (1.3) furnish an upper bound on the integrated autocorrelation, which indeed is a performance measure of Monte Carlo estimators; see, for instance, [2]. On the other hand, the L^2-exponential ergodicity implies characterization of convergence to equilibrium in the other regimes; see, for example, [14, Chapter 8] for a very nice diagram of nine types of ergodicity.

For symmetric Markov processes, one of the powerful tools to investigate ergodicity (under, for example, the variance or the relative entropy) is the functional inequality (e.g., the Poincaré-type inequality and the log-Sobolev inequality). With regarding to a symmetric Markov process under investigation, the corresponding Markov semigroup is L^2-exponentially decaying once the associated Poincaré inequality is valid; see e.g. [35, Theorem 1.1.1, p. 24] and [5, Theorem 4.2.5, p. 183]. Whereas, as far as non-symmetric Markov processes are concerned, the situation will be different drastically. To demonstrate this aspect, we focus on (1.1) with $H(x,v) = U(x) + \Phi(v)$ for some smooth functions U and Φ, $F(x,v) \equiv 1$, and $(Z_t)_{t\geq 0} = (B_t)_{t\geq 0}$, a d-dimensional Brownian motion. More precisely, we work with the following kinetic SDE:

$$\begin{cases} \mathrm{d}X_t = \nabla\Phi(V_t)\,\mathrm{d}t, \\ \mathrm{d}V_t = -(\nabla U(X_t) + \nabla\Phi(V_t))\,\mathrm{d}t + \mathrm{d}B_t. \end{cases} \tag{1.4}$$

If the prerequisite $C_{U,\Phi} := \iint_{\mathbb{R}^d\times\mathbb{R}^d} \mathrm{e}^{-(U(x)+\Phi(v))}\,\mathrm{d}x\,\mathrm{d}v < \infty$ holds true, then the probability measure

$$\mu(\mathrm{d}x,\mathrm{d}v) := C_{U,\Phi}^{-1}\mathrm{e}^{-(U(x)+\Phi(v))}\,\mathrm{d}x\,\mathrm{d}v \tag{1.5}$$

is an IPM of the Markov semigroup $(P_t)_{t\geq 0}$ generated by the Markov process $(X_t, V_t)_{t\geq 0}$. Due to the invariance of μ, we have

$$\partial_t \mu((P_t f)^2) = -2\mu(\Gamma(P_t f)),$$

where $\Gamma(f) := \frac{1}{2}|\nabla_v f|^2$ is the Carré du champ operator; see [5, pp. 20–22 and pp. 122–125]. If there exists a constant $c_0 > 0$ such that the Poincaré inequality:

$$\mathrm{Var}_\mu(f) \leq c_0\,\mu(\Gamma(f)), \quad f \in H^1(\mu), \tag{1.6}$$

is valid, then the L^2-exponential ergodicity of $(X_t, V_t)_{t\geq 0}$ (or the semigroup $(P_t)_{t\geq 0}$ is L^2-exponentially decaying) follows from Gronwall's inequality. Nevertheless, due to $\Gamma(f) = \frac{1}{2}|\nabla_v f|^2$ for $f \in H^1(\mu)$, the energy form $\mathscr{E}(f) := \mu(\Gamma(f))$ is reducible since the x-direction in Γ is missing. Hence, the Poincaré inequality (1.6) is not any more available.

The infinitesimal generator of the Markov process $(X_t, V_t)_{t\geq 0}$ solving (1.4) is given by

$$\begin{aligned} (\mathscr{L}f)(x,v) &= \big(\langle\nabla_x f(x,v), \nabla\Phi(v)\rangle - \langle\nabla_v f(x,v), \nabla U(x)\rangle\big) + \Big(-\langle\nabla_v f(x,v), \nabla\Phi(v)\rangle + \frac{1}{2}\Delta_v f(x,v)\Big) \\ &=: (\mathscr{L}_a f)(x,v) + (\mathscr{L}_s f)(x,v), \quad f \in C_b^2(\mathbb{R}^d), \end{aligned} \tag{1.7}$$

where Δ_v means the Laplacian operator in the variable v. Under appropriate conditions imposed on the potential U, the kinetic Fokker–Planck equation corresponding to (1.4) with $\Phi(v) = \frac{1}{2}|v|^2$:

$$\partial_t h = \mathscr{L}^* h \tag{1.8}$$

is well posed, where $\mathscr{L}^*$ represents the $L^2(\mu)$-adjoint operator of $\mathscr{L}$. In [34], Villani initiated the reputable hypocoercivity approach, which has been applied successfully in coping with the exponential convergence of the solution h to (1.8) in the $H^1(\mu)$-sense, in the $L^2(\mu)$-sense, and in the relative entropy sense, respectively. In particular, in order to obtain the L^2-exponential convergence, an additional L^2-gradient estimate needs to be enforced; see, for example, [8, Remark 3.3]. Later, based on a crucial source of inspiration from [21], Dolbeault, Mouhot, and Schmeiser [16] established a new and simple framework to investigate directly the L^2-exponential convergence of the solution h to (1.8) by examining conveniently coercivity inequalities, an algebraic relation, and boundedness of auxiliary operators. In comparison with the hypocoercivity strategy in [34], the outstanding feature of the abstract setting advanced in [16] lies in its succinctness and directness, and, most importantly, bypassing an examination of the L^2-gradient estimate in short time. Nowadays, the general framework developed in [16] is termed as the DMS framework for hypocoercivity in literature. Subsequently, the DMS framework in the Fokker–Planck setting was extended further in [19] to study the long-time behavior of strongly continuous semigroups generated by Kolmogorov backward operators. As an important application, the L^2-exponential ergodicity of the degenerate spherical velocity Langevin equation was handled in [19]. Furthermore, we refer to [12] for the recent study on more refined explicit estimates of the exponentially decaying rate for underdamped Langevin dynamics. Meanwhile, the authors in [20] went a step further to generalize the general DMS framework and to tackle the L^2-algebraic ergodicity of (1.4). Additionally, Andrieu, Durmus, Nüsken and Roussel [2] and Andrieu, Dobson and Wang [1] formulated a symmetrization-antisymmetrization version of the DMS setup so that the geometric/subgeometric hypocoercivity for piecewise-deterministic Markov process Monte Carlo methods can be established. Also, we refer to [27] for explicit L^2-exponential convergence rates concerning a class of piecewise deterministic-Markov processes for sampling. Regardless of the abstract DMS framework in [16, 34] and the dual counterpart in [19, 20], the densely defined linear operator $\mathscr{L}$ involved, which generates a strongly continuous C_0-contractive semigroup, is assumed to be decomposed into two parts, where one part is symmetric and the other part is anti-symmetric. In (1.7), $\mathscr{L}_a$ is $L^2(\mu)$-antisymmetric and $\mathscr{L}_s$ is $L^2(\mu)$-symmetric so that the DMS setups in [19] and [20] are applicable to investigate the L^2-exponential ergodicity and the L^2-subexponential ergodicity of the Markov semigroup associated with (1.4), respectively.

1.2 Setting

As mentioned previously, in certain scenarios, the deterministic Hamiltonian systems are influenced by random fluctuations with discontinuous sample paths rather than continuous counterparts. In this context, the d-dimensional noise process $(Z_t)_{t\geq 0}$ can be modelled naturally by a pure jump Lévy process (for example, a symmetric α-stable process with $\alpha \in (1,2)$) so the formulation (1.4) needs to be modified correspondingly. More precisely, replacing the Brownian motion $(B_t)_{t\geq 0}$ by a symmetric α-stable process $(L_t)_{t\geq 0}$ prompts us to reformulate (1.4) as below:

$$\begin{cases} \mathrm{d}X_t = \nabla\Phi(V_t)\,\mathrm{d}t, \\ \mathrm{d}V_t = -\nabla U(X_t)\,\mathrm{d}t - \nabla\Phi(V_t)\,\mathrm{d}t + \mathrm{d}L_t. \end{cases} \tag{1.9}$$

Superficially, there are no essential distinctions between the SDE (1.4) and the SDE (1.9) by changing merely noise patterns. Whereas, plenty of intrinsic changes are to be encountered. First of all, the probability measure μ introduced in (1.5) is no longer an IPM of the Markov process $(X_t, V_t)_{t\geq 0}$ solving (1.9). Concerning SDEs with jumps (even for non-degenerating cases), the problem on addressing explicit expressions of IPMs is a tough task and is impossible for almost all of scenarios. This is the prime issue we must be confronted with when we explore the L^2-exponential ergodicity for stochastic Hamiltonian systems with Lévy noises. Whereas, it is still possible to figure out the closed form of IPMs once the jump diffusions under consideration enjoy special structures; see, for instance, [24, 32, 39] for related details. To make sure that μ introduced in (1.5) is still an IPM, we need to alter the drift term of (1.9) in a suitable manner. So far, there are several different ways to amend the drift term in order to achieve our purpose. One of the potential ways is that the drift term $\nabla\Phi$ in the position component is untouched while the drift part $-\nabla\Phi$ in the velocity component is substituted by the following one:

$$b_\Phi(v) := \mathrm{e}^{\Phi(v)}\nabla\big((-\Delta)^{\frac{\alpha}{2}-1}\mathrm{e}^{-\Phi(v)}\big), \quad v \in \mathbb{R}^d, \tag{1.10}$$

when $d > 2-\alpha$. Herein, $(-\Delta)^{\frac{\alpha}{2}-1}$ is the fractional Laplacian operator defined via the inverse of the Riesz potential (see e.g. [26, Definition 2.11]). Subsequently, (1.9) can be rewritten as

$$\begin{cases} \mathrm{d}X_t = \nabla\Phi(V_t)\mathrm{d}t, \\ \mathrm{d}V_t = (-\nabla U(X_t) + b_\Phi(V_t))\mathrm{d}t + \mathrm{d}L_t. \end{cases} \tag{1.11}$$

The detail that μ defined by (1.5) is an IPM of $(X_t, V_t)_{t\geq 0}$ determined by (1.11) will be elaborated in Lemma 3.1 below. In fact, given the local equilibrium $F(v) = \mathrm{e}^{-\Phi(v)}$, the friction force b_Φ defined by (1.10) is the solution to the fractional Fokker–Planck equation

$$(-\Delta)^{\frac{\alpha}{2}}F + \mathrm{div}_v(b_\Phi F) = 0.$$

See e.g. [9, p.1048] for related details. Note that, for the case $\alpha = 2$, it is easy to see that b_Φ defined by (1.10) goes back to $-\nabla\Phi(v)$. This evidently coincides with the counterpart in the Brownian motion setting. In addition, [32, 39] provided another alternative of the drift term b_Φ, where the i-th component $b_{\Phi,i}$ is given by

$$b_{\Phi,i}(v) = \mathrm{e}^{\Phi(v)}\mathcal{D}_{v_i}^{\alpha-2}\big(\mathrm{e}^{-\Phi(v)}\partial_i\Phi(v)\big), \tag{1.12}$$

where $\mathcal{D}$ means the fractional Riesz derivative defined by the aid of the Fourier transform and the inverse Fourier transform, and ∂_i stands for the partial derivative with respect to the i-th component v_i. In contrast to

b_Φ defined in (1.12), b_Φ introduced in (1.10) is much more explicit. Based on this point of view, in this work we are interested in the stochastic Hamiltonian system (1.11), in which b_Φ is defined in (1.10) instead of (1.12).

To proceed, we interpret more backgrounds on stochastic Hamiltonian systems driven by symmetric α-stable noise. The SDE (1.9), with the driven noise $(L_t)_{t\geq 0}$ being a symmetric α-stable process, is named as a fractional underdamped (or kinetic) Langevin dynamic in [32] and a fractional stochastic Hamiltonian Monte Carlo in [39]. Below, let us expound it in the spirit of [39]. In some scenario, the stochastic Hamiltonian Monte Carlo (HMC for short) based on (1.4) exhibits slow mixing during the sampling procedure, and is incapable to provide sufficiently large "jumps" to explore full parameter space efficiently; see e.g. the introductory and preliminary section in [39] for more interpretations. To tackle the aforementioned issues, a variant of (1.11) was initiated in [39] (see, in particular, (12) and (13) therein). In contrast to the HMC on account of (1.4), the experimental results showed that the resulting stochastic fractional HMC could sample the multi-modal and high-dimensional target distribution more efficiently and effectively; see e.g. [31, 32, 39] and the references therein for details. Additionally, in [39] the proposed stochastic fractional HMC was exploited to train deep neural networks with faster convergence speed. In addition to applications on machine learning and computational statistics, the stochastic Hamiltonian systems resembling (1.11) have wide applications in physics. In particular, they have been used to model the anomalous diffusion phenomenon; see e.g. [13, 18] and the monograph [30, Chapter 10] for related details.

The infinitesimal generator $\mathscr{L}$ of $(X_t, V_t)_{t\geq 0}$ solving (1.11) is given by

$$\begin{aligned}(\mathscr{L}f)(x,v) &= \big(\langle\nabla\Phi(v),\nabla_x f(x,v)\rangle - \langle\nabla U(x),\nabla_v f(x,v)\rangle\big) + \big(\langle b_\Phi(v),\nabla_v f(x,v)\rangle - (-\Delta_v)^{\frac{\alpha}{2}} f(x,v)\big)\\ &=: (\mathscr{L}_0 f)(x,v) + (\mathscr{L}_1 f(x,\cdot))(v), \quad f\in C_b^2(\mathbb{R}^{2d}),\end{aligned} \tag{1.13}$$

where for any $g\in C_b^2(\mathbb{R}^d)$,

$$(\mathscr{L}_1 g)(v) := \langle b_\Phi(v),\nabla g(v)\rangle - (-\Delta)^{\frac{\alpha}{2}} g(v). \tag{1.14}$$

By the chain rule, $\mathscr{L}_0$ is $L^2(\mu)$-antisymmetric while $\mathscr{L}_1$ is not $L^2(\mu)$-symmetric so the DMS framework [19] is not applicable to investigate the L^2-exponential ergodicity of (1.11). Therefore, another challenge concerning an establishment of the L^2-exponential ergodicity of (1.11) is attributed to the non-symmetric property of $\mathscr{L}_1$. To deal with the trouble brought on by the non-symmetric property of $\mathscr{L}_1$, we shall establish an improved version of the general DMS framework by following essentially the line of [19, 20]. It is of great importance that the densely defined linear operator involved need not to possess a symmetric part. Once the novel framework is available, as an important application, the L^2-exponential ergodicity of (1.11) can be addressed. The detailed expositions of the aforementioned tasks will be presented sequentially and systematically in the following sections.

At length, we want to stress that the L^2-analytical properties of fractional kinetic equations have received great interest recently, see e.g. [3, 9, 10]. In particular, a newly developed L^2-hypocoercivity approach has been proposed in [9] to establish a decay rate, which is compatible with the fractional diffusion limit for fractional kinetic equations without confinement. However, there are remarkable distinctness between the framework and the approach delivered respectively in [9] and the present paper. For example, the reference measure concerned with the L^2-hypocoercivity in [9] is L^2-$(\mathbb{R}^{2d};\mathrm{d}x\mathrm{d}v)$ accompanying with a proper unbounded weighted function, whilst the reference measure related to the L^2-exponential decay addressed in the present work is the IPM $\mu(\mathrm{d}x,\mathrm{d}v)$. Additionally, the approach in [9] is based on the fractional Nash-type inequality, whereas the cornerstone in our work consists in the Poincaré inequality established for non-local stable-like Dirichlet forms [15, 36, 37].

1.3 Main result

Before proceeding to state our main result, we present assumptions on the coefficients U and Φ in (1.11). Firstly, concerning the potential U, we assume that

(A_U) The term $U:\mathbb{R}^d\to\mathbb{R}_+ := [0,\infty)$ satisfies the following two assumptions:

($\mathrm{A}_{U,1}$) $U\in C^\infty(\mathbb{R}^d;\mathbb{R}_+)$ is a compact function (i.e., for any $r>0$, $\{x\in\mathbb{R}^d : U(x)\leq r\}$ has a compact closure) such that $\mathbb{R}^d\ni x\mapsto \mathrm{e}^{-U(x)}$ is integrable and $\mathbb{R}^d\ni x\mapsto |\nabla U(x)|$ is a compact function (i.e., for any

$r > 0$, $\{x \in \mathbb{R}^d : |\nabla U(x)| \le r\}$ has a compact closure); moreover, there exist constants $c_1, c_2 > 0$ such that for all $x \in \mathbb{R}^d$,

$$\|\nabla^2 U(x)\| \le c_1 |\nabla U(x)| + c_2, \tag{1.15}$$

where ∇^2 stands for the second-order gradient operator (i.e., the Hessian operator) and $\|\cdot\|$ denotes the operator norm.

($A_{U,2}$) We have $\liminf_{|x|\to\infty} \frac{\langle \nabla U(x), x\rangle}{|x|} > 0$.

With regarding to Φ, we suppose that

(A_Φ) The function $\Phi : \mathbb{R}^d \to \mathbb{R}_+$ fulfills the following assumptions:

($A_{\Phi,1}$) $\Phi \in C^3(\mathbb{R}^d; \mathbb{R}_+)$ is radial such that $\Phi(v) = \psi(|v|^2)$ for all $v \in \mathbb{R}^d$ and some $\psi \in C^3(\mathbb{R}_+; \mathbb{R}_+)$; moreover, $|v| \mapsto \Phi(|v|)$ is non-decreasing, and $\mathbb{R}^d \ni v \mapsto \mathrm{e}^{-\Phi(v)}$ is integrable.

($A_{\Phi,2}$) $\|\nabla\Phi\|_\infty + \|\nabla^3\Phi\|_\infty < \infty$ and

$$\sup_{v\in\mathbb{R}^d} (\|\nabla^2\Phi(v)\| \cdot |v|) < \infty, \quad \sup_{v\in\mathbb{R}^d} |\Phi(v) - \Phi(\tfrac{v}{2})| < \infty,$$

where ∇^3 indicates the third-order gradient operator; moreover, there exist constants $c^*, v^* > 0$ such that for all $v \in \mathbb{R}^d$ with $|v| \ge v^*$,

$$\sup_{u\in B_1(v)} \|\nabla^i \Psi(u)\| \le c^* \|\nabla^i \Psi(v)\|, \quad i = 1, 2, 3,$$

where $B_1(v)$ denotes the unite ball with center v and radius 1.

($A_{\Phi,3}$) The map

$$\mathbb{R}^d \ni v \mapsto \frac{\mathrm{e}^{\Phi(v)}}{(1 + |v|)^{2(d+\alpha)}}$$

is integrable.

($A_{\Phi,4}$) We have $\liminf_{|v|\to\infty} \frac{\mathrm{e}^{\Phi(v)}}{|v|^{d+\alpha}} > 0$.

Before we proceed, let us make some comments on Assumptions (A_U) and (A_Φ).

Remark 1.1. Assumption ($A_{U,1}$) is enforced primarily to guarantee that the Poisson equation $(I - \mathscr{L}_{\mathrm{OD}})f = h$ has a unique smooth classical solution $f \in C_b^\infty(\mathbb{R}^d)$ for given $h \in C_b^\infty(\mathbb{R}^d)$. Hereinabove, $\mathscr{L}_{\mathrm{OD}}$, defined in (3.12) below, is the infinitesimal generator of the overdamped Langevin SDE with the potential U. Regularity estimates on the solution to the previous Poisson equation play a vital role in the following analysis. As far as Assumption ($A_{U,2}$) is concerned, it is one of the sufficient conditions to ensure that the x-marginal of the IPM μ defined in (1.5) satisfies the Poincaré inequality (see (3.7) below); see, for instance, [4, Corollary 1.6] for further details.

The structure $\Phi(v) = \psi(|v|^2)$ for some $\psi \in C^3(\mathbb{R}_+; \mathbb{R}_+)$, besides the uniform boundedness of $\nabla\Phi$ and the integrability of the function $x \mapsto \mathrm{e}^{-U(x)}$, ensures that the sufficient criteria (H_1) and (H_2) (see Section 2 below for details) in the general DMS framework (i.e., Theorem 2.1 below) are valid. Under ($A_{\Phi,1}$) and ($A_{\Phi,2}$), it holds that $\lim_{|v|\to\infty} |\nabla \mathrm{e}^{-\Phi(v)}| = 0$, which enables us to establish a crucial link between the operators $\pi\mathscr{L}_0^2\pi$ and $\mathscr{L}_{\mathrm{OD}}$ (see Lemma 3.5 below). This transfers the boundedness of one part of the auxiliary operator into corresponding estimates of the solution to the Poisson equation (see Lemma 3.6). Furthermore, the uniform boundedness and the integrable conditions involved in ($A_{\Phi,2}$) and ($A_{\Phi,3}$) also yields the boundedness of the other part of the auxiliary operator (see Proposition 3.10 below). ($A_{\Phi,4}$) provides a sufficiency so that the v-marginal of the IPM μ in (1.5) satisfies the Poincaré inequality (see (3.8) below), where the corresponding energy is a non-local stable-like Dirichlet form.

The main result in the present paper is presented as below.

Theorem 1.2. *Assume $d > 2 - \alpha$, and suppose further that both* (A_U) *and* (A_Φ) *are satisfied. Then the process $(X_t, V_t)_{t\ge0}$ solving* (1.11) *is L^2-exponentially ergodic, i.e., there exist constants $c, \lambda > 0$ such that for all $f \in L^2(\mu)$ and $t > 0$,*

$$\mathrm{Var}_\mu(P_t f) \le c\,\mathrm{e}^{-\lambda t}\mathrm{Var}_\mu(f), \tag{1.16}$$

where $(P_t)_{t\ge0}$ is the Markov semigroup generated by $(X_t, V_t)_{t\ge0}$ and μ, defined in (1.5)*, is an IPM of $(P_t)_{t\ge0}$.*

As a direct consequence of Theorem 1.2, we have the following statement.

Corollary 1.3. *Assume $d > 2 - \alpha$ and (A_U), and suppose for some $\beta \in [\alpha, 2\alpha)$,*

$$\Phi(v) = \frac{1}{2}(d+\beta)\log(1+|v|^2), \quad x \in \mathbb{R}^d,$$

Then the assertion (1.16) *holds true, i.e., the process $(X_t, V_t)_{t\geq 0}$ solving* (1.11) *is L^2-exponentially ergodic.*

Remark 1.4. Roughly speaking, the process $(X_t, V_t)_{t\geq 0}$ solving (1.4) is L^2-exponentially ergodic provided that the probability measures $\mu_1(\mathrm{d}x) := \frac{1}{C_U}\mathrm{e}^{-U(x)}\,\mathrm{d}x$ and $\mu_2(\mathrm{d}v) := \frac{1}{C_\Phi}\mathrm{e}^{-\Phi(v)}\,\mathrm{d}v$ satisfy respectively the Poincaré inequalities; see e.g. [16, 20, 34]. Hence, in this sense, Assumption (A_U) is reasonable since $(\mathrm{A}_{U,2})$ is a (mild) sufficient condition to ensure that μ_1 fulfils the Poincaré inequality. On the other hand, $(\mathrm{A}_{\Phi,4})$ is a sufficiency making sure μ_2 satisfies the Poincaré inequality as well. However, as stated previously, there are essential distinctness between (1.4) and (1.11). In particular, from the viewpoint of infinitesimal generators, the generator (see (1.13)) corresponding to (1.11) cannot be written into a proper form as that for (1.4), where the associated infinitesimal generator is equal to the $L^2(\mu)$-antisymmetric part plus $L^2(\mu)$-symmetric part. Thus, to apply efficiently the dual version of the DMS framework developed here for the system (1.11), we need to quantify the L^2-estimate on $\mathscr{L}_1^* B_{c^*}^* \pi$; see Lemmas 3.8 and 3.9 for more details. In this sense, the additional Assumptions $(\mathrm{A}_{\Phi,2})$ and $(\mathrm{A}_{\Phi,3})$ are necessary. This in turn requires that $\beta < 2\alpha$ in Corollary 1.3, which leads to an immediate consequence that our main result Theorem 1.2 does not work when μ_2 is of (sub)-exponential decay. We shall emphasize that, for the explicit example provided in Corollary 1.3, $\beta \geq \alpha$ is the optimal condition so that μ_2 satisfies the Poincaré inequality (see e.g. [36]); while $\beta < 2\alpha$ is also sharp to guarantee the L^2-boundedness of the operator $\mathscr{L}_1^* B_{c^*}^* \pi$ via the approach adopted in the present work; see Remarks 3.11 and 3.12 for further comments. Nevertheless, Corollary 1.3 with $\beta = \alpha$ is applicable to fractional underdamped Langevin dynamics with the α-stable kinetic energy, which was studied in [32, Theorem 3]; see also [32, Sections 3.3 and 3.4] for the corresponding Euler discretization and weak convergence analysis. For sure, besides the explicit example of $\Phi(v)$ given in Corollary 1.3, we can also choose its lower-order perturbation as another candidate, e.g.,

$$\Phi(v) = \frac{1}{2}(d+\beta)\log(1+|v|^2) - \theta\log\log(\mathrm{e}+|v|^2) + c_0$$

with $\beta \in [\alpha, 2\alpha)$, $\theta \geq 0$ and $c_0 \geq 0$ such that $\Phi(v) \geq 0$ for all $v \in \mathbb{R}^d$. We also want to mention that such kind conditions are imposed commonly in investigating the analytic properties of fractional Laplacian operator; see, for example, [3, 9, 10] for the fractional hypocoercivity of kinetic equations.

The remainder content of this work is organized as follows. In Section 2, we establish a general DMS framework, where one part of the densely defined linear operator involved is antisymmetric while the other part need not to be symmetric. As an application, we apply the DMS framework developed to complete the proof of Theorem 1.2. This will be addressed in Section 3. Since the proof of Theorem 1.2 is a little bit lengthy, a series of lemmas and propositions are prepared separately in Section 3 so that the paper is much more readable.

2 A general DMS framework

To encompass the non-local kinetic Fokker–Planck operator $\mathscr{L}$ defined in (1.13), in this section we aim to develop a general DMS framework. For this purpose, some warm-up work needs to be carried out in advance. Let $(\mathscr{L}, \mathcal{D}(\mathscr{L}))$ be a densely defined linear operator generating a strongly continuous contraction semigroup $(P_t)_{t\geq 0}$ on a separable Hilbert space $(H, \langle\cdot,\cdot\rangle_H, \|\cdot\|_H)$. Assume that $\mathfrak{D}$ is a core of $(\mathscr{L}, \mathcal{D}(\mathscr{L}))$, and that $\mathscr{L}$ can be written in the following form:

$$\mathscr{L} = \mathscr{L}_0 + \mathscr{L}_1 \quad \text{on } \mathfrak{D},$$

where the linear operator $(\mathscr{L}_0, \mathfrak{D})$ is antisymmetric in H. Since $(\mathscr{L}_0, \mathfrak{D})$ is a densely defined antisymmetric operator on H, it follows that $(\mathscr{L}_0, \mathfrak{D})$ is a closable operator (see e.g. [2, Lemma 26] or [29, Theorem 5.1.5, p. 194]) with the closure $(\mathscr{L}_0, \mathcal{D}(\mathscr{L}_0))$. On the other hand, since the semigroup $(P_t)_{t\geq 0}$ is contractive, the generator $(\mathscr{L}, \mathcal{D}(\mathscr{L}))$ is negative definite on H (i.e., $\langle \mathscr{L}f, f\rangle_H \leq 0$ for all $f \in \mathfrak{D}$). Hence, the antisymmetric property of $(\mathscr{L}_0, \mathfrak{D})$ yields that $\langle \mathscr{L}_1 f, f\rangle_H \leq 0$ for all $f \in \mathfrak{D}$.

Let H_0 be a closed subspace of H so H can be formulated as a direct sum of H_0 and its orthogonal complement $H_0^\perp$ (see e.g. see [25, Theorem 3.3-4, p.146]). Thus, the orthogonal projection operator $\pi : H \to H_0$ is well defined.

Below, we shall assume that:

(H_1) $\mathfrak{D} \subset \mathcal{D}(\mathscr{L}) \cap \mathcal{D}(\mathscr{L}^*)$ and $H_0 \subset \{f \in \mathcal{D}(\mathscr{L}_1) : \mathscr{L}_1 f = 0\} \cap \{f \in \mathcal{D}(\mathscr{L}_1^*) : \mathscr{L}_1^* f = 0\}$, where $\mathscr{L}^*$ and $\mathscr{L}_1^*$ stand for the adjoint operators of $\mathscr{L}$ and $\mathscr{L}_1$ on H, respectively,

(H_2) $\pi\mathfrak{D} \subset \mathcal{D}(\mathscr{L}_0)$ and $\pi\mathscr{L}_0\pi f = 0$ for all $f \in \mathfrak{D}$,

(H_3) there exist constants $\alpha_1, \alpha_2 > 0$ such that

$$\|(I-\pi)f\|_H^2 \le \alpha_1 \langle -\mathscr{L}_1 f, f\rangle_H, \quad f \in \mathfrak{D}, \tag{2.1}$$

and

$$\|\pi f\|_H^2 \le \alpha_2 \|\mathscr{L}_0 \pi f\|_H^2, \quad f \in \mathcal{D}(\mathscr{L}_0\pi). \tag{2.2}$$

From (H_1) and (H_2), it is ready to see that $H_0 \subset \mathcal{D}(\mathscr{L}_1)$ and $\pi\mathfrak{D} \subset \mathcal{D}(\mathscr{L}_0)$. Thus, for all $u \in \mathfrak{D}$, $\pi u \in \mathcal{D}(\mathscr{L}) = \mathcal{D}(\mathscr{L}_0) \cap \mathcal{D}(\mathscr{L}_1)$, and so $(I-\pi)u \in \mathcal{D}(\mathscr{L})$. Consequently, the mapping $\mathfrak{D} \ni u \mapsto \mathscr{L}(I-\pi)u$ is well defined (see (H_4) below).

Due to the fact that $(\mathscr{L}_0, \mathcal{D}(\mathscr{L}_0))$ is a densely defined antisymmetric operator, in addition to $\pi\mathcal{D}(\mathscr{L}_0) \subset \mathcal{D}(\mathscr{L}_0)$ (since $\pi\mathfrak{D} \subset \mathcal{D}(\mathscr{L}_0)$ by invoking (H_2) and $(\mathscr{L}_0, \mathcal{D}(\mathscr{L}_0))$ is a closure of $(\mathscr{L}_0, \mathfrak{D})$), $(\mathscr{L}_0\pi, \mathcal{D}(\mathscr{L}_0))$ is a closable operator (see e.g. [2, Lemma 26]) with the closure $(\mathscr{L}_0\pi, \mathcal{D}(\mathscr{L}_0\pi))$. Because of $(\mathscr{L}_0\pi)^* = \pi\mathscr{L}_0^* = -\pi\mathscr{L}_0$ on $\mathfrak{D}$, $(\mathscr{L}_0\pi)^*$ is a densely defined linear operator. This, along with $\mathscr{L}_0\pi = ((\mathscr{L}_0\pi)^*)^*$ on $\mathfrak{D}$ and [29, Theorem 5.1.5, p. 194], implies that $(\mathscr{L}_0\pi, \mathcal{D}(\mathscr{L}_0\pi))$ is a densely defined closed operator. Next, define $G = (\mathscr{L}_0\pi)^*\mathscr{L}_0\pi$. Then $(G, \mathcal{D}(G))$ is self-adjoint and $\mathcal{D}(G)$ is a core of $\mathscr{L}_0\pi$; and moreover, for $\lambda > 0$, $\lambda I + G$ is bijective from $\mathcal{D}(G)$ to H and the inverse operator $(\lambda I + G)^{-1}$ is a self-adjoint operator with $\|(\lambda I + G)^{-1}\| \le \frac{1}{\lambda}$ (see, instance, [29, Theorem 5.1.9 (i) and (ii), p. 195]), where $\|\cdot\|$ stipulates the operator norm. Accordingly, the operator

$$B_\lambda := (\lambda I + G)^{-1}(\mathscr{L}_0\pi)^*, \quad \mathcal{D}(B_\lambda) := \mathcal{D}((\mathscr{L}_0\pi)^*) = \mathcal{D}(\mathscr{L}_0) \tag{2.3}$$

is well defined. Recalling from [29, Theorem 5.1.5, p. 194] again that $(\mathscr{L}_0\pi)^*$ is a densely defined closed operator, we deduce from [29, Theorem 5.1.9 (iii), p. 195] that

$$\overline{B}_\lambda = (\mathscr{L}_0\pi)^*(\lambda I + G)^{-1}, \quad \mathcal{D}(\overline{B}_\lambda) = H \quad \text{and} \quad \|\overline{B}_\lambda\| \le \lambda^{-\frac{1}{2}}$$

in which $(\overline{B}_\lambda, \mathcal{D}(\overline{B}_\lambda))$ is the closure of $(B_\lambda, \mathcal{D}(B_\lambda))$. Combining the expression of $\overline{B}_\lambda$ given above with the fact that π is a projection operator (so $\pi^* = \pi$ and $\pi^2 = \pi$) on H yields $\pi\overline{B}_\lambda = \overline{B}_\lambda$ right now. For further discussions and more detailed properties on the operator B_λ, one can consult e.g. [20, Section 2] or [2, Appendix B].

On the basis of the preliminary materials concerned with the linear operator B_λ, we further suppose that

(H_4) $\mathfrak{D} \subset \mathcal{D}(G)$, and there exists a constant $\alpha_3 := \alpha_3(\lambda) > 0$ such that for all $f \in \mathfrak{D}$,

$$|\langle B_\lambda \mathscr{L}(I-\pi)f, f\rangle_H| \le \alpha_3 \|\pi f\|_H \|(I-\pi)f\|_H.$$

The main result in this section is stated precisely as follows.

Theorem 2.1. *Assume that* (H_1)–(H_4) *hold true. Then, for all* $f \in H$, $t > 0$ *and* $\lambda > 0$,

$$\|P_t f\|_H^2 \le C \mathrm{e}^{-\lambda_0 t}\|f\|_H^2, \tag{2.4}$$

where for $\alpha_1, \alpha_2 > 0$ *and* $\alpha_3 > 0$ *given in* (H_3) *and* (H_4), *respectively,*

$$\lambda_0 := \frac{\varepsilon_0}{2(1+\varepsilon_0\lambda^{-\frac{1}{2}})(1+\lambda\alpha_2)} \quad \text{and} \quad C := \frac{\lambda^{\frac{1}{2}}+\varepsilon_0}{\lambda^{\frac{1}{2}}-\varepsilon_0} \tag{2.5}$$

with

$$\varepsilon_0 := \frac{1}{2}\Big(\lambda^{\frac{1}{2}} \wedge \frac{1+\lambda\alpha_2}{\alpha_1} \wedge \frac{1}{\alpha_1(1+\lambda\alpha_2)\alpha_3^2}\Big).$$

Proof. Since $(\mathscr{L}, \mathcal{D}(\mathscr{L}))$ is a densely defined linear operator and the associated semigroup $(P_t)_{t\ge0}$ is contractive, it is sufficient to show that (2.4) holds true for any $f \in \mathcal{D}(\mathscr{L})$. Below, we define the modified entropy functional (see e.g. [16, p. 3812])

$$I_\lambda(f) = \frac{1}{2}\|f\|_H^2 + \varepsilon_0\langle B_\lambda f, f\rangle_H, \quad f \in H,$$

where the linear operator B_λ was defined in (2.3) and ε_0 was given in (2.5). By taking advantage of [16, Lemma 1] or [20, (2.7) in Lemma 2.2], it follows from (H_2) that

$$\|B_\lambda u\|_H \le \frac{1}{2\lambda^{\frac12}}\|(I-\pi)u\|_H \le \frac{1}{2\lambda^{\frac12}}\|u\|_H, \quad u\in H.$$

This subsequently implies that

$$\frac12\Big(1-\frac{\varepsilon_0}{\lambda^{\frac12}}\Big)\|u\|_H^2 \le I_\lambda(u) \le \frac12\Big(1+\frac{\varepsilon_0}{\lambda^{\frac12}}\Big)\|u\|_H^2, \quad u\in H, \tag{2.6}$$

where $1-\varepsilon_0\lambda^{-\frac12}\ge\frac12$ by taking the definition of ε_0 into consideration. Recall the basic fact that $f_t := P_t f\in\mathcal{D}(\mathscr{L})$ for any $f\in\mathcal{D}(\mathscr{L})$. Once we can claim that for any $f\in\mathcal{D}(\mathscr{L})$ and $t>0$,

$$\frac{\mathrm{d}}{\mathrm{d}t}I_\lambda(f_t) \le -\frac{\varepsilon_0}{2(1+\varepsilon_0\lambda^{-\frac12})(1+\lambda\alpha_2)}I_\lambda(f_t). \tag{2.7}$$

Then (2.4) is available for any $f\in\mathcal{D}(\mathscr{L})$ by applying Gronwall's inequality and using (2.6). Therefore, to achieve the desired assertion, it remains to prove (2.7).

By invoking the fact that $\frac{\mathrm{d}}{\mathrm{d}t}f_t=\mathscr{L}f_t$ for all $f\in\mathcal{D}(\mathscr{L})$, we deduce that

$$\frac{\mathrm{d}}{\mathrm{d}t}I_\lambda(f_t) = \langle\mathscr{L}f_t,f_t\rangle_H + \varepsilon_0\big(\langle B_\lambda\mathscr{L}f_t,f_t\rangle_H + \langle B_\lambda f_t,\mathscr{L}f_t\rangle_H\big). \tag{2.8}$$

Since $\mathfrak{D}$ is a core of $(\mathscr{L},\mathcal{D}(\mathscr{L}))$, for each fixed $t\ge0$, there is a sequence $(f_t^n)_{n\ge1}\subset\mathfrak{D}$ satisfying

$$\lim_{n\to\infty}\big(\|f_t-f_t^n\|_H + \|\mathscr{L}f_t-\mathscr{L}f_t^n\|_H\big)=0. \tag{2.9}$$

By interpolating $f_t^n\in\mathfrak{D}$ into the right-hand side of (2.8), it is easy to see that

$$\frac{\mathrm{d}}{\mathrm{d}t}I_\lambda(f_t) = \langle\mathscr{L}f_t^n,f_t^n\rangle_H + \varepsilon_0\big(\langle B_\lambda f_t^n,\mathscr{L}f_t^n\rangle_H + \langle B_\lambda\mathscr{L}f_t^n,f_t^n\rangle_H\big) + R_t^{n,\lambda}, \tag{2.10}$$

where the remainder $R_t^{n,\lambda}$ is given as below:

$$\begin{aligned} R_t^{n,\lambda} &:= \langle\mathscr{L}(f_t-f_t^n),f_t\rangle_H + \langle\mathscr{L}f_t^n,f_t-f_t^n\rangle_H + \varepsilon_0\langle B_\lambda(f_t-f_t^n),\mathscr{L}f_t\rangle_H \\ &\quad + \varepsilon_0\langle B_\lambda f_t^n,\mathscr{L}(f_t-f_t^n)\rangle_H + \varepsilon_0\langle B_\lambda\mathscr{L}f_t^n,f_t-f_t^n\rangle_H + \varepsilon_0\langle B_\lambda\mathscr{L}(f_t-f_t^n),f_t\rangle_H. \end{aligned}$$

Owing to $\mathscr{L}_0^*=-\mathscr{L}_0$ (so $\langle\mathscr{L}_0u,u\rangle_H=0$ for $u\in\mathcal{D}(\mathscr{L}_0)$) and (2.1), we find easily that

$$\langle\mathscr{L}f_t^n,f_t^n\rangle_H = -\langle-\mathscr{L}_1f_t^n,f_t^n\rangle_H \le -\frac{1}{\alpha_1}\|(I-\pi)f_t^n\|_H^2. \tag{2.11}$$

Next, by using $\mathscr{L}_0^*=-\mathscr{L}_0$ again, in addition to $\pi B_\lambda=B_\lambda$ and $\mathscr{L}_1^*\pi u=0$ for all $u\in H$ due to (H_1), it follows that for all $u\in\mathfrak{D}$,

$$\langle B_\lambda u,\mathscr{L}u\rangle_H = \langle\mathscr{L}_0^*B_\lambda u,u\rangle_H + \langle\mathscr{L}_1^*\pi B_\lambda u,u\rangle_H = -\langle\mathscr{L}_0B_\lambda u,u\rangle_H.$$

This, together with $\|\mathscr{L}_0B_\lambda u\|_H\le\|(I-\pi)u\|_H$ for all $u\in\mathfrak{D}$ (see e.g. [16, Lemma 1] or [20, (2.8) in Lemma 2.2]), leads to

$$|\langle B_\lambda f_t^n,\mathscr{L}f_t^n\rangle_H| \le \|(I-\pi)f_t^n\|_H\|f_t^n\|_H. \tag{2.12}$$

From (H_1) and (H_2), one obviously has $\mathscr{L}_1\pi u=0$ and $\pi u\subset\mathcal{D}(\mathscr{L}_0)$ for all $u\in\mathfrak{D}$. Whereafter, we derive from (H_4) that for all $u\in\mathfrak{D}$,

$$\begin{aligned} \langle B_\lambda\mathscr{L}u,u\rangle_H &= \langle B_\lambda\mathscr{L}_0\pi u,u\rangle_H + \langle B_\lambda\mathscr{L}_1\pi u,u\rangle_H + \langle B_\lambda\mathscr{L}(I-\pi)u,u\rangle_H \\ &= \langle B_\lambda\mathscr{L}_0\pi u,u\rangle_H + \langle B_\lambda\mathscr{L}(I-\pi)u,u\rangle_H \\ &\le \langle B_\lambda\mathscr{L}_0\pi u,u\rangle_H + \alpha_3\|\pi u\|_H\|(I-\pi)u\|_H. \end{aligned}$$

As a result, we arrive at

$$\langle B_\lambda\mathscr{L}f_t^n,f_t^n\rangle_H \le \langle B_\lambda\mathscr{L}_0\pi f_t^n,f_t^n\rangle_H + \alpha_3\|\pi f_t^n\|_H\|(I-\pi)f_t^n\|_H. \tag{2.13}$$

On the other hand, applying [20, Lemma 2.3] with $A_0 = \mathscr{L}_0\pi$, $\alpha(r) \equiv \alpha_2$, $\Psi_0(r) \equiv 0$ and $\nu(\mathrm{d}s) = \mathrm{e}^{-\lambda s}\mathrm{d}s$, and taking advantage of (2.2), $\pi^2 = \pi$ and $\pi B_\lambda = B_\lambda$ yields that

$$\langle B_\lambda \mathscr{L}_0 \pi f_t^n, f_t^n \rangle_H = \langle B_\lambda \mathscr{L}_0 \pi(\pi f_t^n), \pi f_t^n \rangle_H \le -\frac{1}{1+\lambda\alpha_2}\|\pi f_t^n\|_H^2.$$

Thus, plugging this back into (2.13) gives us that

$$\langle B_\lambda \mathscr{L} f_t^n, f_t^n \rangle_H \le -\frac{1}{1+\lambda\alpha_2}\|\pi f_t^n\|_H^2 + \alpha_3\|\pi f_t^n\|_H\|(I-\pi)f_t^n\|_H. \tag{2.14}$$

Now, combining (2.11) with (2.12) and (2.14) enables us to obtain that

$$\frac{\mathrm{d}}{\mathrm{d}t}I_\lambda(f_t) \le -\frac{1}{\alpha_1}\|(I-\pi)f_t^n\|_H^2 - \frac{\varepsilon_0}{1+\lambda\alpha_2}\|\pi f_t^n\|_H^2 + \varepsilon_0\big(\|(I-\pi)f_t^n\|_H\|f_t^n\|_H + \alpha_3\|\pi f_t^n\|_H\|(I-\pi)f_t^n\|_H\big) + R_t^{n,\lambda}. \tag{2.15}$$

Since B_λ is a bounded linear operator with $\|B_\lambda\| \le \lambda^{-\frac{1}{2}}$, we deduce from (2.9) that $\lim_{n\to\infty} R_t^{n,\lambda} = 0$. Hence, (2.9), (2.10) and the inequality $2ab \le \delta a^2 + \frac{b^2}{\delta}$ for all $a, b \ge 0$ and $\delta > 0$ imply by sending $n \to \infty$ in (2.15) that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}I_\lambda(f_t) &\le -\frac{1}{\alpha_1}\|(I-\pi)f_t\|_H^2 - \frac{\varepsilon_0}{1+\lambda\alpha_2}\|\pi f_t\|_H^2 + \varepsilon_0\big(\|(I-\pi)f_t\|_H\|f_t\|_H + \alpha_3\|\pi f_t\|_H\|(I-\pi)f_t\|_H\big)\\ &\le -\frac{1}{2}\Big(\frac{1}{\alpha_1} - \varepsilon_0\alpha_3^2(1+\lambda\alpha_2)\Big)\|(I-\pi)f_t\|_H^2 - \frac{\varepsilon_0}{2(1+\lambda\alpha_2)}\|\pi f_t\|_H^2 + \frac{1}{2}\alpha_1\varepsilon_0^2\|f_t\|_H^2.\end{aligned} \tag{2.16}$$

By invoking the alternative of ε_0 introduced in (2.5), we infer that

$$\frac{1}{2\alpha_1} - \varepsilon_0\alpha_3^2(1+\lambda\alpha_2) \ge 0, \qquad \frac{\varepsilon_0}{2(1+\lambda\alpha_2)} \le \frac{1}{4\alpha_1} \quad \text{and} \quad -\frac{\varepsilon_0}{4(1+\lambda\alpha_2)} + \frac{1}{2}\alpha_1\varepsilon_0^2 \le 0.$$

Consequently, by leveraging the fact that $\|(I-\pi)f_t\|_H^2 + \|\pi f_t\|_H^2 = \|f_t\|_H^2$, the estimate (2.16) implies that

$$\frac{\mathrm{d}}{\mathrm{d}t}I_\lambda(f_t) \le -\frac{\varepsilon_0}{4(1+\lambda\alpha_2)}\|f_t\|_H^2.$$

Whence, (2.7) follows by taking (2.6) into consideration. The proof is therefore completed. □

Before ending this section, we make some remarks on the comparisons between Theorem 2.1 and the DMS framework in [16, 19, 20].

Remark 2.2. We make the following observations.

(i) In retrospect, the densely defined linear operator $\mathscr{L}$ considered in [16, 19] has to be decomposed into the symmetric part and the antisymmetric part. Nevertheless, the linear operator $\mathscr{L}$ we focus on in this paper has an antisymmetric part, whereas the remaining part needs not to be symmetric.

(ii) In [16, 19], (2.1) and (2.2) in Assumption (H_3) are called the microscopic coercivity and the macroscopic coercivity respectively, which are also referred to as Poincaré inequalities in [20]. Assumption (H_4) is concerned with the boundedness of auxiliary operators.

(iii) Obviously, (H_1) coincides with [20, (H1)] when $\mathscr{L}_1$ is self-adjoint. Assumption (H_4) with $\lambda = 1$ in the present paper is a little bit weaker than [16, Assumption (H4)] and [20, (H3)]. Moreover, the identity operator I involved in the operator B in [16, 20] has been replaced by the operator λI, which plays a tuneable role for our purpose. In addition, we want to demonstrate that parts of [20, (H4)] are unnecessary and that there is no a similar version of [20, (H4)] imposed in the present work. Indeed, to prove [20, Theorem 2.1], the authors first showed that for any $f \in \mathfrak{D}$,

$$\|P_t f\|_H^2 \le \xi(t)\big(\|f\|_H^2 + \Psi(f)\big), \quad t \ge 0, \tag{2.17}$$

where $\xi(t)$ is a decreasing function on $(0,\infty)$, and $\Psi : H \to [0,\infty]$ is a functional such that the set $\{f \in H : \Psi(f) < \infty\}$ is dense in H. Apparently, via the contractive property of $(P_t)_{t\ge0}$, it holds that for any $f \in \mathcal{D}(\mathscr{L})$ and $(f^n)_{n\ge1} \subset \mathfrak{D}$,

$$\begin{aligned}\|P_t f\|_H^2 &= \|P_t(f-f^n)\|_H^2 + 2\langle P_t(f-f^n), P_t f^n\rangle_H + \|P_t f^n\|_H^2\\ &\le \|f-f^n\|_H^2 + 2\|f-f^n\|_H\|f^n\|_H + \xi(t)\big(\|f^n\|_H^2 + \Psi(f^n)\big).\end{aligned}$$

Next, since $\mathfrak{D}$ is a core of $(\mathscr{L}, \mathcal{D}(\mathscr{L}))$, we can choose particularly a sequence $(f^n)_{n\geq 1} \subset \mathfrak{D}$ such that

$$\lim_{n\to\infty} \|f^n - f\|_H = \lim_{n\to\infty} \|\mathscr{L} f^n - \mathscr{L} f\|_H = 0.$$

This, besides an additional assumption $\limsup_{n\to\infty} \Psi(f^n) \leq \Psi(f)$, implies that (2.17) is still valid for any $f \in \mathcal{D}(\mathscr{L})$. Based on the previous analysis, the hypothesis $\limsup_{n\to\infty} \langle -\mathscr{L} f_n, f_n\rangle_H \leq \langle -\mathscr{L} f, f\rangle_H$, which was set in [20, (H4)], can be dropped. Similarly, during the course of the proof for Theorem 2.1, we adopt a different approximation strategy (see in particular (2.10)) to bypass Assumption [20, (H4)].

3 Proof of Theorem 1.2

With the preceding general framework at hand, in this section we aim at implementing the proof of Theorem 1.2. Since it is a little bit cumbersome to finish the proof of Theorem 1.2, we split the associated details and prepare respectively Propositions 3.2, 3.4 and 3.10 below so that the whole proof is much more readable. To this end, several auxiliary lemmas need to be provided simultaneously. We begin with the warm-up statement that the measure μ defined by (1.5) is indeed an IPM of the stochastic system (1.11) with b_Φ therein being given in (1.10).

Lemma 3.1. *Suppose that* $\iint_{\mathbb{R}^d\times\mathbb{R}^d} \mathrm{e}^{-(U(x)+\Phi(v))}\,\mathrm{d}x\,\mathrm{d}v < \infty$. *Then* μ *defined by* (1.5) *is an IPM of the stochastic system* (1.11).

Proof. To show that the probability measure μ defined by (1.5) is an IPM of the system (1.11), it is sufficient to verify

$$(\mathscr{L}^\dagger \mathrm{e}^{-(U(\cdot)+\Phi(\cdot))})(x, v) = 0. \tag{3.1}$$

Herein, $\mathscr{L}^\dagger$ stands for the $L^2(\mathrm{d}x, \mathrm{d}v)$-adjoint of the generator $\mathscr{L}$ associated with the system (1.11). According to (1.13), it is easy to see that

$$\begin{aligned}(\mathscr{L}^\dagger f)(x, v) &= \big(-\mathrm{div}_x(\nabla\Phi(v) f(x, v)) + \mathrm{div}_v(f(x, v)\nabla U(x))\big) + \big(-\mathrm{div}_v(f(x, v) b_\Phi(v)) - (-\Delta_v)^{\frac{\alpha}{2}} f(x, v)\big)\\ &=: (\mathscr{L}_0^\dagger f)(x, v) + (\mathscr{L}_1^\dagger f)(x, v),\end{aligned}$$

where div_x and div_v denote the divergence operators with respect to the x-variable and the v-variable, respectively.

Via the chain rule, we find that

$$\begin{aligned}(\mathscr{L}_0^\dagger \mathrm{e}^{-(U(\cdot)+\Phi(\cdot))})(x, v) &= -\mathrm{e}^{-\Phi(v)}\,\mathrm{div}_x(\mathrm{e}^{-U(x)}\nabla\Phi(v)) + \mathrm{e}^{-U(x)}\,\mathrm{div}_v(\mathrm{e}^{-\Phi(v)}\nabla U(x))\\ &= \mathrm{e}^{-(U(x)+\Phi(v))}\langle \nabla U(x), \nabla\Phi(v)\rangle - \mathrm{e}^{-(U(x)+\Phi(v))}\langle \nabla U(x), \nabla\Phi(v)\rangle = 0,\end{aligned}$$

and that

$$\begin{aligned}(\mathscr{L}_1^\dagger \mathrm{e}^{-(U(\cdot)+\Phi(\cdot))})(x, v) &= -\mathrm{e}^{-U(x)}\big(\mathrm{div}(\mathrm{e}^{-\Phi(v)} b_\Phi(v)) + (-\Delta)^{\frac{\alpha}{2}}\mathrm{e}^{-\Phi(v)}\big)\\ &= -\mathrm{e}^{-U(x)}\big(\mathrm{div}(\nabla((-\Delta)^{-(2-\alpha)/2}\mathrm{e}^{-\Phi(v)})) + (-\Delta)^{\frac{\alpha}{2}}\mathrm{e}^{-\Phi(v)}\big) = 0,\end{aligned}$$

where in the last equality we took the definition of b_Φ into account and used the basic fact that

$$-(-\Delta)^{\frac{\alpha}{2}}\mathrm{e}^{-\Phi(v)} = \mathrm{div}(\nabla((-\Delta)^{-\frac{2-\alpha}{2}}\mathrm{e}^{-\Phi(v)})). \tag{3.2}$$

Putting both equalities together, we conclude that the desired assertion (3.1) follows. □

In the following, from beginning to end, we assume that

$$C_U := \int_{\mathbb{R}^d} \mathrm{e}^{-U(x)}\,\mathrm{d}x \in (0, \infty), \quad C_\Phi := \int_{\mathbb{R}^d} \mathrm{e}^{-\Phi(v)}\,\mathrm{d}v \in (0, \infty),$$

and write $\mu = \mu_1 \times \mu_2$, where

$$\mu_1(\mathrm{d}x) := \frac{1}{C_U}\mathrm{e}^{-U(x)}\,\mathrm{d}x \quad \text{and} \quad \mu_2(\mathrm{d}v) := \frac{1}{C_\Phi}\mathrm{e}^{-\Phi(v)}\,\mathrm{d}v.$$

In order to apply Theorem 2.1 to the stochastic system (1.11) with the coefficient b_Φ being given in (1.10), the principal procedure is to confirm all Assumptions (H_1)-(H_4), step by step. For this purpose, one needs to specify explicitly the Hilbert space H, the closed subspace H_0, the core $\mathfrak{D}$ of $\mathscr{L}$ defined in (1.13), as well as the projection operator π. In detail, for the IPM μ given by (1.5), define

$$H = L_0^2(\mu) := \{f \in L^2(\mu) : \mu(f) = 0\},$$

which is a Hilbert space endowed with the scalar product $\langle f, g\rangle_2 := \mu(fg)$ and the induced norm $\|f\|_2 := \langle f, f\rangle_2^{\frac{1}{2}}$ for $f, g \in L_0^2(\mu)$. Define the mapping

$$(\pi f)(x) = \mu_2(f(x,\cdot)), \quad f \in L_0^2(\mu);$$

i.e., the velocity is drawn afresh from the marginal invariant distribution, while the position is left unchanged. A direct calculation shows that $\pi = \pi^*$ and $\pi^2 = \pi$, so $\pi : L_0^2(\mu) \to H_0$ is an orthogonal projector, where the subspace

$$H_0 := \{f \in L_0^2(\mu) : f(x, v) \text{ is independent of } v\}.$$

Let $C_b^\infty(\mathbb{R}^{2d})$ be the set of bounded functions on $\mathbb{R}^{2d}$ with bounded derivatives of any order. Set

$$C_{b,c}^\infty(\mathbb{R}^{2d}) := \{f \in C_b^\infty(\mathbb{R}^{2d}) : (\nabla_x f, \nabla_v f) \text{ has compact support}\}$$

and

$$\mathfrak{D} := L_0^2(\mu) \cap C_{b,c}^\infty(\mathbb{R}^{2d}) = \{f \in C_{b,c}^\infty(\mathbb{R}^{2d}) : \mu(f) = 0\},$$

which obviously is a core of $\mathscr{L}$.

In the following, the operators $\mathscr{L}$, $\mathscr{L}_0$ and $\mathscr{L}_1$ are given as in (1.13). Let $\mathscr{L}^*$, $\mathscr{L}_0^*$ and $\mathscr{L}_1^*$ be the respective $L^2(\mu)$-adjoint operators of $\mathscr{L}$, $\mathscr{L}_0$ and $\mathscr{L}_1$. Let $(\mathscr{L}, \mathcal{D}(\mathscr{L}))$, $(\mathscr{L}_0, \mathcal{D}(\mathscr{L}_0))$, $(\mathscr{L}_1, \mathcal{D}(\mathscr{L}_1))$ and $(\mathscr{L}_1^*, \mathcal{D}(\mathscr{L}_1^*))$ be the closures in $L_0^2(\mu)$ of $(\mathscr{L}, \mathfrak{D})$, $(\mathscr{L}_0, \mathfrak{D})$, $(\mathscr{L}_1, \mathfrak{D})$, and $(\mathscr{L}_1^*, \mathfrak{D})$, separately.

With the aid of all the previous preliminaries, we present the following several propositions to complete the proof of Theorem 1.2.

Proposition 3.2. *Suppose that* $\Phi(v) = \psi(|v|^2)$ *for some* $\psi \in C^2(\mathbb{R}_+; \mathbb{R}_+)$. *If* $C_U, C_\Phi \in (0, \infty)$ *and* $\mu_2(|\nabla\Phi|) < \infty$, *then Assumptions* ($\mathrm{H}_1$) *and* ($\mathrm{H}_2$) *hold true.*

Proof. We divide the proof into two parts.

(1) Examination of (H_1). By virtue of $C_U, C_\Phi \in (0, \infty)$, both μ_1 and μ_2 are probability measures so $\mu = \mu_1 \times \mu_2$ is also a probability measure. Recall that $\mathscr{L}_0^*$ and $\mathscr{L}_1^*$ are the $L^2(\mu)$-adjoint operators of $\mathscr{L}_0$ and $\mathscr{L}_1$, respectively. Then $\mathscr{L}^* = \mathscr{L}_0^* + \mathscr{L}_1^*$. Note that $\mathscr{L}_0^* = -\mathscr{L}_0$ so $\mathscr{L}_0$ is an $L^2(\mu)$-antisymmetric operator. By the integration by parts formula, it follows that for $f \in \mathfrak{D}$,

$$(\mathscr{L}_1^* f)(x, v) = -\mathrm{e}^{\Phi(v)} \operatorname{div}_v\big(f(x, v)\nabla\big((-\Delta)^{-\frac{2-\alpha}{2}} \mathrm{e}^{-\Phi(v)}\big)\big) - \mathrm{e}^{\Phi(v)}(-\Delta_v)^{\frac{\alpha}{2}}\big(f(x, v)\mathrm{e}^{-\Phi(v)}\big). \tag{3.3}$$

Obviously, $\mathfrak{D} \subset \mathcal{D}(\mathscr{L}) \cap \mathcal{D}(\mathscr{L}^*)$. Whence, to validate Assumption (H_1), it remains to show that for any $f \in L_0^2(\mu)$,

$$\pi f \in \mathcal{D}(\mathscr{L}_1) \cap \mathcal{D}(\mathscr{L}_1^*) \quad \text{and} \quad (\mathscr{L}_1 \pi f)(x, v) = (\mathscr{L}_1^* \pi f)(x, v) = 0. \tag{3.4}$$

In retrospect, $(\mathscr{L}_1, \mathcal{D}(\mathscr{L}_1))$ and $(\mathscr{L}_1^*, \mathcal{D}(\mathscr{L}_1^*))$ are closed operators. Then, according to the closed graph theorem (see [25, Theorem 4.13 (3), p. 293]), (3.4) follows as long as there exists a sequence $(g_n)_{n\geq1} \subset \mathfrak{D}$ satisfying

$$\lim_{n\to\infty} \|g_n - \pi f\|_2 = 0 \quad \text{and} \quad (\mathscr{L}_1 g_n)(x, v) = (\mathscr{L}_1^* g_n)(x, v) = 0 \quad \text{for all } n \geq 1. \tag{3.5}$$

Indeed, for any $f \in L_0^2(\mu)$ (so $\mu_1(\pi f) = 0$), there exists a sequence $(\widetilde{g}_n)_{n\geq1} \subset C_{b,c}^\infty(\mathbb{R}^d)$ such that $\mu_1(\widetilde{g}_n) = 0$ and $\lim_{n\to\infty} \mu_1(|\widetilde{g}_n - \pi f|^2) = 0$. For any $n \geq 1$, set $g_n(x, v) := \widetilde{g}_n(x)$, which is independent of the velocity component. It is easy to see that $(g_n)_{n\geq1} \subset C_{b,c}^\infty(\mathbb{R}^{2d})$, since $(\widetilde{g}_n)_{n\geq1} \subset \mathfrak{D}$ with $\mu_1(\widetilde{g}_n) = 0$ and $\mu = \mu_1 \times \mu_2$. On the other hand, by making use of $\lim_{n\to\infty} \mu_1(|\widetilde{g}_n - \pi f|^2) = 0$, taking the structure of $(g_n)_{n\geq1}$ into account, and noticing that $\mu = \mu_1 \times \mu_2$ again, one can easily see that $\lim_{n\to\infty} \|g_n - \pi f\|_2 = 0$ holds true. Furthermore, because the designed $(g_n)_{n\geq1}$ has nothing to do with the velocity component, it follows from the definitions of $\mathscr{L}_1$ and $\mathscr{L}_1^*$ that, $(\mathscr{L}_1 g_n)(x, v) = 0$ and

$$(\mathscr{L}_1^* g_n)(x, v) = -\mathrm{e}^{\Phi(v)}\widetilde{g}_n(x)\big(\operatorname{div}\big(\nabla\big((-\Delta)^{-\frac{2-\alpha}{2}} \mathrm{e}^{-\Phi(v)}\big)\big) + (-\Delta)^{\frac{\alpha}{2}}\big(\mathrm{e}^{-\Phi(v)}\big)\big) = 0,$$

where the second identity is due to (3.2). Consequently, the requirement (3.5) is verified.

(2) Examination of (H_2). It is obvious to see that $\pi\mathfrak{D} \subset \mathcal{D}(\mathscr{L}_0)$. Furthermore, in accordance with the definitions of $\mathscr{L}_0$ and π, for any $f \in \mathfrak{D}$,

$$(\pi\mathscr{L}_0\pi f)(x) = \int_{\mathbb{R}^d} (\mathscr{L}_0\pi f)(x,v)\,\mu_2(\mathrm{d}v) = \int_{\mathbb{R}^d} \langle \nabla\Phi(v), \nabla(\pi f)(x)\rangle\,\mu_2(\mathrm{d}v)$$
$$= 2\int_{\mathbb{R}^d} \psi'(|v|^2)\langle v, \nabla(\pi f)(x)\rangle\,\mu_2(\mathrm{d}v),$$

where the second identity is valid in view of $\Phi(v) = \psi(|v|^2)$. Then, making use of $\mu_2(|\nabla\Phi|) < \infty$ and the rotationally invariant property of the probability measure μ_2 yields $(\pi\mathscr{L}_0\pi f)(x) = 0$. Therefore, the confirmation of (H_2) is complete. □

Now we proceed to check Assumption (H_3). Before performing this task, we provide the explicit expression of the energy form corresponding to the symmetric operator $\mathscr{L}_1 + \mathscr{L}_1^*$, where the non-local operator $\mathscr{L}_1$ is defined in (1.14).

Lemma 3.3. *For any $f \in C^2_{b,c}(\mathbb{R}^d)$, it holds that*

$$-\mu_2\big(f(\mathscr{L}_1 + \mathscr{L}_1^*)f\big) = c_{d,\alpha}\mathscr{E}_{\alpha,\Phi}(f),$$

where

$$c_{d,\alpha} := \frac{2^\alpha\Gamma(\frac{d+\alpha}{2})}{\pi^{\frac{d}{2}}|\Gamma(-\frac{\alpha}{2})|} \quad \text{and} \quad \mathscr{E}_{\alpha,\Phi}(f) := \iint_{\mathbb{R}^d\times\mathbb{R}^d} \frac{(f(v)-f(\overline{v}))^2}{|v-\overline{v}|^{d+\alpha}}\,\mathrm{d}v\,\mu_2(\mathrm{d}\overline{v}). \tag{3.6}$$

Proof. By invoking the definitions of $\mathscr{L}_1$ and $\mathscr{L}_1^*$, and making use of the chain rule, it follows that for all $f \in C^2_{b,c}(\mathbb{R}^d)$,

$$\mu_2\big(f(\mathscr{L}_1 + \mathscr{L}_1^*)f\big) = \frac{1}{C_\Phi}\int_{\mathbb{R}^d} \langle f(v)\nabla((-\Delta)^{-\frac{2-\alpha}{2}}\mathrm{e}^{-\Phi(v)}), \nabla f(v)\rangle\,\mathrm{d}v - \frac{1}{C_\Phi}\int_{\mathbb{R}^d} f(v)\,\mathrm{div}\big(f(v)\nabla((-\Delta)^{-\frac{2-\alpha}{2}}\mathrm{e}^{-\Phi(v)})\big)\,\mathrm{d}v$$
$$- \frac{1}{C_\Phi}\int_{\mathbb{R}^d} f(v)\big(\mathrm{e}^{-\Phi(v)}(-\Delta)^{\frac{\alpha}{2}}f(v) + (-\Delta)^{\frac{\alpha}{2}}(f(v)\mathrm{e}^{-\Phi(v)})\big)\,\mathrm{d}v$$
$$= \frac{1}{C_\Phi}\int_{\mathbb{R}^d} f^2(v)(-\Delta)^{\frac{\alpha}{2}}\mathrm{e}^{-\Phi(v)}\,\mathrm{d}v - \frac{1}{C_\Phi}\int_{\mathbb{R}^d} f(v)\big(\mathrm{e}^{-\Phi(v)}(-\Delta)^{\frac{\alpha}{2}}f(v) + (-\Delta)^{\frac{\alpha}{2}}(f(v)\mathrm{e}^{-\Phi(v)})\big)\,\mathrm{d}v.$$

This, together with the two facts that

$$(-\Delta)^{\frac{\alpha}{2}}\mathrm{e}^{-\Phi(v)} = c_{d,\alpha}\,\mathrm{p.v.}\int_{\mathbb{R}^d} \frac{\mathrm{e}^{-\Phi(v)} - \mathrm{e}^{-\Phi(\overline{v})}}{|v-\overline{v}|^{d+\alpha}}\,\mathrm{d}\overline{v}$$

and

$$\mathrm{e}^{-\Phi(v)}(-\Delta)^{\frac{\alpha}{2}}f(v) + (-\Delta)^{\frac{\alpha}{2}}(f(v)\mathrm{e}^{-\Phi(v)}) = c_{d,\alpha}\mathrm{e}^{-\Phi(v)}\,\mathrm{p.v.}\int_{\mathbb{R}^d} \frac{f(v)-f(\overline{v})}{|v-\overline{v}|^{d+\alpha}}\,\mathrm{d}\overline{v} + c_{d,\alpha}\,\mathrm{p.v.}\int_{\mathbb{R}^d} \frac{f(v)\mathrm{e}^{-\Phi(v)} - f(\overline{v})\mathrm{e}^{-\Phi(\overline{v})}}{|v-\overline{v}|^{d+\alpha}}\,\mathrm{d}\overline{v},$$

where $c_{d,\alpha}$ was defined in (3.6), yields

$$\mu_2\big(f(\mathscr{L}_1 + \mathscr{L}_1^*)f\big) = -\frac{c_{d,\alpha}}{C_\Phi}\iint_{\mathbb{R}^d\times\mathbb{R}^d} \frac{f(v)(f(v)-f(\overline{v}))}{|v-\overline{v}|^{d+\alpha}}\,\mathrm{d}v\,\mathrm{e}^{-\Phi(\overline{v})}\,\mathrm{d}\overline{v} - \frac{c_{d,\alpha}}{C_\Phi}\iint_{\mathbb{R}^d\times\mathbb{R}^d} \frac{f(v)(f(v)-f(\overline{v}))}{|v-\overline{v}|^{d+\alpha}}\,\mathrm{e}^{-\Phi(v)}\,\mathrm{d}v\mathrm{d}\,\overline{v}.$$

Subsequently, by exchanging the variables v and $\overline{v}$ in the second integral above, we deduce that

$$\mu_2\big(f(\mathscr{L}_1 + \mathscr{L}_1^*)f\big) = -c_{d,\alpha}\iint_{\mathbb{R}^d\times\mathbb{R}^d} \frac{(f(v)-f(\overline{v}))^2}{|v-\overline{v}|^{d+\alpha}}\,\mathrm{d}v\,\mu_2(\mathrm{d}\overline{v}).$$

Therefore, the desired assertion is provable. □

With Lemma 3.3 hand, Assumption (H_3) is verifiable provided that both the marginal μ_1 and the marginal μ_2 fulfil the Poincaré inequalities. This statement is detailed in the following proposition.

Proposition 3.4. *Assume $C_U, C_\Phi \in (0,\infty)$ and $\Phi(v) = \psi(|v|^2)$ with $\mu_2(|\nabla\Phi|^2) \in (0,\infty)$ for some $\psi \in C^2(\mathbb{R}_+;\mathbb{R}_+)$. If μ_1 and μ_2 satisfy respectively the following Poincaré inequalities: for some constants c_1 and $c_2 > 0$,*

$$\mathrm{Var}_{\mu_1}(f) \le c_1\mu_1(|\nabla f|^2), \quad f \in C_b^2(\mathbb{R}^d) \tag{3.7}$$

and

$$\mathrm{Var}_{\mu_2}(f) \le c_2\mathscr{E}_{\alpha,\Phi}(f), \quad f \in C_b^2(\mathbb{R}^d), \tag{3.8}$$

where $\mathscr{E}_{\alpha,\Phi}$ was defined in (3.6)*, then Assumption* (H_3) *holds true.*

So far, there are plenty of sufficient conditions to verify the Poincaré inequality (3.7); see, for instance, [5, Theorem 4.6.2, p. 202] and [4, Theorem 1.4], which are concerned with Lyapunov's criterion. In particular, the generator under consideration therein is $\mathscr{L}_{\mathrm{OD}} := \Delta - \langle\nabla U, \nabla\rangle$. Explicit conditions on the potential term U are, e.g., that there exist constants $\alpha > 0$ and $R \ge 0$ such that $\langle x, \nabla U\rangle \ge \alpha|x|$ for all $|x| \ge R$ (see e.g. [4, Corollary 1.6]) or that U is a convex function (see e.g. [4, Corollary 1.9]). On the other hand, according to [36, Theorem 1.1 (1) and (2)] (see also [15, 37] for more details), the Poincaré inequality (3.8) is available as well in case of $\liminf_{|v|\to\infty} \frac{e^{\Phi(v)}}{|v|^{d+\alpha}} > 0$.

Proof of Proposition 3.4. Via the standard density argument, it is sufficient to show that (2.1) and (2.2) hold respectively for all $f \in \mathfrak{D}$. For any $f \in \mathfrak{D}$, it is easy to see that $\pi f \in C_{b,c}^\infty(\mathbb{R}^d)$. Let $\bar f_x(v) = f(x,v) - (\pi f)(x)$ for $(x,v) \in \mathbb{R}^{2d}$. It is ready to see that

$$\|(I-\pi)f\|_2^2 = \mu_1(\mu_2(|\bar f_\cdot|^2)).$$

Next, by virtue of the Poincaré inequality (3.8) and Lemma 3.3, as well as $\mu_2(\bar f_x) = 0$ for any $x \in \mathbb{R}^d$, we derive that for each fixed $x \in \mathbb{R}^d$,

$$\mu_2(|\bar f_x|^2) = \mathrm{Var}_{\mu_2}(\bar f_x) \le c_2\mathscr{E}_{\alpha,\Phi}(\bar f_x) \le -c_2c_{d,\alpha}^{-1}\mu_2(\langle(\mathscr{L}_1 + \mathscr{L}_1^*)\bar f_x, \bar f_x\rangle).$$

Then, integrating with respect to $\mu_1(\mathrm{d}x)$ on both sides and utilizing $\mu = \mu_1 \times \mu_2$ yields that

$$\|(I-\pi)f\|_2^2 = \mu_1(\mathrm{Var}_{\mu_2}(\bar f_\cdot)) \le -c_2c_{d,\alpha}^{-1}\mu(\langle(\mathscr{L}_1 + \mathscr{L}_1^*)(f-\pi f), f-\pi f\rangle).$$

This, together with the fact that $(\mathscr{L}_1\pi f)(x) = (\mathscr{L}_1^*\pi f)(x) = 0$, leads to

$$\|(I-\pi)f\|_2^2 = \mu_1(\mathrm{Var}_{\mu_2}(\bar f_\cdot)) \le -2c_2c_{d,\alpha}^{-1}\mu(\langle\mathscr{L}_1 f, f\rangle).$$

Hence, we conclude that (2.1) holds true with $\alpha_1 = 2c_2c_{d,\alpha}^{-1}$.

In the sequel, we still fix $f \in \mathfrak{D}$. According to the definition of $\mathscr{L}_0$ and the fact that πf is independent of the velocity variable, as well as that $\Phi(v) = \psi(|v|^2)$ and $\mu = \mu_1 \times \mu_2$,

$$\begin{aligned}\|\mathscr{L}_0\pi f\|_2^2 &= \iint_{\mathbb{R}^d\times\mathbb{R}^d} \langle\nabla\Phi(v), \nabla(\pi f)(x)\rangle^2\,\mu(\mathrm{d}x,\mathrm{d}v)\\ &= 4\sum_{i,j=1}^d \int_{\mathbb{R}^d} \partial_i(\pi f)(x)\partial_j(\pi f)(x)\,\mu_1(\mathrm{d}x)\int_{\mathbb{R}^d}\psi'(|v|^2)^2 v_iv_j\,\mu_2(\mathrm{d}v),\end{aligned}$$

where v_i means the i-th component of v and $\partial_i := \frac{\mathrm{d}}{\mathrm{d}x_i}$. In view of the radial property of $h(v) = h(|v|) := \psi'(|v|^2)$ and the assumption that $\mu_2(|\nabla\Phi|^2) < \infty$,

$$\int_{\mathbb{R}^d}\psi'(|v|^2)^2v_iv_j\,\mu_2(\mathrm{d}v) = 0, \quad i \ne j.$$

This, along with the symmetric property, further results in

$$\begin{aligned}\|\mathscr{L}_0\pi f\|_2^2 &= 4\sum_{i=1}^d\int_{\mathbb{R}^d}(\partial_i(\pi f)(x))^2\,\mu_1(\mathrm{d}x)\int_{\mathbb{R}^d}\psi'(|v|^2)^2v_i^2\,\mu_2(\mathrm{d}v)\\ &= \frac{1}{d}\mu_1(|\nabla(\pi f)|^2)\mu_2(|\nabla\Phi|^2).\end{aligned}$$

Then, by invoking the precondition $\mu_2(|\nabla\Phi|^2) \in (0,\infty)$, it follows from the Poincaré inequality (3.7) that for all $f \in \mathfrak{D}$,

$$\mathrm{Var}_{\mu_1}(\pi f) \le c_1\mu_1(|\nabla(\pi f)|^2) = \frac{4c_1d}{\mu_2(|\nabla\Phi|^2)}\|\mathscr{L}_0\pi f\|_2^2. \tag{3.9}$$

Furthermore, the fact that $\mu_1(\pi f) = \mu(f) = 0$ for $f \in \mathfrak{D}$ implies that for $f \in \mathfrak{D}$,

$$\mathrm{Var}_{\mu_1}(\pi f) = \mu_1((\pi f)^2) - \mu_1(\pi f)^2 = \mu_1((\pi f)^2) = \mu((\pi f)^2) \tag{3.10}$$

by noticing that πf is not related to the velocity component and combining $\mu = \mu_1 \times \mu_2$. As a consequence, (2.2) is verified by combining (3.10) with (3.9). □

Before starting to examine Assumption (H$_4$), some additional work needs to be implemented. The first one is to provide an explicit expression on the operator $\pi\mathscr{L}_0^2\pi$, which is involved in the auxiliary operator B_λ. To achieve this, we recall some facts arising from Assumption (A$_U$). For any $h \in C_b^\infty(\mathbb{R}^d)$, consider the Poisson equation

$$(I - \mathscr{L}_{\mathrm{OD}})f = h, \tag{3.11}$$

where

$$(\mathscr{L}_{\mathrm{OD}}f)(x) := \Delta f(x) - \langle \nabla U(x), \nabla f(x)\rangle, \quad f \in C_b^2(\mathbb{R}^d). \tag{3.12}$$

Under Assumption (A$_U$), in terms of [11, Proposition 4], (3.11) has a unique classical solution $f \in C_b^\infty(\mathbb{R}^d)$, which can be expressed explicitly via Green's formula as below:

$$f(x) = \int_0^\infty \mathrm{e}^{-t}\mathbb{E}h(X_t^x)\,\mathrm{d}t.$$

Herein, $(X_t^x)_{t\ge 0}$ is the solution to the overdamped Langevin dynamics

$$\mathrm{d}X_t^x = -\nabla U(X_t^x)\,\mathrm{d}t + \sqrt{2}\,\mathrm{d}B_t, \quad t > 0, \qquad X_0^x = x,$$

where $(B_t)_{t\ge 0}$ is a standard d-dimensional Brownian motion. Throughout the paper, to emphasize the dependence on h, we shall write the solution f_h in lieu of f. The regularity estimates (see e.g. [11, Lemma 2 and Proposition 5]) on the solution f_h to the Poisson equation (3.11) play a crucial role in the subsequent analysis.

Lemma 3.5. *Assume that $C_U, C_\Phi \in (0,\infty)$, and suppose that $\Phi(v) = \psi(|v|^2)$ for some $\psi \in C^2(\mathbb{R}_+;\mathbb{R}_+)$ satisfying $\mu_2(|\nabla\Phi|^2 + \|\nabla^2\Phi\|) < \infty$ and*

$$\lim_{|v|\to\infty} |\nabla \mathrm{e}^{-\Phi(v)}| = 0. \tag{3.13}$$

Then, for any $f \in C_b^2(\mathbb{R}^{2d})$,

$$(\pi\mathscr{L}_0^2\pi f)(x,v) = c^\star\big((\mathscr{L}_{\mathrm{OD}}\pi)f\big)(x,v). \tag{3.14}$$

Herein, the operator $\mathscr{L}_{\mathrm{OD}}$ was defined in (3.12)*, and*

$$c^\star := \frac{2\omega_d}{C_\Phi}\int_0^\infty u^{\frac{d}{2}}\psi'(u)^2\mathrm{e}^{-\psi(u)}\,\mathrm{d}u, \tag{3.15}$$

where ω_d denotes the volume of the unit ball in $\mathbb{R}^d$.

Proof. According to the definition of the operator $\mathscr{L}_0$, we have

$$(\pi\mathscr{L}_0^2\pi f)(x) = \int_{\mathbb{R}^d} \big(\langle\nabla\Phi(v), \nabla^2(\pi f)(x)\nabla\Phi(v)\rangle - \langle\nabla U(x), \nabla^2\Phi(v)\nabla(\pi f)(x)\rangle\big)\,\mu_2(\mathrm{d}v).$$

Note from $\Phi(v) = \psi(|v|^2)$ that

$$\nabla\Phi(v) = 2\psi'(|v|^2)v \quad \text{and} \quad \nabla^2\Phi(v) = 2\big(\psi'(|v|^2)\mathbb{1}_{d\times d} + 2\psi''(|v|^2)\,v\otimes v\big). \tag{3.16}$$

Thus, we deduce that

$$\begin{aligned}(\pi\mathscr{L}_0^2\pi f)(x) = {} & 4\sum_{i,j=1}^d \int_{\mathbb{R}^d} \psi'(|v|^2)^2 v_i v_j\,\mu_2(\mathrm{d}v)\,(\nabla^2(\pi f))_{ij}(x) - 4\sum_{i,j=1}^d\int_{\mathbb{R}^d}\psi''(|v|^2)v_iv_j\,\mu_2(\mathrm{d}v)\,(\nabla U)_i(x)(\nabla(\pi f))_j(x)\\ & - 2\int_{\mathbb{R}^d}\psi'(|v|^2)\,\mu_2(\mathrm{d}v)\,\langle\nabla U(x), \nabla(\pi f)(x)\rangle.\end{aligned}$$

Furthermore, taking the radial property of $h_1(v) = h_1(|v|) := \psi'(|v|^2)$ and $h_2(v) = h_2(|v|) := \psi''(|v|^2)$ into account, and utilizing the rotationally invariant property of the measure μ_2 as well as $\mu_2(|\nabla\Phi|^2 + \|\nabla^2\Phi\|) < \infty$ further yields that

$$\begin{aligned}(\pi\mathscr{L}_0^2\pi f)(x) &= 4\sum_{i=1}^d \int_{\mathbb{R}^d} \psi'(|v|^2)^2 v_i^2\,\mu_2(\mathrm{d}v)\,(\nabla^2(\pi f))_{ii}(x) - 4\sum_{i=1}^d \int_{\mathbb{R}^d} \psi''(|v|^2)v_i^2\,\mu_2(\mathrm{d}v)\,(\nabla U)_i(x)(\nabla(\pi f))_i(x)\\ &\quad - 2\int_{\mathbb{R}^d} \psi'(|v|^2)\,\mu_2(\mathrm{d}v)\,\langle\nabla U(x), \nabla(\pi f)(x)\rangle\\ &= \frac{4}{d}\int_{\mathbb{R}^d} \psi'(|v|^2)^2|v|^2\,\mu_2(\mathrm{d}v)\,\Delta(\pi f)(x) - 2\int_{\mathbb{R}^d}\Big(\frac{2}{d}\psi''(|v|^2)|v|^2 + \psi'(|v|^2)\Big)\mu_2(\mathrm{d}v)\langle\nabla U(x), \nabla(\pi f)(x)\rangle.\end{aligned}$$

Therefore, to achieve (3.14), it is sufficient to verify

$$2\int_{\mathbb{R}^d}\Big(\frac{2}{d}\psi''(|v|^2)|v|^2 + \psi'(|v|^2)\Big)\mu_2(\mathrm{d}v) = \frac{4}{d}\int_{\mathbb{R}^d}\psi'(|v|^2)^2|v|^2\,\mu_2(\mathrm{d}v) = c^\star < \infty, \tag{3.17}$$

where $c^\star > 0$ was introduced in (3.15).

By invoking Jacobi's transformation formula, we obtain from $\mu(|\nabla\Phi|^2) < \infty$ that

$$\begin{aligned}\frac{4}{d}\int_{\mathbb{R}^d}\psi'(|v|^2)^2|v|^2\,\mu_2(\mathrm{d}v) &= \frac{4\omega_d}{C_\Phi}\int_0^\infty r^{d+1}\psi'(r^2)^2\mathrm{e}^{-\psi(r^2)}\,\mathrm{d}r\\ &= \frac{2\omega_d}{C_\Phi}\int_0^\infty r^{d/2}\psi'(r)^2\mathrm{e}^{-\psi(r)}\,\mathrm{d}r < \infty.\end{aligned}$$

Hence, the second equality in (3.17) is verifiable. On the other hand, by the integration by parts formula, it follows from (3.13) that

$$\begin{aligned}4\int_{\mathbb{R}^d}\psi'(|v|^2)^2|v|^2\,\mu_2(\mathrm{d}v) &= -\frac{1}{C_\Phi}\int_{\mathbb{R}^d}\langle\nabla\psi(|v|^2), \nabla\mathrm{e}^{-\psi(|v|^2)}\rangle\,\mathrm{d}v\\ &= \frac{1}{C_\Phi}\int_{\mathbb{R}^d}\mathrm{e}^{-\psi(|v|^2)}\mathrm{trace}(\nabla^2\psi(|v|^2))\,\mathrm{d}v\\ &= \frac{2}{C_\Phi}\int_{\mathbb{R}^d}\mathrm{e}^{-\psi(|v|^2)}(d\psi'(|v|^2) + 2\psi''(|v|^2)|v|^2)\,\mathrm{d}v,\end{aligned}$$

where the last display holds true due to (3.16). Consequently, the first identity in (3.17) is available. Thus, the proof is complete. □

Lemma 3.6. *Assume that* (A_U) *and Assumptions in Lemma* 3.5 *hold, and suppose further that*

$$\mu_2(|\nabla\Phi|^4 + \|\nabla^2\Phi\|^2) < \infty. \tag{3.18}$$

Then there exists a constant $c > 0$ *such that for all* $f \in \mathfrak{D}$,

$$\|(B_{c^\star}\mathscr{L}_0(I-\pi))^*\pi f\|_2 \le c\,\|\pi f\|_2, \tag{3.19}$$

where $c^\star > 0$ *was defined in* (3.15).

Proof. According to the definition of B_λ and by virtue of the $L^2(\mu)$-antisymmetric property of $\mathscr{L}_0$, it follows readily from Lemma 3.5 that for all $f \in \mathfrak{D}$,

$$\begin{aligned}(B_{c^\star}\mathscr{L}_0(I-\pi))^*\pi f(x,v) &= -(I-\pi)\mathscr{L}_0^2\pi(c^\star I - \pi\mathscr{L}_0^2\pi)^{-1}\pi f(x,v)\\ &= -\frac{1}{c^\star}(I-\pi)\mathscr{L}_0^2\pi u(x,v) = -\frac{1}{c^\star}(I-\pi)\mathscr{L}_0^2 u(x,v),\end{aligned}\tag{3.20}$$

where $u(x,v) := (I-\mathscr{L}_{\mathrm{OD}}\pi)^{-1}\pi f(x,v)$. Note that $u(x,v)$ depends merely on the x-variable so we can write $u(x) = u(x,v)$ in the subsequent analysis. Under Assumption (A_U), one has $u \in C_b^\infty(\mathbb{R}^d)$ by taking advantage of [11, Proposition 4]. Via examining the procedure to derive (3.14), we find that

$$\int_{\mathbb{R}^d} \langle \nabla\Phi(v), \nabla^2 u(x)\nabla\Phi(v)\rangle\, \mu_2(\mathrm{d}v) = c^\star \Delta u(x) \tag{3.21}$$

and

$$\int_{\mathbb{R}^d} \langle \nabla U(x), \nabla^2\Phi(v)\nabla u(x)\rangle\, \mu_2(\mathrm{d}v) = c^\star \langle \nabla U(x), \nabla u(x)\rangle.$$

Then, along with the definition of $\mathscr{L}_0$, we infer from (3.20) that

$$\begin{aligned}(B_{c^\star}\mathscr{L}_0(I-\pi))^*\pi f(x,v) &= -\frac{1}{c^\star}\bigg(\langle \nabla\Phi(v), \nabla^2 u(x)\nabla\Phi(v)\rangle - \langle \nabla U(x), \nabla^2\Phi(v)\nabla u(x)\rangle \\ &\qquad - \int_{\mathbb{R}^d}\big(\langle \nabla\Phi(v), \nabla^2 u(x)\nabla\Phi(v)\rangle - \langle \nabla U(x), \nabla^2\Phi(v)\nabla u(x)\rangle\big)\,\mu_2(\mathrm{d}v)\bigg) \\ &= -\frac{1}{c^\star}\big(\langle \nabla\Phi(v), \nabla^2 u(x)\nabla\Phi(v)\rangle - c^\star\Delta u(x) + c^\star\langle \nabla U(x), \nabla u(x)\rangle - \langle \nabla U(x), \nabla^2\Phi(v)\nabla u(x)\rangle\big).\end{aligned}$$

Subsequently, in addition to (3.18) and the basic inequality $2ab \le a^2 + b^2$ for all $a, b \ge 0$, we deduce that for some constants $C_1, C_2 > 0$,

$$\begin{aligned}\|(B_{c^\star}\mathscr{L}_0(I-\pi))^*\pi f\|_2^2 &\le \frac{2}{(c^\star)^2}\mu_1(\varphi) + 4\Big(1 + \frac{1}{(c^\star)^2}\mu_2(\|\nabla^2\Phi\|^2)\Big)\mu_1(|\nabla U|^2|\nabla u|^2) \\ &\le C_1\mu_1(\varphi) + C_2\mu_1(|\nabla U|^2|\nabla u|^2),\end{aligned} \tag{3.22}$$

where for all $x \in \mathbb{R}^d$,

$$\varphi(x) := \int_{\mathbb{R}^d}\big(\langle \nabla\Phi(v), \nabla^2 u(x)\nabla\Phi(v)\rangle - c^\star\Delta u(x)\big)^2\,\mu_2(\mathrm{d}v).$$

Recall that $\Phi(v) = \psi(|v|^2)$ and $\mu_2(\mathrm{d}v) = \mathrm{e}^{-\Phi(v)}\,\mathrm{d}v$. Thus, (3.18) and (3.21) yield that

$$\begin{aligned}\varphi(x) &= \int_{\mathbb{R}^d}\langle \nabla\Phi(v), \nabla^2 u(x)\nabla\Phi(v)\rangle^2\,\mu_2(\mathrm{d}v) - (c^\star\Delta u(x))^2 \\ &= 4\sum_{i,j,k,\ell=1}^d \int_{\mathbb{R}^d}\psi'(|v|^2)^4 v_i v_j v_k v_\ell\,\mu_2(\mathrm{d}v)\,\partial_{ij}u(x)\partial_{k\ell}u(x) - (c^\star\Delta u(x))^2 \\ &= 4\bigg(\sum_{i,j=1}^d\int_{\mathbb{R}^d}\psi'(|v|^2)^4 v_i^2 v_j^2\,\mu_2(\mathrm{d}v)\big(\partial_{ii}u(x)\partial_{jj}u(x) + 2(\partial_{ij}u(x))^2\big) \\ &\qquad - 2\sum_{i=1}^d\int_{\mathbb{R}^d}\psi'(|v|^2)^4 v_i^4\,\mu_2(\mathrm{d}v)\,(\partial_{ii}u(x))^2\bigg) - (c^\star\Delta u(x))^2 \\ &\le 2\sum_{i,j=1}^d\int_{\mathbb{R}^d}\psi'(|v|^2)^4(v_i^4 + v_j^4)\,\mu_2(\mathrm{d}v)\big(\partial_{ii}u(x)\partial_{jj}u(x) + 2(\partial_{ij}u(x))^2\big),\end{aligned}$$

where $\partial_{ij} := \frac{\partial^2}{\partial x_i \partial x_j}$. The previous estimate, in addition to (3.18), implies that for some constant $C_3 > 0$,

$$\varphi(x) \le \frac{1}{4}\mu_2(|\nabla\Phi|^4)\big((\Delta u(x))^2 + 2\|\nabla^2 u(x)\|_{\mathrm{HS}}^2\big) \le C_3\|\nabla^2 u(x)\|^2,$$

where $\|\cdot\|_{\mathrm{HS}}$ means the Hilbert–Schmidt norm. Accordingly, by applying [11, Proposition 5], there exists a constant $C_4 > 0$ such that

$$\mu_1(\varphi) \le C_4\|\pi f\|_2^2. \tag{3.23}$$

To handle the term $\mu_1(|\nabla U|^2|\nabla u|^2)$, note from [11, p. 1027; line -7] that there exist constants $C_5, C_6 > 0$ satisfying

$$\mu_1(|\nabla U|^2|\nabla u|^2) \le C_5\mu_1(\|\nabla^2 u\|^2) + C_6\mu_1(|\nabla u|^2).$$

Thus, by invoking [11, Proposition 5] and taking advantage of [2, Corollary 30], there exists a constant $C_7 > 0$ such that

$$\mu_1(|\nabla U|^2|\nabla u|^2) \le C_7\|\pi f\|_2^2. \tag{3.24}$$

At length, the assertion (3.19) is reachable by plugging (3.23) and (3.24) back into (3.22). □

Remark 3.7. In order to guarantee that the Poisson equation (3.11) under consideration is well-posed and the associated solution enjoys desired regularity estimates, the compactness of U and $|\nabla U|$, along with the integrability of $\mathbb{R}^d \ni x \mapsto \mathrm{e}^{-U(x)}$ and the growth bound on $\nabla^2 U$, need to be in force; see e.g. [11, Proposition 4]. Accordingly, Assumption $(\mathrm{A}_{U,1})$ has been imposed. Apparently, to implement the proof of Lemma 3.6, we have made full use of plenty of important assertions in [11]. Nevertheless, all assertions applied in the aforementioned proof are taken directly from [11, Proposition 4] without any modification or improvement. So the price we need to pay is to necessitate $U \in C^\infty(\mathbb{R}^d, \mathbb{R}_+)$. By examining the proof of Lemma 3.6, the requirement $u \in C_b^\infty(\mathbb{R}^d)$ involved is unnecessary. In fact, $u \in C_b^2(\mathbb{R}^d)$ is adequate for our purpose. In this case, the smooth assumption on U can be weakened correspondingly. Whereas, this topic is outside the scope of the present article, and is left to purse in our future work.

In order to verify Assumption (H_4), we further need to prepare two additional lemmas.

Lemma 3.8. *Assume that* (A_U) *and assumptions in Lemma* 3.5 *hold. Then, for any* $f \in \mathfrak{D}$ *and* $d > 2-\alpha$,

$$\begin{aligned}\mathscr{L}_1^* B_{c^\star}^* \pi f(x,v) &= \frac{1}{c^\star} C_{d,2-\alpha}(d+\alpha-2)\mathrm{e}^{\Phi(v)}\,\mathrm{p.v.}\int_{\mathbb{R}^d} \frac{\langle v-\bar v, \nabla^2\Phi(v)\nabla u_{\pi f}(x)\rangle \mathrm{e}^{-\Phi(\bar v)}}{|v-\bar v|^{d+\alpha}}\,\mathrm{d}\bar v\\ &\quad - \frac{1}{c^\star} c_{d,\alpha}\mathrm{e}^{\Phi(v)}\,\mathrm{p.v.}\int_{\mathbb{R}^d} \frac{\langle \nabla\Phi(v)-\nabla\Phi(\bar v), \nabla u_{\pi f}(x)\rangle \mathrm{e}^{-\Phi(\bar v)}}{|v-\bar v|^{d+\alpha}}\,\mathrm{d}\bar v,\end{aligned} \tag{3.25}$$

where

$$C_{d,\alpha} := \frac{\Gamma(\frac{d-\alpha}{2})}{2^\alpha \pi^{\frac{d}{2}}\Gamma(\frac{\alpha}{2})}, \quad c_{d,\alpha} := \frac{2^\alpha\Gamma(\frac{d+\alpha}{2})}{\pi^{\frac{d}{2}}|\Gamma(-\frac{\alpha}{2})|}. \tag{3.26}$$

Proof. In view of $\pi^* = \pi$, $\mathscr{L}_0^* = -\mathscr{L}_0$ as well as $\pi B_{c^\star} = B_{c^\star}$, we deduce from Lemma 3.5 that for any $f, g \in \mathfrak{D}$,

$$\begin{aligned}\langle B_{c^\star} f, g\rangle_2 &= \langle \pi\mathscr{L}_0^* f, (c^\star I - \pi\mathscr{L}_0^2\pi)^{-1}\pi g\rangle_2\\ &= \frac{1}{c^\star}\langle \mathscr{L}_0^* f, (I-\mathscr{L}_{OD})^{-1}\pi g\rangle_2,\end{aligned} \tag{3.27}$$

where in the second identity we also used the fact that $(c^\star I - \pi\mathscr{L}_0^2\pi)^{-1}\pi g$ is independent of the v-variable. For $g \in \mathfrak{D}$, in terms of [11, Proposition 4], the Poisson equation

$$(I - \mathscr{L}_{\mathrm{OD}})u_{\pi g} = \pi g$$

has a unique classical solution $u_{\pi g} \in C_b^\infty(\mathbb{R}^d)$. Whereafter, we infer from (3.27) that

$$\langle B_{c^\star} f, g\rangle_2 = \frac{1}{c^\star}\langle f, \mathscr{L}_0 u_{\pi g}\rangle_2.$$

This, combining with the definition of $\mathscr{L}_0$, leads to

$$B_{c^\star}^* f(x,v) = \frac{1}{c^\star}\langle \nabla\Phi(v), \nabla u_{\pi f}(x)\rangle, \quad f \in \mathfrak{D}.$$

Next, employing (3.2) and (3.3), in addition to the chain rule, yields that

$$\begin{aligned}\mathscr{L}_1^* B_{c^\star}^* \pi f(x,v) &= -\frac{1}{c^\star}\mathrm{e}^{\Phi(v)}\,\mathrm{div}_v\big(\langle \nabla\Phi(v), \nabla u_{\pi f}(x)\rangle \nabla\big((-\Delta)^{\frac{\alpha}{2}-1}\mathrm{e}^{-\Phi(v)}\big)\big)\\ &\quad - \frac{1}{c^\star}\mathrm{e}^{\Phi(v)}(-\Delta_v)^{\frac{\alpha}{2}}\big(\langle \nabla\Phi(v), \nabla u_{\pi f}(x)\rangle \mathrm{e}^{-\Phi(v)}\big)\\ &= -\frac{1}{c^\star}\mathrm{e}^{\Phi(v)}\big\langle \nabla^2\Phi(v)\nabla u_{\pi f}(x), \nabla\big((-\Delta)^{\frac{\alpha}{2}-1}\mathrm{e}^{-\Phi(v)}\big)\big\rangle\\ &\quad + \frac{1}{c^\star}\mathrm{e}^{\Phi(v)}\big(\langle \nabla\Phi(v), \nabla u_{\pi f}(x)\rangle(-\Delta)^{\frac{\alpha}{2}}\mathrm{e}^{-\Phi(v)} - (-\Delta_v)^{\frac{\alpha}{2}}\big(\langle \nabla\Phi(v), \nabla u_{\pi f}(x)\rangle \mathrm{e}^{-\Phi(v)}\big)\big)\\ &=: \phi_1(x,v) + \phi_2(x,v).\end{aligned} \tag{3.28}$$

Owing to $d > 2-\alpha$, it follows from [26, Theorem 1.1] that

$$\phi_1(x,v) = \frac{1}{c^\star} C_{d,2-\alpha}(d+\alpha-2)\mathrm{e}^{\Phi(v)}\,\text{p.v.}\int_{\mathbb{R}^d} \frac{\langle v-\overline{v}, \nabla^2\Phi(v)\nabla u_{\pi f}(x)\rangle \mathrm{e}^{-\Phi(\overline{v})}}{|v-\overline{v}|^{d+\alpha}}\,\mathrm{d}\overline{v},$$

and that

$$\begin{aligned}\phi_2(x,v) &= \frac{1}{c^\star} c_{d,\alpha}\mathrm{e}^{\Phi(v)}\,\text{p.v.}\int_{\mathbb{R}^d} \frac{\langle \nabla\Phi(v), \nabla u_{\pi f}(x)\rangle(\mathrm{e}^{-\Phi(v)}-\mathrm{e}^{-\Phi(\overline{v})})}{|v-\overline{v}|^{d+\alpha}}\,\mathrm{d}\overline{v}\\ &\quad - \frac{1}{c^\star} c_{d,\alpha}\mathrm{e}^{\Phi(v)}\,\text{p.v.}\int_{\mathbb{R}^d} \frac{\langle \nabla\Phi(v), \nabla u_{\pi f}(x)\rangle\mathrm{e}^{-\Phi(v)} - \langle \nabla\Phi(\overline{v}), \nabla u_{\pi f}(x)\rangle\mathrm{e}^{-\Phi(\overline{v})}}{|v-\overline{v}|^{d+\alpha}}\,\mathrm{d}\overline{v}\\ &= -\frac{1}{c^\star} c_{d,\alpha}\mathrm{e}^{\Phi(v)}\,\text{p.v.}\int_{\mathbb{R}^d} \frac{\langle \nabla\Phi(v)-\nabla\Phi(\overline{v}), \nabla u_{\pi f}(x)\rangle\mathrm{e}^{-\Phi(\overline{v})}}{|v-\overline{v}|^{d+\alpha}}\,\mathrm{d}\overline{v},\end{aligned}$$

where $C_{d,2-\alpha}$ and $c_{d,\alpha}$ were defined in (3.26). Thus, substituting the explicit expressions above on ϕ_1 and ϕ_2 into (3.28) yields the desired assertion (3.25). □

Lemma 3.9. *Assume that $\Psi \in C^3(\mathbb{R}^d;\mathbb{R}_+)$ such that $C_\Psi := \int_{\mathbb{R}^d}\mathrm{e}^{-\Psi(u)}\,\mathrm{d}u < \infty$ and $\|\nabla\Psi\|_\infty < \infty$. For $\beta,\gamma > 0$ and $v,y\in\mathbb{R}^d$, set*

$$\Psi_{\beta,\gamma}(v,y) := \beta\mathrm{e}^{\Psi(v)}\,\text{p.v.}\int_{\mathbb{R}^d}\frac{\langle u, \nabla^2\Psi(v)y\rangle\mathrm{e}^{-\Psi(v-u)}}{|u|^{d+\alpha}}\,\mathrm{d}u - \gamma\mathrm{e}^{\Psi(v)}\,\text{p.v.}\int_{\mathbb{R}^d}\frac{\langle \nabla\Psi(v)-\nabla\Psi(v-u), y\rangle\mathrm{e}^{-\Psi(v-u)}}{|u|^{d+\alpha}}\,\mathrm{d}u. \quad (3.29)$$

Then $\Psi_{\beta,\gamma}(v,y)$ is well defined so that for any $v,y\in\mathbb{R}^d$, $|\Psi_{\beta,\gamma}(v,y)| \le \Theta(v)|y|$, where $v\mapsto\Theta(v)$ is positive and locally bounded on $\mathbb{R}^d$. Assume further that $\Psi\in C^3(\mathbb{R}^d;\mathbb{R}_+)$ is a radial function so that $|v|\mapsto\Psi(v)=\Psi(|v|)$ is non-decreasing and there exist constants $c^*, v^* > 0$ such that for all $v\in\mathbb{R}^d$ with $|v|\ge v^*$,

$$\sup_{u\in B_1(v)}\|\nabla^i\Psi(u)\| \le c^*\|\nabla^i\Psi(v)\|, \quad i=1,2,3. \quad (3.30)$$

Then there exists a constant $c_0 > 0$ such that for all $v\in\mathbb{R}^d$ with $|v|\ge v^*$ and $y\in\mathbb{R}^d$,

$$|\Psi_{\beta,\gamma}(v,y)| \le c_0\widetilde{\Psi}_{\beta,\gamma}(v)|y|, \quad (3.31)$$

where

$$\begin{aligned}\widetilde{\Psi}_{\beta,\gamma}(v) := {}&\|\nabla^2\Psi(v)\|\mathrm{e}^{\Psi(v)}\big[\mathrm{e}^{-\Psi(v)}|\nabla\Psi(v)| + |v|^{-(d+\alpha-1)} + (\mathbb{1}_{\{\alpha\in(1,2)\}} + \mathbb{1}_{\{\alpha=1\}}\log|v| + |v|^{1-\alpha}\mathbb{1}_{\{\alpha\in(0,1)\}})\mathrm{e}^{-\Psi(\frac{v}{2})}\big]\\ &+ |\nabla\Psi(v)|\,\|\nabla^2\Psi(v)\| + \|\nabla^3\Psi(v)\| + \mathrm{e}^{\Psi(v)}\big(|v|^{-(d+\alpha)} + \mathrm{e}^{-\Psi(\frac{v}{2})}\big).\end{aligned}$$

In particular, if

$$\sup_{v\in\mathbb{R}^d}(\|\nabla^2\Psi(v)\||v|) < \infty, \quad \|\nabla^3\Psi\|_\infty < \infty, \quad \sup_{v\in\mathbb{R}^d}|\Psi(v)-\Psi(\tfrac{v}{2})| < \infty, \quad (3.32)$$

and the integrability

$$\int_{\mathbb{R}^d}\frac{\mathrm{e}^{\Psi(v)}}{(1+|v|)^{2(d+\alpha)}}\,\mathrm{d}v < \infty \quad (3.33)$$

hold respectively, then $\mu_2(\Theta^2) < \infty$.

Proof. The proof is split into three parts.

(i) For $\Psi_{\beta,\gamma}$ introduced in (3.29), via change of variables, it holds that

$$\Psi_{\beta,\gamma}(v,y) = -\beta\mathrm{e}^{\Psi(v)}\,\text{p.v.}\int_{\mathbb{R}^d}\frac{\langle u, \nabla^2\Psi(v)y\rangle\mathrm{e}^{-\Psi(v+u)}}{|u|^{d+\alpha}}\,\mathrm{d}u - \gamma\mathrm{e}^{\Psi(v)}\,\text{p.v.}\int_{\mathbb{R}^d}\frac{\langle \nabla\Psi(v)-\nabla\Psi(v+u), y\rangle\mathrm{e}^{-\Psi(v+u)}}{|u|^{d+\alpha}}\,\mathrm{d}u.$$

Thus, we derive that

$$\begin{aligned}\Psi_{\beta,y}(v,y)&=\frac{1}{2}\beta e^{\Psi(v)}\int_{\mathbb{R}^d}\frac{\langle u,\nabla^2\Psi(v)y\rangle}{|u|^{d+\alpha}}(e^{-\Psi(v-u)}-e^{-\Psi(v+u)})\,du\\&\quad-\frac{1}{2}\gamma e^{\Psi(v)}\int_{\mathbb{R}^d}\frac{1}{|u|^{d+\alpha}}(\langle\nabla\Psi(v)-\nabla\Psi(v-u),y\rangle e^{-\Psi(v-u)}+\langle\nabla\Psi(v)-\nabla\Psi(v+u),y\rangle e^{-\Psi(v+u)})\,du\\&=:\frac{1}{2}\beta e^{\Psi(v)}I_1(v,y)-\frac{1}{2}\gamma e^{\Psi(v)}I_2(v,y).\end{aligned}\tag{3.34}$$

Notice that

$$\begin{aligned}|I_1(v,y)|&\le\|\nabla^2\Psi(v)\||y|\int_{\{|u|\ge1\}}\frac{1}{|u|^{d+\alpha-1}}(e^{-\Psi(v-u)}+e^{-\Psi(v+u)})\,du\\&\quad+\|\nabla^2\Psi(v)\||y|\int_{\{|u|\le1\}}\frac{1}{|u|^{d+\alpha-1}}|e^{-\Psi(v-u)}-e^{-\Psi(v+u)}|\,du\\&=:I_{11}(v,y)+I_{12}(v,y).\end{aligned}\tag{3.35}$$

It is easy to see that

$$I_{11}(v,y)\le 2\,C_\Psi\,\|\nabla^2\Psi(v)\|\,|y|$$

via change of variables, and that

$$I_{12}(v,y)\le 2\Big(\int_{\{|u|\le1\}}\frac{1}{|u|^{d+\alpha-2}}\,du\Big)\|\nabla^2\Psi(v)\|\|\nabla\Psi\|_\infty|y|$$

by the mean value theorem and $\Psi\ge0$.

On the other hand, it is obvious that

$$\begin{aligned}I_2(v,y)&=\int_{\{|u|\le1\}}\frac{1}{|u|^{d+\alpha}}\Psi_1(v,u,y)\,du+\int_{\{|u|\le1\}}\frac{1}{|u|^{d+\alpha}}\Psi_2(v,u,y)\,du\\&\quad+\int_{\{|u|\ge1\}}\frac{1}{|u|^{d+\alpha}}(\Psi_1(v,u,y)+\Psi_2(v,u,y))\,du,\end{aligned}\tag{3.36}$$

where

$$\begin{aligned}\Psi_1(v,u,y)&:=\langle\nabla\Psi(v)-\nabla\Psi(v-u),y\rangle(e^{-\Psi(v-u)}-e^{-\Psi(v+u)}),\\\Psi_2(v,u,y)&:=\langle2\nabla\Psi(v)-\nabla\Psi(v-u)-\nabla\Psi(v+u),y\rangle e^{-\Psi(v+u)}.\end{aligned}$$

Applying the mean value theorem, besides $\Psi\ge0$, yields that

$$\int_{\{|u|\le1\}}\frac{1}{|u|^{d+\alpha}}\Psi_1(v,u,y)\,du\le2\|\nabla\Psi\|_\infty^2\Big(\int_{\{|u|\le1\}}\frac{1}{|u|^{d+\alpha-2}}\,du\Big)|y|.$$

Furthermore, via change of variables again, it is ready to see that

$$\int_{\{|u|>1\}}\frac{1}{|u|^{d+\alpha}}\Psi_1(v,u,y)\,du\le2C_\Psi\|\nabla\Psi\|_\infty|y|.$$

With the aid of the facts that

$$\nabla\Psi(v+u)=\nabla\Psi(v)+\nabla^2\Psi(v)u+\int_0^1\int_0^s\nabla(\nabla^2\Psi(v+\theta u)u)u\,d\theta\,ds$$

and

$$\nabla\Psi(v-u)=\nabla\psi(v)-\nabla^2\Psi(v)u+\int_0^1\int_0^s\nabla(\nabla^2\Psi(v-\theta u)u)u\,d\theta\,ds,$$

we obtain that

$$2\nabla\Psi(v)-\nabla\Psi(v+u)-\nabla\Psi(v-u)=-\int_0^1\int_0^s(\nabla(\nabla^2\Psi(v+\theta u)u)u+\nabla(\nabla^2\Psi(v-\theta u)u)u)\,\mathrm{d}\theta\,\mathrm{d}s. \tag{3.37}$$

This obviously implies that

$$\int_{\{|u|\le1\}}\frac{1}{|u|^{d+\alpha}}\Psi_2(v,u,y)\,\mathrm{d}u\le 2\Big(\sup_{z\in B_1(v)}\|\nabla^3\Psi(z)\|\Big)\bigg(\int_{\{|u|\le1\}}\frac{1}{|u|^{d+\alpha-2}}\,\mathrm{d}u\bigg)|y|.$$

Furthermore, it is obvious that

$$\int_{\{|u|>1\}}\frac{1}{|u|^{d+\alpha}}\Psi_2(v,u,y)\,\mathrm{d}u\le 4C_\Psi\|\nabla\Psi\|_\infty|y|.$$

Putting all the estimates above into (3.34) and taking $\|\nabla\Psi\|_\infty<\infty$ into consideration, we conclude that $\Psi_{\beta,y}(v,y)$ is well defined so that $|\Psi_{\beta,y}(v,y)|\le\Theta(v)|y|$ for any $v,y\in\mathbb{R}^d$, where $v\mapsto\Theta(v)$ is positive and locally bounded on $\mathbb{R}^d$.

(ii) In this part, we shall fix $v\in\mathbb{R}^d$ with $|v|\ge v^*$ and $y\in\mathbb{R}^d$. Let $I_{11}(v,y)$ and $I_{12}(v,y)$ be those defined in (3.35). Taking the non-decreasing property of Ψ into account, we derive that

$$\begin{aligned}
I_{11}(v,y)&\le 2\|\nabla^2\Psi(v)\||y|\bigg(\int_{\{|u|\ge1\}\cap\{\langle u,v\rangle\ge0\}\cap\{|v-u|\le\frac12|v|\}}\frac{1}{|u|^{d+\alpha-1}}\mathrm{e}^{-\Psi(v-u)}\,\mathrm{d}u\\
&\quad+\int_{\{|u|\ge1\}\cap\{\langle u,v\rangle\le0\}\cap\{|v+u|\le\frac12|v|\}}\frac{1}{|u|^{d+\alpha-1}}\mathrm{e}^{-\Psi(v+u)}\,\mathrm{d}u\\
&\quad+\int_{\{|u|\ge1\}\cap\{\langle u,v\rangle\ge0\}\cap\{|v-u|\ge\frac12|v|\}\cap\{|u|\ge|v|\}}\frac{1}{|u|^{d+\alpha-1}}\mathrm{e}^{-\Psi(v-u)}\,\mathrm{d}u\\
&\quad+\int_{\{|u|\ge1\}\cap\{\langle u,v\rangle\le0\}\cap\{|v+u|\ge\frac12|v|\}\cap\{|u|\ge|v|\}}\frac{1}{|u|^{d+\alpha-1}}\mathrm{e}^{-\Psi(v+u)}\,\mathrm{d}u\\
&\quad+\int_{\{|u|\ge1\}\cap\{\langle u,v\rangle\ge0\}\cap\{|v-u|\ge\frac12|v|\}\cap\{|u|\le|v|\}}\frac{1}{|u|^{d+\alpha-1}}\mathrm{e}^{-\Psi(v-u)}\,\mathrm{d}u\\
&\quad+\int_{\{|u|\ge1\}\cap\{\langle u,v\rangle\le0\}\cap\{|v+u|\ge\frac12|v|\}\cap\{|u|\le|v|\}}\frac{1}{|u|^{d+\alpha-1}}\mathrm{e}^{-\Psi(v+u)}\,\mathrm{d}u\bigg)\\
&\le 8\|\nabla^2\Psi(v)\||y|\bigg[\frac{C_\Psi}{(\frac{|v|}{2})^{d+\alpha-1}}+\bigg(\mathbb{1}_{\{1<\alpha<2\}}\int_{\{|u|\ge1\}}\frac{1}{|u|^{d+\alpha-1}}\,\mathrm{d}u+\mathbb{1}_{\{0<\alpha\le1\}}\int_{\{1\le|u|\le|v|\}}\frac{1}{|u|^{d+\alpha-1}}\,\mathrm{d}u\bigg)\mathrm{e}^{-\Psi(\frac{v}{2})}\bigg],
\end{aligned}$$

where the last display is valid due to $|u|\ge|v|-|v\pm u|\ge\frac12|v|$ in case of $|v\pm u|\le\frac12|v|$.

In view of $\|\nabla\Psi\|_\infty<\infty$ and (3.30), we obviously have for some constant $C_0>0$,

$$\sup_{|v|\ge v^*}\sup_{u\in B_1(v)}\bigg(\mathrm{e}^{\Psi(v)-\Psi(u)}\frac{|\nabla\Psi(u)|}{|\nabla\Psi(v)|}\bigg)\le C_0.$$

With the help of this estimate, we arrive at

$$I_{12}(v,y)\le 2C_0\bigg(\int_{\{|u|\le1\}}\frac{1}{|u|^{d+\alpha-2}}\,\mathrm{d}u\bigg)\|\nabla^2\Psi(v)\||\nabla\Psi(v)|\mathrm{e}^{-\Psi(v)}|y|.$$

Subsequently, by invoking the estimates for $I_{11}(v,y)$ and $I_{12}(v,y)$, we infer that there exists a constant $C_1>0$ so that for all $v,y\in\mathbb{R}^d$ with $|v|\ge v^*$,

$$|I_1(v,y)|\le C_1\|\nabla^2\Psi(v)\|\big[|\nabla\Psi(v)|\mathrm{e}^{-\Psi(v)}+|v|^{-(d+\alpha-1)}+\big(\mathbb{1}_{\{\alpha\in(1,2)\}}+\mathbb{1}_{\{\alpha=1\}}\log|v|+|v|^{1-\alpha}\mathbb{1}_{\{\alpha\in(0,1)\}}\big)\mathrm{e}^{-\Psi(\frac{v}{2})}\big]|y|. \tag{3.38}$$

In the sequel, let $I_{21}(v,y)$, $I_{22}(v,y)$ and $I_{23}(v,y)$ be the those terms on the right-hand side of (3.36). Below, we aim to treat them, separately. By virtue of $\|\nabla\Psi\|_\infty < \infty$ and (3.30), we obtain that for some constant $C_2 > 0$,

$$\begin{aligned}|I_{21}(v,y)| &= \Big|\int_{\{|u|\le 1\}}\int_0^1\int_0^1 \frac{1}{|u|^{d+\alpha}}\langle\nabla^2\Psi(v-su)u,y\rangle\big(\mathrm{e}^{-\Psi(v-\theta u)}\langle\nabla\Psi(v-\theta u),u\rangle - \mathrm{e}^{-\Psi(v+\theta u)}\langle\nabla\Psi(v+\theta u),u\rangle\big)\,\mathrm{d}s\,\mathrm{d}\theta\,\mathrm{d}u\Big|\\ &\le C_2\Big(\int_{\{|u|\le 1\}}\frac{1}{|u|^{d+\alpha-2}}\,\mathrm{d}u\Big)\mathrm{e}^{-\Psi(v)}|\nabla\Psi(v)|\,\|\nabla^2\Psi(v)\|\,|y|.\end{aligned}$$

Next, by utilizing (3.37), $\|\nabla\Psi\|_\infty < \infty$ and (3.30), we reach that for some constant $C_3 > 0$,

$$\begin{aligned}|I_{22}(v,y)| &= \Big|\int_{\{|u|\le 1\}}\int_0^1\int_0^s \frac{1}{|u|^{d+\alpha}}\big(\langle\nabla(\nabla^2\Psi(v+\theta u)u+\nabla^2\Psi(v-\theta u)u)u,y\rangle\mathrm{e}^{-\Psi(v+u)}\,\mathrm{d}\theta\,\mathrm{d}s\,\mathrm{d}u\Big|\\ &\le C_3\Big(\int_{\{|u|\le 1\}}\frac{1}{|u|^{d+\alpha-2}}\,\mathrm{d}u\Big)\mathrm{e}^{-\Psi(v)}\|\nabla^3\Psi(v)\||y|.\end{aligned}$$

In accordance with the definitions of Ψ_1 and Ψ_2, along with the non-decreasing property of the mapping $|v| \mapsto \Psi(v) = \Psi(|v|)$, we deduce readily that

$$\begin{aligned}|I_{23}(v,y)| &\le 6\|\nabla\Psi\|_\infty|y|\int_{\{|u|\ge 1\}\cap\{\langle u,v\rangle\ge 0\}}\frac{1}{|u|^{d+\alpha}}\mathrm{e}^{-\Psi(v-u)}\,\mathrm{d}u + 6\|\nabla\Psi\|_\infty|y|\int_{\{|u|\ge 1\}\cap\{\langle u,v\rangle\le 0\}}\frac{1}{|u|^{d+\alpha}}\mathrm{e}^{-\Psi(v+u)}\,\mathrm{d}u\\ &\le 6\|\nabla\Psi\|_\infty|y|\Big(\int_{\{|u|\ge 1\}\cap\{|v-u|\le\frac12|v|\}}\frac{1}{|u|^{d+\alpha}}\mathrm{e}^{-\Psi(v-u)}\,\mathrm{d}u + \int_{\{|u|\ge 1\}\cap\{|v+u|\le\frac12|v|\}}\frac{1}{|u|^{d+\alpha}}\mathrm{e}^{-\Psi(v+u)}\,\mathrm{d}u\Big)\\ &\quad + 6\|\nabla\Psi\|_\infty|y|\Big(\int_{\{|u|\ge 1\}\cap\{|v-u|\ge\frac12|v|\}}\frac{1}{|u|^{d+\alpha}}\mathrm{e}^{-\Psi(v-u)}\,\mathrm{d}u + \int_{\{|u|\ge 1\}\cap\{|v+u|\ge\frac12|v|\}}\frac{1}{|u|^{d+\alpha}}\mathrm{e}^{-\Psi(v+u)}\,\mathrm{d}u\Big).\end{aligned}$$

Consequently, by utilizing the fact that $|u| \ge |v| - |v \pm u| \ge \frac12|v|$ as long as $|v \pm u| \le \frac12|v|$, we derive from $C_\Psi < \infty$ and $\|\nabla\Psi\|_\infty$ that there exists a constant $C_4 > 0$ such that

$$|I_{23}(v,y)| \le C_4\big(|v|^{-(d+\alpha)} + \mathrm{e}^{-\Psi(\frac{v}{2})}\big)|y|.$$

This, in addition to the estimates regarding $I_{21}(v,y)$ and $I_{22}(v,y)$, leads to

$$|I_2(v,y)| \le C_5\mathrm{e}^{-\Psi(v)}\big(|\nabla\Psi(v)|\,\|\nabla^2\Psi(v)\| + \|\nabla^3\Psi(v)\|\big)|y| + C_6\big(|v|^{-(d+\alpha)} + \mathrm{e}^{-\Psi(\frac{v}{2})}\big)|y| \tag{3.39}$$

for some constants $C_5, C_6 > 0$. Correspondingly, the desired assertion (3.31) follows by combining (3.38) with (3.39).

(iii) Since $\Theta(v)$ is locally bounded, the last assertion follows from (3.31) (in particular, the estimate for $\widetilde{\Psi}_{\beta,y}(v)$), (3.32) and (3.33). □

Proposition 3.10. *Assume that* (A_U) *and* (A_Φ) *are satisfied. Then Assumption* (H_4) *holds true.*

Proof. To validate Assumption (H_4), it is sufficient to prove respectively that there exist constants $c_1, c_2 > 0$ such that for all $f \in \mathfrak{D}$,

$$|\langle B_{c^*}\mathscr{L}_0(I-\pi)f,f\rangle_2| \le c_1\|\pi f\|_2\|(I-\pi)f\|_2 \tag{3.40}$$

and

$$|\langle B_{c^*}\mathscr{L}_1(I-\pi)f,f\rangle_2| \le c_2\|\pi f\|_2\|(I-\pi)f\|_2. \tag{3.41}$$

Due to $\pi B_{c^*} = B_{c^*}$ and $(I-\pi)^2 = I-\pi$, it is easy to see that

$$\begin{aligned}|\langle B_{c^*}\mathscr{L}_0(I-\pi)f,f\rangle_2| &= |\langle(I-\pi)f,(B_{c^*}\mathscr{L}_0(I-\pi))^*\pi f\rangle_2|\\ &\le \|(I-\pi)f\|_2\|(B_{c^*}\mathscr{L}_0(I-\pi))^*\pi f\|_2.\end{aligned}$$

Whence, (3.40) follows from Lemma 3.6. Again, by means of $\pi B_{c^\star} = B_{c^\star}$, we infer that for all $f \in \mathfrak{D}$,

$$\langle B_{c^\star}\mathscr{L}_1(I-\pi)f, f\rangle_2 = \langle (I-\pi)f, \mathscr{L}_1^* B_{c^\star}^* \pi f\rangle_2 \le \|(I-\pi)f\|_2 \|\mathscr{L}_1^* B_{c^\star}^* \pi f\|_2.$$

Next, taking Lemma 3.8 into consideration gives that

$$\begin{aligned}\mathscr{L}_1^* B_{c^\star}^* \pi f(x,v) = {} & \frac{1}{c^\star} C_{d,2-\alpha}(d+\alpha-2)\mathrm{e}^{\Phi(v)}\,\mathrm{p.v.}\int_{\mathbb{R}^d} \frac{\langle u, \nabla^2\Phi(v)\nabla u_{\pi f}(x)\rangle \mathrm{e}^{-\Phi(v-u)}}{|u|^{d+\alpha}}\,\mathrm{d}u\\ & - \frac{1}{c^\star} c_{d,\alpha}\mathrm{e}^{\Phi(v)}\,\mathrm{p.v.}\int_{\mathbb{R}^d} \frac{\langle \nabla\Phi(v) - \nabla\Phi(v-u), \nabla u_{\pi f}(x)\rangle \mathrm{e}^{-\Phi(v-u)}}{|u|^{d+\alpha}}\,\mathrm{d}u.\end{aligned}$$

Subsequently, applying Lemma 3.9 with $\beta = \frac{1}{c^\star} C_{d,2-\alpha}(d+\alpha-2)$, $\gamma = \frac{1}{c^\star} c_{d,\alpha}$ and $y = \nabla u_{\pi f}(x)$ yields that for some constant $c_0 > 0$,

$$|\mathscr{L}_1^* B_{c^\star}^* \pi f(x,v)| \le c_0 \Theta(v) |\nabla u_{\pi f}(x)|,$$

where $\Theta(v)$ is given in Lemma 3.9. As a consequence, (3.41) is examinable by combining $\mu(\Theta^2) < \infty$ with [2, Corollary 30]. □

Remark 3.11. To examine Assumption (H_4), we turn to verify the inequalities (3.40) and (3.41) based similarly on the approach in [16, 20]. As for (3.40), we make use of the regularity properties of the Poisson equation (3.11) associated with the Hamiltonian operator (i.e., the anti-symmetric part) $\mathscr{L}_0$ given in (1.13). The approach is inspired by the previous work in the Brownian motion setting; see, for example, [11]. However, to obtain (3.41) it is extremely non-trivial. Note that, since the operator $\mathscr{L}_1$ is not only non-local but also non-symmetric, the expression (3.25) for the dual operator $\mathscr{L}_1^* B_{c^\star}^*$ is a little bit complex, and, in particular, $\mathscr{L}_1^* B_{c^\star}^*$ does not enjoy the chain rule property. On the other hand, in order to establish the bound for $\|\mathscr{L}_1^* B_{c^\star}^*\|_{2\to 2}$, we need some explicit estimates as stated in Lemma 3.9, which in turn require the boundedness condition (3.32) and the integrability condition (3.33). Indeed, by checking carefully the proof of Lemma 3.9, one can see that almost all the calculations here are neat. This partly explains the reason that why we impose $\beta < 2\alpha$ in Corollary 1.3, and that the requirement $\beta < 2\alpha$ is sharp by our approach.

With the previous preparations at hand, we are in position to complete the proof of Theorem 1.2.

Proof of Theorem 1.2. Since $x \mapsto \mathrm{e}^{-U(x)}$ and $v \mapsto \mathrm{e}^{-\Phi(v)}$ are integrable, we have $C_U, C_\Phi \in (0,\infty)$. Due to the uniform boundedness of $\nabla\Phi$ and $\nabla^2\Phi$ (see ($A_{\Phi,2}$)), we have $\mu_2(|\nabla\Phi|^4 + \|\nabla^2\Phi\|^2) < \infty$ and $\mu_2(|\nabla\Phi|^2) \in (0,\infty)$. On the other hand, according to the function that Φ is radial and the boundedness of $|\nabla\Phi|$ as well as the fact that $\mathbb{R}^d \ni v \mapsto \mathrm{e}^{-\Phi(v)}$ is integrable, $\lim_{|v|\to\infty} |\nabla \mathrm{e}^{-\Phi(v)}| = 0$. In particular, (3.13) holds. Under Assumption ($A_{U,2}$), μ_1 satisfies the Poincaré inequality (3.7) (see e.g. [4, Corollary 1.6]); under Assumption ($A_{\Phi,4}$), μ_2 satisfies the Poincaré inequality (3.8) (see e.g. [36, Theorem 1.1]). Therefore, all the assumptions imposed on Propositions 3.2 and 3.4 are fulfilled. Furthermore, under Assumptions (A_U)–(A_Φ), all preconditions in Lemmas 3.5-3.8 are satisfied so that Proposition 3.10 is available. Thus, the proof of Theorem 1.2 is finished by applying Theorem 2.1 and taking Propositions 3.2, 3.4 and 3.10 into account. □

In the end, we finish the proof of Corollary 1.3.

Proof of Corollary 1.3. According to the expression of Φ, we have $\Phi \in C^3(\mathbb{R}^d; \mathbb{R}_+)$, $\psi(r) = \frac{1}{2}(d+\beta)\log(1+r)$, $r \ge 0$, which is non-decreasing, and $v \mapsto \mathrm{e}^{-\Phi(v)}$ is integrable. Hence, Assumption ($A_{\Phi,1}$) is verified. Again, in terms of the form of Φ, we find that

$$\Phi(v) - \Phi\Big(\frac{v}{2}\Big) = \frac{1}{2}(d+\beta)\Big(\log(1+|v|^2) - \log\Big(1 + \frac{|v|^2}{4}\Big)\Big) \le (d+\beta)\log 2.$$

Next, note that

$$\nabla\Phi(v) = \frac{(d+\beta)v}{1+|v|^2} \quad \text{and} \quad \nabla^2\Phi(v) = (d+\beta)\Big(\frac{1}{1+|v|^2} I_{d\times d} - \frac{2(v\otimes v)}{(1+|v|^2)^2}\Big). \tag{3.42}$$

Thus, all assumptions in ($A_{\Phi,2}$) are satisfied. Due to $\beta < 2\alpha$, we deduce that

$$\int_{\mathbb{R}^d} \frac{\mathrm{e}^{\Phi(v)}}{(1+|v|)^{2(d+\alpha)}}\,\mathrm{d}v < \infty.$$

This, together with (3.42), leads to the fulfillment of Assumption ($A_{\Phi,3}$). At last, thanks to $\beta \geq \alpha$, we conclude that Assumption ($A_{\Phi,4}$) is available. Therefore, the proof is complete. □

At the end of this paper, we present one more remark.

Remark 3.12. As mentioned in Remark 3.11, in order to verify the boundedness of $\mathscr{L}_1^* B_{c^*}^*$, the integrability condition ($A_{\Phi,3}$) is required, which subsequently implies that $\beta < 2\alpha$ concerning the explicit example provided in Corollary 1.3. Though such kind assumption looks a little bit strict, it is usually imposed to investigate the analytic properties of non-local operators related to fractional Laplacian operator; see e.g. [3, 9, 10]. On the other hand, the main purpose of this paper is to address the L^2-exponential ergodicity of the SDE (1.11), where the driven noise $(L_t)_{t\geq 0}$ is a symmetric α-stable process. Based on the recent developments on functional inequalities for symmetric Lévy-type Dirichlet forms (see e.g. [15, 36, 37]), one can extend the approach adopted in the present paper to a much more general setting, where the driven noise $(L_t)_{t\geq 0}$ in the SDE (1.11) is a general symmetric Lévy process. With this extension (in particular with more general choices of Lévy-driven noise) there will be more explicit examples beyond that in Corollary 1.3.

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