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Long-time behavior of one-dimensional McKean-Vlasov SDEs with common noise

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ABSTRACT

In this paper, by introducing a new type asymptotic coupling by reflection, we explore the long-time behavior of random probability measure flows associated with a large class of one-dimensional McKean-Vlasov SDEs with common noise. In contrast to the existing literature, the underlying drift term is much more general and of polynomial growth with respect to the state variable. In addition, the idiosyncratic noise is allowed to be of multiplicative type. Most importantly, the theory derived indicates that both the common noise and the idiosyncratic noise facilitate the exponential ergodicity of the associated measure-valued processes.

1. Introduction and main result

1.1. Background

Consider a mean-field game model with N particles evolving in $\mathbb{R}^d$:

$$\mathrm{d}X_t^i = b(X_t^i, \widehat{\mu}_t^N)\mathrm{d}t + \sigma(X_t^i, \widehat{\mu}_t^N)\mathrm{d}B_t^i, \quad i \in \mathbb{S}_N := \{1, \cdots, N\}, \tag{1.1}$$

where $\widehat{\mu}_t^N := \frac{1}{N}\sum_{j=1}^N \delta_{X_t^j}$ and $B^1 := (B_t^1)_{t\geq 0}, \cdots, B^N := (B_t^N)_{t\geq 0}$ are mutually independent d-dimensional Brownian motions on a complete filtered probability space. In (1.12), $(B^1, \cdots, B^N)$ is referred to as an idiosyncratic noise. Under appropriate assumptions (e.g., the distribution of the initial particles is exchangeable), the classical theory on propagation of chaos (see e.g. [43]) demonstrates that all individual particles become asymptotically independent when $N \to \infty$. So, the random probability measure $\widehat{\mu}_t^N$ con-

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verges to a deterministic distribution and the resulting state of a single particle is described by the following McKean-Vlasov SDE:

$$\mathrm{d}X_t = b(X_t, \mu_t)\mathrm{d}t + \sigma(X_t, \mu_t)\mathrm{d}B_t, \tag{1.2}$$

where $\mu_t := \mathscr{L}_{X_t}$ stands for the law of X_t, and $(B_t)_{t\geq 0}$ is a d-dimensional Brownian motion. Initially, the McKean-Vlasov SDE (1.2) was introduced to explore nonlinear Fokker-Planck equations (FPEs for brevity) based on Kac's foundations of kinetic theory [27]. So far, it has been applied widely in various fields (e.g., stochastic control, mean-field games, and mathematical finance) [6]. In the past few decades, as far as McKean-Vlasov SDEs are concerned, significant advancements have been made on behaviors in a finite-time horizon (e.g. strong/weak well-posedness [12,23,26,48] and numerical approximations [13,14]), and long-time asymptotics (e.g. ergodicity [5,16,32,46,47] and uniform-in-time propagation of chaos (PoC for abbreviation) [8,15,22,21,40]).

Nevertheless, in some circumstances, the individual particles involved in a mean-field game model are subject to not only idiosyncratic noises but also random shocks, which are common to all particles. On this occasion, the evolution of underlying particles cannot be modelled by (1.12) any more, and, in turn, is characterized by the following mean-field SDEs:

$$\mathrm{d}X_t^i = b(X_t^i, \widehat{\mu}_t^N)\mathrm{d}t + \sigma(X_t^i, \widehat{\mu}_t^N)\mathrm{d}B_t^i + \sigma_0(X_t^i, \widehat{\mu}_t^N)\mathrm{d}W_t \quad i \in \mathbb{S}_N, \tag{1.3}$$

where the quantities $(b, \sigma, \widehat{\mu}_t^N)$ and $(B^1, \cdots, B^N)$ are defined exactly as in (1.12), and $(W_t)_{t\geq 0}$ is a d-dimensional Brownian motion. In (1.3), $(B^1, \cdots, B^N)$ is also called an idiosyncratic noise (or individual noise) as in (1.12), and $(W_t)_{t\geq 0}$ is named as a common noise, which accounts for the common environment associated with all particles. In the aforementioned setting, all particles are not asymptotically independent any more and the random empirical measure no longer converges to a deterministic distribution as the particle number goes to infinity. Whereas, under suitable conditions, the phenomenon on conditional PoC (see e.g. [7, Theorem 2.12]) illustrates that all particles are asymptotically independent and the corresponding empirical distribution converges to the common conditional distribution of each particle conditioned on the σ-algebra generated by the common noise. Moreover, the subsequent limiting state of each particle can be governed by the McKean-Vlasov SDE with common noise:

$$\mathrm{d}X_t = b(X_t, \mu_t)\mathrm{d}t + \sigma(X_t, \mu_t)\mathrm{d}B_t + \sigma_0(X_t, \mu_t)\mathrm{d}W_t, \tag{1.4}$$

where $\mu_t := \mathscr{L}_{X_t|\mathscr{F}_t^W}$ (the conditional distribution given the σ-algebra $\mathscr{F}_t^W := \sigma\{W_s : s \leq t\}$); $(B_t)_{t\geq 0}$ and $(W_t)_{t\geq 0}$ are mutually independent d-dimensional Brownian motions. In literature, McKean-Vlasov SDEs with common noise are also termed as conditional McKean-Vlasov SDEs (see e.g. [7, Chapter 2]). So far, they have been applied considerably in stochastic optimal control and mean-field games [7,38], and inter-bank borrowing and lending systems [3,31], to name just a few. In fact, the conditional McKean-Vlasov SDE (1.4) arises from many practical applications as shown in e.g. [37,45]. In detail, in order to construct diffusion processes generated by second-order differentiable operators on the Wasserstein space, Wang [45] introduced an image dependent SDEs, which can indeed be reformulated as a special conditional McKean-Vlasov SDE (with $\sigma \equiv 0$ in (1.4)). Moreover, in [37] the authors explored a mean-field game problem with N players in a random environment, which is delineated by a continuous-time Markov chain in lieu of the usual diffusions. In particular, they confirmed that the associated mean-field limiting process solves a conditional McKean-Vlasov SDE, in which the Markov chain involved acts as a common noise.

In contrast to classical McKean-Vlasov SDEs, the research on McKean-Vlasov SDEs with common noise is not too rich. Yet, in the past few years, there are still some progresses on qualitative and quantitative analyses; see, for example, [4,24,41] on well-posedness, and [7,9,17,25,41] concerned with conditional PoC in finite time. According to [7, p. 110-112], the random distribution flow $(\mu_t)_{t>0}$ corresponding to (1.3) solves the nonlinear FPE:

$$\mathrm{d}\mu_t = \Big(- \mathrm{div}\big(b(\cdot,\mu_t)\mu_t\big) + \frac{1}{2}\mathrm{trace}\big(\nabla^2((\sigma\sigma^*)(\cdot,\mu_t)\mu_t)\big)\Big)\,\mathrm{d}t - \mathrm{div}\big((\sigma_0(\cdot,\mu_t)\mathrm{d}W_t)\mu_t\big), \tag{1.5}$$

which is understood in the weak sense. With regard to well-posedness of (1.5), we refer to e.g. [10,18, 33] and references within. Recently, via establishing the superposition principle, [30] built a one-to-one correspondence between the conditional McKean-Vlasov SDE (1.4) and the stochastic FPE (1.5). Moreover, the stochastic PDE (1.5) is also linked closely to the stochastic scalar conservation laws; see e.g. [10, Appendix] for further details. Additionally, we also would like to mention [11], where, concerning first-order scalar conservation laws with stochastic forcing, Freidlin–Wentzell-type large deviation principles were explored. Based on the viewpoints in [10,30], the research on the long-time behavior of the random distribution flow corresponding to (1.4) amounts to the investigation on long-term asymptotics of certain kinds of stochastic FPEs or stochastic scalar conservation laws.

No matter what the conditional McKean-Vlasov SDE (1.4) or the nonlinear FPE (1.5), most of the existing literature (mentioned above) focuses on finite-time behaviors (e.g. well-posedness and conditional PoC in finite time). Nevertheless, the asymptotic analysis in an infinite-time horizon is extremely rare. By comparing (1.2) with (1.4), one of remarkable distinctness between them lies in that the deterministic flow $(\mu_t)_{t>0}$ in (1.2) satisfies a deterministic nonlinear FPE whereas the random counterpart in (1.4) fulfils a stochastic nonlinear FPE. This essential discrepancy brings about major challenges to tackle the long-time behavior of the measure-valued process $(\mu_t)_{t>0}$ solving (1.5).

Concerning (1.4) with $\sigma = 0$, [45] treated exponential ergodicity of the Markov process $(X_t,\mu_t)_{t\geq 0}$ provided that the drift b is globally dissipative with respect to the state variable. Furthermore, as for a special form of (1.4) (or (1.5)), [36] handled the long-term asymptotics of the conditional McKean-Vlasov SDE on $\mathbb{R}$:

$$\mathrm{d}X_t = -\bigg(V'(X_t) + \int_{\mathbb{R}} W'(X_t-y)\mu_t(\mathrm{d}y)\bigg)\mathrm{d}t + \sigma\mathrm{d}B_t + \sigma_0\mathrm{d}W_t, \tag{1.6}$$

where $\mu_t := \mathscr{L}_{X_t|\mathscr{F}_t^0}$, $\sigma,\sigma_0\in\mathbb{R}$, and, $(B_t)_{t\geq 0}$ and $(W_t)_{t\geq 0}$ are independent 1-dimensional Brownian motions. By designing a reflection coupling, exponential ergodicity of the measure-valued process $(\mu_t)_{t>0}$ was investigated in [36] under L^1-Wasserstein distance. Herein, we would like to stress that V' and W' in (1.6) are set to be globally Lipschitz, and moreover that the initial distribution of X_0 is supposed to possess a finite fourth-order moment. In fact, it is quite natural to assume that the initial distribution has a finite first-order moment once the L^1-Wasserstein ergodicity of $(\mu_t)_{t>0}$ is discussed. Hence, the confinement on a finite fourth-order moment concerning the initial distribution is a little bit strict. As revealed in [36], the common noise is beneficial to ergodicity and restoration of uniqueness for invariant probability measures whenever the intensity of the idiosyncratic noise is small enough. In addition, in [36] the synchronous coupling was applied to the idiosyncratic noise so no contributions were made to investigate the ergodic behavior of $(\mu_t)_{t>0}$ even though the intensity of the idiosyncratic noise is big enough.

1.2. Main result

Inspired by the aforementioned literature, in the present work, we make an attempt to investigate ergodicity of the measure-valued process $(\mu_t)_{t>0}$ associated with the following conditional McKean-Vlasov SDE:

$$\mathrm{d}X_t = b(X_t,\mu_t)\,\mathrm{d}t + \sigma(X_t)\,\mathrm{d}B_t + \sigma_0\,\mathrm{d}W_t. \tag{1.7}$$

Herein,

$$b : \mathbb{R} \otimes \mathscr{P}(\mathbb{R}) \to \mathbb{R}, \quad \sigma : \mathbb{R} \to \mathbb{R}, \quad \sigma_0 \in \mathbb{R},$$

where $\mathscr{P}(\mathbb{R})$ is the family of probability measures on $\mathbb{R}$; $(B_t)_{t\geq 0}$ and $(W_t)_{t\geq 0}$ are mutually independent 1-dimensional Brownian motions, where the corresponding probability spaces will be specified explicitly later; $\mu_t := \mathscr{L}_{X_t|\mathscr{F}_t^W}$ is the regular conditional distribution of X_t given the σ-algebra $\mathscr{F}_t^W$. Throughout the paper, we assume that $(W_t)_{t\geq 0}$ is the solely common source of noise (that is, the initial value X_0 is excluded).

Regarding the L^1-Wasserstein ergodicity of the measure-valued Markov process $(\mu_t)_{t>0}$ corresponding to (1.7), we aim to

- allow the drift b to be much more general (rather than the mere convolution form) and of polynomial growth, and permit specifically the idiosyncratic noise to be of multiplicative type;
- permit the initial distribution to admit a finite first-order moment instead of a higher-order one;
- establish a novel asymptotic coupling by reflection, which is not only applied to the common noise part but also to the idiosyncratic noise, so that the idiosyncratic noise can also make contributions to the ergodic behavior of $(\mu_t)_{t>0}$.

The preceding highlights are the important source impelling us to carry out the present work and can also be regarded as the main contributions of the whole paper.

To proceed, we introduce the underlying probability space we are going to work on, and some notations involved in the subsequent analysis. Let $(\Omega^1, \mathscr{F}^1, (\mathscr{F}_t^1)_{t\geq 0}, \mathbb{P}^1)$ and $(\Omega^0, \mathscr{F}^0, (\mathscr{F}_t^0)_{t\geq 0}, \mathbb{P}^0)$ be complete filtered probability spaces, where 1-dimensional Brownian motions $(B_t)_{t\geq 0}$ and $(W_t)_{t\geq 0}$, given in (1.7), are supported respectively on. In the whole paper, we shall focus on the product probability space $(\Omega, \mathscr{F}, \mathbb{F}, \mathbb{P})$, where $\Omega := \Omega^0 \times \Omega^1$, $(\mathscr{F}, \mathbb{P})$ is the completion of $(\mathscr{F}^0 \otimes \mathscr{F}^1, \mathbb{P}^0 \otimes \mathbb{P}^1)$, and $\mathbb{F}$ is the complete and right-continuous augmentation of $(\mathscr{F}_t^0 \otimes \mathscr{F}_t^1)_{t\geq 0}$. Set for $p > 0$,

$$\mathscr{P}_p(\mathbb{R}^d) := \Big\{\mu \in \mathscr{P}(\mathbb{R}^d) : \mu(|\cdot|^p) := \int_{\mathbb{R}^d} |x|^p \mu(\mathrm{d}x) < \infty\Big\}.$$

Under the L^p-Wasserstein distance:

$$\mathbb{W}_p(\mu, \nu) = \inf_{\pi \in \mathscr{C}(\mu,\nu)} \left(\int_{\mathbb{R}^d \times \mathbb{R}^d} |x-y|^p \pi(\mathrm{d}x, \mathrm{d}y) \right)^{\frac{1}{1\vee p}}, \quad \mu, \nu \in \mathscr{P}_p(\mathbb{R}^d), \tag{1.8}$$

where $\mathscr{C}(\mu, \nu)$ denotes the set of couplings for μ and ν, $(\mathscr{P}_p(\mathbb{R}^d), \mathbb{W}_p)$ is a Polish space.

Below, we shall assume that

($\mathbf{H}_{b,1}$) *$b(\cdot, \delta_0)$ is continuous and locally bounded on $\mathbb{R}$, and there exist constants $\lambda_1, \lambda_2, \lambda_3 > 0$ and $\ell_0 \geq 1$ such that for all $x, y \in \mathbb{R}$ and $\mu, \nu \in \mathscr{P}_1(\mathbb{R})$,*

$$2(x-y)(b(x,\mu) - b(y,\mu)) \leq (\lambda_1 + \lambda_2)|x-y|^2 \mathbb{1}_{\{|x-y|\leq \ell_0\}} - \lambda_2 |x-y|^2, \tag{1.9}$$

and

$$|b(x,\mu) - b(x,\nu)| \leq \lambda_3 \mathbb{W}_1(\mu, \nu). \tag{1.10}$$

($\mathbf{H}_{b,2}$) *for any conditionally independent and identically distributed $(X_t^i)_{1\leq i\leq N}$ under the filtration $\mathscr{F}_t^W$, there exists a function $\varphi : [0, \infty) \to [0, \infty)$ satisfying $\lim_{r\to\infty} \varphi(r) = 0$ such that for any $N \geq 1$,*

$$\max_{1\le i\le N}\sup_{t\ge0}\mathbb{E}|b(X_t^i,\mu_t^i)-b(X_t^i,\widetilde{\mu}_t^{N,i})|\le\varphi(N), \tag{1.11}$$

where $\mu_t^i:=\mathscr{L}_{X_t^i|\mathscr{F}_t^W}$ *and* $\widetilde{\mu}_t^{N,i}:=\frac{1}{N-1}\sum_{j=1,j\neq i}^N\delta_{X_t^j}$.

($\mathbf{H}_\sigma$) *there exist constants* $L_\sigma,\kappa_{\sigma,1},\kappa_{\sigma,2}>0$ *such that for all* $x,y\in\mathbb{R}$,

$$|\sigma(x)-\sigma(y)|\le L_\sigma|x-y| \quad \text{and} \quad \kappa_{\sigma,1}\le\sigma(x)^2\le\kappa_{\sigma,2}.$$

In recent years, strong well-posedness of conditional McKean-Vlasov SDEs has been treated in various scenarios in case the drift and diffusion terms are continuous in the measure argument under the L^2-Wasserstein distance; see e.g. [7, Proposition 2.8] and [29, Theorem 2.1]. Under ($\mathbf{H}_{b,1}$) and ($\mathbf{H}_\sigma$), via the fixed point iteration method adopted in [29, Theorem 2.1], the SDE (1.7) is strongly well-posed even for the multidimensional setting (i.e., $d\ge2$), where the drift term involved is uniformly continuous under the L^1-Wasserstein distance.

Below, we make some comments concerned with Assumptions ($\mathbf{H}_{b,1}$), ($\mathbf{H}_{b,2}$) and ($\mathbf{H}_\sigma$).

Remark 1.1. (1.9) and (1.10) show respectively that b is dissipative in long distance with respect to the state variable, and uniformly (with respect to the state variable) continuous in the measure variable under the L^1-Wasserstein distance. Below, we provide an example to demonstrate that ($\mathbf{H}_{b,1}$) is valid. Set for some constant $a>0$,

$$b(x,\mu):=x-x^3+a\int_{\mathbb{R}}z\mu(\mathrm{d}z), \quad x\in\mathbb{R},\mu\in\mathscr{P}_1(\mathbb{R}).$$

It is easy to see that for any $x,y\in\mathbb{R}$ and $\mu\in\mathscr{P}_1(\mathbb{R})$,

$$(x-y)(b(x,\mu)-b(y,\mu))=(x-y)^2\big(1-(x^2+xy+y^2)\big).$$

Whence, (1.9) follows readily. On the other hand, for any $x\in\mathbb{R}$ and $\mu,\nu\in\mathscr{P}_1(\mathbb{R})$,

$$|b(x,\mu)-b(x,\nu)|\le a\int_{\mathbb{R}\times\mathbb{R}}|z_1-z_2|\pi(\mathrm{d}z_1,\mathrm{d}z_2), \tag{1.12}$$

where $\pi\in\mathscr{C}(\mu,\nu)$. As a result, by taking the infimum over all couplings π on both sides of (1.12), (1.10) is available right now. Another example is taken from in [36], where b is given as below:

$$b(x,\mu)=-V'(x)-\int_{\mathbb{R}}W'(x-y)\mu(\mathrm{d}y), \quad x\in\mathbb{R},\mu\in\mathscr{P}_1(\mathbb{R}).$$

Here, V',W' are Lipschitz continuous with Lipschitz constants L_V and L_W respectively, and V' is dissipative in long distance (see [36, Assumption (**A1**)]). Then, by following the arguments above, one can see that Assumption ($\mathbf{H}_{b,1}$) is fulfilled for the drift $b(x,\mu)$ defined above when L_W is small (which is also required in the main result [36, Theorem 2]). In this sense, the framework of the present paper is much more general than [36]; see Remark 3.6 below for further comments. ($\mathbf{H}_{b,1}$) and ($\mathbf{H}_{b,2}$) together will be used in handling the asymptotic PoC in an infinite-time horizon (see Proposition 3.3 below for more details). In particular, ($\mathbf{H}_{b,2}$) is valid when b is of convolution type; that is, $b(x,\mu)=\displaystyle\int_{\mathbb{R}}b_0(x-y)\,\mu(\mathrm{d}y)$ for all $x\in\mathbb{R}$ and $\mu\in\mathscr{P}_1(\mathbb{R})$ with some Lipschitz continuous function $b_0:\mathbb{R}\to\mathbb{R}$. See Lemma 4.1 in the Appendix for more details. Furthermore, the non-degenerate property of σ plays a crucial role in constructing the asymptotic coupling by reflection, as stated in the second paragraph of Section 3. For example, ($\mathbf{H}_\sigma$) holds true obviously for $\sigma(x)=\frac{2+|x|}{1+|x|}, x\in\mathbb{R}$.

Before we present the main result, we introduce some additional notation. Let for $p \geq 1$,

$$L_p(\mathscr{P}(\mathbb{R}^d)) := \Big\{\mu \in \mathscr{P}(\mathscr{P}(\mathbb{R}^d)) : \int_{\mathscr{P}(\mathbb{R}^d)} \nu(|\cdot|^p)\mu(\mathrm{d}\nu) < \infty\Big\},$$

and

$$\mathcal{W}_p(\mu, \nu) := \inf_{\pi \in \mathscr{C}(\mu,\nu)} \int_{\mathscr{P}(\mathbb{R}^d)\times\mathscr{P}(\mathbb{R}^d)} \mathbb{W}_p(\widetilde{\mu}, \widetilde{\nu})\, \pi(\mathrm{d}\widetilde{\mu}, \mathrm{d}\widetilde{\nu}), \quad \mu, \nu \in L_p(\mathscr{P}(\mathbb{R}^d)).$$

The main result in this paper is stated as follows.

Theorem 1.2. *Assume that* $(\mathbf{H}_{b,1})$, $(\mathbf{H}_{b,2})$ *and* $(\mathbf{H}_\sigma)$ *hold. Then, there exist positive constants* C, λ_0^* *and* λ_3^* *such that for any* $t > 0$ *and* $\mu, \nu \in \mathscr{P}_1(\mathbb{R})$,

$$\mathcal{W}_1(\mu_t, \nu_t) \leq C\mathrm{e}^{-\lambda_0^* t}\mathbb{W}_1(\mu, \nu) \tag{1.13}$$

provided that λ_3 *in* $(\mathbf{H}_{b,1})$ *satisfies* $\lambda_3 \in (0, \lambda_3^*]$, *where* $\mu_t := \mathscr{L}_{X_t|\mathscr{F}_t^W}$ *and* $\nu_t := \mathscr{L}_{X_t|\mathscr{F}_t^W}$ *stand for the regular conditional distributions of* X_t, *determined by* (1.7), *with the initial distributions* $\mathscr{L}_{X_0} = \mu$ *and* $\mathscr{L}_{X_0} = \nu$ *respectively; and* $\lambda_3 > 0$ *is the Lipschitz constant of* $b(x, \mu)$ *with respect to the measure variable, given in* (1.10).

Like the setting concerned with classical McKean-Vlasov SDEs, Theorem 1.2 indicates that there is a unique invariant probability measure (IPM for short) for the McKean-Vlasov SDE with common noise (1.7) if the mean-field interaction is not too strong. Indeed, concerning McKean-Vlasov SDEs with common noise, an interesting phenomenon (that is, restoration of uniqueness for IPMs) emerges due to the introduction of common noise; see, for example, [36]. So, in some sense, the common noise would play a positive impact on the long-time behaviors of McKean-Vlasov SDEs with common noise.

In the sequel, let's explain the roles of the common noise and the idiosyncratic noise via the estimate for the convergence rate $\lambda_0^* > 0$ given in (1.13). That is, Theorem 1.2 will further demonstrate in a more convincing way the impact of the common noise.

Remark 1.3. By tracking proofs of Proposition 3.3 and Theorem 1.2 below, λ_0^* given in (1.13) can be calculated explicitly; see, in particular, (3.7). In Remark 3.4 below, it will be stressed once more that the non-degeneracy of σ_0 and σ_1 will facilitate the exponential ergodicity presented in (1.13). Furthermore, it is easily seen from (3.7) that the smaller ℓ_0 or the larger λ_2, introduced in (1.9), yields a faster convergence rate in (1.13). Nevertheless, the larger λ_1, given in (1.9), deteriorates the convergence rate.

Next, we make an explanation on the alternative of the initial value X_0.

Remark 1.4. In the present paper, to emphasize that the noise $(W_t)_{t\geq 0}$ is the unique common noise, the initial value X_0 is set to be supported on the probability space $(\Omega^1, \mathscr{F}_0^1, \mathbb{P}^1)$. This results in that $\mathbb{W}_1(\mu, \nu)$ rather than $\mathcal{W}_1(\mu, \nu)$ appears on the right hand side of (1.13). When (i) X_0 is defined on $(\Omega^0, \mathscr{F}_0^0, \mathbb{P}^0)$ (so $X_0 = X_0^0$), and (ii) X_0 is measurable with respect to $\sigma(X_0^0, X_0^1)$ with X_0^0 and X_0^1 being defined respectively on $(\Omega^0, \mathscr{F}_0^0, \mathbb{P}^0)$ and $(\Omega^1, \mathscr{F}_1^0, \mathbb{P}^1)$, μ_t is a version of the conditional law of X_t given the σ-algebra $\sigma(X_0^0, W_s, s \leq t)$; see e.g. [7, Remark 2.10] for more discussions on various choices of the initial value associated with McKean-Vlasov SDEs with common noise. For the case (i), (X_0, W) is called the "initial condition-common noise"; regarding the setting (ii), (X_0^0, W) plays the role of systemic noise. As far as the cases (i) and (ii) are concerned, the quantity $\mathbb{W}_1(\mu, \nu)$ on the right hand side of (1.13) can be replaced by $\mathcal{W}_1(\mu, \nu)$ so (1.13) can be written in a symmetric form, i.e., $\mathcal{W}_1(\mu_t, \nu_t) \leq C\mathrm{e}^{-\lambda_0^* t}\mathcal{W}_1(\mu, \nu)$ for some constants $C, \lambda_0^* > 0$.

To investigate ergodicity of classical McKean-Vlasov SDEs, which are strongly well-posed, one usually makes very well use of their decoupled versions. However, this routine does not work for the McKean-Vlasov SDEs with common noise due to the essentially different roles played by the common noise and the idiosyncratic noise. Instead, we turn to work with the non-interacting particle system and the corresponding interacting particle system to explore the long-time behaviors of (1.7).

The detailed comparisons between Theorem 1.2 and the counterpart in [36, Section 4] are to be presented in Remark 3.6. Presently, one might be a little bit confused why we are confined to the 1-dimensional SDE (1.7) rather than the associated multi-dimensional version. From now on, we go into detail about the corresponding explanations.

Remark 1.5. Unsatisfactorily, Theorem 1.2 is concerned merely with a kind of 1-dimensional McKean-Vlasov SDEs with common noise. In terms of Proposition 2.4, we can indeed derive via the asymptotic coupling by reflection the associated coupling process for the multi-dimensional McKean-Vlasov SDEs with common noise. Whereas, for the dimension $d \geq 2$, the asymptotic coupling by reflection constructed in Subsection 2.2 is determined by the average difference between the component processes (see Remark 3.7 below for more details). With such a construction at hand, one can derive only an estimate on the quantity $\mathbb{E}|\overline{\mathbf{X}_t^N} - \overline{\mathbf{X}_t^{N,N}}|$ provided that b enjoys a very special structure. Hereinbefore, $\overline{\mathbf{X}_t^N}$ (resp. $\overline{\mathbf{X}_t^{N,N}}$) indicates the arithmetic mean of the non-interacting particles (resp. interacting particles). Furthermore, by following the line in [36, Section 5], to achieve the main result in Theorem 1.2 for the high dimensional setting, one needs to quantify the difference between each component of the interacting particle system and its averaged process. To this end, a very strict condition (i.e., $\sigma \equiv 0$) has to be imposed. When the idiosyncratic noise vanishes (that is, the McKean-Vlasov SDE under consideration is driven merely by common noise), the corresponding issue has been treated in [36, Section 5]. On the other hand, one can apply directly the synchronous coupling and bypass the aforementioned obstacles as long as the coefficients corresponding to the McKean-Vlasov SDEs with common noise are dissipative; see [36, Section 3] for further details. More interpretations related to the restriction on the dimension $d = 1$ will be further elaborated in Remark 2.7 and Remark 3.7.

The rest of this paper is arranged as follows. In Section 2, we address the issue on conditional PoC in a finite-time horizon for McKean-Vlasov SDEs with common noise, and construct via an asymptotic coupling by reflection the coupling process for the associated non-interacting particle system and the interacting particle system. Section 3 is devoted to the proof of Theorem 1.2, which is based on the uniform-in-time PoC for the conditional McKean-Vlasov SDE (1.7).

2. Preliminaries

Let $(B_t^1)_{t\geq 0}$ and $(B_t^2)_{t\geq 0}$ be d-dimensional Brownian motions defined on $(\Omega^1, \mathscr{F}^1, (\mathscr{F}_t^1)_{t\geq 0}, \mathbb{P}^1)$, and $(W_t)_{t\geq 0}$ a d-dimensional Brownian motion supported on $(\Omega^0, \mathscr{F}^0, (\mathscr{F}_t^0)_{t\geq 0}, \mathbb{P}^0)$. Write $\mathbb{E}$, $\mathbb{E}^0$ and $\mathbb{E}^1$ as the expectation operators under $\mathbb{P} := \mathbb{P}^0 \times \mathbb{P}^1$, $\mathbb{P}^0$ and $\mathbb{P}^1$, respectively. In this section, we focus on the McKean-Vlasov SDE with common noise in the following form:

$$\mathrm{d}X_t = b(X_t, \mu_t)\mathrm{d}t + \sigma_1 \mathrm{d}B_t^1 + \overline{\sigma}(X_t)\mathrm{d}B_t^2 + \sigma_0 \mathrm{d}W_t, \tag{2.1}$$

where $\mu_t := \mathscr{L}_{X_t|\mathscr{F}_t^W}$, $\sigma_0, \sigma_1 \in \mathbb{R}$,

$$b : \mathbb{R}^d \times \mathscr{P}(\mathbb{R}^d) \to \mathbb{R}^d, \quad \overline{\sigma} : \mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^d,$$

and the initial value X_0 is an $\mathscr{F}_0^1$-measurable random variable. As the chapter unfolds, the reason why we prefer the SDE formulated in the framework (2.1) will become more and more transparent; see, in particular, the introductory part of Section 3 for more details.

In this section, we suppose that

($\mathbf{A}_b$) *$b(\cdot,\delta_{\mathbf{0}}):\mathbb{R}^d\to\mathbb{R}^d$ is continuous and locally bounded on $\mathbb{R}^d$, and there exist constants $L_1,L_2>0$ such that for all $x,y\in\mathbb{R}^d$ and $\mu,\nu\in\mathscr{P}_1(\mathbb{R}^d)$,*

$$2\langle x-y,b(x,\mu)-b(y,\mu)\rangle\le L_1|x-y|^2, \tag{2.2}$$

and

$$|b(x,\mu)-b(x,\nu)|\le L_2\mathbb{W}_1(\mu,\nu); \tag{2.3}$$

($\mathbf{A}_{\overline{\sigma}}$) *there exists a constant $L_3>0$ such that*

$$\|\overline{\sigma}(x)-\overline{\sigma}(y)\|_{\mathrm{HS}}\le L_3|x-y|,\quad x,y\in\mathbb{R}^d, \tag{2.4}$$

where $\|\cdot\|_{\mathrm{HS}}$ means the Hilbert-Schmidt norm.

Under ($\mathbf{A}_b$) and ($\mathbf{A}_{\overline{\sigma}}$), note that for all $x,y\in\mathbb{R}^d$ and $\mu,\nu\in\mathscr{P}_1(\mathbb{R}^d)$,

$$2\langle x-y,b(x,\mu)-b(y,\nu)\rangle\le L_4\big(|x-y|+\mathbb{W}_1(\mu,\nu)\big)|x-y|, \tag{2.5}$$

where $L_4:=\max\{L_1,2L_2\}$. Then, the SDE (2.1) has a unique strong solution; see, for instance, the proof of [29, Theorem 2.1] for related details.

To handle the issue on PoC concerned with (2.1), we consider the non-interacting particle system and the interacting particle system associated with (2.1): for any $i\in\mathbb{S}_N$,

$$\mathrm{d}X_t^i=b(X_t^i,\mu_t^i)\mathrm{d}t+\sigma_1\mathrm{d}B_t^{1,i}+\overline{\sigma}(X_t^i)\mathrm{d}B_t^{2,i}+\sigma_0\mathrm{d}W_t, \tag{2.6}$$

and

$$\mathrm{d}X_t^{i,N}=b(X_t^{i,N},\widehat{\mu}_t^N)\mathrm{d}t+\sigma_1\mathrm{d}B_t^{1,i}+\overline{\sigma}(X_t^{i,N})\mathrm{d}B_t^{2,i}+\sigma_0\mathrm{d}W_t. \tag{2.7}$$

Herein, $\mu_t^i:=\mathscr{L}_{X_t^i|\mathscr{F}_t^W}$ and $\widehat{\mu}_t^N:=\frac{1}{N}\sum_{j=1}^N\delta_{X_t^{j,N}}$; $((B_t^{1,i})_{t\ge0})_{i\in\mathbb{S}_N}$ and $((B_t^{2,i})_{t\ge0})_{i\in\mathbb{S}_N}$ are mutually independent d-dimensional Brownian motions on $(\Omega^1,\mathscr{F}^1,(\mathscr{F}_t^1)_{t\ge0},\mathbb{P}^1)$; $(W_t)_{t\ge0}$, carried on $(\Omega^0,\mathscr{F}^0,(\mathscr{F}_t^0)_{t\ge0},\mathbb{P}^0)$, is kept untouched as in (2.1). In addition, throughout this section, we assume that $(X_0^i,X_0^{i,N})_{i\in\mathbb{S}_N}$ are i.i.d. $\mathscr{F}_0^1$-measurable random variables. Note that (2.7) can be reformulated as a classical $(\mathbb{R}^d)^N$-valued SDE, where the corresponding coefficients satisfy the locally weak monotonicity and the globally weak coercivity once ($\mathbf{A}_b$) and ($\mathbf{A}_{\overline{\sigma}}$) are available. Thus, (2.7) is strongly well-posed; see, for instance, e.g. [39, Theorem 3.1.1].

To proceed, we make some comments concerning Assumptions ($\mathbf{A}_b$) and ($\mathbf{A}_{\overline{\sigma}}$).

Remark 2.1. In the following subsection, we are concerned merely with behaviors of the SDE (2.1) in a finite horizon. For this, the one-sided Lipschitz condition (2.2) is enough. That is, the dissipativity in long distance (see in particular (1.9)), which is dedicated to the long-term analysis, is unnecessary. In addition, whether σ_1, $\overline{\sigma}$ and σ_0 are degenerate or not does not have impact on the subsequent analysis. Nevertheless, the non-degeneracy of σ_1 and σ_0 is indispensable as far as the establishment of long-time behaviors is concerned.

2.1. Conditional PoC in finite time

In the past few years, there are some progresses on the issue of PoC for conditional McKean-Vlasov SDEs; see, for example, [7, Theorem 2.12] and [25, Theorem 2.3], where the coefficients are Lipschitz continuous with respect to the state variable, and [29, Proposition 2.1], in which the coefficients satisfy the monotone condition. It is worthy to emphasize that, as for McKean-Vlasov SDEs with common noise investigated in [7,29,25], the coefficients are L^2-Wasserstein Lipschitz continuous with respect to the measure variable. Yet, in the present paper, the drift part of the conditional McKean-Vlasov SDE we are interested in is L^1-Wasserstein Lipschitz continuous. In addition, by invoking [19, Theorem 1], the convergence rate of conditional propagation of chaos was provided in [7,29,25] once the initial distribution enjoys the high-order moment. Whereas, for our purpose, the quantitative convergence rate of conditional PoC is unnecessary so the high-order moment of the initial distribution is dispensable as shown in the following proposition.

Concerning the McKean-Vlasov SDE with common noise (2.1), the following proposition addresses PoC in a finite horizon.

Proposition 2.2. *Consider the SDEs* (2.6) *and* (2.7) *with* $X_0^{i,N}=X_0^i$ *for all* $i\in\mathbb{S}_N$. *Assume* $(\mathbf{A}_b)$ *and* $(\mathbf{A}_{\overline{\sigma}})$, *and suppose further* $\mathbb{E}|X_0^1|<\infty$. *Then, for each given* $t\ge 0$ *and any* $i\in\mathbb{S}_N$,

$$\lim_{N\to\infty}\mathbb{E}\,\mathbb{W}_1(\mu_t^i,\widetilde{\mu}_t^N)=0, \tag{2.8}$$

where $\mu_t^i:=\mathscr{L}_{X_t^i|\mathscr{F}_t^W}$ *and* $\widetilde{\mu}_t^N:=\frac{1}{N}\sum_{j=1}^N\delta_{X_t^j}$, *and*

$$\lim_{N\to\infty}\mathbb{E}|X_t^i-X_t^{i,N}|=0. \tag{2.9}$$

Proof. The proof is split into two parts.

(i) First of all, we show that for each given $t\ge0$ and $i\in\mathbb{S}_N$,

$$\mathbb{E}|Z_t^{i,N}|\le\frac{1}{2}L_4t\mathrm{e}^{(L_4+L_3^2/2)t}\int_0^t\mathbb{E}\,\mathbb{W}_1(\mu_s^i,\widetilde{\mu}_s^N)\,\mathrm{d}s, \tag{2.10}$$

where $Z_t^{i,N}:=X_t^i-X_t^{i,N}$ and L_4 was given in (2.5). Once (2.10) is verifiable, by Fatou's lemma, we deduce that

$$\begin{aligned}\limsup_{N\to\infty}\mathbb{E}|Z_t^{i,N}|&\le\frac{1}{2}L_4t\mathrm{e}^{(L_4+L_3^2/2)t}\limsup_{N\to\infty}\int_0^t\mathbb{E}\,\mathbb{W}_1(\mu_s^i,\widetilde{\mu}_s^N)\,\mathrm{d}s\\&\le\frac{1}{2}L_4t\mathrm{e}^{(L_4+L_3^2/2)t}\int_0^t\limsup_{N\to\infty}\mathbb{E}\,\mathbb{W}_1(\mu_s^i,\widetilde{\mu}_s^N)\,\mathrm{d}s.\end{aligned}$$

Consequently, (2.9) follows by taking (2.8) into consideration.

In the sequel, we shall fix the index $i\in\mathbb{S}_N$. For any $\delta\in(0,1]$, define the function V_δ by

$$V_\delta:\mathbb{R}^d\to\mathbb{R}_+,\quad V_\delta(x)=(\delta+|x|^2)^{1/2},\quad x\in\mathbb{R}^d, \tag{2.11}$$

which is a smooth approximation of the function $\mathbb{R}^d\ni x\mapsto|x|$. It is ready to see that

$$\nabla V_\delta(x)=\frac{x}{V_\delta(x)}\quad\text{and}\quad\nabla^2V_\delta(x)=\frac{1}{V_\delta(x)}I_d-\frac{x\otimes x}{V_\delta(x)^3},\quad x\in\mathbb{R}^d, \tag{2.12}$$

where $x\otimes x\in\mathbb{R}^d\otimes\mathbb{R}^d$ with entries $(x\otimes x)_{i,j}=x_ix_j$, and I_d means the $d\times d$-identity matrix. Then, applying Itô's formula, we deduce from (2.4) and (2.5) that

$$\begin{aligned}\mathrm{d}V_\delta(Z_t^{i,N}) &= \big\langle \nabla V_\delta(Z_t^{i,N}), b(X_t^i,\mu_t^i) - b(X_t^{i,N},\widehat{\mu}_t^N)\big\rangle \mathrm{d}t \\ &\quad + \frac{1}{2}\big\langle \nabla^2 V_\delta(Z_t^{i,N}), (\overline{\sigma}(X_t^i)-\overline{\sigma}(X_t^{i,N}))(\overline{\sigma}(X_t^i)-\overline{\sigma}(X_t^{i,N}))^*\big\rangle_{\mathrm{HS}}\mathrm{d}t + \mathrm{d}M_t^i \\ &\le \frac{|Z_t^{i,N}|}{2V_\delta(Z_t^{i,N})}\big((L_4+L_3^2)|Z_t^{i,N}| + L_4\mathbb{W}_1(\mu_t^i,\widehat{\mu}_t^N)\big)\mathrm{d}t + \mathrm{d}M_t^i \\ &\le \frac{1}{2}\big((L_4+L_3^2)|Z_t^{i,N}| + L_4\mathbb{W}_1(\mu_t^i,\widehat{\mu}_t^N)\big)\mathrm{d}t + \mathrm{d}M_t^i,\end{aligned}$$

where

$$\mathrm{d}M_t^i := \big\langle \nabla V_\delta(Z_t^{i,N}), \big(\overline{\sigma}(X_t^i)-\overline{\sigma}(X_t^{i,N})\big)\mathrm{d}B_t^{2,i}\big\rangle.$$

Thus, via Fatou's lemma, in addition to $X_0^i = X_0^{i,N}$, we have

$$\mathbb{E}|Z_t^{i,N}| \le \frac{1}{2}\int_0^t \big((L_4+L_3^2)\mathbb{E}|Z_s^{i,N}| + L_4\mathbb{E}\,\mathbb{W}_1(\mu_s^i,\widehat{\mu}_s^N)\big)\mathrm{d}s. \tag{2.13}$$

Note from the triangle inequality that

$$\begin{aligned}\mathbb{W}_1(\mu_t^i,\widehat{\mu}_t^N) &\le \mathbb{W}_1(\mu_t^i,\widetilde{\mu}_t^N) + \mathbb{W}_1(\widetilde{\mu}_t^N,\widehat{\mu}_t^N) \\ &\le \mathbb{W}_1(\mu_t^i,\widetilde{\mu}_t^N) + \frac{1}{N}\sum_{j=1}^N |Z_t^{j,N}|.\end{aligned}$$

Whence, since $(X_t^i, X_t^{i,N})_{i\in\mathbb{S}_N}$ are identically distributed (see e.g. [7, p. 122–123]) by recalling that $(X_0^i, X_0^{i,N})_{i\in\mathbb{S}_N}$ are i.i.d. $\mathscr{F}_0^1$-measurable random variables, we derive from (2.13) that

$$\mathbb{E}|Z_t^{i,N}| \le \frac{1}{2}\int_0^t \big((2L_4+L_3^2)\mathbb{E}|Z_s^{i,N}| + L_4\mathbb{E}\,\mathbb{W}_1(\mu_s^i,\widetilde{\mu}_s^N)\big)\mathrm{d}s.$$

Accordingly, (2.10) follows from Gronwall's inequality.

(ii) Next, we prove (2.8). We firstly verify that there exists a constant $c_0 > 0$ such that for all $i \in \mathbb{S}_N$ and all $t > 0$,

$$\mathbb{E}|X_t^i| \le \big(1 + c_0 t + \mathbb{E}|X_0^i|\big)\mathrm{e}^{c_0 t}. \tag{2.14}$$

Indeed, applying Itô's formula to the function V_1, defined in (2.11) with $\delta = 1$, and taking advantage of (2.12) with $\delta = 1$, we infer from (2.4), (2.5) and $V_1 \ge 1$ that for some constant $c_1 > 0$,

$$\begin{aligned}\mathrm{d}V_1(X_t^i) &= \langle \nabla V_1(X_t^i), b(X_t^i,\mu_t^i)\rangle \mathrm{d}t + \frac{1}{2}\langle \nabla^2 V_1(X_t^i), (\sigma_1^2+\sigma_0^2)I_d + \overline{\sigma}(X_t^i)(\overline{\sigma}(X_t^i))^*\rangle_{\mathrm{HS}}\mathrm{d}t + \mathrm{d}\overline{M}_t^i \\ &\le \frac{1}{V_1(X_t^i)}\Big(\langle X_t^i, b(X_t^i,\mu_t^i)\rangle + \frac{1}{2}\big((\sigma_1^2+\sigma_0^2)d + \|\overline{\sigma}(X_t^i)\|_{\mathrm{HS}}^2\big)\Big)\mathrm{d}t + \mathrm{d}\overline{M}_t^i \\ &\le c_1\big(1 + |X_t^i| + \mu_t^i(|\cdot|)\big)\mathrm{d}t + \mathrm{d}\overline{M}_t^i,\end{aligned}$$

where

$$\mathrm{d}\overline{M}_t^i := \big\langle \nabla V_1(X_t^i), \sigma_1\mathrm{d}B_t^{1,i} + \overline{\sigma}(X_t^i)\mathrm{d}B_t^{2,i} + \sigma_0\mathrm{d}W_t\big\rangle.$$

Thus, by invoking the fact that

$$\mathbb{E}^0\mu_t^i(|\cdot|)=\mathbb{E}^0\big(\mathbb{E}^1\big(|X_t^i|\big|\mathscr{F}_t^W\big)\big)=\mathbb{E}|X_t^i|,$$

we conclude that

$$\mathbb{E}|X_t^i|\le 1+\mathbb{E}|X_0^i|+2c_1\int_0^t(1+\mathbb{E}|X_s^i|)\mathrm{d}s.$$

Therefore, (2.14) is attainable by applying Gronwall's inequality.

With (2.14) at hand, we proceed to prove (2.8). Since, $\mathbb{P}^0$-almost surely, $\widetilde{\mu}_t^N$ converges weakly to μ_t^i, and

$$\mathbb{P}^1\Big(\lim_{N\to\infty}\widetilde{\mu}_t^N(|\cdot|)=\mu_t^i(|\cdot|)\Big)=1$$

by means of the law of large numbers, [6, Theorem 5.5] yields $\mathbb{P}^0$-almost surely

$$\mathbb{P}^1\Big(\lim_{N\to\infty}\mathbb{W}_1(\mu_t^i,\widetilde{\mu}_t^N)=0\Big)=1.$$

Whereafter, owing to

$$\mathbb{W}_1(\mu_t^i,\widetilde{\mu}_t^N)\le\mu_t^i(|\cdot|)+\widetilde{\mu}_t^N(|\cdot|)$$

and the fact that X_t^i and X_t^j are identically distributed given the filtration $\mathscr{F}_t^W$, the dominated convergence theorem yields that

$$\mathbb{P}^0\Big(\lim_{N\to\infty}\mathbb{E}^1\mathbb{W}_1(\mu_t^i,\widetilde{\mu}_t^N)=0\Big)=1.$$

Next, in the light of

$$\mathbb{E}^1\mathbb{W}_1(\mu_t^i,\widetilde{\mu}_t^N)\le 2\mu_t^i(|\cdot|)\quad\text{and}\quad\mathbb{E}\,\mathbb{W}_1(\mu_t^i,\widetilde{\mu}_t^N)=\mathbb{E}^0\big(\mathbb{E}^1\mathbb{W}_1(\mu_t^i,\widetilde{\mu}_t^N)\big),$$

the verification (2.14) and the dominated convergence theorem enable us to derive (2.8). □

At the end of this part, we make a comment on Proposition 2.2.

Remark 2.3. The statement in Proposition 2.2 can be made quantitatively by applying [19, Theorem 1] provided that the associated initial value has finite moment with order great than 1; see [2, Theorem 4.3] for the recent study on this topic. Indeed, in this case the associated convergence rates can be established exactly as those in [19, Theorem 1]. On the other hand, once $(\mathbf{A}_b)$ is strengthened into a version, given as in (1.9), and $(\mathbf{H}_{b,2})$ is further imposed, Proposition 3.3 below enables us to derive the uniform-in-time conditional PoC. Indeed, Lemma 4.1 provides a sufficiency to guarantee $(\mathbf{H}_{b,2})$; see Section 3 for more details.

2.2. Asymptotic coupling by reflection

For any $\varepsilon>0$, define the cut-off function h_ε by

$$h_\varepsilon(r)=\begin{cases}0, & r\in[0,\varepsilon],\\ 1-\exp\big((r-\varepsilon)/(r-2\varepsilon)\big), & r\in(\varepsilon,2\varepsilon),\\ 1, & r\ge 2\varepsilon.\end{cases}\tag{2.15}$$

Set for any $x \in \mathbb{R}^d$,

$$\mathbf{n}(x) := \frac{x}{|x|}\mathbb{1}_{\{x\neq\mathbf{0}\}} + (1,0,\cdots,0)^\top \mathbb{1}_{\{x=\mathbf{0}\}},$$

where $a^\top$ means the transpose of the d-dimensional row vector a. Below, let $\rho : (\mathbb{R}^d)^N \to [0,\infty)$ and $\phi : (\mathbb{R}^d)^N \to \mathbb{R}^d$, where their precise expressions are unimportant in this subsection, and will be prescribed explicitly in Section 3. Define for $\mathbf{x} := (x_1,\cdots,x_N) \in (\mathbb{R}^d)^N$ and $\varepsilon > 0$,

$$\Pi_\varepsilon(\mathbf{x}) := I_d - 2h_\varepsilon(\rho(\mathbf{x}))\mathbf{n}(\phi(\mathbf{x})) \otimes \mathbf{n}(\phi(\mathbf{x})). \tag{2.16}$$

In particular, for the case $d = 1$, one has

$$\Pi_\varepsilon(\mathbf{x}) = 1 - 2h_\varepsilon(\rho(\mathbf{x})).$$

In order to investigate the issue on uniform-in-time PoC for the SDE (2.1), we construct the asymptotic coupling by reflection associated with the non-interacting particle system (2.6) and the corresponding interacting particle system (2.7). More precisely, we build the following approximate interacting particle systems: for $i \in \mathbb{S}_N$ and $\varepsilon > 0$,

$$\begin{cases} \mathrm{d}X_t^i = b(X_t^i,\mu_t^i)\mathrm{d}t + \sigma_1\mathrm{d}B_t^{1,i} + \overline{\sigma}(X_t^i)\mathrm{d}B_t^{2,i} + \sigma_0\mathrm{d}W_t, \\ \mathrm{d}X_t^{i,N,\varepsilon} = b(X_t^{i,N,\varepsilon},\widehat{\mu}_t^{N,\varepsilon})\mathrm{d}t + \sigma_1\Pi_\varepsilon(\mathbf{X}_t^N - \mathbf{X}_t^{N,N,\varepsilon})\mathrm{d}B_t^{1,i} + \overline{\sigma}\big(X_t^{i,N,\varepsilon}\big)\mathrm{d}B_t^{2,i}, \\ \qquad\qquad +\sigma_0\Pi_\varepsilon(\mathbf{X}_t^N - \mathbf{X}_t^{N,N,\varepsilon})\mathrm{d}W_t, \end{cases} \tag{2.17}$$

where $X_0^{i,N,\varepsilon} = X_0^{i,N}$, $\widehat{\mu}_t^{N,\varepsilon} := \frac{1}{N}\sum_{j=1}^N \delta_{X_t^{j,N,\varepsilon}}$,

$$\mathbf{X}_t^N := \big(X_t^1,\cdots,X_t^N\big), \quad \mathbf{X}_t^{N,N,\varepsilon} := \big(X_t^{1,N,\varepsilon},\cdots,X_t^{N,N,\varepsilon}\big),$$

and $(X_0^i, X_0^{i,N})_{i\in\mathbb{S}_N}$ are i.i.d. $\mathscr{F}_0^1$-measurable random variables.

To proceed, let's say a few words on the construction in (2.17). Roughly speaking, for the additive common noise and the idiosyncratic noise, we employ the asymptotic coupling by reflection. Nevertheless, regarding the multiplicative noise, we exploit the synchronous coupling. The main thesis in this part is presented as follows.

Proposition 2.4. *Fix $N \geq 1$ and $T > 0$. Let $(\mathbf{X}_{[0,T]}^N, \mathbf{X}_{[0,T]}^{N,N,\varepsilon})_{\varepsilon>0} := ((\mathbf{X}_t^N)_{t\in[0,T]}, (\mathbf{X}_t^{N,N,\varepsilon})_{t\in[0,T]})_{\varepsilon>0}$ be the process determined by (2.17) such that the initial value $(\mathbf{X}_0^N, \mathbf{X}_0^{N,N,\varepsilon})_{\varepsilon>0}$ satisfies all properties mentioned above. Under $(\mathbf{A}_b)$ and $(\mathbf{A}_{\overline{\sigma}})$, $(\mathbf{X}_{[0,T]}^N, \mathbf{X}_{[0,T]}^{N,N,\varepsilon})_{\varepsilon>0}$ has a weakly convergent subsequence such that the corresponding weak limit process is the coupling process of $\mathbf{X}_{[0,T]}^N$ and $\mathbf{X}_{[0,T]}^{N,N}$, where $\mathbf{X}_{[0,T]}^{N,N} := (\mathbf{X}_t^{N,N})_{t\in[0,T]}$ with $\mathbf{X}_t^{N,N} := \big(X_t^{1,N},\cdots,X_t^{N,N}\big)$ for any $t \geq 0$.*

For fixed $N \geq 1$ and $T \geq 0$, we first show that $(\mathbf{X}_{[0,T]}^{N,N,\varepsilon})_{\varepsilon>0}$ owns a uniform moment in ε, which plays a crucial role in illustrating the tightness of $(\mathbf{X}_{[0,T]}^{N,N,\varepsilon})_{\varepsilon>0}$.

Lemma 2.5. *Let Assumptions $(\mathbf{A}_b)$ and $(\mathbf{A}_{\overline{\sigma}})$ hold. Fix $N \geq 1$ and $T > 0$. Suppose further $\mathbb{E}|X_0^{1,N}| < \infty$. Then, there is a constant $C_T > 0$ (which is independent of N) such that for any $\varepsilon > 0$,*

$$\mathbb{E}\Big(\sup_{0\leq t\leq T} |\mathbf{X}_t^{N,N,\varepsilon}|\Big) \leq C_T N\big(1 + \mathbb{E}|X_0^{1,N}|\big). \tag{2.18}$$

Proof. Below, we fix the particle number $N \geq 1$ and a finite-time horizon $T > 0$. It is easy to see from $h_\varepsilon \in [0,1]$ that for all $\mathbf{x} \in (\mathbb{R}^d)^N$,

$$\|\Pi_\varepsilon(\mathbf{x})\|_{\mathrm{HS}}^2 = d + 4h_\varepsilon(\rho(\mathbf{x}))(h_\varepsilon(\rho(\mathbf{x})) - 1) \leq d. \tag{2.19}$$

Then, applying Itô's formula to V_1, introduced in (2.11) with $\delta = 1$, and making use of $V_1 \geq 1$, we deduce from (2.2) and (2.4) (see also the arguments below (2.14)) that for some constant $c_1 > 0$,

$$\begin{aligned}\mathrm{d}V_1(X_t^{i,N,\varepsilon}) &\leq \frac{1}{2V_1(X_t^{i,N,\varepsilon})}\big(2\langle X_t^{i,N,\varepsilon}, b(X_t^{i,N,\varepsilon}, \widehat{\mu}_t^{N,\varepsilon})\rangle + \|\overline{\sigma}(X_t^{i,N,\varepsilon})\|_{\mathrm{HS}}^2 + (\sigma_0^2 + \sigma_1^2)d\big)\,\mathrm{d}t + \mathrm{d}M_t^{i,N,\varepsilon} \\ &\leq c_1\big(1 + |X_t^{i,N,\varepsilon}| + \widehat{\mu}_t^{N,\varepsilon}(|\cdot|)\big)\,\mathrm{d}t + \mathrm{d}M_t^{i,N,\varepsilon},\end{aligned}$$

where

$$\mathrm{d}M_t^{i,N,\varepsilon} := \big\langle \nabla V_1(X_t^{i,N,\varepsilon}), \Pi_\varepsilon(\mathbf{X}_t^N - \mathbf{X}_t^{N,N,\varepsilon})(\sigma_1 \mathrm{d}B_t^{1,i} + \sigma_0 \mathrm{d}W_t) + \overline{\sigma}(X_t^{i,N,\varepsilon})\mathrm{d}B_t^{2,i}\big\rangle.$$

Define the stopping time for any integer $n \geq 1$,

$$\tau_n^{N,\varepsilon} = \inf\big\{t \geq 0 : |\mathbf{X}_t^{N,N,\varepsilon}| \geq n\big\}.$$

Employing BDG's inequality and taking (2.4) and (2.19) into consideration yields that for some constants $c_2, c_3 > 0$,

$$\begin{aligned}\gamma_n^{i,N,\varepsilon}(t) &:= \mathbb{E}\Big(\sup_{0\leq s\leq t\wedge\tau_n^{N,\varepsilon}} |X_s^{i,N,\varepsilon}|\Big) \\ &\leq \mathbb{E}|X_0^{i,N,\varepsilon}| + c_2 t + c_2\int_0^t \Big(\gamma_n^{i,N,\varepsilon}(s) + \frac{1}{N}\sum_{j=1}^N \gamma_n^{j,N,\varepsilon}(s)\Big)\,\mathrm{d}s \\ &\quad + c_2\mathbb{E}\Big(\int_0^{t\wedge\tau_n^{N,\varepsilon}} \big(1 + |X_s^{i,N,\varepsilon}|\big)^2\,\mathrm{d}s\Big)^{1/2} \\ &\leq \mathbb{E}|X_0^{i,N,\varepsilon}| + c_3 t + c_3\int_0^t \Big(\gamma_n^{i,N,\varepsilon}(s) + \frac{1}{N}\sum_{j=1}^N \gamma_n^{j,N,\varepsilon}(s)\Big)\,\mathrm{d}s + \frac{1}{2}\gamma_n^{i,N,\varepsilon}(t),\end{aligned}$$

where in the last inequality we used the fact that $2ab \leq \eta^{-1}a^2 + \eta b^2, a, b, \eta > 0$. This obviously implies that for some constant $c_4 > 0$,

$$\frac{1}{N}\sum_{i=1}^N \gamma_n^{i,N,\varepsilon}(t) \leq c_4\Big(\mathbb{E}|X_0^{1,N}| + t + \frac{1}{N}\sum_{j=1}^N \int_0^t \gamma_n^{j,N,\varepsilon}(s)\,\mathrm{d}s\Big),$$

since $(X_0^{i,N,\varepsilon})_{i\in\mathbb{S}_N} = (X_0^{i,N})_{i\in\mathbb{S}_N}$ are i.i.d. $\mathscr{F}_0^1$-measurable random variables. Hence, by applying Gronwall's inequality and Fatou's lemma, there exists a constant $C_T^* > 0$ such that

$$\frac{1}{N}\sum_{i=1}^N \mathbb{E}\Big(\sup_{0\leq t\leq T} |X_t^{i,N,\varepsilon}|\Big) \leq C_T^*\big(1 + \mathbb{E}|X_0^{1,N}|\big).$$

Thus, the assertion (2.18) follows immediately by noting that

$$\mathbb{E}\Big(\sup_{0\le t\le T}|\mathbf{X}_t^{N,N,\varepsilon}|\Big)\le\sum_{i=1}^N\mathbb{E}\Big(\sup_{0\le t\le T}|X_t^{i,N,\varepsilon}|\Big).\quad\square$$

Lemma 2.6. *Let Assumptions* $(\mathbf{A}_b)$ *and* $(\mathbf{A}_{\overline{\sigma}})$ *hold. Fix* $N\ge 1$ *and* $T>0$*. Suppose further* $\mathbb{E}|X_0^{1,N}|<\infty$*. Then,* $(\mathbf{X}_{[0,T]}^{N,N,\varepsilon})_{\varepsilon>0}$ *is tight.*

Proof. In the subsequent analysis, we shall fix $N\ge 1$ and $T>0$. According to [1, Theorem 1], for the sake of tightness of $(\mathbf{X}_{[0,T]}^{N,N,\varepsilon})_{\varepsilon>0}$, it amounts to establishing that

(i) for each $t\in[0,T]$, $(\mathbf{X}_t^{N,N,\varepsilon})_{\varepsilon>0}$ is tight;
(ii) $\mathbf{X}_{\tau_\varepsilon+\delta_\varepsilon}^{N,N,\varepsilon}-\mathbf{X}_{\tau_\varepsilon}^{N,N,\varepsilon}\to 0$ in probability as $\varepsilon\to 0$, where, for each $\varepsilon>0$, $\tau_\varepsilon\in[0,T]$ is a stopping time and $\delta_\varepsilon\in[0,1]$ is a constant such that $\delta_\varepsilon\to 0$ as $\varepsilon\to 0$.

In the sequel, we aim to verify the two statements above, one by one.

For any $r>0$, let $\mathbf{B}_r=B_r\times B_r\cdots\times B_r\subset(\mathbb{R}^d)^N$, where $B_r:=\{x\in\mathbb{R}^d:|x|\le r\}$. Let B_r^c and $\mathbf{B}_r^c$ be the respective complements of B_r and $\mathbf{B}_r$. By the Chebyshev inequality, in addition to (2.18), we find that for any $t\in[0,T]$ and $R>0$,

$$\begin{aligned}\mathbb{P}\big(\mathbf{X}_t^{N,N,\varepsilon}\in\mathbf{B}_R^c\big)&\le(2^N-1)\max_{i\in\mathbb{S}_N}\mathbb{P}\big(X_t^{i,N,\varepsilon}\in B_R^c\big)\\&\le\frac{1}{R}(2^N-1)C_TN\big(1+\mathbb{E}|X_0^{1,N}|\big).\end{aligned}$$

Whence, the statement (i) is valid right now.

For any $\beta>0$, it is easy to notice that

$$\begin{aligned}\mathbb{P}\big(\big|\mathbf{X}_{\tau_\varepsilon+\delta_\varepsilon}^{N,N,\varepsilon}-\mathbf{X}_{\tau_\varepsilon}^{N,N,\varepsilon}\big|\ge\beta\big)\le\sum_{i=1}^N\Bigg(&\mathbb{P}\Big(\int_{\tau_\varepsilon}^{\tau_\varepsilon+\delta_\varepsilon}\big|b(X_s^{i,N,\varepsilon},\widehat{\mu}_s^{N,\varepsilon})\big|\,\mathrm{d}s\ge\frac{\beta}{4N}\Big)\\&+\mathbb{P}\Big(|\sigma_1|\Big|\int_{\tau_\varepsilon}^{\tau_\varepsilon+\delta_\varepsilon}\Pi_\varepsilon(\mathbf{X}_s^N-\mathbf{X}_s^{N,N,\varepsilon})\,\mathrm{d}B_s^{1,i}\Big|\ge\frac{\beta}{4N}\Big)\\&+\mathbb{P}\Big(|\sigma_0|\Big|\int_{\tau_\varepsilon}^{\tau_\varepsilon+\delta_\varepsilon}\Pi_\varepsilon(\mathbf{X}_s^N-\mathbf{X}_s^{N,N,\varepsilon})\,\mathrm{d}W_s\Big|\ge\frac{\beta}{4N}\Big)\\&+\mathbb{P}\Big(\Big|\int_{\tau_\varepsilon}^{\tau_\varepsilon+\delta_\varepsilon}\overline{\sigma}(X_s^{i,N,\varepsilon})\,\mathrm{d}B_s^{2,i}\Big|\ge\frac{\beta}{4N}\Big)\Bigg)\\=:&\sum_{i=1}^N\sum_{j=1}^4\Gamma_i^{j,\varepsilon}.\end{aligned}$$

In the event of $\sigma_1,\sigma_0=0$, $\Gamma_i^{2,\varepsilon}=\Gamma_i^{3,\varepsilon}=0$ holds true trivially for any $i\in\mathbb{S}_N$ so we shall prescribe $\sigma_1,\sigma\neq 0$ in the analysis below. Applying Chebyshev's inequality followed by (2.18) yields that for $R_0>0$,

$$\mathbb{P}\Big(\sup_{0\le t\le T+1}|\mathbf{X}_t^{N,N,\varepsilon}|\ge R_0\Big)\le\frac{1}{R_0}C_{T+1}N\big(1+\mathbb{E}|X_0^{1,N}|\big).$$

Hence, for any $\varepsilon_0>0$, we can take $R_0^*=R_0^*(\varepsilon_0)>0$ large enough satisfying

$$\mathbb{P}\Big(\sup_{0\le t\le T+1}|\mathbf{X}_t^{N,N,\varepsilon}|\ge R_0^*\Big)\le\varepsilon_0.\tag{2.20}$$

For $R_0^* > 0$ stipulated above, we define the stopping time

$$\tau_0^{N,\varepsilon} = \inf\left\{t \geq 0 : |\mathbf{X}_t^{N,N,\varepsilon}| \geq R_0^*\right\}.$$

Whereafter, the term $\Gamma_i^{1,\varepsilon}, i \in \mathbb{S}_N$, can be estimated as below:

$$\begin{aligned}
\Gamma_i^{1,\varepsilon} &\leq \mathbb{P}\bigg(\int_{\tau_\varepsilon}^{\tau_\varepsilon+\delta_\varepsilon} \big|b(X_s^{i,N,\varepsilon}, \widehat{\mu}_s^{N,\varepsilon}) - b(X_s^{i,N,\varepsilon}, \delta_0)\big|\,\mathrm{d}s \geq \frac{\beta}{8N}\bigg) \\
&\quad + \mathbb{P}\bigg(\int_{\tau_\varepsilon}^{\tau_\varepsilon+\delta_\varepsilon} \big|b(X_s^{i,N,\varepsilon}, \delta_0)\big|\,\mathrm{d}s \geq \frac{\beta}{8N}\bigg) \\
&\leq \mathbb{P}\bigg(\int_{\tau_\varepsilon}^{\tau_\varepsilon+\delta_\varepsilon} \mathbb{W}_1(\widehat{\mu}_s^{N,\varepsilon}, \delta_0)\,\mathrm{d}s \geq \frac{\beta}{8NL_2}\bigg) + \mathbb{P}\big(\tau_0^{N,\varepsilon} \leq T+1\big) \\
&\quad + \mathbb{P}\bigg(\int_{\tau_\varepsilon}^{\tau_\varepsilon+\delta_\varepsilon} \big|b(X_s^{i,N,\varepsilon}, \delta_0)\big|\,\mathrm{d}s \geq \frac{\beta}{8N}, \tau_0^{N,\varepsilon} > T+1\bigg) \\
&\leq \mathbb{P}\bigg(\frac{1}{N}\sum_{j=1}^N \int_{\tau_\varepsilon}^{\tau_\varepsilon+\delta_\varepsilon} |X_s^{j,N,\varepsilon}|\,\mathrm{d}s \geq \frac{\beta}{8NL_2}\bigg) + \mathbb{P}\bigg(\sup_{0\leq t\leq T+1} |\mathbf{X}_t^{N,N,\varepsilon}| \geq R_0^*\bigg) \\
&\quad + \mathbb{P}\bigg(\int_{\tau_\varepsilon}^{\tau_\varepsilon+\delta_\varepsilon} \mathbb{1}_{[0,\tau_0^{N,\varepsilon}]}(s)\big|b(X_s^{i,N,\varepsilon}, \delta_0)\big|\,\mathrm{d}s \geq \frac{\beta}{8N}\bigg),
\end{aligned}$$

where the second inequality holds true due to (2.3). As a consequence, by taking (2.18) and (2.20) into account and retrospecting that $b(\cdot, \delta_0)$ is continuous and locally bounded on $\mathbb{R}^d$ (see the Assumption ($\mathbf{A}_b$)) and $\lim_{\varepsilon\downarrow 0} \delta_\varepsilon = 0$, we conclude that $\lim_{\varepsilon\downarrow 0} \Gamma_i^{1,\varepsilon} = 0$.

On the one hand, by applying Chebyshev's inequality and Itô's isometry, along with (2.4) and (2.19), it follows that for any $i \in \mathbb{S}_N$,

$$\begin{aligned}
\Gamma_i^{2,\varepsilon} + \Gamma_i^{3,\varepsilon} &\leq \frac{16N^2}{\beta^2}\bigg((\sigma_0^2 + \sigma_1^2)\mathbb{E}\bigg(\int_{\tau_\varepsilon}^{\tau_\varepsilon+\delta_\varepsilon} \big\|\Pi_\varepsilon(\mathbf{X}_s^N - \mathbf{X}_s^{N,N,\varepsilon})\big\|_{\mathrm{HS}}^2 \mathrm{d}s\bigg)\bigg) \\
&\leq \frac{16N^2}{\beta^2}(\sigma_0^2 + \sigma_1^2)d\delta_\varepsilon.
\end{aligned}$$

On the other hand, in terms of [20, Lemma 2.3], concerning $\varepsilon_0 > 0$ given in (2.20), we find that for any $i \in \mathbb{S}_N$,

$$\begin{aligned}
\Gamma_i^{4,\varepsilon} &\leq \varepsilon_0 + \mathbb{P}\bigg(\bigg|\int_{\tau_\varepsilon}^{\tau_\varepsilon+\delta_\varepsilon} \big\|\overline{\sigma}(X_s^{i,N,\varepsilon})\big\|_{\mathrm{HS}}^2 \mathrm{d}s\bigg| \geq \frac{\beta^2\varepsilon_0}{16N^2}\bigg) \\
&\leq \varepsilon_0 + \mathbb{P}\bigg(\sup_{0\leq t\leq T+1} |\mathbf{X}_t^{N,N,\varepsilon}| \geq R_0^*\bigg) + \mathbb{P}\bigg(\bigg|\int_{\tau_\varepsilon}^{\tau_\varepsilon+\delta_\varepsilon} \mathbb{1}_{[0,\tau_0^{N,\varepsilon}]}(s)\big\|\overline{\sigma}(X_s^{i,N,\varepsilon})\big\|_{\mathrm{HS}}^2 \mathrm{d}s\bigg| \geq \frac{\beta^2\varepsilon_0}{16N^2}\bigg) \\
&\leq 2\varepsilon_0 + \mathbb{P}\bigg(\bigg|\int_{\tau_\varepsilon}^{\tau_\varepsilon+\delta_\varepsilon} \mathbb{1}_{[0,\tau_0^{N,\varepsilon}]}(s)\big\|\overline{\sigma}(X_s^{i,N,\varepsilon})\big\|_{\mathrm{HS}}^2 \mathrm{d}s\bigg| \geq \frac{\beta^2\varepsilon_0}{16N^2}\bigg),
\end{aligned}$$

where the second inequality is obtained by following the line to deal with the term $\Gamma_i^{1,\varepsilon}$, and the last display is owing to (2.20). Consequently, with the aid of $\lim_{\varepsilon\downarrow 0} \delta_\varepsilon = 0$ and the Lipschitz property of $\overline{\sigma}$ (so it is continuous and locally bounded on $\mathbb{R}^d$), the conclusion $\sum_{j=2}^4 \lim_{\varepsilon\downarrow 0} \Gamma_i^{j,\varepsilon} = 0$ is reachable for arbitrary $i \in \mathbb{S}_N$. At length, the statement (ii) is verifiable by recalling $\lim_{\varepsilon\downarrow 0} \Gamma_i^{1,\varepsilon} = 0$ for any $i \in \mathbb{S}_N$. $\square$

With Lemma 2.6 at hand, we intend to complete the proof of Proposition 2.4.

Proof of Proposition 2.4. Let $\mathscr{C}_\infty = C([0,\infty); (\mathbb{R}^d)^N)$ be the collection of continuous functions $\psi : [0,\infty) \to (\mathbb{R}^d)^N$. Define the projection operator $\pi : \mathscr{C}_\infty \to (\mathbb{R}^d)^N$ by $\pi_t\psi = \psi(t)$ for $\psi \in \mathscr{C}_\infty$ and $t \geq 0$, and write $\mathcal{F}_t = \sigma(\pi_s : s \leq t)$ as the σ-algebra on $\mathscr{C}_\infty$ induced by the projections π_s for $s \in [0,t]$.

With the help of Lemma 2.6, the Prohorov theorem yields, for fixed $N \geq 1$ and $T > 0$, that $(\mathbf{X}^N_{[0,T]}, \mathbf{X}^{N,N,\varepsilon}_{[0,T]})_{\varepsilon>0}$ has a weakly convergent subsequence $(\mathbf{X}^N_{[0,T]}, \mathbf{X}^{N,N,\varepsilon_l}_{[0,T]})_{l\geq 0}$ with the associated weak limit $(\mathbf{X}^N_{[0,T]}, \widetilde{\mathbf{X}}^{N,N}_{[0,T]})$, where $(\varepsilon_l)_{l\geq 0}$ is a sequence such that $\lim_{l\to\infty}\varepsilon_l = 0$. To demonstrate that $(\mathbf{X}^N_{[0,T]}, \widetilde{\mathbf{X}}^{N,N}_{[0,T]})$ is indeed a coupling process of $\mathbf{X}^N_{[0,T]}$ and $\mathbf{X}^{N,N}_{[0,T]}$, it is sufficient to verify that $\mathscr{L}_{\widetilde{\mathbf{X}}^{N,N}} = \mathscr{L}_{\mathbf{X}^{N,N}}$, where $\mathscr{L}_{\widetilde{\mathbf{X}}^{N,N}}$ and $\mathscr{L}_{\mathbf{X}^{N,N}}$ are the infinitesimal generators of $(\widetilde{\mathbf{X}}^{N,N}_t)_{t\geq 0}$ and $(\mathbf{X}^{N,N}_t)_{t\geq 0}$, respectively. In particular, we have for $f \in C^2_c((\mathbb{R}^d)^N)$ and $\mathbf{x} := (x_1, \cdots, x_N) \in (\mathbb{R}^d)^N$,

$$
\begin{aligned}
\big(\mathscr{L}_{\mathbf{X}^{N,N}} f\big)(\mathbf{x}) = \sum_{i=1}^N \Big(&\langle \nabla_i f(\mathbf{x}), b(x^i, \widehat{\mu}^N_{\mathbf{x}})\rangle + \frac{1}{2}\sigma_1^2\,\mathrm{trace}(\nabla^2_{ii} f(\mathbf{x})) + \frac{1}{2}\langle \nabla^2_{ii} f(\mathbf{x}), \overline{\sigma}(x^i)(\overline{\sigma}(x^i))^*\rangle_{\mathrm{HS}} \\
&+ \frac{1}{2}\sigma_0^2 \sum_{j=1}^N \mathrm{trace}\big(\nabla^2_{ij} f(\mathbf{x})\big)\Big),
\end{aligned}
$$

where $\widehat{\mu}^N_{\mathbf{x}} := \frac{1}{N}\sum_{j=1}^N \delta_{x_j}$.

To realize this goal, we define for any $f \in C^2_c((\mathbb{R}^d)^N)$,

$$
M^{N,f}_t = f(\widetilde{\mathbf{X}}^{N,N}_t) - f(\widetilde{\mathbf{X}}^{N,N}_0) - \int_0^t \big(\mathscr{L}_{\mathbf{X}^{N,N}} f\big)(\widetilde{\mathbf{X}}^{N,N}_s)\,\mathrm{d}s.
$$

For any $f \in C^2_c((\mathbb{R}^d)^N)$, provided that $(M^{N,f}_t)_{t\geq 0}$ is a martingale with respect to $(\mathcal{F}_t)_{t\geq 0}$, i.e., for any $t \geq s \geq 0$ and $\mathcal{F}_s$-measurable bounded continuous functional $F : \mathscr{C}_\infty \to \mathbb{R}$,

$$
\mathbb{E}\big(M^{N,f}_t F(\widetilde{\mathbf{X}}^{N,N})\big) = \mathbb{E}\big(M^{N,f}_s F(\widetilde{\mathbf{X}}^{N,N})\big), \tag{2.21}
$$

$\mathscr{L}_{\widetilde{\mathbf{X}}^{N,N}} = \mathscr{L}_{\mathbf{X}^{N,N}}$ is available by invoking the weak uniqueness of (2.7).

Below, we intend to prove (2.21). For $\mathbf{x} \in (\mathbb{R}^d)^N$, let $\mathscr{L}^{N,\varepsilon}_{\mathbf{x}}$ be the infinitesimal generator of $(\mathbf{X}^{N,N,\varepsilon}_t)_{t\geq 0}$ provided that the Markov process $(\mathbf{X}^{N,N}_t)_{t\geq 0}$ is known in advance. For $f \in C^2((\mathbb{R}^d)^N)$, $\mathbf{y} \in (\mathbb{R}^d)^N$, and given $\mathbf{x} \in (\mathbb{R}^d)^N$, we find that

$$
\begin{aligned}
\big(\mathscr{L}^{N,\varepsilon}_{\mathbf{x}} f\big)(\mathbf{y}) &= \big(\mathscr{L}_{\mathbf{X}^{N,N}} f\big)(\mathbf{y}) - 2\Big(\sigma_1^2 \sum_{i=1}^N \langle \nabla^2_{ii} f(\mathbf{y}), \mathbf{n}(\phi(\mathbf{x}-\mathbf{y})) \otimes \mathbf{n}(\phi(\mathbf{x}-\mathbf{y}))\rangle_{\mathrm{HS}} \\
&\qquad + \sigma_0^2 \sum_{i,j=1}^N \langle \nabla^2_{ij} f(\mathbf{y}), \mathbf{n}(\phi(\mathbf{x}-\mathbf{y})) \otimes \mathbf{n}(\phi(\mathbf{x}-\mathbf{y}))\rangle_{\mathrm{HS}}\Big) \\
&\qquad \times h_\varepsilon(\rho(\mathbf{x}-\mathbf{y}))\big(1 - h_\varepsilon(\rho(\mathbf{x}-\mathbf{y}))\big) \\
&=: \big(\mathscr{L}_{\mathbf{X}^{N,N}} f\big)(\mathbf{y}) - \big(\mathscr{L}^{N,\varepsilon,*}_{\mathbf{x}} f\big)(\mathbf{y}).
\end{aligned} \tag{2.22}
$$

By Itô's formula, for any $f \in C^2_c((\mathbb{R}^d)^N)$ and $t \geq 0$,

$$
M^{N,f,\varepsilon_l}_t := f(\mathbf{X}^{N,N,\varepsilon_l}_t) - f(\mathbf{X}^{N,N,\varepsilon_l}_0) - \int_0^t \big(\mathscr{L}^{N,\varepsilon_l}_{\mathbf{X}^N_s} f\big)(\mathbf{X}^{N,N,\varepsilon_l}_s)\,\mathrm{d}s
$$

is a martingale with respect to $(\mathcal{F}_t)_{t\geq 0}$. Therefore, for any $t \geq s \geq 0$ and $\mathcal{F}_s$-measurable bounded continuous functional $F : \mathscr{C}_\infty \to \mathbb{R}$, we obviously have

$$\mathbb{E}\big(M_t^{N,f,\varepsilon_l}F(\mathbf{X}^{N,N,\varepsilon_l})\big)=\mathbb{E}\big(M_s^{N,f,\varepsilon_l}F(\mathbf{X}^{N,N,\varepsilon_l})\big). \tag{2.23}$$

Next, owing to (2.22), M_t^{N,f,ε_l} can be rewritten as below

$$M_t^{N,f,\varepsilon_l}=f(\mathbf{X}_t^{N,N,\varepsilon_l})-f(\mathbf{X}_0^{N,N,\varepsilon_l})-\int_0^t(\mathscr{L}_{\mathbf{X}^{N,N}}f)(\mathbf{X}_s^{N,N,\varepsilon_l})\,\mathrm{d}s+\int_0^t\big(\mathscr{L}_{\mathbf{X}_s^N}^{N,\varepsilon_l,*}f\big)(\mathbf{X}_s^{N,N,\varepsilon_l})\,\mathrm{d}s$$

Whence, the assertion (2.21) is attainable by applying (2.23), [42, Lemma A.2] as well as the dominated convergence theorem, and using the fact that

$$\lim_{\varepsilon\to0}\big(\mathscr{L}_{\mathbf{x}}^{N,\varepsilon,*}f\big)(\mathbf{y})=0$$

by making use of

$$\lim_{\varepsilon\downarrow0}\big(h_\varepsilon(r)(1-h_\varepsilon(r))\big)=\lim_{\varepsilon\downarrow0}h_\varepsilon(r)\lim_{\varepsilon\downarrow0}(1-h_\varepsilon(r))=\mathbb{1}_{\{r\neq0\}}(1-\mathbb{1}_{\{r\neq0\}})=0,\quad r\geq0.$$

The proof is therefore complete. □

Before the ending of this section, we make a comment on the asymptotic coupling by reflection constructed in (2.17).

Remark 2.7.

(i) In terms of (2.16), the asymptotic reflection matrix Π_ε embodies the information concerned with all particles, which are common for each single particle. Intuitively, such construction is reasonable since we design the coupling for the system (2.17) determined by all particles rather than the single particle. Indeed, by a close inspection of the proof for Proposition 2.4, one can see that such an observation plays an extremely important role in verifying that the weak limit process of $(\mathbf{X}_{[0,T]}^N,\mathbf{X}_{[0,T]}^{N,N,\varepsilon})_{\varepsilon>0}$ is the coupling process we desire. Once Π_ε contains only partial information associated with all particles, it is impossible to examine that the weak limit process of $(\mathbf{X}_{[0,T]}^N,\mathbf{X}_{[0,T]}^{N,N,\varepsilon})_{\varepsilon>0}$ is the coupling process due to the involvement of the common noise. In particular, inspired by the reflection coupling constructed in [15, Section 3.1] for the independent nonlinear processes and the associated mean-field particle system with common noise, we can naturally take $\rho(\mathbf{x})=|x_i|$ and $\phi(\mathbf{x})=x_i/|x_i|$ for $\mathbf{x}=(x_1,\cdots,x_N)$. Whereafter, for the case $d\geq2$, the intractable term: for $x_i,y_i,x_j,y_j\in\mathbb{R}^d$,

$$\begin{aligned}&h_\varepsilon(|x_i-y_i|)\mathbf{n}(x_i-y_i)(\mathbf{n}(x_i-y_i))^\top+h_\varepsilon(|x_j-y_j|)\mathbf{n}(x_j-y_j)(\mathbf{n}(x_j-y_j))^\top\\&-2h_\varepsilon(|x_i-y_i|)h_\varepsilon(|x_j-y_j|)\langle\mathbf{n}(x_i-y_i),\mathbf{n}(x_j-y_j)\rangle\mathbf{n}(x_i-y_i)(\mathbf{n}(x_j-y_j))^\top\end{aligned}$$

appears naturally in the infinitesimal generator of $(\mathbf{X}_t^{N,N,\varepsilon})_{t\geq0}$. However, the preceding term might not converge to zero as $\varepsilon\to0$. This definitely brings essential difficulties to identify the weak limit process of $(\mathbf{X}_{[0,T]}^{N,N,\varepsilon})_{\varepsilon>0}$.

(ii) In the present framework, the conditional distribution of X_t is given under the σ-algebra $\mathscr{F}_t^W:=\sigma\{W_s,s\leq t\}$ (i.e., the σ-algebra generated by *the common noise* $(W_s)_{s\geq0}$ up to time t). Motivated by the work [15] concerning uniform-in-time PoC for McKean-Vlasov SDEs without common noise, one may decompose formally in the distribution sense the common noise $\sigma_0\mathrm{d}W_t$ into the sum of

$$\sigma_0\mathrm{d}W_t\overset{d}{=}\sigma_0\Big(h_\varepsilon\big(\rho(\mathbf{X}_t^N-\mathbf{X}_t^{N,N,\varepsilon})\big)^{\frac12}\mathrm{d}W_t+\big(1-h_\varepsilon\big(\rho(\mathbf{X}_t^N-\mathbf{X}_t^{N,N,\varepsilon})\big)\big)^{\frac12}\mathrm{d}\widetilde{W}_t\Big),$$

where $(\widetilde{W}_t)_{t\geq 0}$ is an independent copy of $(W_t)_{t\geq 0}$. In particular, one more common noise $(\widetilde{W}_t)_{t\geq 0}$ is brought into being. Accordingly, this will result in the following problems: (i) under which σ-algebra, the conditional distribution involved in the coupled SDE is defined; (ii) the corresponding measure-valued process $(\mu_t)_{t\geq 0}$ might satisfy a different stochastic FPE; see Section 3 for details. In order to bypass the aforementioned difficulties, we will employ the asymptotic coupling by reflection as indicated by the approximate interacting particle systems (2.17), which is essentially different from that in [15].

3. Proof of Theorem 1.2

Our goal in this section is to complete the proof of Theorem 1.2. In particular, we herein are only concerned with the SDE (1.7) with the case $d = 1$. To this end, there are a series of preparations to be carried out.

The non-interacting particle system corresponding to (1.7) is governed by the following SDEs: for each $i \in \mathbb{S}_N$,

$$\mathrm{d}X_t^i = b(X_t^i, \mu_t^i)\,\mathrm{d}t + \sigma(X_t^i)\,\mathrm{d}B_t^i + \sigma_0\,\mathrm{d}W_t, \tag{3.1}$$

where $\mu_t^i := \mathscr{L}_{X_t^i|\mathscr{F}_t^W}$; $(B^i)_{i\in\mathbb{S}_N} := ((B_t^i)_{t\geq 0})_{i\in\mathbb{S}_N}$ are mutually independent 1-dimensional Brownian motions on $(\Omega^1, \mathscr{F}^1, (\mathscr{F}_t^1)_{t\geq 0}, \mathbb{P}^1)$; $(X_0^i)_{i\in\mathbb{S}_N}$ are i.i.d. $\mathscr{F}_0^1$-measurable random variables. According to [7, Proposition 2.11], for any $T > 0$ and $i \in \mathbb{S}_N$,

$$\mathbb{P}^0\big(\mu_t^i = \mu_t^1 \quad \text{for all } t \in [0, T]\big) = 1$$

so that we can write $\mu_t = \mu_t^i$ for all $i \in \mathbb{S}_N$. Moreover, as shown in [7, (2.4)], $(\mu_t)_{t>0}$ solves the nonlinear stochastic FPE:

$$\mathrm{d}\mu_t = -\partial_x(b(\cdot, \mu_t)\mu_t)\,\mathrm{d}t + \frac{1}{2}\partial_{xx}^2\big((\sigma^2(\cdot) + \sigma_0^2)\mu_t\big)\,\mathrm{d}t - \partial_x\big((\sigma_0\mathrm{d}W_t)\mu_t\big). \tag{3.2}$$

The preceding SPDE is understood in the weak sense; namely, for any test function $f \in C_c^2(\mathbb{R})$,

$$\mathrm{d}\mu_t(f) = \mu_t\big(f'(\cdot)b(\cdot, \mu_t)\big)\,\mathrm{d}t + \frac{1}{2}\mu_t\big((\sigma(\cdot)^2 + \sigma_0^2)f''(\cdot)\big)\,\mathrm{d}t + \sigma_0\mu_t\big(f'(\cdot)\mathrm{d}W_t\big).$$

To expound that the idiosyncratic noise might make contributions to ergodicity of the measure-valued Markov process $(\mu_t)_{t>0}$ solving (3.2), we decompose the idiosyncratic noise part in the sense of distribution. Due to $\kappa_{\sigma,1} \leq \sigma(x)^2$ (see Assumption $(\mathbf{H}_\sigma)$), there exists a constant $\alpha > 0$ such that $\inf_{x\in\mathbb{R}} \overline{\sigma}_\alpha(x) > 0$, in which

$$\overline{\sigma}_\alpha(x)^2 := \sigma^2(x) - \alpha\kappa_{\sigma,1}, \quad x \in \mathbb{R}. \tag{3.3}$$

Subsequently, we consider the stochastic particle system:

$$\mathrm{d}\overline{X}_t^i = b(\overline{X}_t^i, \overline{\mu}_t^i)\,\mathrm{d}t + \sqrt{\alpha\kappa_{\sigma,1}}\,\mathrm{d}B_t^{1,i} + \overline{\sigma}_\alpha(\overline{X}_t^i)\,\mathrm{d}B_t^{2,i} + \sigma_0\,\mathrm{d}W_t,$$

where $\overline{\mu}_t^i := \mathscr{L}_{\overline{X}_t^i|\mathscr{F}_t^W}$; $(B^{1,i})_{i\in\mathbb{S}_N} := ((B_t^{1,i})_{t\geq 0})_{i\in\mathbb{S}_N}$ and $(B^{2,i})_{i\in\mathbb{S}_N} := ((B_t^{2,i})_{t\geq 0})_{i\in\mathbb{S}_N}$ are mutually independent 1-dimensional Brownian motions on $(\Omega^1, \mathscr{F}^1, (\mathscr{F}_t^1)_{t\geq 0}, \mathbb{P}^1)$; $(\overline{X}_0^i)_{1\leq i\leq d}$ are i.i.d. $\mathscr{F}_0^1$-measurable random variables. Once more, applying [7, Proposition 2.11], we find that for any $T > 0$ and $i \in \mathbb{S}_N$,

$$\mathbb{P}^0\big(\overline{\mu}_t^i = \overline{\mu}_t^1 \quad \text{for all } t \in [0, T]\big) = 1$$

so we can also write $\overline{\mu}_t = \overline{\mu}_t^i$ for all $i \in \mathbb{S}_N$. Satisfactorily, by noting $\sigma(x)^2 = \overline{\sigma}_\alpha(x)^2 + \alpha\kappa_{\sigma,1}$ and the independence between $(B^{1,i})_{i\in\mathbb{S}_N}$ and $(B^{2,i})_{i\in\mathbb{S}_N}$, $(\overline{\mu}_t)_{t\geq 0}$ also solves the SPDE (3.2). Therefore, to tackle ergodicity of the measure-valued process $(\mu_t)_{t>0}$, it is sufficient to work on the McKean-Vlasov SDE with common noise in the form below:

$$\mathrm{d}\overline{X}_t = b(\overline{X}_t, \overline{\mu}_t)\,\mathrm{d}t + \sqrt{\alpha\kappa_{\sigma,1}}\,\mathrm{d}B_t^1 + \overline{\sigma}_\alpha(\overline{X}_t)\,\mathrm{d}B_t^2 + \sigma_0\,\mathrm{d}W_t. \tag{3.4}$$

The previous interpretations explain roughly why we focus on the McKean-Vlasov SDE with common noise formulated in the form of (2.1).

As pointed out above, in this section, we still take the SDE (2.1) with the dimension $d = 1$ as our research object. Besides Assumptions $(\mathbf{H}_{b,1})$ and $(\mathbf{H}_{b,2})$ presented in the Introduction section, we further assume that

$(\mathbf{H}'_{\overline{\sigma}})$ *there exists a constant* $L_{\overline{\sigma}} > 0$ *such that*

$$|\overline{\sigma}(x) - \overline{\sigma}(y)| \leq L_{\overline{\sigma}}\big(1 \wedge |x-y|\big), \quad x, y \in \mathbb{R}.$$

Under Assumptions $(\mathbf{H}_{b,1})$, $(\mathbf{H}_{b,2})$ and $(\mathbf{H}'_{\overline{\sigma}})$, Assumptions $(\mathbf{A}_b)$ and $(\mathbf{A}_{\overline{\sigma}})$ hold trivially so that all results presented in Section 2 are applicable.

Before the proof of Theorem 1.2, we show that the measure-valued process $(\mu_t)_{t>0}$ associated with (2.1) has exponential decay, which is stated precisely as below.

Theorem 3.1. *Assume* $(\mathbf{H}_{b,1})$, $(\mathbf{H}_{b,2})$ *and* $(\mathbf{H}'_{\overline{\sigma}})$. *Then, there are constants* $C, \lambda, \lambda_3^* > 0$ *satisfying that for all* $\lambda_3 \in [0, \lambda_3^*]$, $t > 0$ *and* $\mu, \nu \in \mathscr{P}_1(\mathbb{R})$,

$$\mathcal{W}_1(\mu_t, \nu_t) \leq C\mathrm{e}^{-\lambda t}\,\mathbb{W}_1(\mu, \nu), \tag{3.5}$$

where $\mu_t := \mathscr{L}_{X_t|\mathscr{F}_t^W}$ *and* $\nu_t := \mathscr{L}_{X_t|\mathscr{F}_t^W}$ *mean the regular conditional distributions of* X_t, *solving* (2.1), *with the initial distributions* $\mathscr{L}_{X_0} = \mu$ *and* $\mathscr{L}_{X_0} = \nu$, *respectively;* $\lambda_3 > 0$ *is the Lipschitz constant of* $b(x, \mu)$ *with respect to the measure variable, given in* (1.10).

Prior to the commencement on the proof of Theorem 3.1, some additional work need to be accomplished. The following lemma shows that, for each $i \in \mathbb{S}_N$, $(X_t^i)_{t\geq 0}$ has finite moment in an infinite-time horizon.

Lemma 3.2. *Assume* $(\mathbf{H}_{b,1})$ *with* $\lambda_2 > 2\lambda_3$ *and* $(\mathbf{H}'_{\overline{\sigma}})$, *and suppose further that* $(X_0^i)_{i\in\mathbb{S}_N}$ *are i.i.d.* $\mathscr{F}_0^1$-*measurable random variables such that* $\mathbb{E}|X_0^1| < \infty$. *Then, there is a constant* $C_0 > 0$ *such that for all* $i \in \mathbb{S}_N$,

$$\sup_{t\geq 0} \mathbb{E}|X_t^i| \leq \mathbb{E}|X_0^1| + C_0, \tag{3.6}$$

where $((X_t^i)_{t>0})_{i\in\mathbb{S}_N}$ *solves* (2.6).

Proof. According to (1.9), (1.10) and $(\mathbf{H}'_{\overline{\sigma}})$, it follows that for all $x \in \mathbb{R}$ and $\mu \in \mathscr{P}_1(\mathbb{R})$,

$$\begin{aligned}
2xb(x,\mu) + \overline{\sigma}(x)^2 &= 2x(b(x,\mu) - b(0,\delta_0)) + 2xb(0,\delta_0) + \overline{\sigma}(x)^2 \\
&= 2x(b(x,\mu) - b(0,\mu)) + 2x(b(0,\mu) - b(0,\delta_0)) + 2xb(0,\delta_0) + \overline{\sigma}(x)^2 \\
&\leq -\lambda_2|x|^2 + 2\lambda_3\,\mathbb{W}_1(\mu,\delta_0)|x| + 2|b(0,\delta_0)||x| + (\lambda_1+\lambda_2)\ell_0 + 2\big(L_{\overline{\sigma}}^2 + \overline{\sigma}(0)^2\big).
\end{aligned}$$

Then, applying Itô's formula to V_1, defined in (2.11) with $\delta = 1$, yields that

$$\begin{aligned}\mathrm{d}\big(\mathrm{e}^{\lambda^* t}V_1(X_t^i)\big) &\le \mathrm{e}^{\lambda^* t}\bigg(\lambda^* V_1(X_t^i) + \frac{1}{2V_1(X_t^i)}\big(2X_t^i b(X_t^i,\mu_t^i) + \overline{\sigma}(X_t^i)^2 + \sigma_1^2 + \sigma_0^2\big)\bigg)\mathrm{d}t + \mathrm{d}M_t^i \\ &\le -\lambda_3 \mathrm{e}^{\lambda^* t}\big(V_1(X_t^i) - \mu_t^i(V_1)\big)\,\mathrm{d}t + C_0^*\mathrm{e}^{\lambda^* t}\,\mathrm{d}t + \mathrm{d}M_t^i\end{aligned}$$

for some constant $C_0^* > 0$ and some martingale $(M_t^i)_{t\ge0}$, where $\lambda^* := \frac{1}{2}(\lambda_2 - 2\lambda_3)$. Since $\mathbb{E}\,\mu_t^i(V_1) = \mathbb{E}V_1(X_t^i)$, and $(X_0^i)_{i\in\mathbb{S}_N}$ are i.i.d. $\mathscr{F}_0^1$-measurable random variables, we derive that

$$\mathbb{E}V_1(X_t^i) \le \mathbb{E}V_1(X_0^1) + C_0^*/\lambda^*.$$

This subsequently implies the desired assertion (3.6). □

Recall that the function ρ involved in (2.16) is free in Section 2. In the subsequent part, we shall stipulate

$$\rho(\mathbf{x}) = \|\mathbf{x}\|_1 := \frac{1}{N}\sum_{j=1}^N |x_j|, \quad \mathbf{x}\in\mathbb{R}^N$$

so that $\Pi_\varepsilon(\mathbf{x}) = 1 - 2h_\varepsilon(\|\mathbf{x}\|_1), \mathbf{x}\in\mathbb{R}^N$, and work with the corresponding stochastic system determined by (2.17). Note that, in Section 2, Proposition 2.4 and Lemma 2.5 are derived for the fixed horizon $T > 0$ and the particle number $N \ge 1$, and Proposition 2.2 is concerned with PoC in finite time. Below, we shall address the asymptotic conditional PoC in an infinite horizon for the conditional McKean-Vlasov SDE (2.1) with the dimension $d = 1$. Such a result plays a crucial role in treating ergodicity of the measure-valued process $(\mu_t)_{t>0}$.

The following statement characterizes the uniform-in-time PoC, where the associated convergence rate with respect to the particle number N is governed by the function φ, introduced in Assumption $(\mathbf{H}_{b,2})$.

Proposition 3.3. *Assume* $(\mathbf{H}_{b,1})$ *with* $\lambda_2 > 2\lambda_3$, $(\mathbf{H}_{b,2})$ *and* $(\mathbf{H}'_{\overline{\sigma}})$, *and suppose*

$$\lambda_0^* := \frac{c_2\ell_0}{1-\mathrm{e}^{-c_1\ell_0}+c_2\ell_0}\big(\lambda_1\wedge(\lambda_2/2)\big) - \Big(1+\frac{c_1}{c_2}\Big)\lambda_3 > 0, \tag{3.7}$$

where $\lambda_1,\lambda_2,\lambda_3 > 0$ *are introduced* $(\mathbf{H}_{b,1})$,

$$c_1 := \frac{\lambda_1\ell_0}{\sigma_0^2+\sigma_1^2} \quad \text{and} \quad c_2 := c_1\mathrm{e}^{-c_1\ell_0}. \tag{3.8}$$

Then, there exists a constant $C_0 > 0$ (*which is independent of* $N \ge 1$) *such that for any* $t \ge 0$,

$$\frac{1}{N}\sum_{i=1}^N \mathbb{E}|X_t^i - X_t^{i,N,\varepsilon}| \le \mathrm{e}^{-\lambda_0^* t}\frac{C_0}{N}\sum_{j=1}^N \mathbb{E}|X_0^i - X_0^{i,N,\varepsilon}| + C_0\bigg(\frac{1}{N}\big(1+\mathbb{E}|X_0^1|\big) + \varphi(N) + \varepsilon\bigg), \tag{3.9}$$

where, for each $\varepsilon > 0$, $((X_t^i)_{t\ge0}, (X_t^{i,N,\varepsilon})_{t\ge0})_{i\in\mathbb{S}_N}$ *solves* (2.17).

Below, we make a remark on the decay rate λ_0^* given in (3.7).

Remark 3.4. Plenty of McKean-Vlasov SDEs (e.g., the φ^4-model [35] and the granular media SDE [34]) can be regarded as the corresponding perturbation versions of classical SDEs. To avoid the occurrence of phase transitions, the perturbation/interaction intensity (which, for instance, can be described by the Lipschitz constant of the drifts with respect to the measure variable) is not strong in general; see e.g. [35, Theorem 3] and [34, Corollary 2.9] for more details. In the same spirit of [34,35], it is quite natural to require the

Lipschitz constant $\lambda_3 > 0$ involved in (3.7) is small. For this setting, the prerequisite $\lambda_0^* > 0$ in (3.7) can be ensured. Note that

$$\begin{aligned}\lambda_0^* =& \frac{c_1 \mathrm{e}^{-c_1\ell_0}\ell_0}{1-\mathrm{e}^{-c_1\ell_0}+c_2\ell_0}\big(\lambda_1\wedge(\lambda_2/2)\big) - \Big(1+\frac{c_1}{c_2}\Big)\lambda_3\\ =&\Big(1+\frac{\mathrm{e}^{c_1\ell_0}-1}{c_1\ell_0}\Big)^{-1}\big(\lambda_1\wedge(\lambda_2/2)\big) - (1+\mathrm{e}^{c_1\ell_0})\lambda_3.\end{aligned} \tag{3.10}$$

It follows from the increasing property of the functions $[0,\infty)\ni r\mapsto \frac{\mathrm{e}^r-1}{r}$ and $c_1\mapsto \mathrm{e}^{c_1\ell_0}$ on $(0,\infty)$ that the non-degenerate property of σ_0 and σ_1 will facilitate ergodicity of the empirical average (i.e., $\frac{1}{N}\sum_{i=1}^N |X_t^i - X_t^{i,N,\varepsilon}|$). On the other hand, it is easily seen from (3.10) that the smaller ℓ_0 or the larger λ_2 implies a faster convergence rate in (3.9). Whereas, the larger λ_1 indicates a slower convergence rate in (3.9).

Proof of Proposition 3.3. Below, for notation brevity, we set $Z_t^{i,N,\varepsilon} := X_t^i - X_t^{i,N,\varepsilon}$ for $t\geq 0$ and $i\in\mathbb{S}_N$. It is easy to see from (2.17) that for all $i\in\mathbb{S}_N$,

$$\begin{aligned}\mathrm{d}Z_t^{i,N,\varepsilon} =& \big(b(X_t^i,\mu_t^i) - b(X_t^{i,N,\varepsilon},\widehat{\mu}_t^{N,\varepsilon})\big)\,\mathrm{d}t + 2h_\varepsilon\big(\|\mathbf{Z}_t^{N,N,\varepsilon}\|_1\big)\big(\sigma_1\,\mathrm{d}B_t^{1,i} + \sigma_0\,\mathrm{d}W_t\big)\\ &+ \big(\overline{\sigma}(X_t^i) - \overline{\sigma}(X_t^{i,N,\varepsilon})\big)\,\mathrm{d}B_t^{2,i},\end{aligned}$$

where $\mathbf{Z}_t^{N,N,\varepsilon} := (Z_t^{1,N,\varepsilon},\cdots,Z_t^{N,N,\varepsilon})$. By Itô's formula, we thus have that for all $i\in\mathbb{S}_N$,

$$\begin{aligned}\mathrm{d}|Z_t^{i,N,\varepsilon}|^2 =& \Big[2Z_t^{i,N,\varepsilon}\big(b(X_t^i,\mu_t^i) - b(X_t^{i,N,\varepsilon},\widehat{\mu}_t^{N,\varepsilon})\big)\\ &+ 4h_\varepsilon\big(\|\mathbf{Z}_t^{N,N,\varepsilon}\|_1\big)^2(\sigma_1^2+\sigma_0^2) + \big(\overline{\sigma}(X_t^i) - \overline{\sigma}(X_t^{i,N,\varepsilon})\big)^2\Big]\,\mathrm{d}t\\ &+ 2Z_t^{i,N,\varepsilon}\Big[2h_\varepsilon\big(\|\mathbf{Z}_t^{N,N,\varepsilon}\|_1\big)\big(\sigma_1\,\mathrm{d}B_t^{1,i}+\sigma_0\,\mathrm{d}W_t\big) + \big(\overline{\sigma}(X_t^i) - \overline{\sigma}(X_t^{i,N,\varepsilon})\big)\,\mathrm{d}B_t^{2,i}\Big].\end{aligned} \tag{3.11}$$

Below, we set

$$f(r) := 1-\mathrm{e}^{-c_1 r} + c_2 r, \qquad r\geq 0,$$

where $c_1, c_2 > 0$ were defined in (3.8). Moreover, we define $F(r) = f(r^{1/2}), r\geq 0$. Subsequently, for $\lambda_0^* > 0$ defined in (3.7), applying Itô-Tanaka's formula (see e.g. [28, Theorem 29.5]) yields that for all $t > 0$,

$$\begin{aligned}\mathrm{e}^{\lambda_0^* t}F(|Z_t^{i,N,\varepsilon}|^2) =& F(|Z_0^{i,N,\varepsilon}|^2) + \lambda_0^*\int_0^t \mathrm{e}^{\lambda_0^* s}F(|Z_s^{i,N,\varepsilon}|^2)\,\mathrm{d}s\\ &+ \int_0^t \mathrm{e}^{\lambda_0^* s}F'_-(|Z_s^{i,N,\varepsilon}|^2)\,\mathrm{d}|Z_s^{i,N,\varepsilon}|^2 + \frac{1}{2}\int_0^t \mathrm{e}^{\lambda_0^* s}\int_0^\infty \mathrm{d}L_s^{i,N,\varepsilon,x}\,\mu_F(\mathrm{d}x),\end{aligned}$$

where F'_- means the left derivative of F, $(L_t^{i,N,\varepsilon,x})_{t\geq 0}$ is the local time of $(|Z_t^{i,N,\varepsilon}|^2)_{t\geq 0}$ at the point x, and μ_F denotes the Lebesgue-Stieltjes measure associated with the left derivative F'_- (i.e., $\mu_F([a,b)) = F'_-(b) - F'_-(a)$ for $a\leq b$). Denote by F'' the almost everywhere defined second derivative of the function F. Since $\mu_F(\mathrm{d}x)\leq F''(x)\,\mathrm{d}x$ (thanks to the fact that F'_- is non-increasing) and $t\mapsto L_t^{i,N,\varepsilon,x}$ is increasing, we infer that

$$\begin{aligned}\mathrm{e}^{\lambda_0^* t}F(|Z_t^{i,N,\varepsilon}|^2) \leq& F(|Z_0^{i,N,\varepsilon}|^2) + \lambda_0^*\int_0^t \mathrm{e}^{\lambda_0^* s}F(|Z_s^{i,N,\varepsilon}|^2)\,\mathrm{d}s\\ &+ \int_0^t \mathrm{e}^{\lambda_0^* s}F'_-(|Z_s^{i,N,\varepsilon}|^2)\,\mathrm{d}|Z_s^{i,N,\varepsilon}|^2 + \frac{1}{2}\int_0^t \mathrm{e}^{\lambda_0^* s}\int_0^\infty \mathrm{d}L_s^{i,N,\varepsilon,x}F''(x)\,\mathrm{d}x.\end{aligned} \tag{3.12}$$

Next, by the chain rule and Fubini's theorem, in addition to the occupation time formula (see e.g. [28, Theorem 29.5]), we find that

$$\begin{aligned}\int_0^t \mathrm{e}^{\lambda_0^* s}\int_0^\infty \mathrm{d}L_s^{i,N,\varepsilon,x}F''(x)\,\mathrm{d}x &= \int_0^\infty \Big(\int_0^t \mathrm{e}^{\lambda_0^* s}\,\mathrm{d}L_s^{i,N,\varepsilon,x}\Big)F''(x)\,\mathrm{d}x\\ &= \int_0^\infty \Big(\mathrm{e}^{\lambda_0^* t}L_t^{i,N,\varepsilon,x} - \lambda_0^*\int_0^t L_s^{i,N,\varepsilon,x}\mathrm{e}^{\lambda_0^* s}\,\mathrm{d}s\Big)F''(x)\,\mathrm{d}x\\ &= \mathrm{e}^{\lambda_0^* t}\int_0^t F''(|Z_s^{i,N,\varepsilon}|^2)\,\mathrm{d}\langle|Z^{i,N,\varepsilon}|^2\rangle_s\\ &\quad - \lambda_0^*\int_0^t \mathrm{e}^{\lambda_0^* s}\Big(\int_0^s F''(|Z_u^{i,N,\varepsilon}|^2)\,\mathrm{d}\langle|Z^{i,N,\varepsilon}|^2\rangle_u\Big)\,\mathrm{d}s\\ &= \mathrm{e}^{\lambda_0^* t}\int_0^t F''(|Z_s^{i,N,\varepsilon}|^2)\,\mathrm{d}\langle|Z^{i,N,\varepsilon}|^2\rangle_s\\ &\quad - \lambda_0^*\int_0^t \Big(\int_u^t \mathrm{e}^{\lambda_0^* s}\,\mathrm{d}s\Big)F''(|Z_u^{i,N,\varepsilon}|^2)\,\mathrm{d}\langle|Z^{i,N,\varepsilon}|^2\rangle_u\\ &= \int_0^t \mathrm{e}^{\lambda_0^* s}F''(|Z_s^{i,N,\varepsilon}|^2)\,\mathrm{d}\langle|Z^{i,N,\varepsilon}|^2\rangle_s,\end{aligned}$$

where $(\langle|Z^{i,N,\varepsilon}|^2\rangle_t)_{t\geq 0}$ stands for the quadratic variation process of $(|Z_t^{i,N,\varepsilon}|^2)_{t\geq 0}$. Plugging the preceding identity into (3.12) enables us to deduce that

$$\begin{aligned}\mathrm{e}^{\lambda_0^* t}F(|Z_t^{i,N,\varepsilon}|^2) \leq & F(|Z_0^{i,N,\varepsilon}|^2) + \lambda_0^*\int_0^t \mathrm{e}^{\lambda_0^* s}F(|Z_s^{i,N,\varepsilon}|^2)\,\mathrm{d}s\\ &+ \int_0^t \mathrm{e}^{\lambda_0^*}F'_-(|Z_s^{i,N,\varepsilon}|^2)\,\mathrm{d}|Z_s^{i,N,\varepsilon}|^2 + \frac{1}{2}\int_0^t \mathrm{e}^{\lambda_0^* s}F''(|Z_s^{i,N,\varepsilon}|^2)\,\mathrm{d}\langle|Z^{i,N,\varepsilon}|^2\rangle_s.\end{aligned} \tag{3.13}$$

Note that for $r>0$,

$$F'(r) = \frac{1}{2}f'(r^{1/2})r^{-1/2},\quad F''(r) = \frac{1}{4}\big(f''(r^{1/2})r^{-1} - f'(r^{1/2})r^{-3/2}\big) = \frac{1}{4}f''(r^{1/2})r^{-1} - \frac{1}{2}F'(r)r^{-1}$$

and that

$$\langle|Z^{i,N,\varepsilon}|^2\rangle_t = 4|Z_t^{i,N,\varepsilon}|^2\Big(4h_\varepsilon\big(\|\mathbf{Z}_t^{N,N,\varepsilon}\|_1\big)^2(\sigma_1^2+\sigma_0^2) + \big(\overline{\sigma}(X_t^i) - \overline{\sigma}(X_t^{i,N,\varepsilon})\big)^2\Big).$$

Whence, along with (3.11) and (3.13), $f''<0$ as well as $f'>0$, we derive that

$$\begin{aligned}\mathrm{e}^{\lambda_0^* t}f(|Z_t^{i,N,\varepsilon}|) = & f(|Z_0^{i,N,\varepsilon}|) + \lambda_0^*\int_0^t \mathrm{e}^{\lambda_0^* s}f(|Z_s^{i,N,\varepsilon}|)\,\mathrm{d}s\\ &+ \frac{1}{2}\int_0^t \mathrm{e}^{\lambda_0^* s}f'(|Z_s^{i,N,\varepsilon}|)|Z_s^{i,N,\varepsilon}|^{-1}\,\mathrm{d}|Z_s^{i,N,\varepsilon}|^2\\ &+ \frac{1}{2}\int_0^t \mathrm{e}^{\lambda_0^* s}\big(f''(|Z_s^{i,N,\varepsilon}|) - f'(|Z_s^{i,N,\varepsilon}|)|Z_s^{i,N,\varepsilon}|^{-1}\big)\\ &\qquad\times\big(4h_\varepsilon\big(\|\mathbf{Z}_s^{N,N,\varepsilon}\|_1\big)^2(\sigma_1^2+\sigma_0^2) + \big(\overline{\sigma}(X_s^i) - \overline{\sigma}(X_s^{i,N,\varepsilon})\big)^2\big)\,\mathrm{d}s + M_t^{i,N,\varepsilon}\\ \leq & f(|Z_0^{i,N,\varepsilon}|) + \int_0^t \mathrm{e}^{\lambda_0^* s}\Big(\lambda_0^* f(|Z_s^{i,N,\varepsilon}|) + 2(\sigma_1^2+\sigma_0^2)f''(|Z_s^{i,N,\varepsilon}|)h_\varepsilon\big(\|\mathbf{Z}_s^{N,N,\varepsilon}\|_1\big)^2\end{aligned} \tag{3.14}$$

$$
\begin{aligned}
&\qquad + f'(|Z_s^{i,N,\varepsilon}|)\frac{Z_s^{i,N,\varepsilon}}{|Z_s^{i,N,\varepsilon}|}\big(b(X_s^i,\mu_s^i) - b(X_s^{i,N,\varepsilon},\widehat{\mu}_s^{N,\varepsilon})\big)\Big)\,\mathrm{d}s + M_t^{i,N,\varepsilon} \\
&\le f(|Z_0^{i,N,\varepsilon}|) + \int_0^t \mathrm{e}^{\lambda_0^* s}\Big(\lambda_0^* f(|Z_s^{i,N,\varepsilon}|) + 2(\sigma_1^2+\sigma_0^2) f''(|Z_s^{i,N,\varepsilon}|) h_\varepsilon\big(\|\mathbf{Z}_s^{N,N,\varepsilon}\|_1\big)^2 \\
&\qquad + f'(|Z_s^{i,N,\varepsilon}|)\frac{Z_s^{i,N,\varepsilon}}{|Z_s^{i,N,\varepsilon}|}\big(b(X_s^i,\widetilde{\mu}_s^N) - b(X_s^{i,N,\varepsilon},\widehat{\mu}_s^{N,\varepsilon})\big)\mathbb{1}_{\{Z_s^{i,N,\varepsilon}\neq 0\}} \\
&\qquad + f'(|Z_s^{i,N,\varepsilon}|)\big|b(X_s^i,\widetilde{\mu}_s^{N,i}) - b(X_s^i,\widetilde{\mu}_s^N)\big| \\
&\qquad + f'(|Z_s^{i,N,\varepsilon}|)\big|b(X_s^i,\mu_s^i) - b(X_s^i,\widetilde{\mu}_s^{N,i})\big|\Big)\,\mathrm{d}s + M_t^{i,N,\varepsilon}
\end{aligned}
$$

for some martingale $(M_t^{i,N,\varepsilon})_{t\ge 0}$, where $x/|x| = \mathrm{sgn}(x) = 0$ if $x = 0$,

$$
\widetilde{\mu}_t^N := \frac{1}{N}\sum_{j=1}^N \delta_{X_t^j} \quad \text{and} \quad \widetilde{\mu}_t^{N,i} := \frac{1}{N-1}\sum_{j=1:j\neq i}^N \delta_{X_t^j}.
$$

By means of (1.9) and (1.10), it follows that

$$
\begin{aligned}
&\frac{Z_t^{i,N,\varepsilon}}{|Z_t^{i,N,\varepsilon}|}\mathbb{1}_{\{Z_t^{i,N,\varepsilon}\neq 0\}}\big(b(X_t^i,\widetilde{\mu}_t^N) - b(X_t^{i,N,\varepsilon},\widehat{\mu}_t^{N,\varepsilon})\big) \\
&= \frac{Z_t^{i,N,\varepsilon}}{|Z_t^{i,N,\varepsilon}|}\mathbb{1}_{\{Z_t^{i,N,\varepsilon}\neq 0\}}\Big[\big(b(X_t^i,\widetilde{\mu}_t^N) - b(X_t^{i,N,\varepsilon},\widetilde{\mu}_t^N)\big) + \big(b(X_t^{i,N,\varepsilon},\widetilde{\mu}_t^N) - b(X_t^{i,N,\varepsilon},\widehat{\mu}_t^{N,\varepsilon})\big)\Big] \\
&\le \frac{1}{2}(\lambda_1+\lambda_2)|Z_t^{i,N,\varepsilon}|\mathbb{1}_{\{|Z_t^{i,N,\varepsilon}|\le \ell_0\}} - \frac{1}{2}\lambda_2|Z_t^{i,N,\varepsilon}| + \frac{\lambda_3}{N}\sum_{j=1}^N |Z_t^{j,N,\varepsilon}|.
\end{aligned}
\tag{3.15}
$$

On the other hand, using the fact that for $\mu \in \mathscr{P}_1(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$,

$$
\mathbb{W}_1\Big(\frac{N-1}{N}\mu + \frac{1}{N}\delta_x, \mu\Big) \le \frac{1}{N}\big(|x| + \mu(|\cdot|)\big),
$$

which can be attainable analogously as [44, Lemma 3.1], in addition to (1.10) and

$$
\widetilde{\mu}_t^N = \frac{N-1}{N}\widetilde{\mu}_t^{N,i} + \frac{1}{N}\delta_{X_t^i}, \quad t\ge 0,\ i\in\mathbb{S}_N,
$$

implies that

$$
\big|b(X_t^i,\widetilde{\mu}_t^N) - b(X_t^i,\widetilde{\mu}_t^{N,i})\big| \le \lambda_3\mathbb{W}_1(\widetilde{\mu}_t^{N,i},\widetilde{\mu}_t^N) \le \frac{\lambda_3}{N}\Bigg(|X_t^i| + \frac{1}{N-1}\sum_{j=1:j\neq i}^N |X_t^j|\Bigg). \tag{3.16}
$$

Thus, plugging (3.15) and (3.16) back into (3.14) yields that

$$
\begin{aligned}
\mathrm{d}\big(\mathrm{e}^{\lambda_0^* t} f(|Z_t^{i,N,\varepsilon}|)\big) &\le \mathrm{e}^{\lambda_0^* t}\Big(\lambda_0^* f(|Z_t^{i,N,\varepsilon}|) + \psi(|Z_t^{i,N,\varepsilon}|) h_\varepsilon\big(\|\mathbf{Z}_t^{N,N,\varepsilon}\|_1\big)^2 + \Upsilon_i(\mathbf{Z}_t^{N,N,\varepsilon}) \\
&\qquad + f'(|Z_t^{i,N,\varepsilon}|)\Big(J_i(\mathbf{X}_t^N) + \frac{\lambda_3}{N}\sum_{j=1}^N |Z_t^{j,N,\varepsilon}|\Big)\Big)\,\mathrm{d}t + \mathrm{d}M_t^{i,N,\varepsilon}.
\end{aligned}
$$

Herein,

$$\psi(r) := \frac{1}{2}f'(r)\big((\lambda_1+\lambda_2)\mathbb{1}_{\{r\le \ell_0\}}-\lambda_2\big)r+2(\sigma_0^2+\sigma_1^2)f''(r), \quad r\ge 0$$
$$\Upsilon_i(\mathbf{Z}_t^{N,N,\varepsilon}) := \frac{1}{2}f'(|Z_t^{i,N,\varepsilon}|)\Big((\lambda_1+\lambda_2)|Z_t^{i,N,\varepsilon}|\mathbb{1}_{\{|Z_t^{i,N,\varepsilon}|\le \ell_0\}}-\lambda_2|Z_t^{i,N,\varepsilon}|\Big)\big(1-h_\varepsilon\big(\|\mathbf{Z}_t^{N,N,\varepsilon}\|_1\big)^2\big)$$

and

$$J_i(\mathbf{X}_t^N) := \frac{\lambda_3}{N}\bigg(|X_t^i|+\frac{1}{N-1}\sum_{j=1:j\ne i}^N |X_t^j|\bigg)\,\mathrm{d}t+\big|b(X_t^i,\mu_t^i)-b(X_t^i,\widetilde{\mu}_t^{N,i})\big|.$$

By virtue of

$$f'(r)=c_1\mathrm{e}^{-c_1 r}+c_2, \qquad f''(r)=-c_1^2\mathrm{e}^{-c_1 r}, \qquad r\ge 0,$$

and the alternatives of c_1 and c_2 given in (3.8), for any $r\le \ell_0$, we have

$$\psi(r)\le -c_1^2\mathrm{e}^{-c_1\ell_0}(\sigma_0^2+\sigma_1^2)\le -\frac{c_1^2\mathrm{e}^{-c_1\ell_0}(\sigma_0^2+\sigma_1^2)}{1-\mathrm{e}^{-c_1\ell_0}+c_2\ell_0}f(r)=-\frac{c_1c_2(\sigma_0^2+\sigma_1^2)}{1-\mathrm{e}^{-c_1\ell_0}+c_2\ell_0}f(r).$$

On the other hand, for the case $r\ge \ell_0$, we infer that

$$\begin{aligned}\psi(r)=-\frac{1}{2}\lambda_2(c_1\mathrm{e}^{-c_1 r}+c_2)r-2c_1^2(\sigma_0^2+\sigma_1^2)\mathrm{e}^{-c_1 r}&\le -\frac{\lambda_2c_2 r}{2(1-\mathrm{e}^{-c_1 r}+c_2 r)}f(r)\\ &\le -\frac{c_2\lambda_2\ell_0}{2(1-\mathrm{e}^{-c_1\ell_0}+c_2\ell_0)}f(r),\end{aligned}$$

where in the last inequality we used the fact that the function $r\mapsto \frac{r}{1-\mathrm{e}^{-c_1 r}+c_2 r}$ is increasing on $(0,\infty)$. Therefore, we arrive at

$$\psi(r)\le -\bigg(\frac{c_1c_2(\sigma_0^2+\sigma_1^2)}{1-\mathrm{e}^{-c_1\ell_0}+c_2\ell_0}\wedge\frac{c_2\lambda_2\ell_0}{2(1-\mathrm{e}^{-c_1\ell_0}+c_2\ell_0)}\bigg)f(r)=:-\lambda_0^{**}f(r), \quad r\ge 0.$$

This, along with $c_2\le f'(r)\le c_1+c_2$, implies that

$$\begin{aligned}\mathrm{e}^{\lambda_0^* t}\mathbb{E}f(|Z_t^{i,N,\varepsilon}|)\le \mathbb{E}f(|Z_0^{i,N,\varepsilon}|)+\int_0^t \mathrm{e}^{\lambda_0^* s}\bigg[&-C_1\bigg(\mathbb{E}f(|Z_s^{i,N,\varepsilon}|)-\frac{1}{N}\sum_{j=1}^N\mathbb{E}f(|Z_s^{j,N,\varepsilon}|)\bigg)\\ &+\lambda_0^{**}\mathbb{E}\Big(f(|Z_s^{i,N,\varepsilon}|)\big(1-h_\varepsilon\big(\|\mathbf{Z}_s^{N,N,\varepsilon}\|_1\big)^2\big)\Big)\\ &+(c_1+c_2)\mathbb{E}J_i(\mathbf{X}_s^N)+\mathbb{E}\Upsilon_i(\mathbf{Z}_s^{N,N,\varepsilon})\bigg]\,\mathrm{d}s,\end{aligned}$$

where $C_1 := \lambda_3(1+c_1/c_2)$. Next, combining with

$$\mathbb{E}J_i(\mathbf{X}_t^N)\le \frac{\lambda_3}{N}\bigg(\mathbb{E}|X_t^i|+\frac{1}{N-1}\sum_{j=1:j\ne i}^N\mathbb{E}|X_t^j|\bigg)+\varphi(N)\le \frac{C_2}{N}\big(1+\mathbb{E}|X_0^1|\big)+\varphi(N)$$

for some constant $C_2>0$, thanks to (1.11) and Lemma 3.2, we deduce that for some $C_3>0$,

$$
\begin{aligned}
\mathrm{e}^{\lambda_0^* t}\frac{1}{N}\sum_{i=1}^N \mathbb{E} f(|Z_t^{i,N,\varepsilon}|) \le{}& \frac{1}{N}\sum_{i=1}^N \mathbb{E} f(|Z_0^{i,N,\varepsilon}|) + C_3\bigg(\frac{1}{N}\big(1+\mathbb{E}|X_0^1|\big)+\varphi(N)\bigg)\int_0^t \mathrm{e}^{\lambda_0^* s}\,\mathrm{d}s \\
&+ \lambda_0^{**}\int_0^t \mathrm{e}^{\lambda_0^* s}\mathbb{E}\bigg(\frac{1}{N}\sum_{i=1}^N f(|Z_s^{i,N,\varepsilon}|)\Big(1-h_\varepsilon\big(\|\mathbf{Z}_s^{N,N,\varepsilon}\|_1\big)^2\Big)\bigg)\,\mathrm{d}s \\
&+ \frac{1}{N}\sum_{i=1}^N\int_0^t \mathrm{e}^{\lambda_0^* s}\mathbb{E}\Upsilon_i(\mathbf{Z}_s^{N,N,\varepsilon})\,\mathrm{d}s.
\end{aligned}
$$

By invoking $c_2 \le f'(r) \le c_1 + c_2$ and $f(0) = 0$, in addition to $h_\varepsilon \in [0, 1]$, we find that for all $s > 0$,

$$
\begin{aligned}
&\frac{1}{N}\sum_{i=1}^N f(|Z_s^{i,N,\varepsilon}|)\Big(1-h_\varepsilon\big(\|\mathbf{Z}_s^{N,N,\varepsilon}\|_1\big)^2\Big) + \frac{1}{N}\sum_{i=1}^N \Upsilon_i(\mathbf{Z}_s^{N,N,\varepsilon}) \\
&\le (c_1+c_2)(2+\lambda_1)\|\mathbf{Z}_s^{N,N,\varepsilon}\|_1\big(1-h_\varepsilon\big(\|\mathbf{Z}_s^{N,N,\varepsilon}\|_1\big)\big) \\
&\le 2(c_1+c_2)(2+\lambda_1)\varepsilon,
\end{aligned}
$$

where in the last display we used the fact that

$$
r(1-h_\varepsilon(r)) \le 2\varepsilon, \qquad r \ge 0
$$

by taking the definition of the function h_ε into consideration. Thus, we derive that for some constant $C_4 > 0$,

$$
\mathrm{e}^{\lambda_0^* t}\frac{1}{N}\sum_{i=1}^N \mathbb{E} f(|Z_t^{i,N,\varepsilon}|) \le \frac{1}{N}\sum_{i=1}^N \mathbb{E} f(|Z_0^{i,N,\varepsilon}|) + C_4\bigg(\frac{1}{N}\big(1+\mathbb{E}|X_0^1|\big)+\varphi(N)+\varepsilon\bigg)\mathrm{e}^{\lambda_0^* t}.
$$

Consequently, according to $c_2 \le f'(r) \le c_1 + c_2$ again and $f(0) = 0$, there is a constant $C_5 > 0$ so that for all $t > 0$,

$$
\frac{1}{N}\sum_{i=1}^N \mathbb{E}|Z_t^{i,N,\varepsilon}| \le \mathrm{e}^{-\lambda_0^* t}\frac{C_5}{N}\sum_{i=1}^N \mathbb{E}|Z_0^{i,N,\varepsilon}| + C_5\bigg(\frac{1}{N}\big(1+\mathbb{E}|X_0^1|\big)+\varphi(N)+\varepsilon\bigg),
$$

and so the desired assertion follows directly. □

Before we proceed, we make an additional comment.

Remark 3.5. We turn to the multi-dimensional case (i.e., $d \ge 2$). Recall that the functions ρ and ϕ involved in Π_ε, defined in (2.16), are undetermined. By applying the Itô-Tanaka formula to the radial process $|Z_t^{i,N,\varepsilon}|$, it is easy to see that the quadratic variation term:

$$
\begin{aligned}
\Upsilon_t^{i,N,\varepsilon} :={}& \frac{2}{|Z_t^{i,N,\varepsilon}|^3}\mathbb{1}_{\{Z_t^{i,N,\varepsilon}\neq\mathbf{0}\}} h_\varepsilon(\rho(\mathbf{Z}_t^{N,N,\varepsilon})))^2 \\
&\times \langle |Z_t^{i,N,\varepsilon}|^2 I_d - Z_t^{i,N,\varepsilon}\otimes Z_t^{i,N,\varepsilon}, \mathbf{n}(\phi(\mathbf{Z}_t^{N,N,\varepsilon}))\otimes\mathbf{n}(\phi(\mathbf{Z}_t^{N,N,\varepsilon}))\rangle_{\mathrm{HS}}
\end{aligned}
$$

arises naturally, where $\mathbf{Z}_t^{N,N,\varepsilon} := \mathbf{X}_t^N - \mathbf{X}_t^{N,N,\varepsilon}$. Due to the appearance of $|Z_t^{i,N,\varepsilon}|^2 I_d - Z_t^{i,N,\varepsilon} \otimes Z_t^{i,N,\varepsilon}$, we cannot choose $\phi(\mathbf{Z}_t^{N,N,\varepsilon})$, which are dependent on the whole particles, to kill the term $\Upsilon_t^{i,N,\varepsilon}$. On the other hand, the proof of Proposition 2.4 is unavailable as soon as we take $\phi(\mathbf{Z}_t^{N,N,\varepsilon}) = \phi(Z_t^{i,N,\varepsilon})$ (i.e., dependent merely on the i-th component); see Remark 2.7 for more explanations. The aforementioned interpretations further demonstrate why we focus just on the 1-dimensional SDE (1.7) rather than the multi-dimensional setting.

Based on the previous warm-up preparations, we start to complete the proof of Theorem 3.1.

Proof of Theorem 3.1. For $\mu,\nu\in\mathscr{P}_1(\mathbb{R})$, via existence of optimal couplings, there is $\pi^*\in\mathscr{C}(\mu,\nu)$ such that

$$\mathbb{W}_1(\mu,\nu)=\int_{\mathbb{R}\times\mathbb{R}}|x-y|\pi^*(\mathrm{d}x,\mathrm{d}y). \tag{3.17}$$

Let $((X_t^{i,\mu})_{t>0})_{i\in\mathbb{S}_N}$ and $((X_t^{i,N,\nu})_{t>0})_{i\in\mathbb{S}_N}$ be solutions to (2.6) and (2.7), respectively. Furthermore, $(X_0^{i,\mu},X_0^{i,N,\nu})_{i\in\mathbb{S}_N}$ are set to be i.i.d. $\mathscr{F}_0^1$-measurable random variables such that $\mathscr{L}_{(X_0^i,X_0^{i,N,\nu})}=\pi^*$. Therefore, $\mathbb{W}_1(\mu,\nu)=\mathbb{E}|X_0^{i,\mu}-X_0^{i,N,\nu}|$, and the common distributions of $X_0^{i,\mu}$ and $X_0^{i,N,\nu}$ are μ and ν, respectively.

Via the triangle inequality, it is easy to see that for all $t>0$,

$$\begin{aligned}\mathcal{W}_1(\mu_t,\nu_t)&\le\mathbb{E}^0\mathbb{W}_1(\mu_t,\nu_t)\\&\le\mathbb{E}^0\Big(\mathbb{E}^1\mathbb{W}_1\Big(\mu_t,\frac1N\sum_{j=1}^N\delta_{X_t^{j,\mu}}\Big)\Big)+\mathbb{E}^0\Big(\mathbb{E}^1\mathbb{W}_1\Big(\frac1N\sum_{j=1}^N\delta_{X_t^{j,\mu}},\frac1N\sum_{j=1}^N\delta_{X_t^{j,N,\nu}}\Big)\Big)\\&\quad+\mathbb{E}^0\Big(\mathbb{E}^1\mathbb{W}_1\Big(\frac1N\sum_{j=1}^N\delta_{X_t^{j,N,\nu}},\frac1N\sum_{j=1}^N\delta_{X_t^{j,\nu}}\Big)\Big)+\mathbb{E}^0\Big(\mathbb{E}^1\mathbb{W}_1\Big(\nu_t,\frac1N\sum_{j=1}^N\delta_{X_t^{j,\nu}}\Big)\Big)\\&=\mathbb{E}\,\mathbb{W}_1\Big(\mu_t,\frac1N\sum_{j=1}^N\delta_{X_t^{j,\mu}}\Big)+\mathbb{E}\,\mathbb{W}_1\Big(\frac1N\sum_{j=1}^N\delta_{X_t^{j,\mu}},\frac1N\sum_{j=1}^N\delta_{X_t^{j,N,\nu}}\Big)\\&\quad+\mathbb{E}\,\mathbb{W}_1\Big(\frac1N\sum_{j=1}^N\delta_{X_t^{j,N,\nu}},\frac1N\sum_{j=1}^N\delta_{X_t^{j,\nu}}\Big)+\mathbb{E}\,\mathbb{W}_1\Big(\nu_t,\frac1N\sum_{j=1}^N\delta_{X_t^{j,\nu}}\Big)\\&=:\Gamma_1(t,N)+\Gamma_2(t,N)+\Gamma_3(t,N)+\Gamma_4(t,N).\end{aligned}\tag{3.18}$$

In the subsequent analysis, we estimate the terms $\Gamma_i(t,N), i=1,2,3,4$, separately. Obviously, Assumption $(\mathbf{H}_{\overline{\sigma}}')$ implies Assumption $(\mathbf{A}_{\overline{\sigma}})$. Since $(X_0^{i,\mu})_{i\in\mathbb{S}_N}$ (resp. $(X_0^{i,\nu})_{i\in\mathbb{S}_N}$) are i.i.d. random variables with $\mathbb{E}|X_0^{1,\mu}|<\infty$ (resp. $\mathbb{E}|X_0^{1,\nu}|<\infty$), an application of Proposition 2.2 yields that

$$\lim_{N\to\infty}\big(\Gamma_1(t,N)+\Gamma_4(t,N)\big)=0.$$

Next, note that

$$\Gamma_3(t,N)\le\frac1N\sum_{j=1}^N\mathbb{E}|X_t^{j,\nu}-X_t^{j,N,\nu}|=\mathbb{E}|X_t^{1,\nu}-X_t^{1,N,\nu}|,$$

where the identity is due to the fact that $(X_t^{i,\nu},X_t^{i,N,\nu})$ and $(X_t^{j,\nu},X_t^{j,N,\nu})$ are identically distributed due to the fact that $(X_0^{i,\mu},X_0^{i,N,\nu})_{1\le i\le N}$ are i.i.d. $\mathscr{F}_0^1$-measurable random variables. Whereafter, applying Proposition 2.2 once more prompts us to derive that

$$\lim_{N\to\infty}\Gamma_3(t,N)=0.$$

Consider the system (2.17) associated with the processes $(X_t^{i,\mu})_{t\ge0}$ and $(X_t^{i,N,\nu})_{t\ge0}$, which are respective solutions to (2.6) and (2.7). For each $\varepsilon>0$, denote by $(\mathbf{X}_t^N,\mathbf{X}_t^{N,N,\varepsilon})_{t\ge0}$ the solution to the system (2.17). Evidently, Assumptions $(\mathbf{H}_{b,1})$ and $(\mathbf{H}_{\overline{\sigma}}')$ imply $(\mathbf{A}_b)$ and $(\mathbf{A}_{\overline{\sigma}})$. So, according to Proposition 2.4, $(\mathbf{X}_{[0,T]}^N,\mathbf{X}_{[0,T]}^{N,N,\varepsilon})_{\varepsilon>0}$ has a weakly convergent subsequence such that the corresponding weak limit process is the coupling process of $\mathbf{X}^N_{[0,T]}$ and $\mathbf{X}^{N,N}_{[0,T]}$ for any $T > 0$. In the following analysis, for the sake of notation simplicity, we shall still write $(\mathbf{X}^N_t, \mathbf{X}^{N,N}_t)_{t\in[0,T]}$ as the associated weak limit process (which is obtained by letting $\varepsilon \downarrow 0$ for fixed $T < \infty$ and $1 \le N < \infty$). Furthermore, it is ready to see that there exists a constant $\lambda_3^* < \lambda_2/2$ such that (3.7) is true for any $\lambda_3 \in [0, \lambda_3^*]$. Thus, by employing Proposition 3.3, there exists a constant $C^\star > 0$ such that

$$\begin{aligned}\Gamma_2(t,N) \le& C^\star\Big(\mathrm{e}^{-\lambda_0^* t}\mathbb{E}|X_0^{i,\mu} - X_0^{i,N,\nu}| + \frac{1}{N}\big(1+\mathbb{E}|X_0^1|\big) + \varphi(N)\Big)\\ =& C^\star\Big(\mathrm{e}^{-\lambda_0^* t}\mathbb{W}_1(\mu,\nu) + \frac{1}{N}\big(1+\mathbb{E}|X_0^1|\big) + \varphi(N)\Big),\end{aligned}$$

where the function $\varphi(N)$ was introduced in (1.11). The estimate above, together with the prerequisite $\lim_{N\to\infty}\varphi(N) = 0$, leads to

$$\limsup_{N\to\infty}\Gamma_2(t,N) \le C^\star \mathrm{e}^{-\lambda_0^* t}\mathbb{W}_1(\mu,\nu).$$

At last, by putting together the estimates concerning $\Gamma_i(t,N)$, $i = 1,\cdots,4$, we accomplish the proof of Theorem 3.1. □

We now can present the proof of Theorem 1.2 on the basis of Theorem 3.1.

Proof of Theorem 1.2. As we elaborated in the second paragraph of this section, in order to investigate ergodicity of the measure-valued process $(\mu_t)_{t>0}$ associated with (1.7), it is sufficient to consider the McKean-Vlasov SDE with common noise (3.4). Based on Theorem 3.1, it remains to examine Assumptions imposed in Theorem 3.1 with $\sigma_1 = \sqrt{\alpha\kappa_{\sigma,1}}$ and $\overline{\sigma}(x) = \overline{\sigma}_\alpha(x)$, separately. Concerning the drift b, the same assumptions are set in Theorems 1.2 and 3.1. So, the validation on the drift b is trivial.

Define the set

$$\Lambda_\sigma = \Big\{\alpha > 0 : \inf_{x\in\mathbb{R}}\overline{\sigma}_\alpha(x) > 0\Big\},$$

where $\overline{\sigma}_\alpha(x) = (\sigma(x)^2 - \alpha\kappa_{\sigma,1})^{1/2}$ (see (3.3) for details). Below, we fix $\alpha \in \Lambda_\sigma$. By virtue of $(\mathbf{H}_\sigma)$, we deduce that for $x, y \in \mathbb{R}$,

$$|\overline{\sigma}_\alpha(x) - \overline{\sigma}_\alpha(y)| \le 2\sqrt{\kappa_{\sigma,2}},$$

and that for $x, y \in \mathbb{R}$,

$$\begin{aligned}|\overline{\sigma}_\alpha(x) - \overline{\sigma}_\alpha(y)| = \frac{|\sigma(x)^2 - \sigma(y)^2|}{\overline{\sigma}_\alpha(x) + \overline{\sigma}_\alpha(y)} &\le \frac{(|\sigma(x)| + |\sigma(y)|)|\sigma(x) - \sigma(y)|}{\overline{\sigma}_\alpha(x) + \overline{\sigma}_\alpha(y)}\\ &\le \frac{L_\sigma\sqrt{\kappa_{\sigma,2}}}{\inf_{x\in\mathbb{R}}\overline{\sigma}_\alpha(x)}|x-y|.\end{aligned}$$

Therefore, we arrive at

$$|\overline{\sigma}_\alpha(x) - \overline{\sigma}_\alpha(y)| \le \bigg(\big(2\sqrt{\kappa_{\sigma,2}}\big) \vee \frac{L_\sigma\sqrt{\kappa_{\sigma,2}}}{\inf_{x\in\mathbb{R}}\overline{\sigma}_\alpha(x)}\bigg)\big(1 \wedge |x-y|\big), \quad x, y \in \mathbb{R}.$$

Whence, Assumption $(\mathbf{H}'_{\overline{\sigma}_\alpha})$ holds true with

$$L'_{\overline{\sigma}_\alpha} = \big(2\sqrt{\kappa_{\sigma,2}}\big) \vee \frac{L_\sigma\sqrt{\kappa_{\sigma,2}}}{\inf_{x\in\mathbb{R}}\overline{\sigma}_\alpha(x)}.$$

Furthermore, with $\sigma_1 = \sqrt{\alpha\kappa_{\sigma,1}}$ and $\overline{\sigma}(x) = \overline{\sigma}_\alpha(x)$ at hand, there exists a positive constant $\lambda_3^* < \lambda_2/2$ such that $\lambda_0^* > 0$ for all $\lambda_3 \in (0, \lambda_3^*]$, where λ_0^* was introduced in (3.7).

In a word, all of the sufficiency conditions in Theorem 3.1 are fulfilled and therefore the proof of Theorem 1.2 is complete. □

Before the end of this section, we make some further comments on the comparisons between our main result and [36, Section 4] for the case $d = 1$, and the approach available for the multi-dimensional setting (i.e., $d \geq 2$).

Remark 3.6. We compare Theorem 1.2 with the counterpart of [36, Section 4] based on the following four aspects:

- *Framework*: In [36, Section 4], the drift $b(x,\mu) = -V'(x) + \int_{\mathbb{R}} W'(x-y)\mu(\mathrm{d}y)$, where both V' and W' are of *linear growth*. Whereas, in our setting, the drift b is much more general and is allowed to be of polynomial growth with respect to the state variable. Moreover, in [36, Section 4], the idiosyncratic noise is additive. However, in the present work, the idiosyncratic noise is multiplicative.
- *Contribution of noises*: As shown in Proposition 3.3 and Remark 3.4, not only the common noise but also the idiosyncratic noise make contributions to ergodicity of the measure-valued process $(\mu_t)_{t>0}$. Nevertheless, in [36, Section 4], the common noise makes the sole contribution to ergodicity of $(\mu_t)_{t>0}$.
- *Construction of the asymptotic coupling by reflection*: In general, we can decompose the noise part in the sense of distribution to construct (asymptotic) coupling by reflection when the underlying SDEs (including McKean-Vlasov SDEs) are partially dissipative as indicated in [36, Section 4]. However, regarding McKean-Vlasov SDEs with common noise, if we adopt the previous procedure, then the common noise will become not explicit and moreover change drastically so the measure-valued process $(\mu_t)_{t>0}$ might satisfy a different nonlinear stochastic FPE. Moreover, in order to carry out the proof of [36, Theorem 2], the identity [36, (26)] is vital. Unfortunately, there is a gap to derive [36, (26)] by invoking the following SDE:

$$\mathrm{d}|E_t^{i,N,\delta}| = -e_t^{i,N,\delta}\big(V'(X_t^{i,\delta}) - V'(X_t^{i,N,\delta})\big)\mathrm{d}t + A_t^{i,N,\delta}\mathrm{d}t + 2\sigma_0\pi_\delta(E_t^{N,\delta})e_t^{i,N,\delta}\mathrm{d}B_t^0, \tag{3.19}$$

  where, particularly, $\pi_\delta(E_t^{N,\delta})^2\mathbb{1}_{\{E_t^{i,N,\delta}\neq 0\}} \neq \pi_\delta(E_t^{N,\delta})^2$. Herein, $E_t^{N,\delta} := (E_t^{1,N,\delta}, \cdots, E_t^{N,N,\delta})$ with $E_t^{i,N,\delta} := X_t^{i,\delta} - X_t^{i,N,\delta}$; $e_t^{i,N,\delta} := \mathrm{sign}(E_t^{i,N,\delta})$; $\pi_\delta : \mathbb{R} \to [0,1]$ is a non-decreasing and continuous function such that $\pi_\delta(x) = 1$ for $\frac{1}{N}\sum_{j=1}^N |x_j| \geq \delta$ and $\pi_\delta(x) = 0$ for $\frac{1}{N}\sum_{j=1}^N |x_j| \leq \delta/2$; for each $i \in \mathbb{S}_N$, $(A_t^{i,N,\delta})_{t>0}$ is an adapted non-negative stochastic process given in [36, Proposition 6]. Most importantly, we would like to emphasize that, unlike [15, Lemma 7], the SDE (3.19) cannot be derived via an approximate strategy as shown in [36, Appendix A.5], where in particular the identity in [36, p. 28] is not valid since the variables involved in functions π_δ and ψ_a are not consistent. Based on previous viewpoints, we build a totally novel asymptotic coupling by reflection as demonstrated in (2.17).
- *Moment on initial distributions*: To investigate ergodicity of $(\mu_t)_{t>0}$ under the Wasserstein distance $\mathcal{W}_1$, it is quite reasonable to require $\mathscr{L}_{\mu_0} \in L_1(\mathscr{P}(\mathbb{R}))$, which is imposed in Theorem 1.2. However, $\mathscr{L}_{\mu_0} \in L_4(\mathscr{P}(\mathbb{R}))$ was set in [36, Corollary 3].

Remark 3.7. The proof for the ergodicity of the measured process $(\mu_t)_{t>0}$ relies on the inequality (3.18), where the terms $\Gamma_1(t,N)$, $\Gamma_3(t,N)$ and $\Gamma_4(t,N)$ can be handled similarly for $d \geq 2$ due to the fact that Proposition 2.4 holds for all $d \geq 1$. Therefore, the main task is to estimate $\Gamma_2(t,N)$. For this, we still make use of the asymptotic coupling by reflection constructed in Subsection 2.2. For the case $d \geq 2$, we take $\varphi(\mathbf{x}) = \overline{\mathbf{x}} := \frac{1}{N}\sum_{j=1}^N x_j$ and $\rho(\mathbf{x}) = |\overline{\mathbf{x}}|$, which is different from the one-dimensional case. Note that the averaged process $\overline{Z}_t^{N,\varepsilon} := \frac{1}{N}\sum_{j=1}^N (X_t^j - X_t^{j,N,\varepsilon})$ solves the following SDE:

$$
\begin{aligned}
\mathrm{d}\overline{Z}_t^{N,\varepsilon} &= \frac{1}{N}\sum_{i=1}^N \big(b(X_t^i,\mu_t^i) - b(X_t^{i,N,\varepsilon},\widehat{\mu}_t^{N,\varepsilon})\big)\,\mathrm{d}t \\
&\quad + 2h_\varepsilon(|\overline{Z}_t^{N,\varepsilon}|)\mathbf{n}(\overline{Z}_t^{N,\varepsilon})\otimes\mathbf{n}(\overline{Z}_t^{N,\varepsilon})\Big(\frac{1}{N}\sum_{i=1}^N \sigma_1\,\mathrm{d}B_t^{1,i} + \sigma_0\,\mathrm{d}W_t\Big).
\end{aligned}
$$

Whence, to derive the long-term estimate on the quantity $|\overline{Z}_t^{N,\varepsilon}|$, a special structure on b (e.g., $b(x,\mu) = -x + b_0(x,\mu)$ for some $b_0 : \mathbb{R}^d\times\mathscr{P}(\mathbb{R}^d)\to\mathbb{R}^d$) need to be enforced. This undoubtedly restrict applications of the theory derived. Furthermore, to achieve our goal, it is also necessary to quantitatively estimate the uniform-in-time moment distance between each component process X_t^i (resp. $X_t^{i,N,\varepsilon}$) and the averaged process $\frac{1}{N}\sum_{j=1}^N X_t^j$ (resp. $\frac{1}{N}\sum_{j=1}^N X_t^{j,N,\varepsilon}$); see [36, Proposition 8] for related details. Unfortunately, such an estimate necessitates to require $\sigma_1 = 0$. This reduces definitely practical applications of the main result. Additionally, for the multi-dimensional setup, [36, Theorem 3] derived Theorem 1.2 by setting specifically $b(x,\mu) := -V'(x) - 2\alpha\int_{\mathbb{R}^d}(x-y)\mu(\mathrm{d}y)$ for $\alpha > 0$, and $\sigma_1 = 0$, where $V' : \mathbb{R}^d\to\mathbb{R}^d$ is globally Lipschitz. For more related discussions, one can refer to [36, Section 5].

4. Appendix

This Appendix section is devoted to providing a sufficiency condition to guarantee that Assumption $(\mathbf{H}_{b,2})$ is valid.

Lemma 4.1. *Let $((X_t^i)_{t\ge0})_{i\in\mathbb{S}_N}$ be conditionally independent and identically distributed under the filtration $\mathscr{F}_t^W$ and $b(x,\mu) = \int_{\mathbb{R}} b_0(x-y)\,\mu(\mathrm{d}y)$ for some Lipschitz continuous function $b_0 : \mathbb{R}\to\mathbb{R}$. Then, there exists a constant $C_0 > 0$ such that for all $i\in\mathbb{S}_N$ and $t\ge0$,*

$$
\mathbb{E}|b(X_t^i,\mu_t^i) - b(X_t^i,\widetilde{\mu}_t^{N,i})|^2 \le \frac{C_0}{N}\big(1+\mathbb{E}|X_t^i|^2\big), \tag{4.1}
$$

where $\widetilde{\mu}_t^{N,i} := \frac{1}{N-1}\sum_{j=1:j\ne i}^N \delta_{X_t^j}$. In particular, $(\mathbf{H}_{b,2})$ holds true with

$$
\varphi(N) := \frac{\sqrt{C_0}}{\sqrt{N}}\Big(1+\sup_{t\ge0}(\mathbb{E}|X_t^i|^2)^{1/2}\Big)
$$

in case of $\sup_{t\ge0}(\mathbb{E}|X_t^i|^2)^{1/2} < \infty$.

Proof. Obviously, Assumption $(\mathbf{H}_{b,2})$ is available provided that (4.1) is attainable plus the validity of $\sup_{t\ge0}(\mathbb{E}|X_t^i|^2)^{1/2} < \infty$.

Below, let $((X_t^i)_{t\ge0})_{i\in\mathbb{S}_N}$ be conditionally independent and identically distributed under the filtration $\mathscr{F}_t^W$ and set $\mu_t^i := \mathscr{L}_{X_t^i|\mathscr{F}_t^W}$. Since

$$
b(X_t^i,\mu_t^i) = \frac{1}{N-1}\sum_{j=1:j\ne i}^N \mathbb{E}\big(b_0(X_t^i - X_t^j)\big|X_t^i,\mathscr{F}_t^W\big),
$$

we thus obtain that

$$
\mathbb{E}|b(X_t^i,\mu_t^i) - b(X_t^i,\widetilde{\mu}^{N,i})|^2 = \frac{1}{(N-1)^2}\Big(\sum_{j=1:j\ne i}^N \mathbb{E}|\Psi_t^{ij}|^2 + \sum_{j,k=1:j,k\ne i,j\ne k}^N \mathbb{E}\big(\Psi_t^{ij}\Psi_t^{ik}\big)\Big),
$$

where

$$\Psi_t^{ij} := \mathbb{E}\big(b_0(X_t^i - X_t^j)\big|X_t^i, \mathscr{F}_t^0\big) - b_0(X_t^i - X_t^j).$$

Notice that for any $j, k \neq i$ and $j \neq k$,

$$\mathbb{E}\big(\Psi_t^{ij}\Psi_t^{ik}\big) = \mathbb{E}\big(\mathbb{E}\big(\Psi_t^{ij}\Psi_t^{ik}\big|X_t^i, \mathscr{F}_t^0\big)\big) = \mathbb{E}\big(\mathbb{E}\big(\mathbb{E}(\Psi_t^{ij}|X_t^i, \mathscr{F}_t^0)\mathbb{E}(\Psi_t^{ik}|X_t^i, \mathscr{F}_t^0)\big)\big) = 0$$

by taking the conditional independency under $\mathscr{F}_t^W$ of the sequence $(X_t^i)_{i\in\mathbb{S}_N}$ into consideration. Subsequently, we derive that

$$\begin{aligned}\mathbb{E}|b(X_t^i, \mu_t^i) - b(X_t^i, \widetilde{\mu}^{N,i})|^2 &\leq \frac{2}{(N-1)^2}\sum_{j=1:j\neq i}^{N}\mathbb{E}\big|b_0(X_t^i - X_t^j)\big|^2 \\ &\leq \frac{C_0}{(N-1)^2}\sum_{j=1}^{N}\big(\mathbb{E}|X_t^j|^2 + |b_0(0)|^2\big),\end{aligned}$$

where in the second inequality we utilized the Lipschitz property of b_0 and the fact that X_t^i and X_t^j are identically distributed given $\mathscr{F}_t^W$. Finally, (4.1) follows directly by using again that, for any $i, j \in \mathbb{S}_N$, X_t^i and X_t^j share the same law. □

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