Log-Harnack Inequality and Bismut Formula for McKean-Vlasov SDEs with Singularities in all Variables

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Abstract

The log-Harnack inequality and Bismut formula are established for McKean-Vlasov SDEs with singularities in all (time, space, distribution) variables, where the drift satisfies an integrability condition in time-space, and the continuity in distribution may be weaker than Dini. The main results considerably improve the existing ones for the case where the drift is $\boldsymbol{L}$-differentiable and Lipschitz continuous in distribution with respect to the 2-Wasserstein distance.

Keywords: McKean-Vlasov SDEs, Log-Harnack inequality, Bismut formula, Dini function, Wasserstein distance

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1 Introduction

Let $\mathscr{P}$ be the set of all probability measures on $\mathbb{R}^d$ equipped with the weak topology, and let W_t be an m-dimensional Brownian motion on a complete filtration probability space $(\Omega, \{\mathscr{F}_t\}_{t\geq 0}, \mathscr{F}, \mathbb{P})$. Consider the following McKean-Vlasov SDE on $\mathbb{R}^d$:

$$\mathrm{d}X_t = b_t(X_t, \mathscr{L}_{X_t})\mathrm{d}t + \sigma_t(X_t)\mathrm{d}W_t, \quad t \in [0,T], \tag{1}$$

where $T>0$ is a fixed time, $\mathscr{L}_{X_t}$ is the distribution of X_t, and

$$b:[0,T]\times\mathbb{R}^d\times\tilde{\mathscr{P}}\to\mathbb{R}^d,\quad \sigma:[0,T]\times\mathbb{R}^d\to\mathbb{R}^d\otimes\mathbb{R}^m$$

are measurable for some non-empty subspace $\tilde{\mathscr{P}}\subset\mathscr{P}$ equipped with a complete distance $\tilde{\rho}$. Because of its wide applications, this type SDE has been intensively investigated, see for instance [4, 5, 8, 9, 16, 17, 23] and the survey [11].

In this paper, we study the regularity of (1) for distributions in

$$\mathscr{P}_k:=\big\{\mu\in\mathscr{P}:\|\mu\|_k:=\mu(|\cdot|^k)^{\frac{1}{k}}<\infty\big\},\quad k\in(1,\infty).$$

Note that $\mathscr{P}_k$ is a Polish space under the Wasserstein distance

$$\mathbb{W}_k(\mu,\nu)=\inf_{\pi\in\mathscr{C}(\mu,\nu)}\bigg(\int_{\mathbb{R}^d\times\mathbb{R}^d}|x-y|^k\pi(\mathrm{d}x,\mathrm{d}y)\bigg)^{\frac{1}{k}},$$

where $\mathscr{C}(\mu,\nu)$ is the set of all couplings of μ and ν. The SDE (1) is called well-posed for distributions in $\mathscr{P}_k$, if for any initial value X_0 with $\mathscr{L}_{X_0}\in\mathscr{P}_k$ (respectively, any initial distribution $\gamma\in\mathscr{P}_k$), it has a unique solution (respectively, a unique weak solution) $X=(X_t)_{t\in[0,T]}$ such that $\mathscr{L}_{X_\cdot}:=(\mathscr{L}_{X_t})_{t\in[0,T]}\in C([0,T];\mathscr{P}_k)$. In this case, for any $\gamma\in\mathscr{P}_k$, let $P_t^*\gamma=\mathscr{L}_{X_t^\gamma}$ for the solution X_t^γ with $\mathscr{L}_{X_0^\gamma}=\gamma$. We study the regularity of the map

$$\mathscr{P}_k\ni\gamma\mapsto P_tf(\gamma):=\mathbb{E}[f(X_t^\gamma)]=\int_{\mathbb{R}^d}f\mathrm{d}\{P_t^*\gamma\}$$

for $t\in(0,T]$ and $f\in\mathscr{B}_b(\mathbb{R}^d)$, where $\mathscr{B}_b(\mathbb{R}^d)$ is the space of bounded measurable functions on $\mathbb{R}^d$.

As powerful tools characterizing the regularity in distribution for stochastic systems, the dimension-free Harnack inequality due to [25], the log-Harnack inequality introduced in [26], and the Bismut (also called Bismut-Elworthy-Li) formula developed from [6, 10], have been intensively investigated. See for instance the monograph [27] for an account of related study on SPDEs.

In recent years, the log-Harnack inequality and Bismut type formula have also been established for McKean-Vlasov SDEs with coefficients regular in the distribution variable. Below we present a brief summary.

Write $b_t(x,\mu)=b_t^{(0)}(x)+b_t^{(1)}(x,\mu)$. According to [29], if $b^{(0)}$ satisfies some integrability condition on (t,x), and there exists a constant $K_b\geq0$ such that

$$|b_t^{(1)}(x,\mu)-b_t^{(1)}(y,\nu)|\leq K_b(|x-y|+\mathbb{W}_2(\mu,\nu)),\quad (x,\mu),(y,\nu)\in\mathbb{R}^d\times\mathscr{P}_2,t\in[0,T],$$

then there exists a constant $c>0$ such that the log-Harnack inequality

$$P_t\log f(\tilde{\gamma})\leq\log P_tf(\gamma)+\frac{c}{t}\mathbb{W}_2(\gamma,\tilde{\gamma})^2,\quad t\in(0,T],\ f\in\mathscr{B}_b^+(\mathbb{R}^d),\ \gamma,\tilde{\gamma}\in\mathscr{P}_2$$

holds, where $\mathscr{B}_b^+(\mathbb{R}^d)$ is the space of positive elements in $\mathscr{B}_b(\mathbb{R}^d)$. This inequality is equivalent to the entropy-cost inequality

$$\mathrm{Ent}(P_t^*\gamma|P_t^*\tilde{\gamma}) \le \frac{c}{t}\mathbb{W}_2(\gamma,\tilde{\gamma})^2, \quad t\in(0,T], \gamma,\tilde{\gamma}\in\mathscr{P}_2,$$

where Ent is the relative entropy, i.e. for any $\mu,\nu\in\mathscr{P}$, $\mathrm{Ent}(\nu|\mu):=\infty$ if ν is not absolutely continuous with respect to μ, while

$$\mathrm{Ent}(\nu|\mu) := \mu(\rho\log\rho) = \int_{\mathbb{R}^d}(\rho\log\rho)\mathrm{d}\mu, \quad \text{if } \rho := \frac{\mathrm{d}\nu}{\mathrm{d}\mu} \text{ exists.}$$

See also [14, 20, 28] for log-Harnack inequalities with more regular $b^{(0)}$, and see [18] for the dimension-free Harnack inequality with power.

If furthermore $b_t^{(1)}(x,\mu)$ is L-differentiable in $\mu\in\mathscr{P}_k$, the following Bismut type formula has been established in [30] for the intrinsic derivative D_ϕ^I (see Definition 1 below):

$$D_\phi^I P_t f(\mu) = \mathbb{E}\big[f(X_t^\mu)M_t^{\mu,\phi}\big], \quad t\in(0,T], f\in\mathscr{B}_b(\mathbb{R}^d), \mu\in\mathscr{P}_k, \phi\in L^k(\mathbb{R}^d\to\mathbb{R}^d;\mu),$$

where $M_t^{\mu,\phi}$ is an explicit martingale. See [3, 5, 12, 19] for earlier results with more regular $b^{(0)}$. See [2] for the case where $\mu=\delta_x$ is the Dirac measure at $x\in\mathbb{R}^d$, and see [22, 24] for a less explicit Bismut formula involving in the inverse of the Malliavin matrix of the solution.

We emphasize that existing results on log-Harnack inequality and Bismut formula for McKean-Vlasov SDEs only apply to the case with coefficients regular in the distribution variable, i.e. either $\mathbb{W}_2$-Lipschitz continuous or L-differentiable. The reason is that the Zvonkin transform technique [34] used in these references only kills singularities in the time-spatial variables (t,x), but not the distribution variable.

On the other hand, a derivative estimate has been presented in [7] for the heat kernel when the drift is of type $b_t(x,\mu(V))$, where V is a Hölder continuous function, and $\mu(V):=\int_{\mathbb{R}^d}V\mathrm{d}\mu$. In this case, the drift is only Lipschitz continuous in distribution with respect to

$$\mathbb{W}_\varepsilon(\mu,\nu) := \sup\big\{\big|\mu(f)-\nu(f)\big| : |f(x)-f(y)|\le|x-y|^\varepsilon\big\}$$

for some $\varepsilon\in(0,1)$ rather than $\mathbb{W}_1$, and hence also has certain singularity in the distribution variable. This result encourages us to establish the log-Harnack inequality and Bismut formula for McKean-Vlasov SDEs with coefficients singular in all time-spatial-distribution variables.

Indeed, we will establish the log-Harnack inequality and Bismut formula for McKean-Vlasov SDEs with stronger singularity in the distribution variable: the drift is only Lipschitz continuous with respect to

$$\mathbb{W}_\alpha(\mu,\nu) := \sup\big\{\big|\mu(f)-\nu(f)\big| : |f(x)-f(y)|\le\alpha(|x-y|)\big\},$$

where α is the square root of a Dini function, i.e. it belongs to class

$$\mathscr{A} := \Big\{\alpha : [0,\infty) \to [0,\infty) \text{ is increasing and concave,} \\ \alpha(0) = 0,\ \alpha(r) > 0 \text{ for } r > 0,\ \int_0^1 \frac{\alpha(r)^2}{r}\mathrm{d}r \in (0,\infty)\Big\}.$$

Noting that $\int_0^1 \frac{\alpha(r)^2}{r}\mathrm{d}r < \infty$ is the Dini condition for α^2, the continuity in the distribution variable is even weaker than Dini, so that the existing study in the literature is considerably improved.

The log-Harnack inequality is established in Section 2, where a key step is to derive the estimate (Lemma 5 for $k = 2$):

$$\mathbb{W}_\alpha(P_t^*\gamma, P_t^*\tilde{\gamma}) \le c\mathbb{W}_2(\tilde{\gamma},\gamma)\frac{\alpha(t^{\frac{1}{2}})}{\sqrt{t}}, \quad \gamma,\tilde{\gamma} \in \mathscr{P}_2, t \in (0,T]$$

for some constant $c > 0$.

The Bismut formula for the intrinsic derivative of $P_t f$ is presented in Section 3, for which we develop new techniques to control the intrinsic derivative D^I and the extrinsic derivative D^E of the drift term in the distribution variable (Theorem 7(1)):

$$\|D^I P_t[D^E b_t(y,\nu)(\cdot)](\mu)\|_{L^{\frac{k}{k-1}}(\mu)} \le \frac{c\,\alpha(t^{\frac{1}{2}})}{\sqrt{t}}, \quad t \in (0,T], \mu \in \mathscr{P}_k, y \in \mathbb{R}^d, \nu \in \mathscr{P}_k.$$

2 Log-Harnack Inequality

Since $\mathbb{W}_2$ is involved in the log-Harnack inequality, in this section we mainly consider (1) for $(\tilde{\mathscr{P}},\tilde{\rho}) = (\mathscr{P}_2,\mathbb{W}_2)$, but the drift may be not Lipschitz continuous in $\mathbb{W}_k$ for any $k > 0$. We first state the concrete assumption and the main result on the log-Harnack inequality, then present a complete proof in a separate subsection.

2.1 Assumption and main result

We will allow $b_t(x,\cdot)$ to be merely Lipschitz continuous in the sum of $\mathbb{W}_2$ and the Wasserstein distance induced by the square root of a Dini function.

Let $\alpha \in \mathscr{A}$. Then it holds

$$\alpha(s+t) \le \alpha(s) + \alpha(t), \quad \alpha(rt) \le r\alpha(t), \quad s,t > 0, r \ge 1. \tag{2}$$

These inequalities follow from $\alpha(0) = 0$ and the decreasing monotonicity of α' such that

$$\alpha'(s+t) \le \alpha'(s), \quad \frac{\mathrm{d}}{\mathrm{d}t}\alpha(rt) = r\alpha'(rt) \le r\alpha'(t), \quad s,t \ge 0, r \ge 1.$$

The second estimate in (2) with $r = t^{-1}$ yields

$$\alpha(t) \ge \alpha(1)t > 0, \quad t \in (0,1]. \tag{3}$$

To measure the singularity in $(t,x) \in [0,T] \times \mathbb{R}^d$, we recall locally integrable functional spaces presented in [31]. For any $t > s \geq 0$ and $p, q \in (1, \infty]$, we write $f \in \tilde{L}_p^q([s,t])$ if $f : [s,t] \times \mathbb{R}^d \to \mathbb{R}$ is measurable with

$$\|f\|_{\tilde{L}_p^q([s,t])} := \sup_{y \in \mathbb{R}^d} \left\{ \int_s^t \left(\int_{B(y,1)} |f(r,x)|^p \mathrm{d}x \right)^{\frac{q}{p}} \mathrm{d}r \right\}^{\frac{1}{q}} < \infty,$$

where $B(y,1) := \{x \in \mathbb{R}^d : |x-y| \leq 1\}$ is the unit ball centered at the point y. When $s = 0$, we simply denote

$$\tilde{L}_p^q(t) = \tilde{L}_p^q([0,t]), \quad \|f\|_{\tilde{L}_p^q(t)} = \|f\|_{\tilde{L}_p^q([0,t])}.$$

We take (p,q) from the space

$$\mathscr{K} := \Big\{ (p,q) \in (2,\infty]^2 : \ \frac{d}{p} + \frac{2}{q} < 1 \Big\},$$

and make the following assumption where ∇ is the gradient in $x \in \mathbb{R}^d$.

(A) Let $(\tilde{\mathscr{P}}, \tilde{\rho}) = (\mathscr{P}_k, \mathbb{W}_k)$ for some $k \in (1,\infty)$. There exist $K \in (0,\infty), l \in \mathbb{N}, \alpha \in \mathscr{A}$ and

$$1 \leq f_i \in \tilde{L}_{p_i}^{q_i}(T), \quad (p_i,q_i) \in \mathscr{K}, \quad 0 \leq i \leq l$$

such that the following conditions hold.

(A_1) $(\sigma_t \sigma_t^*)(x)$ is invertible and $\sigma_t(x)$ is weakly differentiable in x such that

$$\|\sigma\sigma^*\|_\infty + \|(\sigma\sigma^*)^{-1}\|_\infty < \infty, \quad |\nabla\sigma| \leq \sum_{i=1}^l f_i,$$

$$\lim_{\varepsilon \downarrow 0} \sup_{t \in [0,T], |x-x'| \leq \varepsilon} \|(\sigma_t\sigma_t^*)(x) - (\sigma_t\sigma_t^*)(x')\| = 0.$$

(A_2) $b_t(x,\mu) = b_t^{(0)}(x) + b_t^{(1)}(x,\mu)$, where for any $t \in [0,T], x, y \in \mathbb{R}^d, \mu, \nu \in \mathscr{P}_k$,

$$|b_t^{(0)}(x)| \leq f_0(t,x), \quad |b_t^{(1)}(0,\delta_0)| \leq K,$$
$$|b_t^{(1)}(x,\mu) - b_t^{(1)}(y,\nu)| \leq K\big\{|x-y| + \mathbb{W}_\alpha(\mu,\nu) + \mathbb{W}_k(\mu,\nu)\big\}.$$

We first observe that **(A)** implies the well-posedness of (1) for distributions in $\mathscr{P}_k$. Let $[\cdot]_\alpha$ be the α-continuity modulus defined by

$$[f]_\alpha := \sup_{x \neq y} \frac{|f(x) - f(y)|}{\alpha(|x-y|)}.$$

Since $\alpha(0) = 0$ and α is concave, there exists a constant $c > 0$ such that

$$\sup_{[f]_\alpha \leq 1} |f(x) - f(0)| \leq \alpha(|x|) \leq \alpha(1)(1+|x|) \leq c + c|x|^k, \quad x \in \mathbb{R}^d.$$

Thus,

$$\frac{1}{c}\mathbb{W}_\alpha(\mu,\nu) \le \mathbb{W}_{k,var}(\mu,\nu) := \sup_{|f|\le 1+|\cdot|^k} |\mu(f)-\nu(f)|. \tag{4}$$

So, by [29, Theorem 3.1(1)] for $D=\mathbb{R}^d$, under assumption **(A)**, (1) is well-posed for distributions in $\mathscr{P}_k$, and for any $n \ge 1$ there exists a constant $c_n > 0$ such that

$$\mathbb{E}\Big[\sup_{t\in[0,T]} |X_t|^n \Big| \mathscr{F}_0\Big] \le c_n(1+|X_0|^n). \tag{5}$$

Consequently,

$$\sup_{t\in[0,T]} \|P_t^*\gamma\|_k^k = \sup_{t\in[0,T]} (P_t^*\gamma)(|\cdot|^k) \le \mathbb{E}\Big[\sup_{t\in[0,T]} |X_t^\gamma|^k\Big] \le c_k(1+\|\gamma\|_k^k). \tag{6}$$

Theorem 1. *Assume* **(A)** *with* $k=2$. *Then there exists a constant* $c>0$ *such that*

$$\mathrm{Ent}(P_t^*\gamma|P_t^*\tilde\gamma) \le \frac{c}{t}\,\mathbb{W}_2(\gamma,\tilde\gamma)^2, \quad t\in(0,T], \gamma,\tilde\gamma\in\mathscr{P}_2. \tag{7}$$

Example 1. *Let* $h:\mathbb{R}^d\times\mathbb{R}^d\to\mathbb{R}^d$ *satisfy*

$$|h(x_1,y_1)-h(x_2,y_2)| \le K_h|x_1-x_2| + \alpha(|y_1-y_2|), \quad x_1,x_2,y_1,y_2\in\mathbb{R}^d$$

for some $\alpha\in\mathscr{A}$ *and* $K_h\ge 0$. *Then* $b_t^{(1)}(x,\mu)=\int_{\mathbb{R}^d} h(x,y)\mu(\mathrm{d}y)$ *satisfies* (A_2).

2.2 Proof of Theorem 1

Although in Theorem 1 we assume **(A)** for $k=2$, for later use we will also consider general $k\in(1,\infty)$. For any $\gamma\in\mathscr{P}_k$, consider the decoupled SDE of (1):

$$\mathrm{d}X_t^{x,\gamma} = b_t(X_t^{x,\gamma}, P_t^*\gamma)\mathrm{d}t + \sigma_t(X_t^{x,\gamma})\mathrm{d}W_t, \quad X_0^{x,\gamma}=x. \tag{8}$$

By [29, Theorem 3.1(1)] for $D=\mathbb{R}^d$, this SDE is well-posed and (5) also holds for $X_t^{x,\gamma}$ in place of X_t, i.e. for any $n\ge 1$ there exists a constant $c_n(\gamma)>0$ such that

$$\mathbb{E}\Big[\sup_{t\in[0,T]} |X_t^{x,\gamma}|^n\Big] \le c_n(\gamma)(1+|x|^n), \quad x\in\mathbb{R}^d. \tag{9}$$

Let P_t^γ be the associated Markov semigroup, i.e.

$$P_t^\gamma f(x) := \mathbb{E}[f(X_t^{x,\gamma})], \quad t\in[0,T], x\in\mathbb{R}^d, f\in\mathscr{B}_b(\mathbb{R}^d).$$

We first present the following generalized Hölder inequality with a concave function α.

Lemma 2. *Let $\alpha : [0,\infty) \to [0,\infty)$ be concave. Then for any non-negative random variables ξ and η,*

$$\mathbb{E}[\alpha(\xi)\eta] \le \|\eta\|_{L^p(\mathbb{P})}\alpha\Big(\|\xi\|_{L^{\frac{p}{p-1}}(\mathbb{P})}\Big), \quad p>1. \tag{10}$$

Consequently, for any random variable $\bar{\xi}$ on $\mathbb{R}^d$, $f \in C(\mathbb{R}^d;\mathbb{B})$ for a Banach space $(\mathbb{B},\|\cdot\|_{\mathbb{B}})$ with $[f]_\alpha < \infty$, and any real random variable $\bar{\eta}$ with $\mathbb{E}[\bar{\eta}]=0$,

$$\big\|\mathbb{E}[f(\bar{\xi})\bar{\eta}]\big\|_{\mathbb{B}} \le [f]_\alpha \|\bar{\eta}\|_{L^p(\mathbb{P})}\alpha\Big(\|\bar{\xi}-z\|_{L^{\frac{p}{p-1}}(\mathbb{P})}\Big), \quad p>1, z\in\mathbb{R}^d. \tag{11}$$

Proof. Since the assertion holds trivially for $p=\infty$, we only prove for $p<\infty$. It suffices to prove for $\mathbb{E}[\eta^p]\in(0,\infty)$. Let $\mathbb{Q} := \frac{\eta}{\mathbb{E}[\eta]}\mathbb{P}$. By Jensen's and Hölder's inequalities, and using the second inequality in (2), we obtain

$$\begin{aligned}
&\mathbb{E}[\alpha(\xi)\eta] = \mathbb{E}[\eta]\mathbb{E}_{\mathbb{Q}}[\alpha(\xi)] \le \mathbb{E}[\eta]\alpha(\mathbb{E}_{\mathbb{Q}}[\xi]) \le \mathbb{E}[\eta]\alpha\bigg(\frac{(\mathbb{E}[\eta^p])^{\frac{1}{p}}}{\mathbb{E}[\eta]}\big(\mathbb{E}[\xi^{\frac{p}{p-1}}]\big)^{\frac{p-1}{p}}\bigg)\\
&\le \mathbb{E}[\eta]\bigg\{\frac{(\mathbb{E}[\eta^p])^{\frac{1}{p}}}{\mathbb{E}[\eta]}\ \alpha\Big(\big(\mathbb{E}[\xi^{\frac{p}{p-1}}]\big)^{\frac{p-1}{p}}\Big)\bigg\} = \big(\mathbb{E}[\eta^p]\big)^{\frac{1}{p}}\alpha\Big(\big(\mathbb{E}[\xi^{\frac{p}{p-1}}]\big)^{\frac{p-1}{p}}\Big).
\end{aligned}$$

Then the second inequality follows by noting that $\mathbb{E}[\bar{\eta}]=0$ implies

$$\big\|\mathbb{E}[f(\bar{\xi})\bar{\eta}]\big\|_{\mathbb{B}} = \big\|\mathbb{E}[\{f(\bar{\xi})-f(z)\}\bar{\eta}]\big\|_{\mathbb{B}} \le [f]_\alpha \mathbb{E}[\alpha(|\bar{\xi}-z|)|\bar{\eta}|].$$

Therefore, the proof is completed. □

To characterize properties of (8), consider the following PDE for $u:[0,T]\times\mathbb{R}^d\to\mathbb{R}^d$:

$$\frac{\partial}{\partial t}u_t(x) + (\mathscr{L}_t^\gamma u_t)(x) + b_t^{(0)}(x) = \lambda u_t(x), \quad u_T = 0, \tag{12}$$

where $\lambda>0$ is a constant, and

$$\mathscr{L}_t^\gamma := \frac{1}{2}\mathrm{tr}\big\{(\sigma_t\sigma_t^*)\nabla^2\big\} + b_t(\cdot, P_t^*\gamma)\cdot\nabla. \tag{13}$$

By [33, Theorem 2.1] and **(A)**, for large enough constants $\lambda, c>0$ independent of γ, (12) has a unique solution $u^{\lambda,\gamma}$ satisfying

$$\|u^{\lambda,\gamma}\|_\infty + \|\nabla u^{\lambda,\gamma}\|_\infty \le \frac{1}{5}, \quad \|\nabla^2 u^{\lambda,\gamma}\|_{\tilde{L}_{p_0}^{q_0}(T)} \le c. \tag{14}$$

So, for any $t\in[0,T]$,

$$x \mapsto \Theta_t^{\lambda,\gamma}(x) := x + u_t^{\lambda,\gamma}(x), \quad x\in\mathbb{R}^d \tag{15}$$

is a homeomorphism on $\mathbb{R}^d$.

Moreover, for any $\gamma \in \mathscr{P}_k$, $t \in [0,T]$, consider

$$\mathrm{d}\theta_t^{\lambda,\gamma}(x) = b_t^{(1)}((\Theta_t^{\lambda,\gamma})^{-1}(\theta_t^{\lambda,\gamma}(x)), P_t^*\gamma)\mathrm{d}t, \quad \theta_0^{\lambda,\gamma}(x) = \Theta_0^{\lambda,\gamma}(x), x \in \mathbb{R}^d, \tag{16}$$

and let

$$\tilde{\theta}_t^{\lambda,\gamma}(x) = (\Theta_t^{\lambda,\gamma})^{-1}(\theta_t^{\lambda,\gamma}(x)), \quad t \in [0,T], x \in \mathbb{R}^d. \tag{17}$$

Then we have

$$\mathrm{d}\Theta_t^{\lambda,\gamma}(\tilde{\theta}_t^{\lambda,\gamma}(x)) = b_t^{(1)}(\tilde{\theta}_t^{\lambda,\gamma}(x), P_t^*\gamma)\mathrm{d}t, \quad t \in [0,T], \tilde{\theta}_0^{\lambda,\gamma}(x) = x \in \mathbb{R}^d. \tag{18}$$

Lemma 3. *Let σ and b satisfy* **(A)**. *Then the following assertions hold.*

(1) *For any $p \geq 1$, there exists a constant $c_p > 0$ such that*

$$\mathbb{E}[|X_t^{x,\gamma} - \tilde{\theta}_t^{\lambda,\gamma}(x)|^p] \leq c_p t^{\frac{p}{2}}, \quad t \in [0,T], x \in \mathbb{R}^d, \gamma \in \mathscr{P}_k. \tag{19}$$

(2) *For any $\alpha \in \mathscr{A}$, there exists a constant $c > 0$ such that the gradient estimate holds:*

$$\begin{aligned} |\nabla P_t^\gamma f|(x) &:= \limsup_{|y-x|\to 0} \frac{|P_t^\gamma f(y) - P_t^\gamma f(x)|}{|y-x|} \\ &\leq \frac{c\alpha(t^{\frac{1}{2}})}{\sqrt{t}}, \qquad [f]_\alpha \leq 1, \ x \in \mathbb{R}^d, \gamma \in \mathscr{P}_k, t \in (0,T]. \end{aligned} \tag{20}$$

Proof. (1) We will use Zvonkin's transform defined in (15). By Itô's formula (see [33, Lemma 3.3]), we derive

$$\mathrm{d}\Theta_t^{\lambda,\gamma}(X_t^{x,\gamma}) = \big\{\lambda u_t^{\lambda,\gamma}(X_t^{x,\gamma}) + b_t^{(1)}(X_t^{x,\gamma}, P_t^*\gamma)\big\}\mathrm{d}t + \big\{(\nabla\Theta_t^{\lambda,\gamma})\sigma_t\big\}(X_t^{x,\gamma})\,\mathrm{d}W_t. \tag{21}$$

By **(A)**, (14), there exists a constant $C > 1$ such that

$$\begin{aligned} &C^{-1}|X_t^{x,\gamma} - \tilde{\theta}_t^{\lambda,\gamma}(x)| \leq \big|\Theta_t^{\lambda,\gamma}(X_t^{x,\gamma}) - \Theta_t^{\lambda,\gamma}(\tilde{\theta}_t^{\lambda,\gamma}(x))\big| \leq C|X_t^{x,\gamma} - \tilde{\theta}_t^{\lambda,\gamma}(x)|, \\ &\big|b_t^{(1)}(X_t^{x,\gamma}, P_t^*\gamma) - b_t^{(1)}(\tilde{\theta}_t^{\lambda,\gamma}(x), P_t^*\gamma)\big| \leq C|X_t^{x,\gamma} - \tilde{\theta}_t^{\lambda,\gamma}(x)|, \\ &\big|\lambda u_t^{\lambda,\gamma}(X_t^{x,\gamma})\big| + \big\|\big\{(\nabla\Theta_t^{\lambda,\gamma})\sigma_t\big\}(X_t^{x,\gamma})\big\| \leq C, \quad (t,x,\gamma) \in [0,T] \times \mathbb{R}^d \times \mathscr{P}_k. \end{aligned}$$

This together with (18), (21) and Gronwall's inequality implies (19).

(2) For any measurable $f : \mathbb{R}^d \to \mathbb{R}$ with $[f]_\alpha \leq 1$, take

$$f_n := [(-n) \vee f] \wedge n, \quad n \geq 1.$$

By an approximation technique, it is sufficient to prove (20) for $f \in \mathscr{B}_b(\mathbb{R}^d)$ with $[f]_\alpha \leq 1$. According to [33, Theorem 4.1], there exists a constant $c_0 > 0$ such that for any $\gamma \in \mathscr{P}_k$, the log-Harnack inequality

$$P_t^\gamma \log f(x) \leq \log P_t^\gamma f(y) + \frac{c_0}{t}|x-y|^2, \quad x, y \in \mathbb{R}^d, t \in (0,T], f \in \mathscr{B}_b^+(\mathbb{R}^d)$$

holds, so that [27, Proposition 1.3.8] implies

$$|\nabla P_t^\gamma f| \le \frac{\sqrt{2c_0}}{\sqrt{t}}\{P_t^\gamma |f|^2\}^{\frac{1}{2}}, \quad f \in \mathscr{B}_b(\mathbb{R}^d), t \in (0,T], \gamma \in \mathscr{P}_k.$$

Observe that for any $f \in \mathscr{B}_b(\mathbb{R}^d)$ with $[f]_\alpha \le 1$,

$$\begin{aligned} |\nabla P_t^\gamma f|(x) &\le \inf_{z\in\mathbb{R}} \frac{\sqrt{2c_0}}{\sqrt{t}}\{P_t^\gamma(|f-z|^2)(x)\}^{\frac{1}{2}} \\ &\le \frac{\sqrt{2c_0}}{\sqrt{t}}\{\mathbb{E}(\alpha(|X_t^{x,\gamma} - \tilde\theta_t^{\lambda,\gamma}(x)|)^2)\}^{\frac{1}{2}} \\ &\le \frac{\sqrt{2c_0}}{\sqrt{t}}\alpha(\{\mathbb{E}(|X_t^{x,\gamma} - \tilde\theta_t^{\lambda,\gamma}(x)|^2)\}^{\frac{1}{2}}), \quad x \in \mathbb{R}^d, t \in (0,T], \end{aligned} \tag{22}$$

where in the last step, we used (10) for $\eta = \alpha(\xi)$ with $\xi = |X_t^{x,\gamma} - \tilde\theta_t^{\lambda,\gamma}(x)|$ and $p = 2$. Therefore, (20) follows from (22), (19) and (2). □

To verify (7), in the following Lemma 4 and Lemma 5 we will prove

$$\int_0^t \{\mathbb{W}_\alpha(P_s^*\gamma, P_s^*\tilde\gamma) + \mathbb{W}_2(P_s^*\gamma, P_s^*\tilde\gamma)\}^2 \mathrm{d}s \le c\mathbb{W}_2(\gamma,\tilde\gamma)^2, \quad t \in [0,T], \gamma, \tilde\gamma \in \mathscr{P}_2 \tag{23}$$

for some constant $c > 0$.

Lemma 4. *Assume* **(A)**. *Then there exists a constant $c > 0$ such that*

$$\mathbb{W}_k(P_t^*\gamma, P_t^*\tilde\gamma) \le c\mathbb{W}_k(\gamma,\tilde\gamma) + c\int_0^t \mathbb{W}_\alpha(P_s^*\gamma, P_s^*\tilde\gamma)\mathrm{d}s, \quad t \in [0,T], \gamma, \tilde\gamma \in \mathscr{P}_k. \tag{24}$$

Proof. We take $\mathscr{F}_0$-measurable random variables $X_0^\gamma, X_0^{\tilde\gamma}$ such that

$$\mathscr{L}_{X_0^\gamma} = \gamma, \quad \mathscr{L}_{X_0^{\tilde\gamma}} = \tilde\gamma, \quad \mathbb{W}_k(\gamma,\tilde\gamma)^k = \mathbb{E}[|X_0^\gamma - X_0^{\tilde\gamma}|^k]. \tag{25}$$

Recall that $\Theta_t^{\lambda,\gamma}$ is defined in (15). By (12), (13) and Itô's formula, we derive

$$\mathrm{d}\Theta_t^{\lambda,\gamma}(X_t^\gamma) = \big\{\lambda u_t^{\lambda,\gamma}(X_t^\gamma) + b_t^{(1)}(X_t^\gamma, P_t^*\gamma)\big\}\mathrm{d}t + \big\{(\nabla\Theta_t^{\lambda,\gamma})\sigma_t\big\}(X_t^\gamma)\,\mathrm{d}W_t, \tag{26}$$

and

$$\begin{aligned} \mathrm{d}\Theta_t^{\lambda,\gamma}(X_t^{\tilde\gamma}) &= \big\{\lambda u_t^{\lambda,\gamma}(X_t^{\tilde\gamma}) + b_t^{(1)}(X_t^{\tilde\gamma}, P_t^*\gamma)\big\}\mathrm{d}t \\ &\quad + \nabla\Theta_t^{\lambda,\gamma}(X_t^{\tilde\gamma})[b_t(X_t^{\tilde\gamma}, P_t^*\tilde\gamma) - b_t(X_t^{\tilde\gamma}, P_t^*\gamma)]\mathrm{d}t + \big\{(\nabla\Theta_t^{\lambda,\gamma})\sigma_t\big\}(X_t^{\tilde\gamma})\,\mathrm{d}W_t. \end{aligned} \tag{27}$$

Combining this with (26) and **(A)**, we prove the desired estimate by using the maximal functional inequality, Khasminskii's estimate and stochastic Gronwall's inequality, see

for instance the proof of [15, Lemma 2.1] for details. Below we simply outline the procedure.

By (A_2) we have

$$\begin{aligned}&|b_t(X_t^{\tilde\gamma}, P_t^*\tilde\gamma) - b_t(X_t^{\tilde\gamma}, P_t^*\gamma)| + |b_t^{(1)}(X_t^{\tilde\gamma}, P_t^*\gamma) - b_t^{(1)}(X_t^{\gamma}, P_t^*\gamma)|\\ &\le K\big\{|X_t^\gamma - X_t^{\tilde\gamma}| + \mathbb{W}_\alpha(P_t^*\gamma, P_t^*\tilde\gamma) + \mathbb{W}_k(P_t^*\gamma, P_t^*\tilde\gamma)\big\}.\end{aligned}$$

Combining this with (26), (27), (A_1), the maximal functional inequality and Khasminskii's estimate (see [31, Lemma 2.1 and Lemma 4.1]), we derive

$$\begin{aligned}\mathrm{d}\big|\Theta_t^{\lambda,\gamma}(X_t^\gamma) - \Theta_t^{\lambda,\gamma}(X_t^{\tilde\gamma})\big|^{k+1} &\le \mathrm{d}M_t + |X_t^\gamma - X_t^{\tilde\gamma}|^{k+1}\mathrm{d}\mathscr{L}_t\\ &\quad + c_1\big\{\mathbb{W}_\alpha(P_t^*\gamma, P_t^*\tilde\gamma) + \mathbb{W}_k(P_t^*\gamma, P_t^*\tilde\gamma)\big\}|\Theta_t^{\lambda,\gamma}(X_t^\gamma) - \Theta_t^{\lambda,\gamma}(X_t^{\tilde\gamma})\big|^k\mathrm{d}t,\end{aligned}$$

where $c_1 > 0$ is a constant, $\mathscr{L}_t$ is an adapted increasing process with $\mathbb{E}[\mathrm{e}^{\delta\mathscr{L}_T}] < \infty$ for any $\delta > 0$, and M_t is a local martingale. Since (14) implies

$$\frac{1}{2}|X_t^{\tilde\gamma} - X_t^\gamma| \le |\Theta_t^{\lambda,\gamma}(X_t^\gamma) - \Theta_t^{\lambda,\gamma}(X_t^{\tilde\gamma})| \le 2|X_t^{\tilde\gamma} - X_t^\gamma|,$$

by the stochastic Gronwall inequality (see [32, Lemma 3.7]), we find a constant $c_2 > 1$ such that

$$\begin{aligned}&\bigg\{\mathbb{E}\Big[\sup_{s\in[0,t]}|X_s^{\tilde\gamma} - X_s^\gamma|^k\Big|\mathscr{F}_0\Big]\bigg\}^{1+k^{-1}} - c_2|X_0^\gamma - X_0^{\tilde\gamma}|^{k+1}\\ &\le c_2\int_0^t\big\{\mathbb{W}_\alpha(P_s^*\gamma, P_s^*\tilde\gamma) + \mathbb{W}_k(P_s^*\gamma, P_s^*\tilde\gamma)\big\}\mathbb{E}\Big[|X_s^{\tilde\gamma} - X_s^\gamma|^k\Big|\mathscr{F}_0\Big]\mathrm{d}s, \quad t\in[0,T].\end{aligned}$$

So, there exists a constant $c_3 > 0$ such that for any $t \in [0,T]$,

$$\begin{aligned}&\mathbb{E}\Big[\sup_{s\in[0,t]}|X_s^{\tilde\gamma} - X_s^\gamma|^k\Big|\mathscr{F}_0\Big] - c_2|X_0^\gamma - X_0^{\tilde\gamma}|^k\\ &\le c_2\bigg(\int_0^t\big\{\mathbb{W}_\alpha(P_s^*\gamma, P_s^*\tilde\gamma) + \mathbb{W}_k(P_s^*\gamma, P_s^*\tilde\gamma)\big\}\mathbb{E}\Big[|X_s^{\tilde\gamma} - X_s^\gamma|^k\Big|\mathscr{F}_0\Big]\mathrm{d}s\bigg)^{\frac{k}{k+1}}\\ &\le \frac{1}{2}\mathbb{E}\Big[\sup_{s\in[0,t]}|X_s^{\tilde\gamma} - X_s^\gamma|^k\Big|\mathscr{F}_0\Big] + c_3\bigg(\int_0^t\big\{\mathbb{W}_\alpha(P_s^*\gamma, P_s^*\tilde\gamma) + \mathbb{W}_k(P_s^*\gamma, P_s^*\tilde\gamma)\big\}\mathrm{d}s\bigg)^k.\end{aligned}$$

This together with (25) yields

$$\begin{aligned}&\mathbb{W}_k(P_t^*\gamma, P_t^*\tilde\gamma) \le \sup_{s\in[0,t]}\big(\mathbb{E}[|X_s^{\tilde\gamma} - X_s^\gamma|^k]\big)^{\frac{1}{k}}\\ &\le (2c_2)^{\frac{1}{k}}\mathbb{W}_k(\gamma,\tilde\gamma) + (2c_3)^{\frac{1}{k}}\int_0^t\big\{\mathbb{W}_\alpha(P_s^*\gamma, P_s^*\tilde\gamma) + \mathbb{W}_k(P_s^*\gamma, P_s^*\tilde\gamma)\big\}\mathrm{d}s, \quad t\in[0,T].\end{aligned}$$

By Gronwall's inequality, this implies the desired estimate for some constant $c > 0$. □

Noting that $X_t^{x,\gamma}$ solves (1) if the initial value x is random with distribution γ, by the standard Markov property of $X_t^{x,\gamma}$, we have

$$P_t f(\gamma) := \int_{\mathbb{R}^d} f(x)(P_t^*\gamma)(\mathrm{d}x) = \int_{\mathbb{R}^d} P_t^\gamma f(x)\gamma(\mathrm{d}x), \quad f \in \mathscr{B}_b(\mathbb{R}^d). \tag{28}$$

The following lemma provides a regularity estimate on P_t^*, which together with Lemma 4 implies the desired (23).

Lemma 5. *Assume* **(A)**. *Then there exists a constant $c > 0$ such that*

$$\mathbb{W}_\alpha(P_t^*\gamma, P_t^*\tilde{\gamma}) \le c\mathbb{W}_k(\tilde{\gamma}, \gamma)\frac{\alpha(t^{\frac{1}{2}})}{\sqrt{t}}, \quad t \in (0, T], \gamma, \tilde{\gamma} \in \mathscr{P}_k. \tag{29}$$

Consequently, there exists a constant $c > 0$ such that for any $\gamma, \tilde{\gamma} \in \mathscr{P}_k$,

$$\sup_{t\in[0,T]} \mathbb{W}_k(P_t^*\gamma, P_t^*\tilde{\gamma}) \le c\mathbb{W}_k(\gamma, \tilde{\gamma}). \tag{30}$$

Proof. By Lemma 4, (30) follows from (29) and the fact $\int_0^T \frac{\alpha(t^{\frac{1}{2}})}{\sqrt{t}}\mathrm{d}t < \infty$. So, we only need to prove (29).

Let X_0^γ and $X_0^{\tilde{\gamma}}$ be in (25). For any $\varepsilon \in [0, 2]$, let

$$X_0^{\gamma^\varepsilon} := X_0^\gamma + \varepsilon(X_0^{\tilde{\gamma}} - X_0^\gamma), \quad \gamma^\varepsilon := \mathscr{L}_{X_0^{\gamma^\varepsilon}},$$

and let $X_t^{\gamma^\varepsilon}$ solve (1) with initial value $X_0^{\gamma^\varepsilon}$. Then

$$\gamma^\varepsilon(|\cdot|) \le 2\|\gamma\|_k + 2\|\tilde{\gamma}\|_k, \quad \varepsilon \in [0, 2], \tag{31}$$

$$\mathbb{W}_k(\gamma^\varepsilon, \gamma^{\varepsilon+r})^k \le \mathbb{E}\big[|X_0^{\gamma^\varepsilon} - X_0^{\gamma^{\varepsilon+r}}|^k\big] = r^k\mathbb{W}_k(\gamma, \tilde{\gamma})^k, \quad \varepsilon, r \in [0, 1]. \tag{32}$$

For any $\varepsilon \ge 0$, consider the SDE

$$\mathrm{d}X_t^{x,\gamma^\varepsilon} = b_t(X_t^{x,\gamma^\varepsilon}, P_t^*\gamma^\varepsilon)\mathrm{d}t + \sigma_t(X_t^{x,\gamma^\varepsilon})\mathrm{d}W_t, \quad X_0^{x,\gamma^\varepsilon} = x, t \in [0, T]. \tag{33}$$

For any $r \in (0, 1)$, let

$$\eta_t^{\varepsilon,r} = [\sigma_t^*(\sigma_t\sigma_t^*)^{-1}](X_t^{x,\gamma^\varepsilon})[b_t(X_t^{x,\gamma^\varepsilon}, P_t^*\gamma^{\varepsilon+r}) - b_t(X_t^{x,\gamma^\varepsilon}, P_t^*\gamma^\varepsilon)], \quad t \in [0, T].$$

By **(A)**, there exists a constant $c_1 > 0$ such that

$$\sup_{t\in[0,T]} |\eta_t^{\varepsilon,r}| \le c_1\big\{\mathbb{W}_\alpha(P_t^*\gamma^\varepsilon, P_t^*\gamma^{\varepsilon+r}) + \mathbb{W}_k(P_t^*\gamma^\varepsilon, P_t^*\gamma^{\varepsilon+r})\big\}, \quad r, \varepsilon \in [0, 1]. \tag{34}$$

By Girsanov's theorem,

$$R_t^{\varepsilon,r} := \exp\left\{\int_0^t \langle \eta_s^{\varepsilon,r}, \mathrm{d}W_s\rangle - \frac{1}{2}\int_0^t |\eta_s^{\varepsilon,r}|^2 \mathrm{d}s\right\}, \quad t \in [0,T]$$

is a martingale, and

$$W_t^{\varepsilon,r} = W_t - \int_0^t \eta_s^{\varepsilon,r}\mathrm{d}s, \quad t \in [0,T]$$

is a Brownian motion under the probability measure $\mathbb{Q}^{\varepsilon,r} := R_T^{\varepsilon,r}\mathbb{P}$. Rewrite (33) as

$$\mathrm{d}X_t^{x,\gamma^\varepsilon} = b_t(X_t^{x,\gamma^\varepsilon}, P_t^*\gamma^{\varepsilon+r})\mathrm{d}t + \sigma_t(X_t^{x,\gamma^\varepsilon})\mathrm{d}W_t^{\varepsilon,r}, \quad X_0^{x,\gamma^\varepsilon} = x, \quad t \in [0,T].$$

By the weak uniqueness we obtain

$$\mathscr{L}_{\{X_t^{x,\gamma^\varepsilon}\}_{t\in[0,T]}|\mathbb{Q}^{\varepsilon,r}} = \mathscr{L}_{\{X_t^{x,\gamma^{\varepsilon+r}}\}_{t\in[0,T]}},$$

where $\mathscr{L}_{\cdot|\mathbb{Q}^{\varepsilon,r}}$ is the law under $\mathbb{Q}^{\varepsilon,r}$, so that

$$P_t^{\gamma^{\varepsilon+r}} f(x) - P_t^{\gamma^\varepsilon} f(x) = \mathbb{E}\left[f(X_t^{x,\gamma^\varepsilon})(R_t^{\varepsilon,r}-1)\right], \quad f \in \mathscr{B}_b(\mathbb{R}^d), \varepsilon, r \in (0,1].$$

Hence, by (28), we have

$$\begin{aligned}
&P_t f(\gamma^{\varepsilon+r}) - P_t f(\gamma^\varepsilon) = \gamma^{\varepsilon+r}(P_t^{\gamma^{\varepsilon+r}} f) - \gamma^\varepsilon(P_t^{\gamma^\varepsilon} f)\\
&= \gamma^{\varepsilon+r}(P_t^{\gamma^{\varepsilon+r}} f - P_t^{\gamma^\varepsilon} f) + \gamma^{\varepsilon+r}(P_t^{\gamma^\varepsilon} f) - \gamma^\varepsilon(P_t^{\gamma^\varepsilon} f)\\
&= \int_{\mathbb{R}^d} \mathbb{E}\left[f(X_t^{x,\gamma^\varepsilon})(R_t^{\varepsilon,r}-1)\right]\gamma^{\varepsilon+r}(\mathrm{d}x) + \mathbb{E}\big[P_t^{\gamma^\varepsilon} f(X_0^{\gamma^{\varepsilon+r}}) - P_t^{\gamma^\varepsilon} f(X_0^{\gamma^\varepsilon})\big],
\end{aligned}$$

so that

$$\begin{aligned}
&\mathbb{W}_\alpha(P_t^*\gamma^{\varepsilon+r}, P_t^*\gamma^\varepsilon)^2 = \sup_{[f]_\alpha\le 1}\big|P_t f(\gamma^{\varepsilon+r}) - P_t f(\gamma^\varepsilon)\big|^2 \le I_1 + I_2,\\
&I_1 := 2\sup_{[f]_\alpha\le 1}\left|\int_{\mathbb{R}^d}\mathbb{E}\left[f(X_t^{x,\gamma^\varepsilon})(R_t^{\varepsilon,r}-1)\right]\gamma^{\varepsilon+r}(\mathrm{d}x)\right|^2,\\
&I_2 := 2\sup_{[f]_\alpha\le 1}\left|\mathbb{E}\big[P_t^{\gamma^\varepsilon} f(X_0^{\gamma^{\varepsilon+r}}) - P_t^{\gamma^\varepsilon} f(X_0^{\gamma^\varepsilon})\big]\right|^2.
\end{aligned} \tag{35}$$

Below we estimate I_1 and I_2 respectively.

By (34), we obtain

$$\begin{aligned}
&\mathbb{E}|R_t^{\varepsilon,r}-1|^2 = \mathbb{E}\big[(R_t^{\varepsilon,r})^2 - 1\big] \le \mathrm{esssup}_\Omega(\mathrm{e}^{\int_0^t|\eta_s^{\varepsilon,r}|^2\mathrm{d}s} - 1)\\
&\le \mathrm{esssup}_\Omega\left(\mathrm{e}^{\int_0^t|\eta_s^{\varepsilon,r}|^2\mathrm{d}s}\int_0^t|\eta_s^{\varepsilon,r}|^2\mathrm{d}s\right)
\end{aligned} \tag{36}$$

$$\leq \psi(\varepsilon,r)\int_0^t \big\{\mathbb{W}_\alpha(P_s^*\gamma^\varepsilon, P_s^*\gamma^{\varepsilon+r})^2 + \mathbb{W}_k(P_s^*\gamma^\varepsilon, P_s^*\gamma^{\varepsilon+r})^2\big\}\mathrm{d}s,$$

where for $c_2 := 2c_1^2$,

$$\psi(\varepsilon,r) := c_2 \mathrm{e}^{c_2\int_0^T \{\mathbb{W}_\alpha(P_s^*\gamma^\varepsilon, P_s^*\gamma^{\varepsilon+r})^2 + \mathbb{W}_k(P_s^*\gamma^\varepsilon, P_s^*\gamma^{\varepsilon+r})^2\}\mathrm{d}s}. \tag{37}$$

By (4) and (6), we have

$$\bar{\psi} := \sup_{\varepsilon,r\in[0,1]} \psi(\varepsilon,\gamma) < \infty. \tag{38}$$

Combining this with (2), (19), (36) and (11) with $z = \tilde{\theta}_t^{\lambda,\gamma^\varepsilon}(x)$, where $\tilde{\theta}_t^{\lambda,\gamma^\varepsilon}(x)$ is defined in (17) with γ^ε replacing γ, we can find constants $k_1, k_2 > 1$ such that

$$\begin{aligned}
&\bigg(\int_{\mathbb{R}^d}\sup_{[f]_\alpha\leq 1}\Big|\mathbb{E}\Big[f(X_t^{x,\gamma^\varepsilon})(R_t^{\varepsilon,r}-1)\Big]\Big|\gamma^{\varepsilon+r}(\mathrm{d}x)\bigg)^2\\
&\leq \bigg(\int_{\mathbb{R}^d}\alpha\big(k_1 t^{\frac{1}{2}}\big)\sup_x\big(\mathbb{E}[|R_t^{\varepsilon,r}-1|^2]\big)^{\frac{1}{2}}\gamma^{\varepsilon+r}(\mathrm{d}x)\bigg)^2\\
&\leq \alpha\big(k_1 t^{\frac{1}{2}}\big)^2\sup_x\mathbb{E}[|R_t^{\varepsilon,r}-1|^2]\\
&\leq k_2\alpha\big(t^{\frac{1}{2}}\big)^2\psi(\varepsilon,r)\int_0^t\big\{\mathbb{W}_\alpha(P_s^*\gamma^\varepsilon, P_s^*\gamma^{\varepsilon+r})^2 + \mathbb{W}_k(P_s^*\gamma^\varepsilon, P_s^*\gamma^{\varepsilon+r})^2\big\}\mathrm{d}s,\quad t\in[0,T].
\end{aligned}$$

Combining this with (2), (24), (32), (31), and letting

$$\Gamma_t(\varepsilon,r) := \mathbb{W}_\alpha(P_t^*\gamma^\varepsilon, P_t^*\gamma^{\varepsilon+r})^2 + \int_0^t \mathbb{W}_\alpha(P_s^*\gamma^\varepsilon, P_s^*\gamma^{\varepsilon+r})^2\mathrm{d}s,$$

we find a constant $c_4 > 0$ such that

$$I_1 \leq c_4\alpha\big(\sqrt{T}\big)^2\psi(\varepsilon,r)\bigg(r^2\mathbb{W}_k(\gamma,\tilde{\gamma})^2 + \int_0^t\Gamma_s(\varepsilon,r)\mathrm{d}s\bigg),\quad t\in[0,T]. \tag{39}$$

By (20), we find a constant $c_5 > 0$ such that

$$\sup_{[f]_\alpha\leq 1}\big|\nabla P_t^{\gamma^\varepsilon} f\big|(x) \leq \frac{c_5}{\sqrt{t}}\alpha\big(t^{\frac{1}{2}}\big).$$

Combining this with (2), we find a constant $c_6 > 0$ such that

$$\begin{aligned}
I_2 &\leq 2\sup_{[f]_\alpha\leq 1}\bigg(\mathbb{E}\Big[|X_0^\gamma - X_0^{\tilde{\gamma}}|\int_0^r|\nabla P_t^{\gamma^\varepsilon} f(X_0^{\gamma^{\varepsilon+\theta}})|\mathrm{d}\theta\Big]\bigg)^2\\
&\leq \frac{c_6\alpha\big(t^{\frac{1}{2}}\big)^2}{t}r^2(\mathbb{E}|X_0^\gamma - X_0^{\tilde{\gamma}}|)^2
\end{aligned} \tag{40}$$

$$\leq \frac{c_6\alpha\big(t^{\frac{1}{2}}\big)^2 r^2}{t}\big(\mathbb{E}[|X_0^\gamma - X_0^{\tilde{\gamma}}|^k]\big)^{\frac{2}{k}}.$$

Let

$$\tilde{\alpha}(r) := \bigg(\int_0^r \frac{\alpha(t)^2}{t}\mathrm{d}t\bigg)^{\frac{1}{2}}, \quad r \geq 0. \tag{41}$$

By (2), we find some constant $c' > 0$ such that

$$\int_0^T \frac{\alpha(rt^{\frac{1}{2}})^2}{t}\mathrm{d}t = 2\int_0^{rT^{\frac{1}{2}}} \frac{\alpha(s)^2}{s}\mathrm{d}s \leq c'\tilde{\alpha}(r)^2 < \infty, \quad r \geq 1. \tag{42}$$

So, (40) together with (35) and (39) yields that for some constant $c_7 > 0$,

$$\begin{aligned} &\Gamma_t(\varepsilon, r) \leq c_7 r^2 \mathbb{W}_k(\gamma, \tilde{\gamma})^2 H_t(\varepsilon, r) + c_7\psi(\varepsilon, r)\int_0^t \Gamma_s(\varepsilon, r)\mathrm{d}s, \\ &H_t(\varepsilon, r) := \psi(\varepsilon, r) + \tilde{\alpha}\big(1\big)^2 + \frac{\alpha\big(t^{\frac{1}{2}}\big)^2}{t}, \quad \varepsilon, r \in [0,1], t \in [0,T]. \end{aligned} \tag{43}$$

By Gronwall's inequality and (43), for any $\varepsilon, r \in [0,1]$ we have

$$\begin{aligned} &\mathbb{W}_\alpha(P_t^*\gamma^\varepsilon, P_t^*\gamma^{\varepsilon+r})^2 \leq \Gamma_t(\varepsilon, r) \\ &\leq c_7 r^2 \mathbb{W}_k(\gamma, \tilde{\gamma})^2 \bigg\{H_t(\varepsilon, r) + c_7\psi(\varepsilon, r)\mathrm{e}^{c_7\psi(\varepsilon,r)T}\int_0^t H_s(\varepsilon, r)\mathrm{d}s\bigg\}, \quad t \in [0,T]. \end{aligned}$$

This together with (24), (42) and (37)-(38) implies that $\psi(\varepsilon, r)$ is bounded in $(\varepsilon, r) \in [0,1]^2$ with $\psi(\varepsilon, r) \to c_2$ as $r \to 0$, so that by the dominated convergence theorem we find a constant $c > 0$ such that

$$\limsup_{r\downarrow 0} \frac{\mathbb{W}_\alpha(P_t^*\gamma^\varepsilon, P_t^*\gamma^{\varepsilon+r})}{r} \leq c\mathbb{W}_k(\tilde{\gamma}, \gamma)\bigg\{\frac{\alpha(t^{\frac{1}{2}})}{\sqrt{t}} + 1\bigg\}. \tag{44}$$

By the triangle inequality,

$$\big|\mathbb{W}_\alpha(P_t^*\gamma, P_t^*\gamma^\varepsilon) - \mathbb{W}_\alpha(P_t^*\gamma, P_t^*\gamma^{\varepsilon+r})\big| \leq \mathbb{W}_\alpha(P_t^*\gamma^\varepsilon, P_t^*\gamma^{\varepsilon+r}), \quad \varepsilon, r \in [0,1],$$

so that (44) implies that $\mathbb{W}_\alpha(P_t^*\gamma, P_t^*\gamma^\varepsilon)$ is Lipschitz continuous (hence a.e. differentiable) in $\varepsilon \in [0,1]$ for any $t \in (0,T]$, and

$$\Big|\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\mathbb{W}_\alpha(P_t^*\gamma, P_t^*\gamma^\varepsilon)\Big| \leq \limsup_{r\downarrow 0} \frac{\mathbb{W}_\alpha(P_t^*\gamma^\varepsilon, P_t^*\gamma^{\varepsilon+r})}{r} \leq c\mathbb{W}_k(\tilde{\gamma}, \gamma)\bigg\{\frac{\alpha(t^{\frac{1}{2}})}{\sqrt{t}} + 1\bigg\}, \quad \varepsilon \in [0,1].$$

This implies (29) by noting that $\gamma^1 = \tilde{\gamma}$ and $\sup_{t\in[0,T]} \frac{\sqrt{t}}{\alpha(t^{\frac{1}{2}})} \leq \frac{\sqrt{T}\vee 1}{\alpha(1)}$ due to (3). □

Proof of Theorem 1. Let $k=2$. According to [29, Theorem 2.5] for $D=\mathbb{R}^d$, see also [33, Theorem 4.1], **(A)** implies the following log-Harnack inequality for some constant $c_0>0$ and any $\gamma\in\mathscr{P}_2$:

$$P_t^\gamma \log f(x) \le \log P_t^\gamma f(y) + \frac{c_0}{t}|x-y|^2, \quad x,y\in\mathbb{R}^d, t\in(0,T], f\in\mathscr{B}_b^+(\mathbb{R}^d).$$

Then by [29, (4.13)], see also [13, Theorem 2.1], it suffices to find a constant $c>0$ such that

$$\sup_{t\in(0,T]} \log \mathbb{E}[|R_t^{\gamma,\tilde\gamma}|^2] \le c\mathbb{W}_2(\gamma,\tilde\gamma)^2, \quad \gamma,\tilde\gamma\in\mathscr{P}_2, \tag{45}$$

where

$$\begin{aligned} R_t^{\gamma,\tilde\gamma} &:= \mathrm{e}^{\int_0^t\langle \eta_s^{\gamma,\tilde\gamma}, \mathrm{d}W_s\rangle - \frac{1}{2}\int_0^t |\eta_s^{\gamma,\tilde\gamma}|^2\mathrm{d}s},\\ \eta_s^{\gamma,\tilde\gamma} &:= \big\{\sigma_s^*(\sigma_s\sigma_s^*)^{-1}\big\}(X_s^\gamma)\big\{b_s(X_s^\gamma,P_s^*\tilde\gamma)-b_s(X_s^\gamma,P_s^*\gamma)\big\}, \quad s\le t\le T.\end{aligned}$$

Noting that **(A)** implies

$$|\eta_s^{\gamma,\tilde\gamma}|^2 \le c_1\big\{\mathbb{W}_\alpha(P_s^*\gamma,P_s^*\tilde\gamma)^2+\mathbb{W}_2(P_s^*\gamma,P_s^*\tilde\gamma)^2\big\}, \quad s\in[0,T]$$

for some constant $c_1>0$, we have

$$\mathbb{E}[|R_t^{\gamma,\tilde\gamma}|^2] \le \mathrm{e}^{c_1\int_0^t\{\mathbb{W}_\alpha(P_s^*\gamma,P_s^*\tilde\gamma)^2+\mathbb{W}_2(P_s^*\gamma,P_s^*\tilde\gamma)^2\}\mathrm{d}s}.$$

Moreover, by (42) and Lemma 5, there exists a constant $c>0$ such that

$$\sup_{t\in[0,T]}\int_0^t \big\{\mathbb{W}_\alpha(P_s^*\gamma,P_s^*\tilde\gamma)^2+\mathbb{W}_2(P_s^*\gamma,P_s^*\tilde\gamma)^2\big\}\mathrm{d}s \le c\mathbb{W}_2(\gamma,\tilde\gamma)^2.$$

Therefore, (45) holds for some constant $c>0$.

□

3 Bismut Formula

Let $k\in(1,\infty)$ and denote $k^*:=\frac{k}{k-1}$. In this part, we consider the SDE (1) with $(\tilde{\mathscr{P}},\tilde\rho)=(\mathscr{P}_k,\mathbb{W}_k)$, where as in (A_2) the drift b is decomposed as

$$b_t(x,\nu)=b_t^{(0)}(x)+b_t^{(1)}(x,\nu), \quad t\in[0,T], x\in\mathbb{R}^d,\ \nu\in\mathscr{P}_k. \tag{46}$$

We aim to establish Bismut type formula for the intrinsic derivative of $\mathscr{P}_k\ni\mu\mapsto P_tf(\mu)$ for bounded measurable functions f on $\mathbb{R}^d$, by only assuming that the extrinsic derivative $D^Eb_t(x,\mu)(z)$ of the drift has a half-Dini continuity in $z\in\mathbb{R}^d$.

To this end, we first recall the notions of intrinsic and extrinsic derivatives which go back to [1], see [3] and [21].

Definition 1. *Let $f \in C(\mathscr{P}_k;\mathbb{B})$ for a Banach space $\mathbb{B}$. The function f is called intrinsically differentiable at a point $\mu \in \mathscr{P}_k$, if*

$$T_{\mu,k} := L^k(\mathbb{R}^d \to \mathbb{R}^d;\mu) \ni \phi \mapsto D^I_\phi f(\mu) := \lim_{\varepsilon \downarrow 0} \frac{f(\mu \circ (id + \varepsilon\phi)^{-1}) - f(\mu)}{\varepsilon} \in \mathbb{B}$$

is a well defined bounded linear operator. In this case, the norm of the intrinsic derivative $D^I f(\mu)$ is given by

$$\|D^I f(\mu)\|_{L^{k^*}(\mu)} := \sup_{\|\phi\|_{L^k(\mu)} \le 1} \|D^I_\phi f(\mu)\|_{\mathbb{B}}.$$

The function f is called intrinsically differentiable on $\mathscr{P}_k$, if it is so at any $\mu \in \mathscr{P}_k$.

Next, we recall the (convexity) extrinsic derivative, see e.g. [21, Definition 1.2].

Definition 2. *A real function f on $\mathscr{P}_k$ is called extrinsically differentiable on $\mathscr{P}_k$ with derivative $D^E f$ if*

$$D^E f(\mu)(x) = \lim_{\varepsilon \downarrow 0} \frac{f((1-\varepsilon)\mu + \varepsilon\delta_x) - f(\mu)}{\varepsilon} \in \mathbb{R}$$

exists for all $(x,\mu) \in \mathbb{R}^d \times \mathscr{P}_k$. When $f = (f^1, f^2, \cdots, f^d)$ is an $\mathbb{R}^d$-valued function on $\mathscr{P}_k$, we denote $D^E f = (D^E f^1, D^E f^2, \cdots, D^E f^d)$.

3.1 Main result

We will establish a Bismut formula for the intrinsic derivative of $P_t f$ under the following assumption.

(B) Let $k \in (1,\infty)$ and let b in (46).

(B_1) $b^{(0)}$ and σ satisfy the corresponding conditions in **(A)**.

(B_2) For any $t \in [0,T], y \in \mathbb{R}^d$, $b_t^{(1)}(y,\cdot)$ is extrinsically differentiable in $\mathscr{P}_k$ with the extrinsic derivative $D^E b_t^{(1)}(y,\nu)(z)$ being continuous in $(y,\nu,z) \in \mathbb{R}^d \times \mathscr{P}_k \times \mathbb{R}^d$. Moreover, there exists $\alpha \in \mathscr{A}$ with $\alpha \le c_0(1 + |\cdot|^{k-1})$ for some $c_0 > 0$ such that

$$|D^E b_t^{(1)}(y,\nu)(z) - D^E b_t^{(1)}(y,\nu)(\bar{z})| \le \alpha(|z-\bar{z}|),$$
$$z,\bar{z} \in \mathbb{R}^d, t \in [0,T], y \in \mathbb{R}^d, \nu \in \mathscr{P}_k.$$

(B_3) For any $t \in [0,T]$, $\nu \in \mathscr{P}_k$, $b_t^{(1)}(\cdot,\nu)$ is differentiable and there exists a constant $\tilde{K} > 0$ such that

$$|b_t^{(1)}(0,\delta_0)| \le \tilde{K}, \quad |\nabla b_t^{(1)}(y,\nu)| \le \tilde{K}, \quad (t,y,\nu) \in [0,T] \times \mathbb{R}^d \times \mathscr{P}_k.$$

As indicated in Introduction that existing results on Bismut type formulas for the intrinsic derivative of $P_t f(\mu)$ are established under upper bound conditions on the L-derivative of $b_t(y,\nu)$ in ν. Noting that under a mild condition, the L-derivative equals

to the gradient of the extrinsic derivative, so the above condition on the α-continuity of $D^E b_t^{(1)}(y,\nu)(z)$ in z is much weaker. To see this, we present below a simple example.

Example 2. *Let $\alpha(s) = s^\varepsilon$ for some $\varepsilon \in (0, 1 \wedge (k-1))$. Let $g : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ satisfy*

$$|g(y,z) - g(y,\bar{z})| \le \alpha(|z-\bar{z}|), \quad |\nabla g(\cdot, z)| \le K, \quad y, z, \bar{z} \in \mathbb{R}^d$$

for some $K \ge 0$. Let $b_t^{(1)}(y,\nu) = \int_{\mathbb{R}^d} g(y,z)\nu(\mathrm{d}z)$. By Definition 2, it holds

$$D^E b_t^{(1)}(y,\nu)(z) = g(y,z) - \int_{\mathbb{R}^d} g(y,z)\nu(\mathrm{d}z), \quad y \in \mathbb{R}^d, \nu \in \mathscr{P}_k, z \in \mathbb{R}^d.$$

However, by Definition 1, $b_t^{(1)}(y,\nu)$ is not intrinsically differentiable in ν. In fact, since $g(y,z)$ is not differentiable in z, for any $y \in \mathbb{R}^d$, $\nu \in \mathscr{P}_k$, $\phi \in L^k(\mathbb{R}^d \to \mathbb{R}^d; \nu)$, the limit

$$\lim_{r \downarrow 0} \frac{\int_{\mathbb{R}^d} g(y, z + r\phi(z))\nu(\mathrm{d}z) - \int_{\mathbb{R}^d} g(y,z)\nu(\mathrm{d}z)}{r}$$

does not exist. Moreover, it holds

$$\begin{aligned}
&|D^E b_t^{(1)}(y,\nu)(z) - D^E b_t^{(1)}(\bar{y},\bar{\nu})(\bar{z})| \\
&= |g(y,z) - g(\bar{y},\bar{z})| + \left| \int_{\mathbb{R}^d} g(y,z)\nu(\mathrm{d}z) - \int_{\mathbb{R}^d} g(\bar{y},z)\bar{\nu}(\mathrm{d}z) \right| \\
&\le \alpha(|z-\bar{z}|) + 2K|y-\bar{y}| + \mathbb{W}_\alpha(\nu,\bar{\nu}), \quad y, \bar{y} \in \mathbb{R}^d, \nu, \bar{\nu} \in \mathscr{P}_k, z, \bar{z} \in \mathbb{R}^d.
\end{aligned}$$

Note that Jensen's inequality implies that

$$\mathbb{W}_\alpha(\mu,\nu) \le \inf_{\pi \in \mathscr{C}(\mu,\nu)} \int_{\mathbb{R}^d \times \mathbb{R}^d} \alpha(|x-y|)\pi(\mathrm{d}x, \mathrm{d}y) \le \alpha(\mathbb{W}_1(\mu,\nu)) \le \alpha(\mathbb{W}_k(\mu,\nu)), \quad \mu, \nu \in \mathscr{P}_k.$$

So, $D^E b_t^{(1)}(y,\nu)(z)$ is continuous in $(y,\nu,z) \in \mathbb{R}^d \times \mathscr{P}_k \times \mathbb{R}^d$. Finally, by the dominated convergence theorem, we have

$$\nabla b_t^{(1)}(\cdot,\nu) = \int_{\mathbb{R}^d} \nabla g(\cdot, z)\nu(\mathrm{d}z), \quad \nu \in \mathscr{P}_k.$$

Therefore, $b^{(1)}$ satisfies (B_2)-(B_3).

Since **(B)** implies **(A)**, as explained before that under this assumption (1) is well-posed for distributions in $\mathscr{P}_k$.

For $\mu \in \mathscr{P}_k$, consider the decoupled SDE

$$\begin{aligned}
\mathrm{d}X_t^{x,\mu} &= \big\{ b_t^{(0)}(X_t^{x,\mu}) + b_t^{(1)}(X_t^{x,\mu}, P_t^*\mu) \big\}\mathrm{d}t + \sigma_t(X_t^{x,\mu})\mathrm{d}W_t, \\
X_0^{x,\mu} &= x, \quad t \in [0,T].
\end{aligned} \tag{47}$$

Let

$$\mathscr{B}_{k,b}(\mathbb{R}^d) := \left\{ f : \ \frac{f}{1 + |\cdot|^k} \in \mathscr{B}_b(\mathbb{R}^d) \right\}.$$

We first give a lemma on Bismut formula of $P_t^\mu f$ for $f \in \mathscr{B}_{k,b}(\mathbb{R}^d)$.

Lemma 6. *Let σ and b satisfy* **(B)**. *Then for any $v \in \mathbb{R}^d, \gamma \in \mathscr{P}_k, x \in \mathbb{R}^d$, the limit*

$$\nabla_v X_t^{x,\gamma} := \lim_{\varepsilon \downarrow 0} \frac{X_t^{x+\varepsilon v,\gamma} - X_t^{x,\gamma}}{\varepsilon}, \quad t \in [0,T]$$

exists in $L^p(\Omega \to C([0,T];\mathbb{R}^d);\mathbb{P})$ for any $p \geq 1$, and there exists a constant $c_p > 0$ such that

$$\mathbb{E}\Big[\sup_{t\in[0,T]} |\nabla_v X_t^{x,\gamma}|^p\Big] \leq c_p |v|^p, \quad v \in \mathbb{R}^d, \gamma \in \mathscr{P}_k, x \in \mathbb{R}^d. \tag{48}$$

Moreover, the Bismut formula for P_t^γ holds:

$$\begin{aligned} \nabla_v P_t^\gamma f(x) &= \mathbb{E}\left[f(X_t^{x,\gamma}) \int_0^t \frac{1}{t} \big\langle \zeta_s(X_s^{x,\gamma}) \nabla_v X_s^{x,\gamma}, \mathrm{d}W_s \big\rangle\right], \\ \zeta_s &:= \sigma_s^*(\sigma_s\sigma_s^*)^{-1}, \quad f \in \mathscr{B}_{k,b}(\mathbb{R}^d),\ x, v \in \mathbb{R}^d, \gamma \in \mathscr{P}_k, t \in (0,T]. \end{aligned} \tag{49}$$

Proof. By [30, Theorem 2.1] for $\beta_s = \frac{s}{t}$, **(B)** implies (48) and (49) for $f \in \mathscr{B}_b(\mathbb{R}^d)$. To deduce (49) for any $f \in \mathscr{B}_{b,k}(\mathbb{R}^d)$, let

$$f_n := [(-n) \vee f] \wedge n, \quad n \geq 1.$$

By (9), (48) and the boundedness of ζ_s, we find constants $c_0, c_1(\gamma) > 0$ such that

$$\begin{aligned} &\mathbb{E}\bigg[\big(1 + |X_t^{x+rv,\gamma}|^k\big) \bigg| \int_0^t \big\langle \zeta_s(X_s^{x+rv,\gamma}) \nabla_v X_s^{x+rv,\gamma}, \mathrm{d}W_s \big\rangle \bigg|\bigg] \\ &\leq c_0 \sqrt{t} \Big(\mathbb{E}\big[2 + |X_t^{x+rv,\gamma}|^{2k}\big]\Big)^{\frac{1}{2}} \Big(\mathbb{E}\big[\sup_{s\in[0,T]} \big|\nabla_v X_s^{x+rv,\gamma}\big|^2\big]\Big)^{\frac{1}{2}} \\ &\leq c_1(\gamma)\sqrt{t}|v|\big(1 + |x|^k + |v|^k\big), \quad t \in (0,T], x, v \in \mathbb{R}^d, r \in [0,1]. \end{aligned} \tag{50}$$

By (49) for f_n in place of f, we obtain

$$\begin{aligned} &\frac{P_t^\gamma f_n(x+\varepsilon v) - P_t^\gamma f_n(x)}{\varepsilon} \\ &= \frac{1}{\varepsilon} \int_0^\varepsilon \mathbb{E}\left[f_n(X_t^{x+rv,\gamma}) \int_0^t \frac{1}{t} \big\langle \zeta_s(X_s^{x+rv,\gamma}) \nabla_v X_s^{x+rv,\gamma}, \mathrm{d}W_s \big\rangle\right] \mathrm{d}r. \end{aligned}$$

Since $f \in \mathscr{B}_{k,b}(\mathbb{R}^d)$, by (9), (48) and (50), we may apply the dominated convergence theorem such that the above formula with $n \to \infty$ implies

$$\begin{aligned} &\frac{P_t^\gamma f(x+\varepsilon v) - P_t^\gamma f(x)}{\varepsilon} \\ &= \frac{1}{\varepsilon} \int_0^\varepsilon \mathbb{E}\left[f(X_t^{x+rv,\gamma}) \int_0^t \frac{1}{t} \big\langle \zeta_s(X_s^{x+rv,\gamma}) \nabla_v X_s^{x+rv,\gamma}, \mathrm{d}W_s \big\rangle\right] \mathrm{d}r, \\ &\quad f \in \mathscr{B}_{k,b}(\mathbb{R}^d), \varepsilon > 0,\ x, v \in \mathbb{R}^d, \gamma \in \mathscr{P}_k, t \in (0,T]. \end{aligned} \tag{51}$$

Note that (49) for f_n in place of f yields

$$\begin{aligned}
&\lim_{n\to\infty}\limsup_{\varepsilon\to 0}\left|\frac{P_t^\gamma f_n(x+\varepsilon v)-P_t^\gamma f_n(x)}{\varepsilon}-\mathbb{E}\left[f(X_t^{x,\gamma})\int_0^t\frac{1}{t}\big\langle\zeta_s(X_s^{x,\gamma})\nabla_v X_s^{x,\gamma},\mathrm{d}W_s\big\rangle\right]\right|\\
&=\lim_{n\to\infty}\bigg|\mathbb{E}\left[f_n(X_t^{x,\gamma})\int_0^t\frac{1}{t}\big\langle\zeta_s(X_s^{x,\gamma})\nabla_v X_s^{x,\gamma},\mathrm{d}W_s\big\rangle\right]\\
&\qquad-\mathbb{E}\left[f(X_t^{x,\gamma})\int_0^t\frac{1}{t}\big\langle\zeta_s(X_s^{x,\gamma})\nabla_v X_s^{x,\gamma},\mathrm{d}W_s\big\rangle\right]\bigg|=0,
\end{aligned}$$

where the last step follows from the dominated convergence theorem due to $f\in\mathscr{B}_{k,b}(\mathbb{R}^d)$ and (50). This together with (51) for $f-f_n$ in place of f implies

$$\begin{aligned}
&\limsup_{\varepsilon\to 0}\left|\frac{P_t^\gamma f(x+\varepsilon v)-P_t^\gamma f(x)}{\varepsilon}-\mathbb{E}\left[f(X_t^{x,\gamma})\int_0^t\frac{1}{t}\big\langle\zeta_s(X_s^{x,\gamma})\nabla_v X_s^{x,\gamma},\mathrm{d}W_s\big\rangle\right]\right|\\
&\le\lim_{n\to\infty}\limsup_{\varepsilon\to 0}\left|\frac{P_t^\gamma(f-f_n)(x+\varepsilon v)-P_t^\gamma(f-f_n)(x)}{\varepsilon}\right|\\
&+\lim_{n\to\infty}\limsup_{\varepsilon\to 0}\bigg|\frac{P_t^\gamma f_n(x+\varepsilon v)-P_t^\gamma f_n(x)}{\varepsilon}\\
&\qquad-\mathbb{E}\left[f(X_t^{x,\gamma})\int_0^t\frac{1}{t}\big\langle\zeta_s(X_s^{x,\gamma})\nabla_v X_s^{x,\gamma},\mathrm{d}W_s\big\rangle\right]\bigg|\\
&\le\lim_{n\to\infty}\limsup_{\varepsilon\to 0}\frac{1}{t\varepsilon}\int_0^\varepsilon\mathbb{E}\bigg|(f_n-f)(X_t^{x+rv,\gamma})\\
&\qquad\times\int_0^t\big\langle\zeta_s(X_s^{x+rv,\gamma})\nabla_v X_s^{x+rv,\gamma},\mathrm{d}W_s\big\rangle\bigg|\mathrm{d}r.
\end{aligned}\tag{52}$$

Since $f\in\mathscr{B}_{k,b}(\mathbb{R}^d)$ implies

$$|(f_n-f)(x)|\le c(1+|x|^k)1_{\{c(1+|x|^k)\ge n\}},\quad n\ge 1$$

for some constant $c>0$, by the same reason leading to (50), we find constants $\tilde{c},c_2(\gamma)>0$ such that

$$\begin{aligned}
&\sup_{r\in[0,1]}\mathbb{E}\bigg|(f_n-f)(X_t^{x+rv,\gamma})\int_0^t\big\langle\zeta_s(X_s^{x+rv,\gamma})\nabla_v X_s^{x+rv,\gamma},\mathrm{d}W_s\big\rangle\bigg|\\
&\le\tilde{c}\sqrt{t}|v|\sup_{r\in[0,1]}\Big(\mathbb{E}\big[1+|X_t^{x+rv,\gamma}|^{4k}\big]\Big)^{\frac{1}{2}}n^{-1}\\
&\le c_2(\gamma)\sqrt{t}|v|\big(1+|x|^{2k}+|v|^{2k}\big)n^{-1}.
\end{aligned}$$

Therefore, (49) follows form (52). □

To state the Bismut formula for $P_t f$, we introduce the quantity I_t^f: for fixed $t \in (0,T]$, let

$$\begin{aligned} I_t^f(\mu,\phi) &:= \frac{1}{t}\int_{\mathbb{R}^d} \mathbb{E}\bigg[f(X_t^{x,\mu})\int_0^t \big\langle \zeta_s(X_s^{x,\mu})\nabla_{\phi(x)}X_s^{x,\mu}, \mathrm{d}W_s\big\rangle\bigg]\mu(\mathrm{d}x), \\ &s\in[0,t], \mu\in\mathscr{P}_k, \phi\in T_{\mu,k}, f\in\mathscr{B}_{k-1,b}(\mathbb{R}^d). \end{aligned} \tag{53}$$

By **(B)** and (48), we find a constant $c>0$ such that

$$|I_t^f(\mu,\phi)| \le \frac{c}{\sqrt{t}}\big(P_t|f|^{k^*}(\mu)\big)^{\frac{1}{k^*}}\|\phi\|_{L^k(\mu)}, \quad \mu\in\mathscr{P}_k, \phi\in T_{\mu,k}, f\in\mathscr{B}_{k-1,b}(\mathbb{R}^d). \tag{54}$$

Next, let X_0^μ be $\mathscr{F}_0$-measurable such that $\mathscr{L}_{X_0^\mu}=\mu$, and let X_t^μ solve (1) with initial value X_0^μ. For any $\varepsilon\ge 0$, denote

$$\mu_\varepsilon := \mu\circ(id+\varepsilon\phi)^{-1}, \quad X_0^{\mu_\varepsilon} := X_0^\mu + \varepsilon\phi(X_0^\mu).$$

Let $X_t^{\mu_\varepsilon}$ solve (1) with initial value $X_0^{\mu_\varepsilon}$. So,

$$X_t^\mu = X_t^{\mu_0}, \quad P_t^*\mu_\varepsilon = \mathscr{L}_{X_t^{\mu_\varepsilon}}, \quad t\in[0,T], \varepsilon\ge 0.$$

Now, we present the main result of this part.

Theorem 7. *Assume* **(B)** *and let* ζ_s *and* I_t^f *be in* (49) *and* (53) *respectively. Then the following assertions hold.*

(1) *For any* $t\in(0,T]$, $y\in\mathbb{R}^d$, $\nu\in\mathscr{P}_k$, $P_t[D^E b_t^{(1)}(y,\nu)(\cdot)](\mu)$ *is intrinsically differentiable on* $\mu\in\mathscr{P}_k$, *and there exists a constant* $c>0$ *such that*

$$\sup_{(y,\nu)\in\mathbb{R}^d\times\mathscr{P}_k}\|D^I P_t[D^E b_t^{(1)}(y,\nu)(\cdot)](\mu)\|_{L^{\frac{k}{k-1}}(\mu)} \le \frac{c\,\alpha(t^{\frac12})}{\sqrt{t}}, \quad t\in(0,T], \mu\in\mathscr{P}_k.$$

(2) *For any* $t\in(0,T]$ *and* $f\in\mathscr{B}_{k-1,b}(\mathbb{R}^d)$, $P_t f$ *is intrinsically differentiable on* $\mathscr{P}_k$. *Moreover, for any* $\mu\in\mathscr{P}_k$ *and* $\phi\in T_{\mu,k}$,

$$\begin{aligned} D_\phi^I P_t f(\mu) &= I_t^f(\mu,\phi) \\ &\quad + \int_{\mathbb{R}^d}\mathbb{E}\bigg[f(X_t^{x,\mu})\int_0^t\Big\langle \zeta_s(X_s^{x,\mu})N_s(\mu,\phi),\ \mathrm{d}W_s\Big\rangle\bigg]\mu(\mathrm{d}x), \\ N_s(\mu,\phi) &:= \big\{D_\phi^I P_s[D^E b_s^{(1)}(y,\nu)(\cdot)](\mu)\big\}|_{y=X_s^{x,\mu},\nu=P_s^*\mu}, \end{aligned} \tag{55}$$

where $X_t^{x,\mu}$ *solves* (47) *with initial value* $x\in\mathbb{R}^d$.

By (54)-(55), we find a constant $c>0$ such that

$$\|D^I P_t f(\mu)\|_{L^{k^*}(\mu)} \le c\frac{\{P_t|f|^{k^*}(\mu)\}^{\frac{1}{k^*}}}{\sqrt{t}}, \quad t\in(0,T], f\in\mathscr{B}_{k-1,b}(\mathbb{R}^d), \mu\in\mathscr{P}_k.$$

To explain the main idea in the proof of Theorem 7, we first figure out a sketch. By the definition of the intrinsic derivative, we intend to calculate for any $f \in \mathscr{B}_{k-1,b}(\mathbb{R}^d)$,

$$D_\phi^I P_t f(\mu) := \lim_{\varepsilon \downarrow 0} \frac{P_t f(\mu_\varepsilon) - P_t f(\mu)}{\varepsilon} = \lim_{\varepsilon \downarrow 0} \frac{\mathbb{E}[f(X_t^{\mu_\varepsilon}) - f(X_t^\mu)]}{\varepsilon}. \tag{56}$$

To this end, for any $\mu \in \mathscr{P}_k, x \in \mathbb{R}^d$, recall that $X_t^{x,\mu}$ solves the decoupled SDE (47), and

$$P_t^\mu f(x) = \mathbb{E}[f(X_t^{x,\mu})], \quad x \in \mathbb{R}^d.$$

Define

$$P_t^\mu f(\tilde{\mu}) := \int_{\mathbb{R}^d} P_t^\mu f \mathrm{d}\tilde{\mu}, \quad t \geq 0, f \in \mathscr{B}_{k-1,b}(\mathbb{R}^d), \mu, \tilde{\mu} \in \mathscr{P}_k.$$

For $\varepsilon \geq 0$, let $X_t^{\mu_\varepsilon,\mu}$ be the solution of (47) with initial value X_0^ε, i.e,

$$\begin{aligned} \mathrm{d}X_t^{\mu_\varepsilon,\mu} = \big\{b_t^{(0)}(X_t^{\mu_\varepsilon,\mu}) + b_t^{(1)}\big(X_t^{\mu_\varepsilon,\mu}, P_t^*\mu\big)\big\}\mathrm{d}t + \sigma_t(X_t^{\mu_\varepsilon,\mu})\mathrm{d}W_t, \\ t \in [0,T], \ X_0^{\mu_\varepsilon,\mu} = X_0^\varepsilon. \end{aligned}$$

Then $X_t^{\mu_\varepsilon,\mu_\varepsilon}$ solves (1) with initial value X_0^ε, so that

$$P_t f(\mu_\varepsilon) = P_t^{\mu_\varepsilon} f(\mu_\varepsilon) = \mathbb{E}[f(X_t^{\mu_\varepsilon,\mu_\varepsilon})], \quad \varepsilon \geq 0, t \in [0,T], f \in \mathscr{B}_{k-1,b}(\mathbb{R}^d).$$

Noting that $\mu_0 = \mu$, (56) reduces to

$$\begin{aligned} D_\phi^I P_t f(\mu) &= \lim_{\varepsilon \downarrow 0} \frac{P_t^{\mu_\varepsilon} f(\mu_\varepsilon) - P_t^\mu f(\mu)}{\varepsilon} \\ &= \lim_{\varepsilon \downarrow 0} \Big\{ \frac{P_t^\mu f(\mu_\varepsilon) - P_t^\mu f(\mu)}{\varepsilon} + \frac{P_t^{\mu_\varepsilon} f(\mu_\varepsilon) - P_t^\mu f(\mu_\varepsilon)}{\varepsilon} \Big\}. \end{aligned} \tag{57}$$

So, to calculate $D_\phi^I P_t f(\mu)$, we only need to study the limits of

$$J_1 f(t,\varepsilon) := \frac{P_t^\mu f(\mu_\varepsilon) - P_t^\mu f(\mu)}{\varepsilon}, \quad J_2 f(t,\varepsilon) := \frac{P_t^{\mu_\varepsilon} f(\mu_\varepsilon) - P_t^\mu f(\mu_\varepsilon)}{\varepsilon}.$$

By Lemma 6, for any $t \in (0,T], \varepsilon \geq 0$ and $f \in \mathscr{B}_{k-1,b}(\mathbb{R}^d)$, we have

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}\varepsilon} P_t^\mu f(\mu_\varepsilon) := \lim_{r \downarrow 0} \frac{P_t^\mu f(\mu_{\varepsilon+r}) - P_t^\mu f(\mu_\varepsilon)}{r} \\ &= \int_{\mathbb{R}^d} \mathbb{E}\bigg[f(X_t^{x+\varepsilon\phi(x),\mu}) \frac{1}{t} \int_0^t \big\langle \zeta_s(X_s^{x+\varepsilon\phi(x),\mu}) \nabla_{\phi(x)} X_s^{x+\varepsilon\phi(x),\mu}, \mathrm{d}W_s \big\rangle \bigg] \mu(\mathrm{d}x). \end{aligned}$$

In particular,

$$\lim_{\varepsilon\downarrow 0} J_1 f(t,\varepsilon) = \lim_{\varepsilon\downarrow 0} \frac{P_t^\mu f(\mu_\varepsilon) - P_t^\mu f(\mu)}{\varepsilon} = I_t^f(\mu,\phi), \quad t\in(0,T]. \tag{58}$$

Consequently, it remains to prove

$$\lim_{\varepsilon\to 0} J_2 f(t,\varepsilon) = \int_{\mathbb{R}^d} \mathbb{E}\bigg[f(X_t^{x,\mu}) \int_0^t \Big\langle \zeta_s(X_s^{x,\mu}) N_s(\mu,\phi),\ \mathrm{d}W_s \Big\rangle\bigg] \mu(\mathrm{d}x)$$

for $N_s(\mu,\phi)$ defined in (55), which involves in $D_\phi^I\{P_s[D^E b_s^{(1)}(y,\nu)(\cdot)](\mu)\}$. Therefore, we will first study $D_\phi^I\{P_s[D^E b_s^{(1)}(y,\nu)(\cdot)](\mu)\}$.

Recall that $\tilde{\alpha}$ is defined in (41). For any $V\in\mathscr{B}_b([0,T]\times\mathbb{R}^d\times\mathscr{P}_k;\mathbb{R}^d)$, the set of bounded and measurable $\mathbb{R}^d$-valued functions on $[0,T]\times\mathbb{R}^d\times\mathscr{P}_k$ and $t\in[0,T], y\in\mathbb{R}^d, \nu\in\mathscr{P}_k$, we write

$$\begin{aligned}\tilde{I}_t^V(y,\nu) := \int_{\mathbb{R}^d} \mathbb{E}\bigg[& D^E b_t^{(1)}(y,\nu)(X_t^{x,\mu}) \\ &\times \int_0^t \frac{\alpha(s^{\frac12})}{\{\tilde{\alpha}(s^{\frac12})s\}^{\frac12}} \Big\langle \zeta_s(X_s^{x,\mu}) V_s(X_s^{x,\mu}, P_s^*\mu),\ \mathrm{d}W_s \Big\rangle\bigg] \mu(\mathrm{d}x).\end{aligned} \tag{59}$$

By (B_2), for any $t\in[0,T], y\in\mathbb{R}^d, \nu\in\mathscr{P}_k$, we have

$$\begin{aligned}|D^E b_t^{(1)}(y,\nu)(\cdot)| &\le \alpha(|\cdot|) + |D^E b_t^{(1)}(y,\nu)(0)| \\ &\le c_0(1+|\cdot|^{k-1}) + |D^E b_t^{(1)}(y,\nu)(0)|.\end{aligned}$$

So, $I_t^f(\mu,\phi)$ for $f = D^E b_t^{(1)}(y,\nu)(\cdot)$ in (53) is well-defined, and we denote

$$\begin{aligned}I_t^{\mu,\phi}(y,\nu) := \frac{1}{t}\int_{\mathbb{R}^d} \mathbb{E}\bigg[& D^E b_t^{(1)}(y,\nu)(X_t^{x,\mu}) \\ &\times \int_0^t \Big\langle \zeta_s(X_s^{x,\mu}) \nabla_{\phi(x)} X_s^{x,\mu}, \mathrm{d}W_s \Big\rangle\bigg] \mu(\mathrm{d}x).\end{aligned} \tag{60}$$

Consider the following equation for $V\in\mathscr{B}_b([0,T]\times\mathbb{R}^d\times\mathscr{P}_k;\mathbb{R}^d)$:

$$V_t(y,\nu) = \frac{\{t\tilde{\alpha}(t^{\frac12})\}^{\frac12}}{\alpha(t^{\frac12})}\big\{I_t^{\mu,\phi}(y,\nu) + \tilde{I}_t^V(y,\nu)\big\}, \quad t\in[0,T], y\in\mathbb{R}^d, \nu\in\mathscr{P}_k. \tag{61}$$

If this equation has a unique solution, we denote it by $V_t(y,\nu) = v_t^{\mu,\phi}(y,\nu)$ for $(t,y,\nu)\in[0,T]\times\mathbb{R}^d\times\mathscr{P}_k$, to emphasize the dependence on μ and ϕ.

In the following two subsections, we prove the well-posedness of (61) and establish the formula

$$D_\phi^I\{P_t[D^E b_t^{(1)}(y,\nu)(\cdot)]\}(\mu) = \frac{\alpha(t^{\frac{1}{2}})}{\{t\tilde{\alpha}(t^{\frac{1}{2}})\}^{\frac{1}{2}}} v_t^{\mu,\phi}(y,\nu), \quad t\in(0,T], y\in\mathbb{R}^d, \nu\in\mathscr{P}_k. \tag{62}$$

3.2 Well-posedness of (61)

Lemma 8. *Assume* **(B)**. *For any $\mu\in\mathscr{P}_k$ and $\phi\in T_{\mu,k}$, the equation (61) has a unique solution, which is denoted by $\{v_t^{\mu,\phi}(y,\nu)\}_{t\in[0,T],y\in\mathbb{R}^d,\nu\in\mathscr{P}_k}$, and there exists a constant $c>0$ such that*

$$\sup_{\|\phi\|_{L^k(\mu)}\le 1}\sup_{y\in\mathbb{R}^d,\nu\in\mathscr{P}_k}|v_t^{\mu,\phi}(y,\nu)| \le c\sqrt{\tilde{\alpha}(t^{\frac{1}{2}})}, \quad \mu\in\mathscr{P}_k, t\in[0,T]. \tag{63}$$

Proof. Let

$$\mathscr{V}_0 := \big\{V\in\mathscr{B}_b([0,T]\times\mathbb{R}^d\times\mathscr{P}_k;\mathbb{R}^d):\ V_0=0\big\},$$

which is a Banach space under the uniform norm. For $V\in\mathscr{V}_0$, let

$$\|V_t\|_\infty = \sup_{y\in\mathbb{R}^d,\nu\in\mathscr{P}_k}|V_t(y,\nu)|, \quad t\in[0,T]$$

and for any $t\in[0,T], y\in\mathbb{R}^d, \nu\in\mathscr{P}_k$, let

$$\{H(V)\}_t(y,\nu) := \frac{\{t\tilde{\alpha}(t^{\frac{1}{2}})\}^{\frac{1}{2}}}{\alpha(t^{\frac{1}{2}})}\big\{I_t^{\mu,\phi}(y,\nu)+\tilde{I}_t^V(y,\nu)\big\}. \tag{64}$$

Then it suffices to prove

(i) The map $H:\ \mathscr{V}_0\to\mathscr{V}_0$ is well-defined and has a unique fixed point $v^{\mu,\phi}$ which turns out to be the unique solution of (61).
(ii) There exists a constant $c>0$ such that

$$\sup_{\|\phi\|_{L^k(\mu)}\le 1}\|v_t^{\mu,\phi}\|_\infty \le c\sqrt{\tilde{\alpha}(t^{\frac{1}{2}})}, \quad (t,\mu)\in[0,T]\times\mathscr{P}_k.$$

Next, we will prove (i) and (ii) one by one.

(1) Proof of (i).

(a) We first verify

$$H:\mathscr{V}_0\to\mathscr{V}_0. \tag{65}$$

Recall that $\theta^{\lambda,\gamma}$ and $\tilde{\theta}^{\lambda,\gamma}$ are defined in (16)-(18). Since **(B)** implies **(A)**, we conclude that (19) still holds.

By $[D^E b_t^{(1)}(y,\nu)(\cdot)]_\alpha \le 1$ due to (B_2), (11) in Lemma 2 for $p=2$ and $z=\tilde\theta_t^{\lambda,\mu}(x)$, (48), (19), (2) and (60), we find a constant $c_1>0$ such that

$$\begin{aligned}|I_t^{\mu,\phi}(y,\nu)| &\le \frac{c_1}{\sqrt{t}}\int_{\mathbb{R}^d}\alpha\big(t^{\frac12}\big)|\phi(x)|\mu(\mathrm{d}x)\\ &\le \frac{c_1}{\sqrt{t}}\alpha\big(t^{\frac12}\big)\|\phi\|_{L^k(\mu)},\quad t\in(0,T],\mu\in\mathscr{P}_k,\phi\in T_{\mu,k},y\in\mathbb{R}^d,\nu\in\mathscr{P}_k.\end{aligned} \tag{66}$$

So, by (59), (B_2), (11) in Lemma 2 for $p=2$ and $z=\tilde\theta_t^{\lambda,\mu}(x)$, (2) and (19), we find a constant $c_2>0$ such that

$$|\tilde I_t^V(y,\nu)| \le c_2\alpha\big(t^{\frac12}\big)\bigg(\int_0^t \frac{\alpha(s^{\frac12})^2}{s\tilde\alpha(s^{\frac12})}\|V_s\|_\infty^2\mathrm{d}s\bigg)^{\frac12},\quad y\in\mathbb{R}^d,\nu\in\mathscr{P}_k. \tag{67}$$

Combining this with (64) and (66), we find a constant $c_3>0$ such that

$$\|\{H(V)\}_t\|_\infty \le c_3\|\phi\|_{L^k(\mu)}\sqrt{\tilde\alpha\big(t^{\frac12}\big)} + c_3\sqrt{t\tilde\alpha\big(t^{\frac12}\big)}\bigg(\int_0^t\frac{\alpha(s^{\frac12})^2}{s\tilde\alpha\big(s^{\frac12}\big)}\|V_s\|_\infty^2\mathrm{d}s\bigg)^{\frac12}. \tag{68}$$

Then (65) follows from the fact that (41) implies

$$\int_0^t\frac{\alpha(rs^{\frac12})^2}{s\tilde\alpha(rs^{\frac12})}\mathrm{d}s = 2\int_0^{rt^{\frac12}}\frac{\alpha(s)^2}{s\tilde\alpha(s)}\mathrm{d}s = 4\int_0^{rt^{\frac12}}\tilde\alpha'(s)\mathrm{d}s = 4\tilde\alpha(rt^{\frac12}),\quad r\ge0. \tag{69}$$

(b) We intend to prove that H in (i) has a unique fixed point in $\mathscr{V}_0$. Obviously, for any $\delta>0$, $\mathscr{V}_0$ is complete under the metric

$$\rho_\delta(V,U) := \sup_{t\in[0,T]}\mathrm{e}^{-\delta t}\|V_t-U_t\|_\infty,\quad V,U\in\mathscr{V}_0.$$

So, it suffices to prove the contraction of H in ρ_δ for large enough $\delta>0$.

By (64), (66) and (67), we find a constant $c_4>0$ such that

$$\begin{aligned}|\{H(V)\}_t(y,\nu)-\{H(U)\}_t(y,\nu)| &= \frac{\{t\tilde\alpha\big(t^{\frac12}\big)\}^{\frac12}}{\alpha\big(t^{\frac12}\big)}|\tilde I_t^{V-U}(y,\nu)|\\ &\le c_4\bigg(\int_0^t\frac{\alpha(s^{\frac12})^2}{s\tilde\alpha(s^{\frac12})}\|V_s-U_s\|_\infty^2\mathrm{d}s\bigg)^{\frac12},\quad V,U\in\mathscr{V}_0,t\in[0,T].\end{aligned}$$

Combining this with (69), we conclude that H is contractive in the complete metric space $(\mathscr{V}_0,\rho_\delta)$ for large enough $\delta>0$, and hence has a unique fixed point denoted by $v^{\mu,\phi}$.

(2) Proof of (ii). By (68) and noting that $H(v^{\mu,\phi}) = v^{\mu,\phi}$, we derive

$$\|v_t^{\mu,\phi}\|_\infty^2 \le 2c_3^2\tilde{\alpha}\big(t^{\frac{1}{2}}\big)\|\phi\|_{L^k(\mu)}^2 + 2c_3^2 t\tilde{\alpha}\big(t^{\frac{1}{2}}\big)\int_0^t \frac{\alpha(s^{\frac{1}{2}})^2}{s\tilde{\alpha}(s^{\frac{1}{2}})}\|v_s^{\mu,\phi}\|_\infty^2 \mathrm{d}s, \quad t \in [0,T].$$

Combining this with (69) and Gronwall's inequality, we find a constant $c_5 > 0$ such that for any $t \in [0,T]$,

$$\|v_t^{\mu,\phi}\|_\infty \le c_5\|\phi\|_{L^k(\mu)}\sqrt{\tilde{\alpha}(t^{\frac{1}{2}})}, \quad \mu \in \mathscr{P}_k,\ \phi \in T_{\mu,k}.$$

This proves (ii). □

3.3 Proof of Theorem 7

By Lemma 8, the proof of Theorem 7(1) is completed by the following lemma.

Lemma 9. *Assume* **(B)**. *Then for any $\mu \in \mathscr{P}_k$, $\phi \in T_{\mu,k}$, the function $h : (0,T] \times \mathbb{R}^d \times \mathscr{P}_k \to \mathbb{R}^d$ defined by*

$$h_t(y,\nu) := D_\phi^I\big\{P_t[D^E b_t^{(1)}(y,\nu)(\cdot)](\mu)\big\}, \quad t \in (0,T], y \in \mathbb{R}^d, \nu \in \mathscr{P}_k$$

exists in $\mathscr{B}((0,T] \times \mathbb{R}^d \times \mathscr{P}_k; \mathbb{R}^d)$ such that (62) holds. Consequently, there exists a constant $c > 0$ such that for any $\mu \in \mathscr{P}_k$,

$$\sup_{\|\phi\|_{L^k(\mu)} \le 1}\left\{\sup_{t\in(0,T]}\frac{\sqrt{t}}{\alpha(t^{\frac{1}{2}})}\sup_{y\in\mathbb{R}^d,\nu\in\mathscr{P}_k}|D_\phi^I\big\{P_t[D^E b_t^{(1)}(y,\nu)(\cdot)](\mu)\big\}|\right\} \le c.$$

Proof. (a) By Lemma 8, it suffices to prove (62). For simplicity, for any $t \in [0,T], y \in \mathbb{R}^d, \nu \in \mathscr{P}_k$, let

$$U_t(y,\nu,z) := D^E b_t^{(1)}(y,\nu)(z), \quad z \in \mathbb{R}^d. \tag{70}$$

Moreover, simply denote

$$\begin{aligned} v_t^\varepsilon(y,\nu) &:= v_t^{\varepsilon,1}(y,\nu) + v_t^{\varepsilon,2}(y,\nu),\\ v_t^{\varepsilon,1}(y,\nu) &:= \frac{P_t^\mu U_t(y,\nu,\cdot)(\mu_\varepsilon) - P_t^\mu U_t(y,\nu,\cdot)(\mu)}{\varepsilon},\\ v_t^{\varepsilon,2}(y,\nu) &:= \frac{P_t^{\mu_\varepsilon} U_t(y,\nu,\cdot)(\mu_\varepsilon) - P_t^\mu U_t(y,\nu,\cdot)(\mu_\varepsilon)}{\varepsilon}. \end{aligned}$$

Next, for $v_t^{\mu,\phi}$ in Lemma 8, let

$$\hat{v}_t(y,\nu) := \frac{\alpha(t^{\frac{1}{2}})}{\{\tilde{\alpha}(t^{\frac{1}{2}})t\}^{\frac{1}{2}}}v_t^{\mu,\phi}(y,\nu), \quad t \in (0,T], y \in \mathbb{R}^d, \nu \in \mathscr{P}_k, \tag{71}$$

and

$$\hat{v}_t^1(y,\nu) := I_t^{U_t(y,\nu,\cdot)}(\mu,\phi), \quad \hat{v}_t^2(y,\nu) := \hat{v}_t(y,\nu) - I_t^{U_t(y,\nu,\cdot)}(\mu,\phi). \tag{72}$$

Noting that

$$\begin{aligned} &P_t^\mu U_t(y,\nu,\cdot)(\mu_\varepsilon) - P_t^\mu U_t(y,\nu,\cdot)(\mu) \\ &= \int_{\mathbb{R}^d} \Big[P_t^\mu U_t(y,\nu,\cdot)(x+\varepsilon\phi(x)) - P_t^\mu U_t(y,\nu,\cdot)(x)\Big]\mu(\mathrm{d}x), \end{aligned} \tag{73}$$

by (49) for $f = U_t(y,\nu,\cdot)$, we obtain

$$\lim_{\varepsilon\to 0} \big|v_t^{\varepsilon,1}(y,\nu) - \hat{v}_t^1(y,\nu)\big| = 0, \quad t\in(0,T], y\in\mathbb{R}^d, \nu\in\mathscr{P}_k. \tag{74}$$

Since (61) holds for $V_t = v_t^{\mu,\phi}$, (70)-(72) imply that

$$\begin{aligned} &\hat{v}_t^2(y,\nu) = \hat{v}_t(y,\nu) - I_t^{U_t(y,\nu,\cdot)}(\mu,\phi) \\ &= \int_{\mathbb{R}^d} \mathbb{E}\Big[U_t(y,\nu,X_t^{x,\mu}) \\ &\qquad \times \int_0^t \Big\langle \zeta_s(X_s^{x,\mu})[\hat{v}_s^2(X_s^{x,\mu},P_s^*\mu) + \hat{v}_s^1(X_s^{x,\mu},P_s^*\mu)],\ \mathrm{d}W_s\Big\rangle\Big]\mu(\mathrm{d}x). \end{aligned} \tag{75}$$

In view of (74), to prove (62), it remains to verify

$$\lim_{\varepsilon\to 0}\sup_{t\in(0,T]} \frac{\sqrt{t}}{\alpha(t^{\frac{1}{2}})}|v_t^{\varepsilon,2}(y,\nu) - \hat{v}_t^2(y,\nu)| = 0, \quad y\in\mathbb{R}^d, \nu\in\mathscr{P}_k. \tag{76}$$

In the following, we first estimate $\|v_t^{\varepsilon,i}\|_\infty$ and $|v_t^{\varepsilon,i} - \hat{v}_t^i|(i=1,2)$ in steps (b)-(c), then verify (76) in step (d).

(b) Estimates on $\|v_t^{\varepsilon,i}\|_\infty, i=1,2$ and $\|v_t^{\varepsilon,1} - \hat{v}_t^1\|_\infty$.

By (51) for $f = U_t(y,\nu,\cdot)$ and (73), we obtain

$$\begin{aligned} v_t^{\varepsilon,1}(y,\nu) &= \frac{P_t^\mu U_t(y,\nu,\cdot)(\mu_\varepsilon) - P_t^\mu U_t(y,\nu,\cdot)(\mu)}{\varepsilon} \\ &= \frac{1}{\varepsilon}\int_0^\varepsilon \int_{\mathbb{R}^d} \mathbb{E}\Big[\Big(U_t\big(y,\nu,X_t^{x+r\phi(x),\mu}\big) - U_t\big(y,\nu,\tilde{\theta}_t^{\lambda,\mu}(x+r\phi(x))\big)\Big) \\ &\qquad \times \int_0^t \frac{1}{t}\big\langle \zeta_s(X_s^{x+r\phi(x),\mu})\nabla_{\phi(x)}X_s^{x+r\phi(x),\mu}, \mathrm{d}W_s\big\rangle\Big]\mu(\mathrm{d}x)\mathrm{d}r \\ &= \int_0^1 \int_{\mathbb{R}^d} \mathbb{E}\Big[\Big(U_t\big(y,\nu,X_t^{x+\varepsilon u\phi(x),\mu}\big) - U_t\big(y,\nu,\tilde{\theta}_t^{\lambda,\mu}(x+\varepsilon u\phi(x))\big)\Big) \\ &\qquad \times \int_0^t \frac{1}{t}\big\langle \zeta_s(X_s^{x+\varepsilon u\phi(x),\mu})\nabla_{\phi(x)}X_s^{x+\varepsilon u\phi(x),\mu}, \mathrm{d}W_s\big\rangle\Big]\mu(\mathrm{d}x)\mathrm{d}u, \end{aligned} \tag{77}$$

where in the last step, we used the integral transform $r = \varepsilon u$. Similar to (66), noting that **(B)** implies $[U_t(y,\nu,\cdot)]_\alpha \le 1$, by (11) in Lemma 2 for $p=2$ and $z = \tilde{\theta}_t^{\lambda,\mu}(x+$

$\varepsilon r\phi(x))$, (48), (19) and (2), we find a constant $c(\mu,\phi)>0$ depending on ϕ,μ such that

$$\begin{aligned}\|v_t^{\varepsilon,1}\|_\infty &= \sup_{y,\nu}\frac{|P_t^\mu U_t(y,\nu,\cdot)(\mu_\varepsilon)-P_t^\mu U_t(y,\nu,\cdot)(\mu)|}{\varepsilon}\\ &\le \sup_{y,\nu}\int_0^1\int_{\mathbb{R}^d}\bigg|\mathbb{E}\bigg[\Big(U_t\big(y,\nu,X_t^{x+\varepsilon r\phi(x),\mu}\big)-U_t\big(y,\nu,\tilde\theta_t^{\lambda,\mu}(x+\varepsilon r\phi(x))\big)\Big)\\ &\qquad\times\int_0^t\frac{1}{t}\big\langle\zeta_s(X_s^{x+\varepsilon r\phi(x),\mu})\nabla_{\phi(x)}X_s^{x+\varepsilon r\phi(x),\mu},\mathrm{d}W_s\big\rangle\bigg]\bigg|\mu(\mathrm{d}x)\mathrm{d}r\\ &\le \frac{c(\mu,\phi)\alpha(t^{\frac12})}{\sqrt{t}},\quad \varepsilon\in(0,1],t\in(0,T].\end{aligned}\tag{78}$$

This together with (74) and (66) implies that for a constant $c(\mu,\phi)>0$

$$h_{t,1}^\varepsilon(y,\nu):=\big\{v_t^{\varepsilon,1}(y,\nu)-\hat v_t^1(y,\nu)\big\}\frac{\{t\tilde\alpha(t^{\frac12})\}^{\frac12}}{\alpha(t^{\frac12})}$$

satisfies

$$\lim_{\varepsilon\to0}|h_{t,1}^\varepsilon(y,\nu)|=0,\quad \sup_{\varepsilon\in(0,1]}\sup_{y,\nu}|h_{t,1}^\varepsilon(y,\nu)|\le c(\mu,\phi)\sqrt{\tilde\alpha(t^{\frac12})},\quad t\in(0,T].\tag{79}$$

Next, we estimate $\|v_t^{\varepsilon,2}\|_\infty$. Recall that $X_t^{x+\varepsilon\phi(x),\mu}$ solves (47) with initial value $x+\varepsilon\phi(x)$. For any $x\in\mathbb{R}^d, s,t\in[0,T]$, let

$$\begin{aligned}R_t^{\varepsilon,x}&:=\mathrm{e}^{\int_0^t\langle\eta_s^{\varepsilon,x},\mathrm{d}W_s\rangle-\frac12\int_0^t|\eta_s^{\varepsilon,x}|^2\mathrm{d}s},\\ \eta_s^{\varepsilon,x}&:=\zeta_s(X_s^{x+\varepsilon\phi(x),\mu})\\ &\quad\times\big\{b_s^{(1)}(X_s^{x+\varepsilon\phi(x),\mu},P_s^*\mu_\varepsilon)-b_s^{(1)}(X_s^{x+\varepsilon\phi(x),\mu},P_s^*\mu)\big\}.\end{aligned}\tag{80}$$

By [21, Lemma 3.2], we have

$$\begin{aligned}&b_t^{(1)}(y,P_t^*\mu_\varepsilon)-b_t^{(1)}(y,P_t^*\mu)\\ &=\int_0^1\frac{\mathrm{d}}{\mathrm{d}r}b_t^{(1)}(y,(1-r)P_t^*\mu+rP_t^*\mu_\varepsilon)\mathrm{d}r\\ &=\int_0^1\int_{\mathbb{R}^d}D^E b_t^{(1)}(y,(1-r)P_t^*\mu+rP_t^*\mu_\varepsilon)(z)(P_t^*\mu_\varepsilon-P_t^*\mu)(\mathrm{d}z)\mathrm{d}r.\end{aligned}\tag{81}$$

Since **(B)** implies **(A)**, Lemma 5 holds so that we find constants $c_0,c>0$ such that

$$|\eta_s^{\varepsilon,x}|\le c_0\mathbb{W}_\alpha(P_s^*\mu_\varepsilon,P_s^*\mu)\le c\varepsilon\|\phi\|_{L^k(\mu)}\frac{\alpha(s^{\frac12})}{\sqrt{s}},\quad s\in[0,T],\ \varepsilon\in[0,1],x\in\mathbb{R}^d.\tag{82}$$

Then by Girsanov's theorem, for any $x \in \mathbb{R}^d$,

$$W_t^{\varepsilon,x} := W_t - \int_0^t \eta_s^{\varepsilon,x} \mathrm{d}s, \quad s \in [0,T]$$

is a Brownian motion under $\mathbb{Q} := R_T^{\varepsilon,x}\mathbb{P}$. Reformulate (47) with $x+\varepsilon\phi(x)$ replacing x as

$$\begin{aligned} \mathrm{d}X_t^{x+\varepsilon\phi(x),\mu} &= \Big\{b_t^{(0)}(X_t^{x+\varepsilon\phi(x),\mu}) + b_t^{(1)}(X_t^{x+\varepsilon\phi(x),\mu}, P_t^*\mu)\Big\}\mathrm{d}t + \sigma_t(X_t^{x+\varepsilon\phi(x),\mu})\mathrm{d}W_t \\ &= \Big\{b_t^{(0)}(X_t^{x+\varepsilon\phi(x),\mu}) + b_t^{(1)}(X_t^{x+\varepsilon\phi(x),\mu}, P_t^*\mu_\varepsilon)\Big\}\mathrm{d}t + \sigma_t(X_t^{x+\varepsilon\phi(x),\mu})\mathrm{d}W_t^{\varepsilon,x}, \\ X_0^{x+\varepsilon\phi(x),\mu} &= x+\varepsilon\phi(x),\ x \in \mathbb{R}^d. \end{aligned}$$

By the weak uniqueness of (47) with $\mu = \mu_\varepsilon$, we get

$$\begin{aligned} v_t^{\varepsilon,2}(y,\nu) &= \frac{P_t^{\mu_\varepsilon}U_t(y,\nu,\cdot)(\mu_\varepsilon) - P_t^{\mu}U_t(y,\nu,\cdot)(\mu_\varepsilon)}{\varepsilon} \\ &= \frac{\int_{\mathbb{R}^d}[P_t^{\mu_\varepsilon}U_t(y,\nu,\cdot)(x+\varepsilon\phi(x)) - P_t^{\mu}U_t(y,\nu,\cdot)(x+\varepsilon\phi(x))]\mu(\mathrm{d}x)}{\varepsilon} \\ &= \frac{1}{\varepsilon}\int_{\mathbb{R}^d}\mathbb{E}[U_t(y,\nu,X_t^{x+\varepsilon\phi(x),\mu})(R_t^{\varepsilon,x}-1)]\mu(\mathrm{d}x), \quad t\in[0,T]. \end{aligned} \tag{83}$$

By (82), for any $p \geq 1$ there exists a constant $c(p,\mu,\phi) > 0$ such that

$$\mathbb{E}[|R_t^{\varepsilon,x}-1|^p] \leq c(p,\mu,\phi)\varepsilon^p\bigg(\int_0^t \frac{\alpha(s^{\frac{1}{2}})^2}{s}\mathrm{d}s\bigg)^{\frac{p}{2}}, \quad t\in[0,T], \varepsilon\in[0,1], x\in\mathbb{R}^d. \tag{84}$$

Again by (11) in Lemma 2 for $p=2$ and $z = \tilde{\theta}_t^{\lambda,\mu}(x+\varepsilon\phi(x))$, (83), (84), (48), (19) and (2), we find a constant $c_1(\mu,\phi) > 0$ such that

$$\|v_t^{\varepsilon,2}\|_\infty \leq c_1(\mu,\phi)\alpha(t^{\frac{1}{2}})\bigg(\int_0^t \frac{\alpha(s^{\frac{1}{2}})^2}{s}\mathrm{d}s\bigg)^{\frac{1}{2}}, \quad t\in[0,T], \varepsilon\in(0,1]. \tag{85}$$

This together with (78) yields that for some constant $c_2(\mu,\phi) > 0$,

$$\begin{aligned} \|v_t^\varepsilon\|_\infty^2 &\leq 2\|v_t^{\varepsilon,1}\|_\infty^2 + 2\|v_t^{\varepsilon,2}\|_\infty^2 \\ &\leq c_2(\mu,\phi)\bigg(\frac{\alpha(t^{\frac{1}{2}})^2}{t} + \alpha(t^{\frac{1}{2}})\int_0^t \frac{\alpha(s^{\frac{1}{2}})^2}{s}\mathrm{d}s\bigg), \quad t\in(0,T], \varepsilon\in(0,1]. \end{aligned}$$

By the definition of α and (42), we find a constant $c_3(\mu,\phi) > 0$ such that

$$\|v_t^\varepsilon\|_\infty^2 \leq c_3(\mu,\phi)\frac{\alpha(t^{\frac{1}{2}})^2}{t}, \quad t\in(0,T], \quad \varepsilon\in(0,1]. \tag{86}$$

(c) Estimate on $\|v_t^{\varepsilon,2} - \hat{v}_t^2\|_\infty$. Similarly to (b), we have

$$\begin{aligned}
&\frac{R_t^{\varepsilon,x} - 1}{\varepsilon} = \int_0^t R_s^{\varepsilon,x} \big\langle \varepsilon^{-1} \eta_s^{\varepsilon,x}, \mathrm{d}W_s \big\rangle \\
&= \int_0^t R_s^{\varepsilon,x} \Big\langle \frac{\zeta_s(X_s^{x+\varepsilon\phi(x),\mu})[b_s^{(1)}(\cdot, P_s^*\mu_\varepsilon) - b_s^{(1)}(\cdot, P_s^*\mu)](X_s^{x+\varepsilon\phi(x),\mu})}{\varepsilon}, \ \mathrm{d}W_s \Big\rangle \qquad (87) \\
&= h_t(\varepsilon, x) + \int_0^t \Big\langle \zeta_s(X_s^{x,\mu}) v_s^\varepsilon(X_s^{x,\mu}, P_s^*\mu), \ \mathrm{d}W_s \Big\rangle, \quad x \in \mathbb{R}^d,
\end{aligned}$$

where

$$\begin{aligned}
h_t(\varepsilon, x) := \int_0^t \Big\langle &\zeta_s(X_s^{x+\varepsilon\phi(x),\mu}) R_s^{\varepsilon,x} \frac{[b_s^{(1)}(\cdot, P_s^*\mu_\varepsilon) - b_s^{(1)}(\cdot, P_s^*\mu)](X_s^{x+\varepsilon\phi(x),\mu})}{\varepsilon} \\
&- \zeta_s(X_s^{x,\mu}) v_s^\varepsilon(X_s^{x,\mu}, P_s^*\mu), \ \mathrm{d}W_s \Big\rangle, \quad x \in \mathbb{R}^d
\end{aligned}$$

satisfies

$$\lim_{\varepsilon \to 0} \mathbb{E}\Big[\sup_{t \in [0,T]} |h_t(\varepsilon, x)|^2 \Big] = 0, \quad x \in \mathbb{R}^d. \qquad (88)$$

Indeed, by (81) and the definition of v_s^ε, we have

$$\begin{aligned}
&\frac{[b_s^{(1)}(\cdot, P_s^*\mu_\varepsilon) - b_s^{(1)}(\cdot, P_s^*\mu)]((X_s^{x+\varepsilon\phi(x),\mu})}{\varepsilon} \\
&= \frac{1}{\varepsilon} \int_0^1 \int_{\mathbb{R}^d} D^E b_s^{(1)}(X_s^{x+\varepsilon\phi(x),\mu}, (1-r)P_s^*\mu + rP_s^*\mu_\varepsilon)(z)(P_s^*\mu_\varepsilon - P_s^*\mu)(\mathrm{d}z)\mathrm{d}r \\
&= \int_0^1 v_s^\varepsilon(X_s^{x+\varepsilon\phi(x),\mu}, (1-r)P_s^*\mu + rP_s^*\mu_\varepsilon)\mathrm{d}r.
\end{aligned}$$

This together with the BDG inequality implies

$$\begin{aligned}
&\mathbb{E}\Big[\sup_{t \in [0,T]} |h_t(\varepsilon, x)|^2 \Big] \\
&\le 2 \int_0^T \mathbb{E}\Big| \zeta_s(X_s^{x+\varepsilon\phi(x),\mu}) R_s^{\varepsilon,x} \int_0^1 v_s^\varepsilon(X_s^{x+\varepsilon\phi(x),\mu}, (1-r)P_s^*\mu + rP_s^*\mu_\varepsilon)\mathrm{d}r \qquad (89) \\
&\qquad - \zeta_s(X_s^{x,\mu}) v_s^\varepsilon(X_s^{x,\mu}, P_s^*\mu) \Big|^2 \mathrm{d}s.
\end{aligned}$$

By (48), for any $p > 1$, we can find a constant $c_p > 0$ such that

$$\mathbb{E}\Big[\sup_{t \in [0,T]} |X_t^{x+\varepsilon\phi(x),\mu} - X_t^{x,\mu}|^p \Big] \le c_p |\phi(x)|^p \varepsilon^p, \quad \varepsilon \in [0,1], \mu \in \mathscr{P}_k. \qquad (90)$$

By the boundedness and continuity of ζ due to **(B)**, $\int_0^T \frac{\alpha(t^{\frac{1}{2}})^2}{t} \mathrm{d}t < \infty$, (90), (84), (86), (89), and the dominated convergence theorem, to prove (88), it is sufficient to

prove that for $(s,x,r)\in(0,T]\times\mathbb{R}^d\times[0,1]$,

$$\lim_{\varepsilon\to 0}\mathbb{E}\Big|v_s^\varepsilon(X_s^{x+\varepsilon\phi(x),\mu},(1-r)P_s^*\mu+rP_s^*\mu_\varepsilon)-v_s^\varepsilon(X_s^{x,\mu},P_s^*\mu)\Big|=0. \tag{91}$$

For any $(\omega,\omega')\in\Omega\times\Omega$, let

$$\begin{aligned}
&U_r^{1,\varepsilon}(x,y,s,u,\omega,\omega')=U_s(X_s^{x+\varepsilon\phi(x),\mu}(\omega'),(1-r)P_s^*\mu+rP_s^*\mu_\varepsilon,X_s^{y+\varepsilon u\phi(y),\mu}(\omega)),\\
&U_r^{2,\varepsilon}(x,y,s,u,\omega,\omega')=U_s(X_s^{x+\varepsilon\phi(x),\mu}(\omega'),(1-r)P_s^*\mu+rP_s^*\mu_\varepsilon,\tilde\theta_s^{\lambda,\mu}(y+\varepsilon u\phi(y))),\\
&\tilde U_r^{1,\varepsilon}(x,y,s,u,\omega,\omega')=U_s(X_s^{x,\mu}(\omega'),P_s^*\mu,X_s^{y+\varepsilon u\phi(y),\mu}(\omega)),\\
&\tilde U_r^{2,\varepsilon}(x,y,s,u,\omega,\omega')=U_s(X_s^{x,\mu}(\omega'),P_s^*\mu,\tilde\theta_s^{\lambda,\mu}(y+\varepsilon u\phi(y))).
\end{aligned}$$

Since **(B)** implies **(A)**, (30) holds such that

$$\mathbb{W}_k((1-r)P_s^*\mu+rP_s^*\mu_\varepsilon,P_s^*\mu)\le r\mathbb{W}_k(P_s^*\mu_\varepsilon,P_s^*\mu)\le cr\varepsilon\|\phi\|_{L^k(\mu)}. \tag{92}$$

By (77), (48) and Hölder's inequality, we conclude that for any $\beta\in(1,k)$,

$$\begin{aligned}
&\mathbb{E}|v_s^{\varepsilon,1}(X_s^{x+\varepsilon\phi(x),\mu},(1-r)P_s^*\mu+rP_s^*\mu_\varepsilon)-v_s^{\varepsilon,1}(X_s^{x,\mu},P_s^*\mu)|\\
&\le\int_0^1\int_{\mathbb{R}^d}\int_{\Omega\times\Omega}\Big|\big[(U_r^{1,\varepsilon}(x,y,s,u,\omega,\omega')-U_r^{2,\varepsilon}(x,y,s,u,\omega,\omega'))\\
&\qquad-(\tilde U_r^{1,\varepsilon}(x,y,s,u,\omega,\omega')-\tilde U_r^{2,\varepsilon}(x,y,s,u,\omega,\omega'))\big]\\
&\qquad\times\int_0^s\frac{1}{s}\big\langle\zeta_v(X_v^{y+\varepsilon u\phi(y),\mu})\nabla_{\phi(y)}X_v^{y+\varepsilon u\phi(y),\mu},\mathrm{d}W_v\big\rangle\Big|\mathrm{d}\mathbb{P}(\omega)\mathrm{d}\mathbb{P}(\omega')\mu(\mathrm{d}y)\mathrm{d}u\\
&\le c_0\int_0^1\int_{\mathbb{R}^d}\frac{1}{\sqrt{s}}|\phi(y)|\Big\{\int_{\Omega\times\Omega}\Big|(U_r^{1,\varepsilon}(x,y,s,u,\omega,\omega')-U_r^{2,\varepsilon}(x,y,s,u,\omega,\omega'))\\
&\qquad-(\tilde U_r^{1,\varepsilon}(x,y,s,u,\omega,\omega')-\tilde U_r^{2,\varepsilon}(x,y,s,u,\omega,\omega'))\Big|^\beta\mathrm{d}\mathbb{P}(\omega)\mathrm{d}\mathbb{P}(\omega')\Big\}^{\frac{1}{\beta}}\mu(\mathrm{d}y)\mathrm{d}u.
\end{aligned}$$

By (10) for $\eta=\alpha(\xi)^{k-1}$ and $p=\frac{k}{k-1}$, we obtain $\|\alpha(\xi)\|_{L^k(\mathbb{P})}\le\alpha(\|\xi\|_{L^k(\mathbb{P})})$, which together with (19) implies

$$\begin{aligned}
&\int_{\Omega\times\Omega}\Big|(U_r^{1,\varepsilon}(x,y,s,u,\omega,\omega')-U_r^{2,\varepsilon}(x,y,s,u,\omega,\omega'))\\
&\qquad-(\tilde U_r^{1,\varepsilon}(x,y,s,u,\omega,\omega')-\tilde U_r^{2,\varepsilon}(x,y,s,u,\omega,\omega'))\Big|^k\mathrm{d}\mathbb{P}(\omega)\mathrm{d}\mathbb{P}(\omega')\\
&\le 2^k\mathbb{E}\alpha(|X_s^{y+\varepsilon u\phi(y),\mu}-\tilde\theta_s^{\lambda,\mu}(y+\varepsilon u\phi(y))|)^k\\
&\le 2^k\alpha\Big((\mathbb{E}|X_s^{y+\varepsilon u\phi(y),\mu}-\tilde\theta_s^{\lambda,\mu}(y+\varepsilon u\phi(y))|^k)^{\frac{1}{k}}\Big)^k\\
&\le c_k\alpha(\sqrt{s})^k
\end{aligned}$$

for some constant $c_k > 0$. So, it follows from the fact that $D^E b_t^{(1)}(y,\nu)(z)$ is continuous in $(y,\nu,z) \in \mathbb{R}^d \times \mathscr{P}_k \times \mathbb{R}^d$ due to (B_2), (92), (90), (70) and the dominated convergence theorem that

$$\lim_{\varepsilon\to 0} \mathbb{E}\Big| v_s^{\varepsilon,1}(X_s^{x+\varepsilon\phi(x),\mu}, (1-r)P_s^*\mu + rP_s^*\mu_\varepsilon)\mathrm{d}r - v_s^{\varepsilon,1}(X_s^{x,\mu}, P_s^*\mu)\Big| = 0.$$

Similarly, by (83) and (84), we have

$$\lim_{\varepsilon\to 0} \mathbb{E}\Big| v_s^{\varepsilon,2}(X_s^{x+\varepsilon\phi(x),\mu}, (1-r)P_s^*\mu + rP_s^*\mu_\varepsilon)\mathrm{d}r - v_s^{\varepsilon,2}(X_s^{x,\mu}, P_s^*\mu)\Big| = 0.$$

Therefore, (91) holds, which implies (88) as explained before (91).

Moreover, by (88), (79), (19), (86), (89) and the argument leading to (66), we obtain from the dominated convergence theorem that

$$\lim_{\varepsilon\to 0} \sup_{t\in(0,T]} \frac{\sqrt{t}}{\alpha(t^{\frac{1}{2}})} \int_{\mathbb{R}^d} \|\mathbb{E}[U_t(\cdot,\cdot,X_t^{x,\mu})h_t(\varepsilon,x)]\|_\infty \mu(\mathrm{d}x) = 0, \tag{93}$$

and

$$\begin{aligned}\lim_{\varepsilon\to 0} \sup_{t\in(0,T]} \frac{\sqrt{t}}{\alpha(t^{\frac{1}{2}})} \int_{\mathbb{R}^d} \Bigg\| \mathbb{E}\Bigg[& U_t(\cdot,\cdot,X_t^{x,\mu}) \\ & \times \int_0^t \Big\langle \zeta_s(X_s^{x,\mu})\big\{[v_s^{\varepsilon,1} - \hat{v}_s^1](X_s^{x,\mu}, P_s^*\mu)\big\},\ \mathrm{d}W_s \Big\rangle \Bigg] \Bigg\|_\infty \mu(\mathrm{d}x) = 0. \end{aligned} \tag{94}$$

Moreover, combining (83) with $[U_t(y,\nu,\cdot)]_\alpha \le 1$, and (10) for $p = k^*$, we obtain

$$\begin{aligned} & \Big\| v_t^{\varepsilon,2} - \int_{\mathbb{R}^d} \frac{1}{\varepsilon}\mathbb{E}[U_t(\cdot,\cdot,X_t^{x,\mu})(R_t^{\varepsilon,x}-1)]\mu(\mathrm{d}x)\Big\|_\infty \\ & \le \frac{1}{\varepsilon}\int_{\mathbb{R}^d} \mathbb{E}[\alpha(|X_t^{x,\mu} - X_t^{x+\varepsilon\phi(x),\mu}|)|R_t^{\varepsilon,x}-1|]\mu(\mathrm{d}x) \\ & \le \frac{1}{\varepsilon}\int_{\mathbb{R}^d} \Big\{ \Big(\mathbb{E}[|R_t^{\varepsilon,x}-1|^{k^*}]\Big)^{\frac{1}{k^*}} \alpha\Big(\big(\mathbb{E}[|X_t^{x,\mu} - X_t^{x+\varepsilon\phi(x),\mu}|^k]\big)^{\frac{1}{k}}\Big)\Big\}\mu(\mathrm{d}x). \end{aligned}$$

This together with (90) and (84) yields that for some constant $k_1(\mu,\phi) > 0$,

$$\Big\| v_t^{\varepsilon,2} - \int_{\mathbb{R}^d} \frac{1}{\varepsilon}\mathbb{E}[U_t(\cdot,\cdot,X_t^{x,\mu})(R_t^{\varepsilon,x}-1)]\mu(\mathrm{d}x)\Big\|_\infty \le k_1(\mu,\phi)\alpha(\varepsilon), \quad t\in[0,T],\ \varepsilon\in(0,1].$$

Combining this with (75), (83), (88), (94), (87), (93), and the argument leading to (67), we find a constant $k_2(\mu,\phi)$ and a measurable function $\tilde{h} : (0,T]\times(0,1] \to (0,\infty)$ with

$$\sup_{\varepsilon\in(0,1],t\in(0,T]} \frac{\sqrt{t}}{\alpha(t^{\frac{1}{2}})}\tilde{h}_t(\varepsilon) \le k_2(\mu,\phi), \quad \lim_{\varepsilon\to 0}\sup_{t\in(0,T]} \frac{\sqrt{t}}{\alpha(t^{\frac{1}{2}})}\tilde{h}_t(\varepsilon) = 0 \tag{95}$$

such that

$$\begin{aligned}\left\|v_t^{\varepsilon,2}-\hat{v}_t^2\right\|_\infty &\le \tilde{h}_t(\varepsilon)+\int_{\mathbb{R}^d}\left\|\mathbb{E}\Big[U_t(\cdot,\cdot,X_t^{x,\mu})\right.\\ &\qquad\left.\times\int_0^t\Big\langle\{\zeta_s[v_s^{\varepsilon,2}-\hat{v}_s^2]\}(X_s^{x,\mu},P_s^*\mu),\ \mathrm{d}W_s\Big\rangle\Big]\right\|_\infty\mu(\mathrm{d}x) \qquad (96)\\ &\le \tilde{h}_t(\varepsilon)+k_2(\mu,\phi)\bigg(\int_0^t\|v_s^{\varepsilon,2}-\hat{v}_s^2\|_\infty^2\mathrm{d}s\bigg)^{\frac{1}{2}},\quad t\in(0,T].\end{aligned}$$

(d) Proof of (76). Let

$$\beta_t:=\limsup_{\varepsilon\to0}\sup_{s\in(0,t]}\frac{\sqrt{s}}{\alpha(s^{\frac{1}{2}})}\|v_s^{\varepsilon,2}-\hat{v}_s^2\|_\infty.$$

Noting that (63), (71), (72), (66) and (85) imply that β_t satisfies

$$\sup_{t\in(0,T]}\beta_t\le\sup_{\varepsilon\in(0,1]}\sup_{s\in(0,T]}\frac{\sqrt{s}}{\alpha(s^{\frac{1}{2}})}\|v_s^{\varepsilon,2}-\hat{v}_s^2\|_\infty=:\tilde{c}(\mu,\phi)<\infty,$$

so that by Fatou's lemma in (96) we derive from (95) that

$$\beta_t^2\le Ck_2(\mu,\phi)^2\int_0^t\frac{\alpha(s^{\frac{1}{2}})^2}{s}\beta_s^2\mathrm{d}s,\quad t\in(0,T],$$

where by (3),

$$C:=\sup_{t\in(0,T]}\frac{t}{\alpha(t^{\frac{1}{2}})^2}<\infty.$$

Combining this with $\int_0^T\frac{\alpha(t^{\frac{1}{2}})^2}{t}\mathrm{d}t<\infty$, and applying Gronwall's inequality, we prove (76), which together with (79) completes the proof. □

We are now ready to prove Theorem 7.

Proof of Theorem 7. By (57) and (58), it suffices to prove that for any $t\in(0,T]$ and $f\in\mathscr{B}_{k-1,b}(\mathbb{R}^d)$,

$$\begin{aligned}&\lim_{\varepsilon\downarrow0}\frac{P_t^{\mu_\varepsilon}f(\mu_\varepsilon)-P_t^\mu f(\mu_\varepsilon)}{\varepsilon}\\ &=\int_{\mathbb{R}^d}\mathbb{E}\bigg[f(X_t^{x,\mu})\int_0^t\Big\langle\zeta_s(X_s^{x,\mu})N_s(\mu,\phi),\ \mathrm{d}W_s\Big\rangle\bigg]\mu(\mathrm{d}x).\end{aligned}\qquad(97)$$

Let $R_t^{\varepsilon,x}$ be in (80). By (83) for f replacing $U_t(y,\nu,\cdot)$, we obtain

$$\frac{P_t^{\mu_\varepsilon}f(\mu_\varepsilon)-P_t^\mu f(\mu_\varepsilon)}{\varepsilon}=\int_{\mathbb{R}^d}\frac{1}{\varepsilon}\mathbb{E}\big[f(X_t^{x+\varepsilon\phi(x),\mu})(R_t^{\varepsilon,x}-1)\big]\mu(\mathrm{d}x),\quad t\in(0,T].\qquad(98)$$

Noting that (90) implies

$$\lim_{\varepsilon\to 0}\mathbb{E}\Big[\sup_{t\in[0,T]}|X_t^{x+\varepsilon\phi(x),\mu}-X_t^{x,\mu}|^k\Big]=0,$$

while (88), (87), Lemma 9, (74), (71), (62) and (76) lead to

$$\lim_{\varepsilon\to 0}\frac{R_t^{\varepsilon,x}-1}{\varepsilon}=\int_0^t\Big\langle\zeta_s(X_s^{x,\mu})N_s(\mu,\phi),\ \mathrm{d}W_s\Big\rangle$$

in $L^2(\mathbb{P})$, by taking $\varepsilon\to 0$ in (98) and using the dominated convergence theorem, we deduce (97) for $f\in C_b(\mathbb{R}^d)$. By an approximation argument as in [30, Proof of (2.3)] for $f\in\mathscr{B}_b(\mathbb{R}^d)$, this implies (97) for $f\in\mathscr{B}_b(\mathbb{R}^d)$. By the approximation argument used in the proof of (49), we may further extend (97) to $f\in\mathscr{B}_{k-1,b}(\mathbb{R}^d)$. □

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Declarations

- The authors declare that they have no conflict of interest.
- Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

References

[1] Albeverio, S., Kondratiev, Y. G., Röckner, M.: Differential geometry of Poisson spaces. C R Acad Sci Paris Sér I Math. 323, 1129-1134(1996).

[2] Baños, D.: The Bismut-Elworthy-Li formula for mean-field stochastic differential equations. Ann. Inst. Henri Poincaré Probab. Stat. 54, 220-233(2018) .

[3] Bao, J., Ren,P., Wang, F.-Y.: Bismut formula for Lions derivative of distribution-path dependent SDEs. J. Differential Equations 282, 285-329(2021).

[4] Bauer, M., Meyer-Brandis, T.: Existence and regularity of solutions to multi-dimensional mean-field stochastic differential equations with irregular drift. arXiv:1912.05932.

[5] Bauer, M., Meyer-Brandis, T., Proske, F.: Strong solutions of mean-field stochastic differential equations with irregular drift. Electron. J. Probab. 23, 1-35(2018).

[6] Bismut, J. M.: Large Deviations and the Malliavin Calculus. Boston: Birkhäuser, MA(1984).

[7] Chaudru de Raynal, P.-E.: Strong well-posedness of McKean-Vlasov stochastic differential equation with Hölder drift. Stochatic Process. Appl. 130, 79-107(2020).

[8] Crippa,G., De Lellis, C.: Estimates and regularity results for the Di Perna- Lions flow, J. Reine Angew. Math. 616, 15-46(2008).

[9] Csiszár, I., Körne, J.: Information Theory: Coding Theorems for Discrete Memory-less Systems. Academic Press, New York(1981).

[10] Elworthy, K. D., Li, X.-M.: Formulae for the derivatives of heat semigroups. J. Funct. Anal. 125, 252-286(1994).

[11] Huang, X., Ren, P., Wang, F.-Y.: Distribution dependent stochastic differential equations. Front. Math. China 16, 257-301(2021).

[12] Huang, X., Song, Y., Wang,F.-Y.: Bismut formula for intrinsic/Lions derivatives of distribution dependent SDEs with singular coefficients. Discrete Contin. Dyn. Syst. 42, 4597-4614(2022).

[13] Huang, X., Lv, W.: Exponential Ergodicity and propagation of chaos for path-distribution dependent stochastic Hamiltonian system. Electron. J. Probab. 28, Paper No. 134, 20 pp(2023).

[14] Huang, X., Wang, F.-Y.: Distribution dependent SDEs with singular coefficients. Stochatic Process. Appl. 129, 4747-4770(2019).

[15] Huang, X., Wang, F.-Y.: Singular McKean-Vlasov (reflecting) SDEs with distribution dependent noise. J. Math. Anal. Appl. 514, 126301 21pp(2022).

[16] Lacker, D.: On a strong form of propagation of chaos for McKean-Vlasov equations. Electron. Commun. Probab. 23, 1-11(2018).

[17] Mishura, Yu. S., Veretennikov, A. Yu.: Existence and uniqueness theorems for solutions of McKean-Vlasov stochastic equations. Theo. Probab. Math. Statist. 103, 59-101(2020).

[18] Ren, P.: Singular McKean-Vlasov SDEs: well-posedness, regularities and Wang's Harnack inequality. Stochastic Process. Appl. 156, 291-311(2023).

[19] Ren, P., Wang, F.-Y.: Bismut Formula for Lions Derivative of Distribution Dependent SDEs and Applications. J. Differential Equations 267, 4745-4777(2019).

[20] Ren, P., Wang, F.-Y.: Exponential convergence in entropy and Wasserstein for McKean-Vlasov SDEs. Nonlinear Anal. 206, 112259(2021).

[21] Ren, P., Wang, F.-Y.: Derivative formulas in measure on Riemannian manifolds. Bull. Lond. Math. Soc. 53, 1786-1800(2021).

[22] Song, Y.: Gradient estimates and exponential ergodicity for Mean-Field SDEs with jumps. J. Theoret. Probab. 33, 201-238(2020).

[23] Sznitman,A.-S.: Topics in propagation of chaos. In "École d'Été de Probabilités de Sain-Flour XIX-1989", Lecture Notes in Mathematics 1464, p. 165-251, Springer, Berlin(1991).

[24] Tahmasebi, M.: The Bismut-Elworthy-Li formula for semi-linear distribution-dependent SDEs driven by fractional Brownian motion. arXiv:2209.05586.

[25] Wang, F.-Y.: Logarithmic Sobolev inequalities on noncompact Riemannian manifolds. Probab. Theory Relat. Fields 109, 417-424(1997).

[26] Wang, F.-Y.: Harnack inequalities on manifolds with boundary and applications. J. Math. Pures Appl. 94, 304-321(2010).

[27] Wang, F.-Y.: Harnack Inequality and Applications for Stochastic Partial Differential Equations. Springer, New York(2013).

[28] Wang, F.-Y.: Distribution-dependent SDEs for Landau type equations. Stochatic Process. Appl. 128, 595-621(2018).

[29] Wang, F.-Y.: Distribution dependent reflecting stochastic differential equations. Sci. China Math. 66, 2411-2456(2023).

[30] Wang, F.-Y.: Derivative formula for singular McKean-Vlasov SDEs. Commun. Pure Appl. Anal. 22, 1866-1898(2023).

[31] Xia, P., Xie, L., Zhang, X., Zhao, G.: $L^q(L^p)$-theory of stochastic differential equations. Stochatic Process. Appl. 130, 5188-5211(2020).

[32] Xie, L., Zhang, X.: Ergodicity of stochastic differential equations with jumps and singular coefficients. Ann. Inst. Henri Poincaré Probab. Stat. 56, 175-229(2020).

[33] Yuan, C., Zhang, S.-Q.: A Zvonkin's transformation for stochastic differential equations with singular drift and applications. J. Differential Equations 297, 277-319(2021).

[34] Zvonkin, A. K.: A transformation of the phase space of a diffusion process that removes the drift. Math. Sb. 93, 129-149(1974).