Induced Subgraphs with Large Degrees at End-vertices for Hamiltonicity of Claw-free Graphs

Roman ČADA

Department of Mathematics,
European Centre of Excellence – NTIS,
University of West Bohemia,
30614 Pilsen, Czech Republic
E-mail: *cadar@kma.zcu.cz*

Binlong LI

Department of Applied Mathematics,
School of Science,
Northwestern Polytechnical University,
Xi'an, Shaanxi 710072, P. R. China
E-mail: *binlongli@nwpu.edu.cn*

Bo NING

Center for Applied Mathematics,
Tianjin University,
Tianjin 300072, P. R. China
E-mail: *bo.ning@tju.edu.cn*

Shenggui ZHANG

Department of Applied Mathematics,
School of Science,
Northwestern Polytechnical University,
Xi'an, Shaanxi 710072, P. R. China
E-mail: *sgzhang@nwpu.edu.cn*

Abstract   A graph is called *claw-free* if it contains no induced subgraph isomorphic to $K_{1,3}$. Matthews and Sumner proved that a 2-connected claw-free graph G is hamiltonian if every vertex of it has degree at least $(|V(G)| - 2)/3$. At the workshop C&C (Novy Smokovec, 1993), Broersma conjectured the degree condition of this result can be restricted only to end-vertices of induced copies of N (the graph obtained from a triangle by adding three disjoint pendant edges). Fujisawa and Yamashita showed that the degree condition of Matthews and Sumner can be restricted only to end-vertices of induced copies of Z_1 (the graph obtained from a triangle by adding one pendant edge). Our main result in this paper is a characterization of all graphs H such that a 2-connected claw-free graph G is hamiltonian if each

end-vertex of every induced copy of H in G has degree at least $|V(G)|/3 + 1$. This gives an affirmative solution of the conjecture of Broersma up to an additive constant.

Keywords   Induced subgraph; large degree; end-vertex; claw-free graph; hamiltonian graph

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1   Introduction

We use Bondy and Murty [2] for terminology and notation not defined here and consider finite simple graphs only.

Let G be a graph. For a vertex $v \in V(G)$ and a subgraph H of G, we use $N_H(v)$ to denote the set, and $d_H(v)$ the number, of neighbors of v in H, respectively. We call $d_H(v)$ the *degree* of v in H. For $x, y \in V(G)$, an (x, y)-*path* is a path connecting x and y. If $x, y \in V(H)$, the *distance* between x and y in H, denoted $d_H(x, y)$, is the length of a shortest (x, y)-path in H. When no confusion occurs, we will denote $N_G(v)$, $d_G(v)$ and $d_G(x, y)$ by $N(v)$, $d(v)$ and $d(x, y)$, respectively.

Let G be a graph and G' a subgraph of G. If G' contains all edges $xy \in E(G)$ with $x, y \in V(G')$, then G' is called an *induced subgraph* of G (or a subgraph *induced by* $V(G')$). For a given graph H, we say that G is H-*free* if G contains no induced copy of H. If G is H-free, then we call H a *forbidden subgraph* of G. Note that if H_1 is an induced subgraph of a graph H_2, then an H_1-free graph is also H_2-free.

We first give a fundamental sufficient condition for hamiltonicity of graphs.

Theorem 1.1 (Dirac [6])   *Let G be a graph on $n \geq 3$ vertices. If every vertex of G has degree at least $n/2$, then G is hamiltonian.*

The graph $K_{1,3}$ is called the *claw*, and its only vertex of degree 3 is called its *center*. For a given graph H, we call a vertex v of H an *end-vertex* of H if $d_H(v) = 1$. Thus a claw has three end-vertices. In this paper, we use the term claw-free graphs for $K_{1,3}$-free graphs.

Hamiltonian properties of claw-free graphs have been well studied by many graph theorists. The lower bound on the degrees in Dirac's theorem can be lowered to roughly $n/3$ in the case of (2-connected) claw-free graphs.

Theorem 1.2 (Matthews and Sumner [8])   *Let G be a 2-connected claw-free graph on n vertices. If every vertex of G has degree at least $(n-2)/3$, then G is hamiltonian.*

Forbidden subgraph conditions for hamiltonicity of graphs also have received much attention. As K_2-free graphs are precisely the edgeless graphs, it is natural to assume that, throughout this paper, all forbidden subgraphs under consideration will have at least three vertices. We also note that every connected P_3-free graph is a complete graph, and then is trivially hamiltonian if it has at least 3 vertices. It is in fact easy to show that P_3 is the only connected graph R such that every 2-connected R-free graph is hamiltonian.

Bedrossian [1] characterized all the pairs of forbidden subgraphs for hamiltonicity, excluding P_3.

Theorem 1.3 (Bedrossian [1])   *Let R and S be connected graphs with $R, S \neq P_3$ and let G be a 2-connected graph. Then G being R-free and S-free implies G is hamiltonian if and only if (up to symmetry) $R = K_{1,3}$ and $S = P_4, P_5, P_6, C_3, Z_1, Z_2, B, N$ or W (see Fig. 1).*
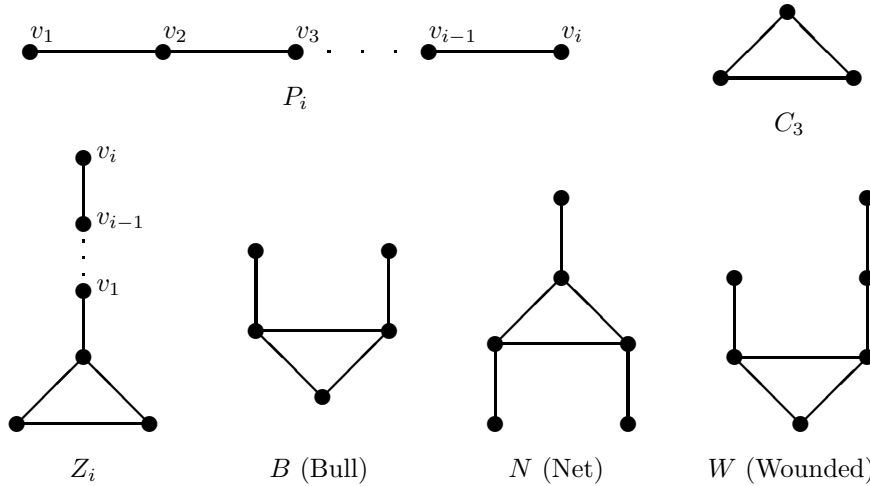
Fig. 1. Graphs P_i, C_3, Z_i, B, N and W.

Note here that the claw is always one subgraph of each forbidden pairs. Also recall that a P_4-free graph is P_5-free, etc., so the relevant graphs for S (in Theorem 1.3) are in fact P_6, N and W. All the other listed graphs are induced subgraphs of P_6, N or W.

At the workshop Cycles and Colourings 93 (Slovakia), Broersma [3] proposed the following conjecture.

Conjecture 1.4 (Broersma [3])   *Let G be a 2-connected claw-free graph on n vertices. If every vertex of G which is an end-vertex of an induced copy of N in G, has degree at least $(n-2)/3$, then G is hamiltonian.*

This conjecture is still open. Whereas, Fujisawa and Yamashita [7] obtained a similar result as follows.

Theorem 1.5 (Fujisawa and Yamashita [7])   *Let G be a 2-connected claw-free graph on n vertices. If every vertex which is an end-vertex of an induced copy of Z_1 in G has degree at least $(n-2)/3$, then G is hamiltonian.*

Let G be a graph on n vertices and H a given graph. We say that G satisfies $\Phi(H, k)$ if for every vertex v which is an end-vertex of an induced copy of H in G, $d(v) \geq (n+k)/3$.

In any connected graph, a vertex which is not an end-vertex of an induced P_3 will be adjacent to all other vertices. Thus a graph satisfying $\Phi(P_3, -2)$ implies that every vertex of it has degree at least $(n-2)/3$. By Theorem 1.2, such a graph is hamiltonian if it is 2-connected and claw-free. Also note that Theorem 1.5 implies that every 2-connected claw-free graph satisfying $\Phi(Z_1, -2)$ is hamiltonian. Motivated by Conjecture 1.4 and Theorem 1.5, in this paper, we consider the following question: For which graphs H, every 2-connected claw-free graph satisfying $\Phi(H, -2)$ is hamiltonian?

First, for a given connected graph H, note that if a graph is H-free, then it naturally satisfies $\Phi(H, -2)$. To guarantee a 2-connected claw-free graph satisfying $\Phi(H, -2)$ is hamiltonian, by Theorem 1.3, we can get that H must be one of the graphs in $\{P_3, P_4, P_5, P_6, C_3, Z_1, Z_2, B, N, W\}$ (to avoid the discussion of trivial cases, we assume that H has at least

three vertices). Note that C_3 has no end-vertex, and every graph satisfies $\Phi(C_3, -2)$ naturally. Since not every 2-connected claw-free graph is hamiltonian, C_3 does not meet our result. Another counterexample is Z_2. The graph in Fig. 2 is 2-connected claw-free and satisfies $\Phi(Z_2, -2)$ but it is not hamiltonian. Thus we have the following result.
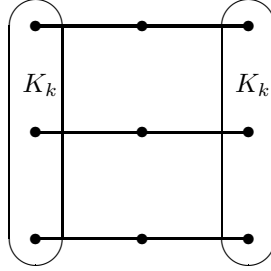

Fig. 2. A graph satisfying $\Phi(Z_2, -2)$.

Proposition 1.6 *Let H be a connected graph on at least 3 vertices and let G be a 2-connected claw-free graph. If G satisfying $\Phi(H, -2)$ implies G is hamiltonian, then $H = P_3, P_4, P_5, P_6, Z_1, B, N$ or W.*

What about the converse? Is every 2-connected claw-free graph satisfying $\Phi(H, -2)$ hamiltonian for all the graphs H listed in Proposition 1.6?

Note that if a graph G satisfies $\Phi(P_i, k)$, then it also satisfies $\Phi(P_j, k)$ for $j \geq i$. Also note that if G satisfies $\Phi(Z_1, k)$, then it also satisfies $\Phi(B, k)$; and if G satisfies $\Phi(B, k)$, then it also satisfies $\Phi(N, k)$. (We remark that a graph satisfying $\Phi(Z_2, k)$ cannot ensure it satisfies $\Phi(W, k)$, although Z_2 is an induced subgraph of W.) So, in the following, we just consider the three graphs P_6, N and W. We propose the following problem:

Problem 1.7 Let $H = P_6$, N or W. Is every 2-connected claw-free graph satisfying $\Phi(H, -2)$ hamiltonian?

We believe that the answer to Problem 1.7 is positive, but the proof may need more technical discussion. However, we can prove a slightly weaker result as follows.

Theorem 1.8 *Let $H = P_6$, N or W, and let G be a 2-connected claw-free graph. If G satisfies $\Phi(H, 3)$, then G is hamiltonian.*

Note that the graph in Fig. 2 satisfies $\Phi(Z_2, 3)$ when $k \geq 6$. Combining with Proposition 1.6 and Theorem 1.8 yields our main theorem.

Theorem 1.9 *Let H be a connected graph on at least 3 vertices and let G be a 2-connected claw-free graph. Then G satisfying $\Phi(H, 3)$ implies G is hamiltonian, if and only if $H = P_3, P_4, P_5, P_6, Z_1, B, N$ or W.*

Note that the case of $H = N$ in Theorem 1.9 shows that every 2-connected claw-free graph G is hamiltonian if every vertex of G which is an end-vertex of an induced copy of N, has degree at least $|V(G)|/3 + 1$. This gives an affirmative solution of the conjecture of Broersma up to an additive constant.

2 Some preliminaries

Two famous conjectures in the field of hamiltonicity of graphs are Thomassen's conjecture [10] that every 4-connected line graph is hamiltonian and Matthews and Sumner's conjecture [8] that every 4-connected claw-free graph is hamiltonian. Ryjáček proved these two conjectures are equivalent. One major tool for the proof is his closure theory [9]. Now we introduce Ryjáček's closure theory, which we will use in our proof.

Let G be a claw-free graph and x a vertex of G. Following the terminology of Ryjáček [9], we call x an *eligible* vertex if $N(x)$ induces a connected graph but is not a clique in G. The *completion* of G at x, denoted by G'_x, is the graph obtained from G by adding all missing edges uv with $u, v \in N(x)$.

Note that if a vertex, say v, has a complete neighborhood in G, i.e., $G[N(v)]$ is complete, then it also has a complete neighborhood in G'_x; also note that if P' is an induced path in G'_x, then there is an induced path P in G with the same end-vertices such that $V(P) \subset V(P') \cup \{x\}$.

Let G be a claw-free graph. The *closure* of G, denoted by cl(G), is the graph defined by a sequence of graphs $G_1, G_2, \ldots, G_t$, and vertices $x_1, x_2, \ldots, x_{t-1}$ such that

(1) $G_1 = G$, $G_t = $ cl(G);

(2) x_i is an eligible vertex of G_i, $G_{i+1} = (G_i)'_{x_i}$, $1 \le i \le t - 1$; and

(3) G_t has no eligible vertices.

By $c(G)$ we denote the length of a longest cycle of G.

Theorem 2.1 (Ryjáček [9]) *Let G be a claw-free graph. Then*

(1) the closure cl(G) *is well-defined;*

(2) there is a triangle-free graph H such that cl(G) *is the line graph of H; and*

(3) $c(G) = c(\text{cl}(G))$.

Clearly every vertex has degree in cl(G) no less than that in G. Ryjáček proved that if G is claw-free, then so is cl(G). A claw-free graph is said to be *closed* if it has no eligible vertices. The following properties of a closed claw-free graph are obvious, and we omit the proofs.

Lemma 2.2 *Let G be a closed claw-free graph. Then*

(1) every vertex is contained in exactly one or two maximal cliques;

(2) two distinct maximal cliques have at most one common vertex;

(3) if two vertices are nonadjacent, then they have at most two common neighbors; and

(4) if a vertex has two neighbors in a maximal clique, then it is contained in the clique.

Now we introduce some new terminology which is useful for our proof. Let G be a claw-free graph and K a maximal clique of cl(G). We call $G[K]$ a *region* of G. For a vertex v of G, we call v an *interior vertex* if it is contained in only one region, and a *frontier vertex* if it is contained in two distinct regions. For two vertices u, v of G, we say that they are *associated* if they are in a common region, and *dissociated* otherwise. We use the notations $u \sim v$ ($u \nsim v$) to express the statement that u and v are associated (dissociated). So two vertices are associated in G if and only if they are adjacent in cl(G). Now we can reformulate Lemma 2.2 as follows.

Lemma 2.3 *Let G be a claw-free graph. Then*

(1) every vertex is either an interior vertex of a region, or a frontier vertex of two regions;

(2) every two regions are either disjoint or have only one common vertex;

(3) every two dissociated vertices have at most two common neighbors; and

(4) if a vertex is associated with two vertices in a common region, then it is also contained in the region.

We can also get the following

Lemma 2.4   *Let G be a claw-free graph. Then*

    (1) if v is a frontier vertex of two regions R, R', then $N_R(v), N_{R'}(v)$ are cliques;

    (2) if R is a region of G, then cl(R) is complete;

    (3) if v is a frontier vertex and R is a region containing v, then v has an interior neighbor in R or R is complete and has no interior vertices; and

    (4) if $u \sim v$, then there is an induced path from u to v such that all internal vertices are interior vertices in the region containing u and v.

Proof   (1) If there are two neighbors x, x' of v in R such that $xx' \notin E(G)$, then let y be a neighbor of v in R'. Note that y is nonadjacent to x, x'; otherwise it will be contained in R. Now the subgraph induced by $\{v, x, x', y\}$ is a claw, a contradiction. Thus $N_R(v)$, and similarly, $N_{R'}(v)$, is a clique.

(2) Let $K = V(R)$. Let $G_1, G_2, \ldots, G_t$ be the sequence of graphs, and $x_1, x_2, \ldots, x_{t-1}$ the sequence of vertices in the definition of cl(G). Note that for every $i \leq t - 1$, x_i has a complete neighborhood in G_{i+1}, and then in cl(G). This implies that x_i is an interior vertex. Thus if $x_i \notin K$, then the completion of G_i at x_i does not change the structure of $G_i[K]$. Let $x_{k_1}, \ldots, x_{k_{t'-1}}$ be the subsequence of $x_1, \ldots, x_{t-1}$ containing all vertices $x_{k_i} \in K$. Note that $N_{G_{k_i}}(x_{k_i}) \subset K$. Thus x_{k_i} is an eligible vertex of $G_{k_i}[K]$ and $(G_{k_i}[K])'_{x_{k_i}} = G_{k_i+1}[K]$. Thus we have that cl(R) = cl(G)[K] is the complete subgraph of cl(G) corresponding to R.

(3) If R is complete in G, then either v has an interior neighbor in R or R has no interior vertices. Now we assume that R is not complete. By (2), cl(R) = cl(G)[V(R)] is complete. This implies that R has at least one eligible vertex, and then, R has at least one interior vertex. If v is nonadjacent to any interior vertex in R, then the completion of an eligible vertex in R does not change the neighborhood of v. Thus v will have no interior neighbors in R in the closure cl(R), a contradiction to that cl(R) is a clique.

(4) Let R be the region of G containing u and v. We use the notation in the proof of (2). Note that for an induced path P' in $G_{k_{i+1}}[V(R)]$ connecting u and v, there is also an induced path P in $G_{k_i}[V(R)]$ connecting u and v such that $V(P) \subset V(P') \cup \{x_{k_i}\}$. This implies that there is an induced path P in R connecting u and v such that $V(P) \subset \{u, v\} \cup \{x_{k_i} : 1 \leq i \leq t' - 1\}$. Note that every x_{k_i} is an interior vertex of R. The proof is complete. $\square$

In the case that $u \sim v$, we use $\Pi[uv]$ to denote an induced path from u to v such that all internal vertices are interior vertices in the region containing u and v. For an induced path $P = v_0 v_1 v_2 \cdots v_k$ in cl(G), we denote $\Pi[P] = \Pi[v_0 v_1] v_1 \Pi[v_1 v_2] v_2 \cdots v_{k-1} \Pi[v_{k-1} v_k]$ (note that $\Pi[P]$ is an induced path of G).

Following [4], we denote by $\mathcal{P}$ the class of all graphs that are obtained by taking two disjoint triangles $a_1 a_2 a_3 a_1$, $b_1 b_2 b_3 b_1$, and by joining every pair of vertices $\{a_i, b_i\}$ by a path $P_{k_i} = a_i c_i^1 c_i^2 \cdots c_i^{k_i-2} b_i$ for $k_i \geq 3$ or by a triangle $a_i b_i c_i a_i$. We denote a graph from $\mathcal{P}$ by

P_{x_1,x_2,x_3}, where $x_i = k_i$ if a_i, b_i are joined by a path P_{k_i}, and $x_i = T$ if a_i, b_i are joined by a triangle.

Theorem 2.5 (Brousek [4]) *Every non-hamiltonian 2-connected claw-free graph contains an induced subgraph in $\mathcal{P}$.*

We list the following result deduced from Brousek et al. [5] to complete this section.

Theorem 2.6 (Brousek et al. [5]) *Let G be a claw-free graph. If G is N-free, then $\mathrm{cl}(G)$ is also N-free.*

3   Proof of Theorem 1.8

Assume that G is not hamiltonian. By Theorems 2.1 and 2.5, $\mathrm{cl}(G)$ contains an induced subgraph $P_{x_1,x_2,x_3} \in \mathcal{P}$. We use the notation a_i, b_i, c_i and c_i^j defined in Section 2. If $x_i = k_i$, then let P^i be the path $a_i c_i^1 c_i^2 \cdots c_i^{k_i-2} b_i$; if $x_i = T$, then let $P^i = a_i b_i$. Let A be the region of G containing the vertices a_1, a_2, a_3, B be the region of G containing the vertices b_1, b_2, b_3. Note that A and B are possibly not disjoint. If they are not disjoint, then let c be the common vertex of A and B. Clearly, a_i, b_i and c (if exists) are all frontier vertices. If $x_i = T$, then let a_i' be the successor of a_i in $\Pi[a_i c_i]$ and b_i' be the successor of b_i in $\Pi[b_i c_i]$; if $x_i = k_i$, then let a_i' be the successor of a_i in $\Pi[a_i c_i^1]$ and b_i' be the successor of b_i in $\Pi[b_i c_i^{k_i-2}]$.

In this section, we say that a vertex is *hefty* if it has degree at least $n/3 + 1$.

Claim 1   Let v_1, v_2, v_3 be three pairwise nonadjacent vertices of G.
   (1) If $v_1 \nsim v_2$, $v_1 \nsim v_3$ and v_2, v_3 have at most one common neighbor, then one of v_1, v_2, v_3 is not hefty.
   (2) If $v_1 \nsim v_2$, $v_1 \nsim v_3$ and $v_2 \nsim v_3$, then one of v_1, v_2, v_3 is not hefty.

Proof   (1) By Lemma 2.3 (3), $|N(v_1) \cap N(v_2)| \leq 2$ and $|N(v_1) \cap N(v_3)| \leq 2$. Note that $|N(v_2) \cap N(v_3)| \leq 1$. If all these three vertices are hefty, i.e., $d(v_i) \geq n/3 + 1$ for $i = 1, 2, 3$, then

$$n \geq 3 + \sum_{1 \leq i \leq 3} d(v_i) - \sum_{1 \leq i < j \leq 3} |N(v_i) \cap N(v_j)| \geq 3 + 3\left(\frac{n}{3} + 1\right) - 5 = n + 1,$$

a contradiction.

(2) By (1) and Lemma 2.3 (3), each of $\{v_1, v_2\}, \{v_1, v_3\}, \{v_2, v_3\}$ has exactly two common neighbors. Let u_{ij} and u_{ij}' be the two common neighbors of v_i and v_j. By Lemma 2.3 (4), $u_{ij} \nsim u_{ij}'$. This implies that all the three vertices v_1, v_2, v_3 are frontier vertices. Moreover, by applying a similar argument as in (1), we have

$$n \geq 3 + d(v_1) + d(v_2) + d(v_3) - 6 \geq 3 \cdot \left(\frac{n}{3} + 1\right) - 3 = n.$$

This implies that every vertex of G is adjacent to at least one vertex in $\{v_1, v_2, v_3\}$. Thus G consists of the six regions containing v_1, v_2 and v_3, and all the six regions are cliques by Lemma 2.4 (1).

Since $u_{12} \nsim u_{12}'$ and $u_{13} \nsim u_{13}'$ and all the four vertices are adjacent to v_1, we have either $u_{12} \sim u_{13}$ or $u_{12} \sim u_{13}'$. We assume without loss of generality that $u_{12} \sim u_{13}$, which implies that $u_{12}u_{13} \in E(G)$. Now we can begin with the cycle $C_0 = v_1 u_{12}' v_2 u_{12} u_{13} v_3 u_{13}' v_1$, and add other vertices, one by one, to the cycle at the place between two associated vertices, and finally obtain a Hamilton cycle of G, a contradiction.   □

The case $H = P_6$

Let $P = a_1' a_1 \Pi[a_1 a_2] a_2 \Pi[P^2] b_2 \Pi[b_2 b_3] b_3 b_3'$. Note that P is an induced copy of P_l with $l \geq 6$. This implies that a_1', and similarly, a_2', a_3', are hefty. Note that a_1', a_2' and a_3' are pairwise dissociated in G, a contradiction to Claim 1.

The case $H = N$

Claim 2   There are at least two hefty vertices in A (and similarly, in B).

Proof   Let $G' = G[V(A) \cup \{a_1', a_2', a_3'\}]$. From Lemma 2.4 (2), we can see that $\text{cl}(\text{G}') = \text{cl}(\text{G})[V(\text{G}')]$. Note that the subgraph of $\text{cl}(\text{G})[V(\text{G}')]$ induced by $\{a_1, a_1', a_2, a_2', a_3, a_3'\}$ is an N. By Theorem 2.6, G' contains an induced N. This implies that $V(G')$ contains at least three pairwise nonadjacent hefty vertices. If two of them are not in A, then we assume without loss of generality that a_1', a_2' are hefty. Note that the third hefty vertex is in $(V(A) \cup \{a_3'\}) \backslash \{a_1, a_2\}$. This implies that the three hefty vertices are pairwise dissociated, a contradiction to Claim 1. $\square$

Let b, b' be two hefty vertices in B. Set

$$N_i = \{v \in V(A) : d_A(a_1, v) = i\} \text{ and } j = \max\{i : N_i \neq \emptyset\}.$$

Note that $N_0 = \{a_1\}$ and $N_1 = N_A(a_1)$. In addition, we define that $N_{-1} = \{a_1'\}$. Note that for any vertex $v \in N_i$, with $1 \leq i \leq j$, v has a neighbor in N_{i-1}. Also note that if v has a neighbor in N_{i+1}, $1 \leq i \leq j - 1$, then by Lemma 2.4 (1), v is an interior vertex, especially, v is not a_2, a_3 and c.

Claim 3 N_i is a clique for all $1 \leq i \leq j$.

Proof   We use induction on i. By Lemma 2.4 (1), N_1 is a clique. Now we assume that $2 \leq i \leq j$. Note that N_{i-1}, N_{i-2} and N_{i-3} are nonempty.

Assume that there are two vertices y, y' in N_i with $yy' \notin E(G)$. If y and y' have a common neighbor in N_{i-1}, then let x be a common neighbor of y and y' in N_{i-1}, and w be a neighbor of x in N_{i-2}. Then the subgraph induced by $\{x, w, y, y'\}$ is a claw, a contradiction. This implies that y and y' have no common neighbors in N_{i-1}. Now let x be a neighbor of y in N_{i-1} and x' be a neighbor of y' in N_{i-1}. Note that $xy', x'y \notin E(G)$. Let w be a neighbor of x in N_{i-2} and let v be a neighbor of w in N_{i-3}. By the induction hypothesis, $xx' \in E(G)$. If $wx' \notin E(G)$, then the subgraph induced by $\{x, w, x', y\}$ is a claw, a contradiction. This implies that $wx' \in E(G)$. Now the subgraph induced by $\{w, v, x, y, x', y'\}$ is an N. Thus the three vertices v, y and y' are all hefty.

By Lemma 2.3 (4), $v \nsim b$ or $v \nsim b'$. We assume without loss of generality that $v \nsim b$. Similarly $b \nsim y$ or $b \nsim y'$, we assume without loss of generality that $b \nsim y$. Note that b, v, y are all hefty, $b \nsim v$, $b \nsim y$ and v, y have no common neighbors. We get a contradiction. $\square$

If both a_2 and a_3 are in N_j, then let w be a neighbor of a_2 in N_{j-1}, v be a neighbor of w in N_{j-2}. By Claim 3 and Lemma 2.4 (1), $a_2 a_3, w a_3 \in E(G)$. Thus the subgraph induced by $\{w, v, a_2, a_2', a_3, a_3'\}$ is an N. Thus v, a_2' and a_3' are three hefty vertices. Note that v, a_2' and a_3' are pairwise dissociated, a contradiction. So we assume without loss of generality that $a_2 \notin N_j$.

Let $a_2 \in N_i$, where $1 \leq i \leq j - 1$. Let y be a vertex in N_{i+1}. Recall that a_2 has no neighbors in N_{i+1}. Let x be a neighbor of y in N_i, w be a neighbor of a_2 in N_{i-1} and v be a neighbor

of w in N_{i-2}. By Claim 3 and Lemma 2.4 (1), $a_2x, wx \in E(G)$, and the subgraph induced by $\{w, v, x, y, a_2, a_2'\}$ is an N. Thus v, y and a_2' are three hefty vertices. Note that $a_2' \nsim v$, $a_2' \nsim y$, and v, y have no common neighbors, a contradiction.

The case $H = W$

Claim 4 For i, j, $1 \le i < j \le 3$, one of the edges in $\{a_ia_j, b_ib_j, a_ib_i, a_jb_j\}$ is not in $E(G)$.

Proof We assume that $a_ia_j, b_ib_j, a_ib_i, a_jb_j \in E(G)$. By Lemma 2.4 (1), $a_i'b_i, a_j'b_j \in E(G)$. Let a be the successor of a_j in the path $\Pi[a_ja_k]$, where $k \ne i, j$. Then the subgraph induced by $\{a_j', a_j, a, b_j, b_i, a_i'\}$ is a W. Thus a, a_i', and similarly a_j', are hefty. Note that a, a_i' and a_j' are pairwise dissociated, a contradiction. $\square$

As in the case of N, we set

$$N_i = \{v \in V(A) : d_A(a_1, v) = i\} \text{ and } j = \max\{i : N_i \ne \emptyset\}.$$

Note that $N_0 = \{a_1\}$, $N_1 = N_A(a_1)$ and we define additionally $N_{-1} = \{a_1'\}$.

Claim 5 There is a hefty vertex in $A\backslash\{a_1, a_2, a_3, c\}$ (and similarly, in $B\backslash\{b_1, b_2, b_3, c\}$).

Proof We assume on the contrary that there are no hefty vertices in $A\backslash\{a_1, a_2, a_3, c\}$.

Claim 5.1 N_i is a clique for all $1 \le i \le j$.

Proof We use induction on i. By Lemma 2.4 (1), N_1 is a clique. Now we assume that $2 \le i \le j$. Note that N_{i-1}, N_{i-2} and N_{i-3} are nonempty.

Assume that there are two vertices y, y' in N_i with $yy' \notin E(G)$. Note that y and y' have no common neighbors in N_{i-1}. Let x be a neighbor of y in N_{i-1}, x' be a neighbor of y' in N_{i-1}, w be a neighbor of x in N_{i-2} and v be a neighbor of w in N_{i-3}. By the induction hypothesis, $xx' \in E(G)$. Note that $wx' \in E(G)$; otherwise the subgraph induced by $\{x, w, x', y\}$ is a claw.

If $y = a_2$, then the subgraph induced by $\{x', w, v, x, a_2, a_2'\}$ and the subgraph induced by $\{w, x', y', x, a_2, a_2'\}$ are W's. Thus v, y' and a_2' are three hefty vertices. Note that $a_2' \nsim v$ $a_2' \nsim y'$, and v, y' have no common neighbors, a contradiction. So we assume that $y \ne a_2$, and similarly, $y \ne a_3$, $y' \ne a_2$, $y' \ne a_3$. This implies that either y or y' is in $A\backslash\{a_1, a_2, a_3, c\}$.

We assume without loss of generality that $y \in A\backslash\{a_1, a_2, a_3, c\}$. Let P' be a shortest path from w to a_1 (note that P' consists of the vertex a_1 if $w = a_1$). Let w, v and u be the first three vertices in the path $P = P'a_1\Pi[P^1]b_1\Pi[b_1b_2]$. Then the subgraph induced by $\{x', x, y, w, v, u\}$ is a W. Thus y is a hefty vertex, a contradiction. $\square$

If both a_2 and a_3 are in N_j, then let w be a neighbor of a_2 in N_{j-1}, v be a neighbor of w in N_{j-2}. By Claim 5.1 and Lemma 2.4 (1), $a_2a_3, wa_3 \in E(G)$. Let a_2, y and z be the first three vertices in the path $P = \Pi[P^2]b_2\Pi[b_2b_3]$. By Claim 4, $a_3z \notin E(G)$. Then the subgraph induced by $\{a_3, w, v, a_2, y, z\}$ is a W. Let a_3, y', z' be the first three vertices in the path $P = \Pi[P^2]b_2\Pi[b_2b_1]$. By Claim 4, $wz' \notin E(G)$. Then the subgraph induced by $\{w, a_2, a_2', a_3, y', z'\}$ is a W. Thus v, a_2', and similarly, a_3', are hefty. Note that v, a_2' and a_3' are pairwise dissociated, a contradiction. So we assume without loss of generality that $a_2 \notin N_j$.

Let $a_2 \in N_i$, where $1 \le i \le j-1$. Let y be a vertex in N_{i+1}. Recall that a_2 has no neighbors in N_{i+1}. Let x be a neighbor of y in N_i, w be a neighbor of a_2 in N_{i-1} and v be a neighbor of w in N_{i-2}. Note that $a_2x, wx \in E(G)$.

If $y = a_3$, then let $z = a_3'$; and if $y = c$, then let z be the successor of c in $\Pi[cb_3]$. Then the subgraph induced by $\{a_2, w, v, x, y, z\}$ and the subgraph induced by $\{w, a_2, a_2', x, y, z\}$ are W's. Thus v, a_2' and z are hefty. Note that v, a_2' and z are pairwise dissociated, a contradiction. Now we assume that $y \neq c, a_3$. Let a_2, y', z' be the first three vertices in the path $P = \Pi[P^2]b_2\Pi[b_2b_3]$. Then the subgraph induced by $\{w, x, y, a_2, y', z'\}$ is a W. This implies that y is hefty, a contradiction. $\square$

Now let a and b be two hefty vertices in $A \backslash \{a_1, a_2, a_3, c\}$ and $B \backslash \{b_1, b_2, b_3, c\}$, respectively. Since a, b and a_i' are pairwise dissociated, a_i' is not hefty.

By Lemma 2.4 (3), a_1 has an interior neighbor in A or $a_1a \in E(G)$. In any case, a_1 has a neighbor in $A \backslash \{a_2, a_3, c\}$. If $a_1a_2 \in E(G)$, then let v be a neighbor of a_1 in $A \backslash \{a_2, a_3, c\}$. By Lemma 2.4, $a_2v \in E(G)$. Let a_2, x and y be the first three vertices in the path $P = \Pi[P^2]b_2\Pi[b_2b_3]$. Then the subgraph induced by $\{v, a_1, a_1', a_2, x, y\}$ is a W. Thus a_1' is hefty, a contradiction. This implies that a_1a_2, and similarly, a_1a_3, a_2a_3, is not in $E(G)$.

Claim 6 N_i is a clique for all $1 \leq i \leq j$.

Proof We use induction on i. By Lemma 2.4 (1), N_1 is a clique.

We first consider the case $i = 2$. Recall that $a_1a_2 \notin E(G)$, which implies that $a_2 \notin N_1$. If $a_2 \in N_2$, then let $z = a_2', y = a_2$; and if $a_2 \notin N_2$, then $(j \geq 3$ and$)$ let z be a vertex in N_3, and y be a neighbor of z in N_2.

We claim that y is adjacent to every vertex in $N_2 \backslash \{y\}$. Assume that $yy' \notin E(G)$ for $y' \in N_2 \backslash \{y\}$. Then y and y' have no common neighbors in N_1. Let x be a neighbor of y in N_1 and x' be a neighbor of y' in N_1. Then $xy', x'y \notin E(G)$. Since $xx' \in E(G)$, the subgraph induced by $\{x', a_1, a_1', x, y, z\}$ is a W, and this implies that a_1' is hefty, a contradiction. Thus, as we claimed, y is adjacent to every vertex in $N_2 \backslash \{y\}$. Now let y', y'' be two vertices in $N_2 \backslash \{y\}$. We claim that $y'y'' \in E(G)$. If $y'z \in E(G)$, then $(z \neq a_2'$ and$)$ similarly as the case of y, we can see that y' is adjacent to every vertex in $N_2 \backslash \{y'\}$, including y''. So we assume that $y'z$, and similarly, $y''z$, is not in $E(G)$. Then the subgraph induced by $\{y, y', y'', z\}$ is a claw, a contradiction. Thus, as we claimed, N_2 is a clique.

Now we assume that $3 \leq i \leq j$. Note that $N_{i-1}, N_{i-2}, N_{i-3}$ and N_{i-4} are nonempty.

Assume that there are two vertices z and z' in N_i with $zz' \notin E(G)$. Note that z and z' have no common neighbors in N_{i-1}. Let y be a neighbor of z in N_{i-1} and y' be a neighbor of z' in N_{i-1}. Then $yz', y'z \notin E(G)$. Let x be a neighbor of y in N_{i-2}, w be a neighbor of x in N_{i-3} and v be a neighbor of w in N_{i-4}. Then $yy', xy' \in E(G)$. Now the subgraph induced by $\{y', y, z, x, w, v\}$ is a W. Thus v and z are hefty. Note that $b \nsim v, b \nsim z$, and v, z have no common neighbors, a contradiction. $\square$

Recall that $a_2a_3 \notin E(G)$, which implies that either a_2 or $a_3 \notin N_j$. Also recall that $a_2, a_3 \notin N_1$. We assume without loss of generality that $a_2 \in N_i$, where $2 \leq i \leq j - 1$. Let z be a vertex in N_{i+1}, y be a neighbor of z in N_i, x be a neighbor of a_2 in N_{i-1}, w be a neighbor of x in N_{i-2} and v be a neighbor of w in N_{i-3}. By Claim 6 and Lemma 2.4 (1), $a_2y, xy \in E(G)$. Then the subgraph induced by $\{y, a_2, a_2', x, w, v\}$ is a W. This implies that a_2' is hefty, a contradiction.

The proof is complete. $\square$

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