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ON GENERALIZED KISSING NUMBERS OF CONVEX BODIES

Dedicated to the memory of Professor Delin REN

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Abstract In 1694, Gregory and Newton proposed the problem to determine the kissing number of a rigid material ball. This problem and its higher dimensional generalization have been studied by many mathematicians, including Minkowski, van der Waerden, Hadwiger, Swinnerton-Dyer, Watson, Levenshtein, Odlyzko, Sloane and Musin. In this paper, we introduce and study a further generalization of the kissing numbers for convex bodies and obtain some exact results, in particular for balls in dimensions three, four and eight.

Keywords convex body; generalized kissing number; E_8 lattice

MSC2020 52C17; 11H31

1 Introduction

In 1694, Gregory and Newton discussed the following problem: Can a rigid material ball be brought into contact with thirteen other such balls of the same size? Gregory believed "yes", while Newton thought "no". The solution of this problem has a complicated history! Several authors claimed proofs that the largest number of nonoverlapping unit balls which can be brought into contact with a fixed one is twelve (see Hoppe [7], Günter [4], Schütte and van der Waerden [16] and Leech [8]). However, only Schütte and van der Waerden's proof is complete!

Let K be an n-dimensional convex body and let C be an n-dimensional centrally symmetric convex body centered at the origin of $\mathbb{E}^n$. Let $\kappa(K)$ and $\kappa^*(K)$ denote the translative kissing number and the lattice kissing number of K, respectively. In other words, $\kappa(K)$ is the maximum number of nonoverlapping translates $K+\mathbf{x}$ that can touch K at its boundary, and $\kappa^*(K)$ is defined similarly, with the restriction that the translates are members of a lattice packing of K. It is easy to see that $\kappa^*(K)\le\kappa(K)$ holds for every convex body K.

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In 1904, Minkowski [10] defined

$$D(K) = \{\mathbf{x} - \mathbf{y} : \ \mathbf{x} \in K, \ \mathbf{y} \in K\}$$

and proved that

$$\kappa^*(K) = \kappa^*(D(K))$$

and

$$\kappa(K) = \kappa(D(K)).$$

Clearly, $D(K)$ is always centrally symmetric and centered at the origin of $\mathbb{E}^n$. Three years later, Minkowski [11] proved that

$$\kappa^*(K) \leq 3^n - 1$$

holds for every n-dimensional convex body K. In 1957, Hadwiger [5] improved Minkowski's upper bound to

$$\kappa(K) \leq 3^n - 1.$$

Let B^n denote the n-dimensional unit ball centered at the origin of $\mathbb{E}^n$. The kissing numbers $\kappa^*(B^n)$ and $\kappa(B^n)$ have been studied by many authors (see [1, 9, 12–14, 18]). The known exact results are summarized in the following Table 1.

Table 1

n	2	3	4	5	6	7	8	9	24
$\kappa^*(B^n)$	6	12	24	40	72	126	240	272	196560
$\kappa(B^n)$	6	12	24	??	??	??	240	??	196560

It is well-known that each centrally symmetric convex body C centered at the origin defines a metric $\|\cdot\|_C$ on $\mathbb{R}^n$ by

$$\|\mathbf{x}, \mathbf{y}\|_C = \|\mathbf{x} - \mathbf{y}\|_C = \min\{r : \ r > 0, \ \mathbf{x} - \mathbf{y} \in rC\}.$$

Especially, we use $||\cdot||$ to denote the metric defined by B^n.

Clearly, the kissing numbers $\kappa(C)$ and $\kappa^*(C)$ only consider the closest neighbours of C in translative packings and lattice packings, respectively. In fact, in many physic situations the neighbours in a larger region also have effect on C. For example, in some potential energy models. Therefore, it is reasonable to make following generalizations: For $\alpha \geq 0$, we define $\kappa_\alpha(C)$ to be the maximum number of translates $C + \mathbf{x}$ which can be packed into the region $(3+\alpha)C \setminus \mathrm{int}(C)$ and define $\kappa^*_\alpha(C)$ to be the maximum number of translates $C + \mathbf{x}$ which can be packed into the region $(3+\alpha)C \setminus \mathrm{int}(C)$ where all the translative vectors simultaneously belong to a lattice, where $\mathrm{int}(C)$ denotes the interior of C.

In this paper, among other things, we will prove the following results:

Theorem 1 In $\mathbb{E}^2$, we have

$$\kappa^*_\alpha(B^2) = \begin{cases} 6, & 0 \leq \alpha < 2\sqrt{2} - 2, \\ 8, & 2\sqrt{2} - 2 \leq \alpha < 2\sqrt{3} - 2, \\ 12, & \alpha = 2\sqrt{3} - 2. \end{cases}$$

Theorem 2 In $\mathbb{E}^3$, we have

$$\kappa_\alpha^*(B^3) = \begin{cases} 12, & 0 \le \alpha < \frac{4}{3}\sqrt{3} - 2, \\ 14, & \frac{4}{3}\sqrt{3} - 2 \le \alpha < 2\sqrt{2} - 2, \\ 20, & \alpha = 2\sqrt{2} - 2. \end{cases}$$

Theorem 3 In $\mathbb{E}^4$, we have

$$\kappa_\alpha^*(B^4) = \begin{cases} 30, & \sqrt{6} - 2 \le \alpha < 2\sqrt{2} - 2, \\ 50, & \alpha = 2\sqrt{2} - 2. \end{cases}$$

Theorem 4 In $\mathbb{E}^8$, when $\alpha = 2\sqrt{2} - 2$ we have $\kappa_\alpha^*(B^8) = 2400$.

2 Some Basic Lemmas

In 1907, Minkowski [11] studied the lattice kissing number of an n-dimensional convex body and proved the following result.

Lemma 2.1 If K is an n-dimensional convex body, then $\kappa^*(K) \le 3^n - 1$, where the equality holds if and only if K is a parallelepiped. If C is an n-dimensional centrally symmetric strictly convex body centered at $\mathbf{o}$, then $\kappa^*(C) \le 2(2^n - 1)$.

For a non-negative number α and a packing lattice Λ of B^n, we define

$$X(\alpha, \Lambda) = \{\mathbf{v} :\ 2 \le ||\mathbf{v}|| \le 2 + \alpha,\ \mathbf{v} \in \Lambda\}.$$

Next we introduce two technical results which will be frequently used in this paper.

Lemma 2.2 When $0 \le \alpha < 2\sqrt{2} - 2$, we have

$$\kappa_\alpha^*(B^n) \le 2(2^n - 1).$$

Proof On the contrary, suppose that there is such a positive α which is less than $2\sqrt{2}-2$ and a suitable lattice Λ satisfying

$$\mathrm{card}\{X(\alpha, \Lambda)\} \ge 2^{n+1}.$$

For convenience, we assume that $\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_n$ is a basis of Λ and say two lattice vectors

$$\mathbf{v} = z_1\mathbf{a}_1 + z_2\mathbf{a}_2 + \cdots + z_n\mathbf{a}_n$$

and

$$\mathbf{v}' = z_1'\mathbf{a}_1 + z_2'\mathbf{a}_2 + \cdots + z_n'\mathbf{a}_n$$

are equivalent if $z_i - z_i' \equiv 0 \pmod 2$ for all $i = 1, 2, \cdots, n$. In other words, $\mathbf{v}$ and $\mathbf{v}'$ are equivalent if and only if $\frac{1}{2}(\mathbf{v} - \mathbf{v}') \in \Lambda$. Clearly, this relation divides the points of Λ into 2^n classes.

Since $X(\alpha, \Lambda)$ is centrally symmetric and $\mathrm{card}\{X(\alpha, \Lambda)\} \ge 2^{n+1}$, it contains 2^n lattice points $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_{2^n}$ satisfying

$$\mathbf{v}_i \ne \pm\mathbf{v}_j, \quad i \ne j.$$

If one of the 2^n points, say $\mathbf{v}_1$, is equivalent to $\mathbf{o}$, then we get $\frac{1}{2}\mathbf{v}_1 \in \Lambda$ and $1 \le ||\frac{1}{2}\mathbf{v}_1|| < \sqrt{2} < 2$, which contradicts the assumption that $B^n + \Lambda$ is a packing.

Since all $\mathbf{o}$, $\mathbf{v}_1$, $\mathbf{v}_2$, $\cdots$, $\mathbf{v}_{2^n}$ belong to at most 2^n classes, two of them must belong to the same class. Without loss of generality, we may assume that $\mathbf{v}_1$ and $\mathbf{v}_2$ are equivalent. Then, we have

$$2 \le ||\mathbf{v}_i|| < 2\sqrt{2}, \quad i = 1, 2 \tag{2.1}$$

and

$$||\tfrac{1}{2}(\mathbf{v}_1 \pm \mathbf{v}_2)|| \ge 2. \tag{2.2}$$

By (2.1), we get

$$||\mathbf{v}_1||^2 + ||\mathbf{v}_2||^2 < 16. \tag{2.3}$$

By (2.2), we get

$$||\mathbf{v}_1||^2 + ||\mathbf{v}_2||^2 - 2\langle \mathbf{v}_1, \mathbf{v}_2 \rangle \ge 16 \tag{2.4}$$

and

$$||\mathbf{v}_1||^2 + ||\mathbf{v}_2||^2 + 2\langle \mathbf{v}_1, \mathbf{v}_2 \rangle \ge 16, \tag{2.5}$$

where $\langle \mathbf{v}_1, \mathbf{v}_2 \rangle$ denotes the inner product of $\mathbf{v}_1$ and $\mathbf{v}_2$. Then, by (2.4) and (2.5) we obtain

$$||\mathbf{v}_1||^2 + ||\mathbf{v}_2||^2 \ge 16,$$

which contradicts (2.3). Therefore, for $0 \le \alpha < 2\sqrt{2} - 2$, we have

$$\kappa^*_\alpha(B^n) \le 2(2^n - 1).$$

Lemma 2.2 is proved. □

Remark 2.1 Writing

$$X = \{\mathbf{v} : \ 2 \le ||\mathbf{v}|| \le 2\sqrt{2}, \ \mathbf{v} \in \Lambda\}$$

and repeating the calculations (2.1)–(2.5) one can deduce that two lattice points $\mathbf{v}_1, \mathbf{v}_2 \in X$ satisfying $\mathbf{v}_1 \ne \pm\mathbf{v}_2$ belong to the same equivalent class if and only if

$$||\mathbf{v}_1|| = 2\sqrt{2}, \quad ||\mathbf{v}_2|| = 2\sqrt{2} \quad \text{and} \quad \langle \mathbf{v}_1, \mathbf{v}_2 \rangle = 0.$$

Lemma 2.3 When $\alpha < 2\sqrt{3} - 2$, the set $X(\alpha, \Lambda)$ contains no four collinear points.

Proof On the contrary, suppose $X(\alpha, \Lambda)$ has four collinear points $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ and $\mathbf{v}_4$. Without loss of generality, we may assume that $n = 2$ and all $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ and $\mathbf{v}_4$ have the same x-coordinates, namely $\mathbf{v}_1 = (x_0, y_1)$, $\mathbf{v}_2 = (x_0, y_2)$, $\mathbf{v}_3 = (x_0, y_3)$ and $\mathbf{v}_4 = (x_0, y_4)$. Furthermore, we may also assume that $x_0 \ge 0$ and

$$y_1 - y_2 = y_2 - y_3 = y_3 - y_4 \ge 2, \tag{2.6}$$

since $B^2 + \Lambda$ is a packing.

If $x_0 \ge \sqrt{3}$, we get

$$y_1 - y_4 \le 2\sqrt{(\alpha+2)^2 - x_0^2} < 2\sqrt{(2\sqrt{3})^2 - 3} = 6 \tag{2.7}$$

and therefore

$$y_1 - y_2 = y_2 - y_3 = y_3 - y_4 < 2,$$

which contradicts (2.6).

If $x_0 < \sqrt{3}$, we get

$$y_2 - y_3 \geq 2\sqrt{4 - x_0^2} \tag{2.8}$$

since both $\mathbf{v}_2$ and $\mathbf{v}_3$ belong to Λ. On the other hand, since both $\mathbf{v}_1$ and $\mathbf{v}_4$ belong to $\mathrm{int}(2\sqrt{3}B^2)$, we get

$$y_1 - y_4 < 2\sqrt{12 - x_0^2}. \tag{2.9}$$

By (2.8) and (2.9) one can easily deduce

$$y_1 - y_4 < 2\sqrt{12 - x_0^2} < 6\sqrt{4 - x_0^2} \leq 3(y_2 - y_3), \tag{2.10}$$

which contradicts (2.6).

As a conclusion of the two cases, Lemma 2.3 is proved. □

Remark 2.2 Writing

$$X = \left\{\mathbf{v} :\ 2 \leq ||\mathbf{v}|| \leq 2\sqrt{3},\ \mathbf{v} \in \Lambda\right\}$$

and repeating the calculations (2.6)-(2.10) one can deduce that X has four collinear points $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ and $\mathbf{v}_4$ if and only if, up to some rotation,

$$\mathbf{v}_1 = (\sqrt{3}, 3), \quad \mathbf{v}_2 = (\sqrt{3}, 1), \quad \mathbf{v}_3 = (\sqrt{3}, -1) \quad \text{and} \quad \mathbf{v}_4 = (\sqrt{3}, -3).$$

3 Proof of Theorem 1

Theorem 1 In $\mathbb{E}^2$, we have

$$\kappa_\alpha^*(B^2) = \begin{cases} 6, & 0 \leq \alpha < 2\sqrt{2} - 2, \\ 8, & 2\sqrt{2} - 2 \leq \alpha < 2\sqrt{3} - 2, \\ 12, & \alpha = 2\sqrt{3} - 2. \end{cases}$$

Proof When $0 \leq \alpha < 2\sqrt{2} - 2$, by Lemma 2.2 we have $\kappa_\alpha^*(B^2) \leq 6$. Combining with $\kappa_\alpha^*(B^2) \geq \kappa^*(B^2) = 6$, for $0 \leq \alpha < 2\sqrt{2} - 2$ we get

$$\kappa_\alpha^*(B^2) = 6. \tag{3.1}$$

By Remark 2.1, when $\alpha = 2\sqrt{2} - 2$ one can deduce that

$$\kappa_\alpha^*(B^2) = 8, \tag{3.2}$$

where the equality holds when the corresponding lattice Λ is generated by $\mathbf{a}_1 = (2, 0)$, $\mathbf{a}_2 = (0, 2)$. In fact, the optimal lattice is unique up to some rotation.

When $2\sqrt{2} - 2 \leq \alpha < 2\sqrt{3} - 2$, we assume that $B^2 + \Lambda$ is a lattice packing attaining $\kappa_\alpha^*(B^2)$. Then we have

$$\mathrm{card}\{X(\alpha, \Lambda)\} \geq 8.$$

Without loss of generality, by a routine argument we may assume that $\{\mathbf{a}_1, \mathbf{a}_2\}$ is a basis of Λ, $\|\mathbf{a}_1\| = 2$ and $\{\mathbf{a}_1, \mathbf{a}_2\} \subset X(\alpha, \Lambda)$.

For an arbitrary vector $\mathbf{v} \in X(\alpha, \Lambda)$ which is not $\pm\mathbf{a}_1$, the lattice Λ' generated by $\{\mathbf{a}_1, \mathbf{v}\}$ is a sublattice of Λ. Therefore, let $\det(\Lambda)$ denote the determinant of the lattice Λ, we have

$$\det(\Lambda') = g \det(\Lambda),$$

where g is a positive integer. It is easy to see that

$$\det(\Lambda') \leq ||\mathbf{a}_1|| \cdot ||\mathbf{v}|| < 2 \cdot 2\sqrt{3} = 4\sqrt{3} \tag{3.3}$$

and

$$\det(\Lambda) \geq \frac{\omega(B^2)}{\delta^*(B^2)} = 2\sqrt{3}, \tag{3.4}$$

where $\omega(B^2)$ denotes the area of B^2 and $\delta^*(B^2) = \pi/\sqrt{12}$ is the density of the densest lattice packing of B^2. By (3.3) and (3.4) one can easily deduce that $g = 1$. Consequently, if

$$\mathbf{v} = z_1\mathbf{a}_1 + z_2\mathbf{a}_2,$$

then we must have

$$z_2 = \pm g = \pm 1. \tag{3.5}$$

Since $\pm\mathbf{a}_2 \in X(\alpha, \Lambda)$, by Lemma 2.3 one can deduce that $|z_1| \leq 2$, which means that

$$\mathbf{v} = \pm\mathbf{a}_2,\ \pm(\mathbf{a}_1 + \mathbf{a}_2),\ \pm(\mathbf{a}_1 - \mathbf{a}_2),\ \pm(2\mathbf{a}_1 + \mathbf{a}_2) \text{ or } \pm(2\mathbf{a}_1 - \mathbf{a}_2).$$

If both $(\mathbf{a}_1 + \mathbf{a}_2)$ and $(\mathbf{a}_1 - \mathbf{a}_2)$ belong to $X(\alpha, \Lambda)$, since $2\mathbf{a}_1 + \mathbf{a}_2,\ \mathbf{a}_1 + \mathbf{a}_2,\ \mathbf{a}_2,\ -\mathbf{a}_1 + \mathbf{a}_2$ are collinear, $(2\mathbf{a}_1 + \mathbf{a}_2)$ can not belong to $X(\alpha, \Lambda)$. Similarly, since $2\mathbf{a}_1 - \mathbf{a}_2,\ \mathbf{a}_1 - \mathbf{a}_2,\ -\mathbf{a}_2,\ -\mathbf{a}_1 - \mathbf{a}_2$ are collinear, $(2\mathbf{a}_1 - \mathbf{a}_2) \notin X(\alpha, \Lambda)$. On the other hand, if both $(2\mathbf{a}_1 + \mathbf{a}_2)$ and $(2\mathbf{a}_1 - \mathbf{a}_2)$ belong to $X(\alpha, \Lambda)$, by convexity one can deduce that $2\mathbf{a}_1 \in X(\alpha, \Lambda)$, which contradicts to $X(\alpha, \Lambda) \subset \mathrm{int}(2\sqrt{3}B^2)$. As a conclusion of these two cases, we get

$$\mathrm{card}\{X(\alpha, \Lambda)\} \leq 8.$$

Combining with (3.2), for $2\sqrt{2} - 2 \leq \alpha < 2\sqrt{3} - 2$ we get

$$\kappa^*_\alpha(B^2) = 8. \tag{3.6}$$

Finally, we deal with the case $\alpha = 2\sqrt{3} - 2$. If $X(\alpha, \Lambda)$ contains no four collinear points and (3.5) holds, by previous arguments we still obtain

$$\mathrm{card}\{X(\alpha, \Lambda)\} \leq 8.$$

Therefore, the necessary condition for $\mathrm{card}\{X(\alpha, \Lambda)\} \geq 10$ is either $X(\alpha, \Lambda)$ contains four collinear points or (3.5) does not hold.

If $X(\alpha, \Lambda)$ has four collinear vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ and $\mathbf{v}_4$, by Remark 2.2 we may assume that

$$\mathbf{v}_1 = (\sqrt{3}, 3), \quad \mathbf{v}_2 = (\sqrt{3}, 1), \quad \mathbf{v}_3 = (\sqrt{3}, -1), \quad \mathbf{v}_4 = (\sqrt{3}, -3).$$

In this case, it is easy to verify that Λ is generated by $\mathbf{a}_1 = (\sqrt{3}, 1),\ \mathbf{a}_2 = (\sqrt{3}, -1)$ and

$$\mathrm{card}\{X(\alpha, \Lambda)\} = 12.$$

If there is a point $\mathbf{v} = z_1\mathbf{a}_1 + z_2\mathbf{a}_2 \in X(\alpha, \Lambda)$ with $z_2 = \pm 2$, by repeating (3.3) and (3.4) we get

$$\det(\Lambda) = \frac{\omega(B^2)}{\delta^*(B^2)} = 2\sqrt{3}.$$

In this case, Λ is the densest packing lattice of B^2 and therefore

$$\mathrm{card}\{X(\alpha, \Lambda)\} = 12.$$

As a conclusion of the two cases, for $\alpha = 2\sqrt{3} - 2$ we have

$$\kappa_\alpha^*(B^2) = 12, \tag{3.7}$$

and the corresponding lattice Λ is generated by $\mathbf{a}_1 = (\sqrt{3}, 1)$ and $\mathbf{a}_2 = (\sqrt{3}, -1)$, up to some rotation.

The theorem follows from (3.1), (3.6) and (3.7). □

In 2003, Zong [23] proved the following result.

Lemma 3.1 For every two-dimensional centrally symmetric convex domain C there is a parallelogram with vertices $\mathbf{o}$, $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_1 + \mathbf{v}_2$ such that

$$||\mathbf{o}, \mathbf{v}_1||_C = ||\mathbf{o}, \mathbf{v}_2||_C = 2$$

and

$$2 \le ||\mathbf{v}_1, \mathbf{v}_2||_C = ||\mathbf{o}, \mathbf{v}_1 + \mathbf{v}_2||_C \le 2\sqrt{2}.$$

This result has the following corollary.

Corollary 3.1 When $\alpha = 2\sqrt{2} - 2$, $\kappa_\alpha^*(C) \ge 8$ holds for every two-dimensional centrally symmetric convex domain C.

4 Proof of Theorem 2

Lemma 4.1 When $\frac{4}{3}\sqrt{3} - 2 \le \alpha < 2\sqrt{2} - 2$, we have $\kappa_\alpha^*(B^3) = 14$.

Proof Let Λ be the lattice generated by $\mathbf{a}_1 = (-\frac{2}{3}\sqrt{6}, \frac{2}{3}\sqrt{3}, 0)$, $\mathbf{a}_2 = (\frac{2}{3}\sqrt{6}, \frac{2}{3}\sqrt{3}, 0)$ and $\mathbf{a}_3 = (0, \frac{2}{3}\sqrt{3}, \frac{2}{3}\sqrt{6})$. When $\alpha = \frac{4}{3}\sqrt{3} - 2$, one can verify that $\mathrm{card}\{X(\alpha, \Lambda)\} = 14$. Combining with Lemma 2.2, for $\frac{4}{3}\sqrt{3} - 2 \le \alpha < 2\sqrt{2} - 2$ we have $\kappa_\alpha^*(B^3) = 14$. Lemma 4.1 is proved. □

Lemma 4.2 Let $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ be three linearly independent vectors of $X(\alpha, \Lambda)$ and let Λ' denote the lattice generated by them. If $0 \le \alpha < \frac{4}{3}\sqrt{3} - 2$, then $\det(\Lambda')/\det(\Lambda) \le 2$.

Proof Since $\alpha < \frac{4}{3}\sqrt{3} - 2$, we have

$$\det(\Lambda') \le ||\mathbf{v}_1|| \cdot ||\mathbf{v}_2|| \cdot ||\mathbf{v}_3|| < (\frac{4}{3}\sqrt{3})^3.$$

On the other hand, let $\delta^*(B^3)$ denote the density of the densest three-dimensional lattice sphere packing which is $\pi/\sqrt{18}$, we have

$$\det(\Lambda) \ge \frac{\mathrm{vol}(B^3)}{\delta^*(B^3)} = \frac{\frac{4}{3}\pi}{\frac{\pi}{\sqrt{18}}} = 4\sqrt{2}.$$

Therefore, we get

$$\det(\Lambda')/\det(\Lambda) < \left(\frac{4}{3}\sqrt{3}\right)^3/4\sqrt{2} < 3.$$

In other words, $\det(\Lambda')/\det(\Lambda)$ only can take two values, one or two. Lemma 4.2 is proved. □

Lemma 4.3 If $\alpha < \frac{4}{3}\sqrt{3} - 2$ and $\mathrm{card}\{X(\alpha, \Lambda)\} = 14$, then Λ has a basis $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ such that

$$X(\alpha, \Lambda) = \{\pm\mathbf{a}_1,\ \pm\mathbf{a}_2,\ \pm\mathbf{a}_3,\ \pm(\mathbf{a}_1 + \mathbf{a}_2),\ \pm(\mathbf{a}_1 + \mathbf{a}_3),\ \pm(\mathbf{a}_2 + \mathbf{a}_3),\ \pm(\mathbf{a}_1 + \mathbf{a}_2 + \mathbf{a}_3)\}.$$

Proof Suppose

$$X(\alpha, \Lambda) = \{\pm\mathbf{v}_1,\ \pm\mathbf{v}_2,\ \pm\mathbf{v}_3,\ \pm\mathbf{v}_4,\ \pm\mathbf{v}_5,\ \pm\mathbf{v}_6,\ \pm\mathbf{v}_7\}.$$

By Remark 2.1, $\mathbf{v}_i$ cannot be equivalent to $\pm\mathbf{v}_j$ for $i \neq j$, since $\alpha < \frac{4}{3}\sqrt{3} - 2$. Therefore, for a basis $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ of Λ, we may assume $\mathbf{v}_1,\ \mathbf{v}_2,\ \mathbf{v}_3,\ \mathbf{v}_4,\ \mathbf{v}_5,\ \mathbf{v}_6,\ \mathbf{v}_7$ are equivalent to $\mathbf{a}_1,\ \mathbf{a}_2,\ \mathbf{a}_3,\ \mathbf{a}_1 + \mathbf{a}_2,\ \mathbf{a}_1 + \mathbf{a}_3,\ \mathbf{a}_2 + \mathbf{a}_3,\ \mathbf{a}_1 + \mathbf{a}_2 + \mathbf{a}_3$, respectively.

It is easy to verify that, we can expand to a basis of Λ based on $\{\mathbf{v}_1, \mathbf{v}_2\}$. Without loss of generality, we assume $\mathbf{v}_1 = \pm\mathbf{a}_1,\ \mathbf{v}_2 = \pm\mathbf{a}_2$. By Lemma 4.2, we have $\mathbf{v}_3 = z_1\mathbf{a}_1 + z_2\mathbf{a}_2 \pm \mathbf{a}_3$, where z_1, z_2 are even. Since $\{\mathbf{a}_1,\ \mathbf{a}_2,\ z_1\mathbf{a}_1 + z_2\mathbf{a}_2 \pm \mathbf{a}_3\}$ is also a basis of Λ, we assume $\mathbf{v}_3 = \pm\mathbf{a}_3$ without loss of generality.

Therefore, for $\mathbf{v} = z_1\mathbf{a}_1 + z_2\mathbf{a}_2 + z_3\mathbf{a}_3$ belonging to $X(\alpha, \Lambda)$, we have $||z_i|| \leq 2$ for $i = 1, 2, 3$, by Lemma 4.2. Thus, we have

$$\begin{aligned}
\mathbf{v}_4 &= \pm\mathbf{a}_1 \pm \mathbf{a}_2 + z\mathbf{a}_3, && z = 0 \text{ or } \pm 2,\\
\mathbf{v}_5 &= \pm\mathbf{a}_1 + z'\mathbf{a}_2 \pm \mathbf{a}_3, && z' = 0 \text{ or } \pm 2,\\
\mathbf{v}_6 &= z''\mathbf{a}_1 \pm \mathbf{a}_2 \pm \mathbf{a}_3, && z'' = 0 \text{ or } \pm 2,
\end{aligned}$$

and $\mathbf{v}_7 = \pm\mathbf{a}_1 \pm \mathbf{a}_2 \pm \mathbf{a}_3$.

Suppose one of $z,\ z',\ z''$ is ± 2, without loss of generality, say $z = \pm 2$. Furthermore, we may assume $\mathbf{v}_4 = \pm(\mathbf{a}_1 + \mathbf{a}_2 + 2\mathbf{a}_3)$, since the sign of $\mathbf{a}_1,\ \mathbf{a}_2,\ \mathbf{a}_3$ does not change $\pm\mathbf{v}_1,\ \pm\mathbf{v}_2,\ \pm\mathbf{v}_3$.

It is easy to verify that

$$\det(\mathbf{a}_1 - \mathbf{a}_3,\ \mathbf{a}_2,\ \mathbf{a}_1 + \mathbf{a}_2 + 2\mathbf{a}_3) = \begin{vmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{vmatrix} \det(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3) = 3\det(\Lambda),$$

$$\det(\mathbf{a}_1,\ \mathbf{a}_1 + 2\mathbf{a}_2 + \mathbf{a}_3,\ \mathbf{a}_1 + \mathbf{a}_2 + 2\mathbf{a}_3) = \begin{vmatrix} 1 & 0 & 0 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{vmatrix} \det(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3) = 3\det(\Lambda),$$

$$\det(\mathbf{a}_1,\ \mathbf{a}_1 + 2\mathbf{a}_2 - \mathbf{a}_3,\ \mathbf{a}_1 + \mathbf{a}_2 + 2\mathbf{a}_3) = \begin{vmatrix} 1 & 0 & 0 \\ 1 & 2 & -1 \\ 1 & 1 & 2 \end{vmatrix} \det(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3) = 5\det(\Lambda),$$

$$\det(\mathbf{a}_1,\ -\mathbf{a}_1 + 2\mathbf{a}_2 + \mathbf{a}_3,\ \mathbf{a}_1 + \mathbf{a}_2 + 2\mathbf{a}_3) = \begin{vmatrix} 1 & 0 & 0 \\ -1 & 2 & 1 \\ 1 & 1 & 2 \end{vmatrix} \det(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3) = 3\det(\Lambda),$$

$$\det(\mathbf{a}_1,\ -\mathbf{a}_1 + 2\mathbf{a}_2 - \mathbf{a}_3,\ \mathbf{a}_1 + \mathbf{a}_2 + 2\mathbf{a}_3) = \begin{vmatrix} 1 & 0 & 0 \\ -1 & 2 & -1 \\ 1 & 1 & 2 \end{vmatrix} \det(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3) = 5\det(\Lambda),$$

where $\det(\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3)$ denotes the determinant of the lattice which is generated by $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$. By Lemma 4.2, $\mathbf{v}_5$ cannot be $\pm(\mathbf{a}_1 - \mathbf{a}_3),\ \pm(\mathbf{a}_1 + 2\mathbf{a}_2 + \mathbf{a}_3),\ \pm(\mathbf{a}_1 + 2\mathbf{a}_2 - \mathbf{a}_3),\ \pm(-\mathbf{a}_1 + 2\mathbf{a}_2 + \mathbf{a}_3),\ \pm(-\mathbf{a}_1 + 2\mathbf{a}_2 - \mathbf{a}_3)$. Therefore, $\mathbf{v}_5 = \pm(\mathbf{a}_1 + \mathbf{a}_3)$. By the same deduction, we have $\mathbf{v}_6 = \pm(\mathbf{a}_2 + \mathbf{a}_3)$.

For $\mathbf{v}_7$, since

$$\det(\mathbf{a}_1,\ \mathbf{a}_1+\mathbf{a}_2-\mathbf{a}_3,\ \mathbf{a}_1+\mathbf{a}_2+2\mathbf{a}_3)=\begin{vmatrix}1&0&0\\1&1&-1\\1&1&2\end{vmatrix}\det(\mathbf{a}_1,\mathbf{a}_2,\mathbf{a}_3)=3\det(\Lambda),$$

$$\det(-\mathbf{a}_1,\ \mathbf{a}_1-\mathbf{a}_2+\mathbf{a}_3,\ \mathbf{a}_1+\mathbf{a}_2+2\mathbf{a}_3)=\begin{vmatrix}-1&0&0\\1&-1&1\\1&1&2\end{vmatrix}\det(\mathbf{a}_1,\mathbf{a}_2,\mathbf{a}_3)=3\det(\Lambda),$$

$$\det(-\mathbf{a}_2,\ \mathbf{a}_1-\mathbf{a}_2-\mathbf{a}_3,\ \mathbf{a}_1+\mathbf{a}_2+2\mathbf{a}_3)=\begin{vmatrix}0&-1&0\\1&-1&-1\\1&1&2\end{vmatrix}\det(\mathbf{a}_1,\mathbf{a}_2,\mathbf{a}_3)=3\det(\Lambda),$$

$\mathbf{v}_7$ cannot be $\pm(\mathbf{a}_1+\mathbf{a}_2-\mathbf{a}_3),\ \pm(\mathbf{a}_1-\mathbf{a}_2+\mathbf{a}_3),\ \pm(\mathbf{a}_1-\mathbf{a}_2-\mathbf{a}_3)$, by Lemma 4.2. Therefore, $\mathbf{v}_7=\pm(\mathbf{a}_1+\mathbf{a}_2+\mathbf{a}_3)$, which means that

$$X(\alpha,\Lambda)=\{\pm\mathbf{a}_1,\pm\mathbf{a}_2,\pm\mathbf{a}_3,\pm(\mathbf{a}_1+\mathbf{a}_2+2\mathbf{a}_3),\pm(\mathbf{a}_1+\mathbf{a}_3),\pm(\mathbf{a}_2+\mathbf{a}_3),\pm(\mathbf{a}_1+\mathbf{a}_2+\mathbf{a}_3)\}.$$

By choose $\mathbf{a}_1'=-(\mathbf{a}_1+\mathbf{a}_3),\ \mathbf{a}_2'=\mathbf{a}_3,\ \mathbf{a}_3'=\mathbf{a}_1+\mathbf{a}_2+\mathbf{a}_3$, one can verify that Lemma 4.3 holds in this case.

On the other hand, if $z=z'=z''=0$, we have

$$\mathbf{v}_4=\pm(\mathbf{a}_1+\mathbf{a}_2)\text{ or }\pm(\mathbf{a}_1-\mathbf{a}_2),$$

$$\mathbf{v}_5=\pm(\mathbf{a}_1+\mathbf{a}_3)\text{ or }\pm(\mathbf{a}_1-\mathbf{a}_3),$$

$$\mathbf{v}_6=\pm(\mathbf{a}_2+\mathbf{a}_3)\text{ or }\pm(\mathbf{a}_2-\mathbf{a}_3).$$

Since the sign of $\mathbf{a}_2$ does not change $\pm\mathbf{v}_1,\ \pm\mathbf{v}_2,\ \pm\mathbf{v}_3$, we assume $\mathbf{v}_4=\pm(\mathbf{a}_1+\mathbf{a}_2)$, without loss of generality. Furthermore, since the sign of $\mathbf{a}_3$ does not change $\pm\mathbf{v}_1,\ \pm\mathbf{v}_2,\ \pm\mathbf{v}_3,\pm\mathbf{v}_4$, we may assume

$$\mathbf{v}_5=\pm(\mathbf{a}_1+\mathbf{a}_3),\ \mathbf{v}_6=\pm(\mathbf{a}_2+\mathbf{a}_3)$$

or

$$\mathbf{v}_5=\pm(\mathbf{a}_1+\mathbf{a}_3),\ \mathbf{v}_6=\pm(\mathbf{a}_2-\mathbf{a}_3).$$

Combining with

$$\mathbf{v}_7=\pm(\mathbf{a}_1+\mathbf{a}_2+\mathbf{a}_3),\ \pm(\mathbf{a}_1+\mathbf{a}_2-\mathbf{a}_3),\ \pm(\mathbf{a}_1-\mathbf{a}_2+\mathbf{a}_3)\text{ or }\pm(\mathbf{a}_1-\mathbf{a}_2-\mathbf{a}_3),$$

we obtain that in this case, $X(\alpha,\Lambda)$ is one of the following sets:

(1) $\{\pm\mathbf{a}_1,\pm\mathbf{a}_2,\pm\mathbf{a}_3,\pm(\mathbf{a}_1+\mathbf{a}_2),\pm(\mathbf{a}_1+\mathbf{a}_3),\pm(\mathbf{a}_2+\mathbf{a}_3),\pm(\mathbf{a}_1+\mathbf{a}_2+\mathbf{a}_3)\}$;
(2) $\{\pm\mathbf{a}_1,\pm\mathbf{a}_2,\pm\mathbf{a}_3,\pm(\mathbf{a}_1+\mathbf{a}_2),\pm(\mathbf{a}_1+\mathbf{a}_3),\pm(\mathbf{a}_2+\mathbf{a}_3),\pm(\mathbf{a}_1+\mathbf{a}_2-\mathbf{a}_3)\}$;
(3) $\{\pm\mathbf{a}_1,\pm\mathbf{a}_2,\pm\mathbf{a}_3,\pm(\mathbf{a}_1+\mathbf{a}_2),\pm(\mathbf{a}_1+\mathbf{a}_3),\pm(\mathbf{a}_2+\mathbf{a}_3),\pm(\mathbf{a}_1-\mathbf{a}_2+\mathbf{a}_3)\}$;
(4) $\{\pm\mathbf{a}_1,\pm\mathbf{a}_2,\pm\mathbf{a}_3,\pm(\mathbf{a}_1+\mathbf{a}_2),\pm(\mathbf{a}_1+\mathbf{a}_3),\pm(\mathbf{a}_2+\mathbf{a}_3),\pm(\mathbf{a}_1-\mathbf{a}_2-\mathbf{a}_3)\}$;
(5) $\{\pm\mathbf{a}_1,\pm\mathbf{a}_2,\pm\mathbf{a}_3,\pm(\mathbf{a}_1+\mathbf{a}_2),\pm(\mathbf{a}_1+\mathbf{a}_3),\pm(\mathbf{a}_2-\mathbf{a}_3),\pm(\mathbf{a}_1+\mathbf{a}_2+\mathbf{a}_3)\}$;
(6) $\{\pm\mathbf{a}_1,\pm\mathbf{a}_2,\pm\mathbf{a}_3,\pm(\mathbf{a}_1+\mathbf{a}_2),\pm(\mathbf{a}_1+\mathbf{a}_3),\pm(\mathbf{a}_2-\mathbf{a}_3),\pm(\mathbf{a}_1+\mathbf{a}_2-\mathbf{a}_3)\}$;
(7) $\{\pm\mathbf{a}_1,\pm\mathbf{a}_2,\pm\mathbf{a}_3,\pm(\mathbf{a}_1+\mathbf{a}_2),\pm(\mathbf{a}_1+\mathbf{a}_3),\pm(\mathbf{a}_2-\mathbf{a}_3),\pm(\mathbf{a}_1-\mathbf{a}_2+\mathbf{a}_3)\}$;
(8) $\{\pm\mathbf{a}_1,\pm\mathbf{a}_2,\pm\mathbf{a}_3,\pm(\mathbf{a}_1+\mathbf{a}_2),\pm(\mathbf{a}_1+\mathbf{a}_3),\pm(\mathbf{a}_2-\mathbf{a}_3),\pm(\mathbf{a}_1-\mathbf{a}_2-\mathbf{a}_3)\}$.

Since

$$\det(\mathbf{a}_1+\mathbf{a}_3,\ \mathbf{a}_2+\mathbf{a}_3,\ -(\mathbf{a}_1+\mathbf{a}_2-\mathbf{a}_3)) = \begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ -1 & -1 & 1 \end{vmatrix} \det(\mathbf{a}_1,\mathbf{a}_2,\mathbf{a}_3) = 3\det(\Lambda),$$

$$\det(\mathbf{a}_1+\mathbf{a}_2,\ \mathbf{a}_2+\mathbf{a}_3,\ \mathbf{a}_1-\mathbf{a}_2+\mathbf{a}_3) = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & -1 & 1 \end{vmatrix} \det(\mathbf{a}_1,\mathbf{a}_2,\mathbf{a}_3) = 3\det(\Lambda),$$

$$\det(\mathbf{a}_1+\mathbf{a}_2,\ \mathbf{a}_1+\mathbf{a}_3,\ \mathbf{a}_1-\mathbf{a}_2-\mathbf{a}_3) = \begin{vmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & -1 & -1 \end{vmatrix} \det(\mathbf{a}_1,\mathbf{a}_2,\mathbf{a}_3) = 3\det(\Lambda),$$

$$\det(\mathbf{a}_2-\mathbf{a}_3,\ \mathbf{a}_1+\mathbf{a}_3,\ \mathbf{a}_1-\mathbf{a}_2-\mathbf{a}_3) = \begin{vmatrix} 0 & 1 & -1 \\ 1 & 0 & 1 \\ 1 & -1 & -1 \end{vmatrix} \det(\mathbf{a}_1,\mathbf{a}_2,\mathbf{a}_3) = 3\det(\Lambda),$$

by Lemma 4.2, $X(\alpha,\Lambda)$ cannot be the sets (2), (3), (4), (8).

For set (1), Lemma 4.3 already holds; For set (5), by choosing $\mathbf{a}_1'=\mathbf{a}_1+\mathbf{a}_2,\ \mathbf{a}_2'=-\mathbf{a}_2,\ \mathbf{a}_3'=\mathbf{a}_3$, it can be verified that Lemma 4.3 holds; For set (6), by choosing $\mathbf{a}_1'=\mathbf{a}_1,\ \mathbf{a}_2'=\mathbf{a}_2-\mathbf{a}_3,\ \mathbf{a}_3'=\mathbf{a}_3$, it can be verified that Lemma 4.3 holds; For set (7), by choosing $\mathbf{a}_1'=\mathbf{a}_1,\ \mathbf{a}_2'=\mathbf{a}_2,\ \mathbf{a}_3'=-(\mathbf{a}_1+\mathbf{a}_3)$, it can be verified that Lemma 4.3 holds.

As a conclusion of two cases, Lemma 4.3 is proved. □

Lemma 4.4 When $0\le\alpha<\frac{4}{3}\sqrt{3}-2$, we have $\kappa_\alpha^*(B^3)=12$.

Proof On the contrary, suppose there exists a packing lattice Λ of B^3 and $0\le\alpha<\frac{4}{3}\sqrt{3}-2$ satisfies $\mathrm{card}\{X(\alpha,\Lambda)\}=14$. By Lemma 4.3, there exist a basis $\{\mathbf{a}_1,\mathbf{a}_2,\mathbf{a}_3\}$ of Λ such that

$$X(\alpha,\Lambda)=\{\pm\mathbf{a}_1,\ \pm\mathbf{a}_2,\ \pm\mathbf{a}_3,\ \pm(\mathbf{a}_1+\mathbf{a}_2),\ \pm(\mathbf{a}_1+\mathbf{a}_3),\ \pm(\mathbf{a}_2+\mathbf{a}_3),\ \pm(\mathbf{a}_1+\mathbf{a}_2+\mathbf{a}_3)\}.$$

Without loss of generality, we suppose that $\mathbf{a}_1,\ \mathbf{a}_2,\ \mathbf{a}_1+\mathbf{a}_2$ lie in the plane

$$\{(v_1,v_2,v_3):\ v_3=0\}$$

and

$$\mathbf{a}_1+\mathbf{a}_2=(0,c,0),\quad \mathbf{a}_1=(-a,b,0),\quad \mathbf{a}_2=(a,c-b,0).$$

Then we have:

$$\frac{16}{3}>||\mathbf{a}_1+\mathbf{a}_2||^2=c^2\ge4, \tag{4.1}$$

$$\frac{16}{3}>||\mathbf{a}_1||^2=a^2+b^2\ge4, \tag{4.2}$$

$$\frac{16}{3}>||\mathbf{a}_2||^2=a^2+b^2+c^2-2bc\ge4. \tag{4.3}$$

Let $\mathbf{a}_3=(v_1,v_2,v_3)$ and denote $||\mathbf{a}_3||^2=D_1$, $||\mathbf{a}_1+\mathbf{a}_2+\mathbf{a}_3||^2=D_2$, then we have

$$\frac{16}{3}>D_1=v_1^2+v_2^2+v_3^2\ge4, \tag{4.4}$$

$$\frac{16}{3} > D_2 = v_1^2 + v_2^2 + v_3^2 + c^2 + 2cv_2 \ge 4. \tag{4.5}$$

By (4.4) and (4.5), we have

$$v_2 = \frac{D_2 - D_1 - c^2}{2c}. \tag{4.6}$$

By (4.4) and (4.6), we have

$$\begin{aligned}\frac{16}{3} > ||\mathbf{a}_1 + \mathbf{a}_3||^2 &= (v_1 - a)^2 + (v_2 + b)^2 + v_3^2 \\ &= D_1 + a^2 + b^2 - 2av_1 + 2bv_2\end{aligned}$$

and

$$\begin{aligned}\frac{16}{3} > ||\mathbf{a}_2 + \mathbf{a}_3||^2 &= (v_1 + a)^2 + (v_2 + c - b)^2 + v_3^2 \\ &= D_1 + a^2 + 2av_1 + 2v_2(c - 2b) + 2bv_2 + b^2 + c^2 - 2bc \\ &= D_1 + a^2 + b^2 + 2av_1 + 2bv_2 + (2v_2 + c)(c - 2b) \\ &= D_1 + a^2 + b^2 + 2av_1 + 2bv_2 + \frac{D_2 - D_1}{c}(c - 2b).\end{aligned}$$

Using (4.6) again, we obtain

$$\begin{aligned}\frac{16}{3} &> \frac{||\mathbf{a}_1 + \mathbf{a}_3||^2 + ||\mathbf{a}_2 + \mathbf{a}_3||^2}{2} \\ &= D_1 + a^2 + b^2 + 2bv_2 + \frac{D_2 - D_1}{2c}(c - 2b) \\ &= D_1 + a^2 + b^2 + \frac{b}{c}(D_2 - D_1) - bc + \frac{D_2 - D_1}{2} - \frac{b}{c}(D_2 - D_1) \\ &= \frac{D_1 + D_2}{2} + a^2 + b^2 - bc. \end{aligned} \tag{4.7}$$

On the other hand, by (4.2) and (4.3), we have $2(a^2+b^2-bc)+c^2 \ge 8$. Therefore, by (4.1), we get

$$a^2 + b^2 - bc \ge 4 - \frac{c^2}{2} > \frac{4}{3}, \tag{4.8}$$

together with (4.4) and (4.5), we have

$$\frac{D_1 + D_2}{2} + a^2 + b^2 - bc > 4 + \frac{4}{3} = \frac{16}{3},$$

which contradicts (4.7).

Therefore, when $0 \le \alpha < \frac{4}{3}\sqrt{3} - 2$, we have $\kappa_\alpha^*(B^3) \le 12$. Combining with $\kappa_\alpha^*(B^3) \ge \kappa^*(B^3) = 12$, for $0 \le \alpha < \frac{4}{3}\sqrt{3} - 2$ we have $\kappa_\alpha^*(B^3) = 12$. Lemma 4.4 is proved. □

Remark 4.1 For $\alpha = \frac{4}{3}\sqrt{3} - 2$, by repeating (4.1)–(4.8) one can deduce that

$$\mathrm{card}\{X(\alpha, \Lambda)\} = 14$$

if and only if the lattice Λ is generated by $\mathbf{a}_1 = (-\frac{2}{3}\sqrt{6}, \frac{2}{3}\sqrt{3}, 0)$, $\mathbf{a}_2 = (\frac{2}{3}\sqrt{6}, \frac{2}{3}\sqrt{3}, 0)$ and $\mathbf{a}_3 = (0, \frac{2}{3}\sqrt{3}, \frac{2}{3}\sqrt{6})$, up to some rotation.

Lemma 4.5 When $\alpha = 2\sqrt{2} - 2$, we have $\kappa_\alpha^*(B^3) = 20$.

Proof Let Λ be the lattice generated by $\mathbf{a}_1 = (2, 0, 0), \mathbf{a}_2 = (0, 2, 0)$ and $\mathbf{a}_3 = (1, 0, \sqrt{3})$. When $\alpha = 2\sqrt{2} - 2$, one can verify that

$$\mathrm{card}\{X(\alpha, \Lambda)\} = 20.$$

By Remark 2.1, for a packing lattice Λ of B^3 and $\alpha = 2\sqrt{2} - 2$, to let $\text{card}\{X(\alpha, \Lambda)\} \geq 20$, a necessary condition is there exist $\mathbf{a}_1, \mathbf{a}_2 \in \Lambda$ such that

$$||\mathbf{a}_1|| = ||\mathbf{a}_2|| = 2, \quad ||\mathbf{a}_1 + \mathbf{a}_2|| = ||\mathbf{a}_1 - \mathbf{a}_2|| = 2\sqrt{2}.$$

Without loss of generality, we suppose $\mathbf{a}_1 = (2, 0, 0),\ \mathbf{a}_2 = (0, 2, 0)$. It is obvious to see that we can expand a basis of Λ based on $\{\mathbf{a}_1, \mathbf{a}_2\}$.

Suppose $\mathbf{a}_1, \mathbf{a}_2$ and $\mathbf{a}_3 = (v_1, v_2, v_3)$ is a basis of Λ. We assume $v_3 > 0$, without loss of generality. Since

$$\det(\Lambda) \geq \frac{\text{vol}(B^3)}{\delta^*(B^3)} = \frac{\frac{4}{3}\pi}{\frac{\pi}{\sqrt{18}}} = 4\sqrt{2},$$

we have $v_3 \geq \sqrt{2}$ and the equality holds if and only if Λ is the densest packing lattice of B^3. In this case one can verify that $\text{card}\{X(\alpha, \Lambda)\} = 18 < 20$. Therefore, we have $v_3 > \sqrt{2}$. Which means that, for

$$\mathbf{v} = z_1\mathbf{a}_1 + z_2\mathbf{a}_2 + z_3\mathbf{a}_3 \in X(\alpha, \Lambda),$$

we have $z_3 = 0$ or ± 1.

Since

$$\text{card}\{\mathbf{v} :\ \mathbf{v} = z_1\mathbf{a}_1 + z_2\mathbf{a}_2 \in X(\alpha, \Lambda)\} = 8$$

and $X(\alpha, \Lambda)$ is centrally symmetric, to let $\text{card}\{X(\alpha, \Lambda)\} \geq 20$, we have

$$\text{card}\{\mathbf{v} :\ \mathbf{v} = z_1\mathbf{a}_1 + z_2\mathbf{a}_2 + \mathbf{a}_3 \in X(\alpha, \Lambda)\} \geq 6.$$

Therefore, there exist two of them which are equivalent. Replace $\mathbf{a}_3$ by the mid-point of them, we may further assume $||\mathbf{a}_3||^2 = v_1^2 + v_2^2 + v_3^2 = 4$, by Remark 2.1.

Without loss of generality, we suppose $v_1, v_2 \geq 0$. Since $||\mathbf{a}_3 - \mathbf{a}_1|| \geq 2, \quad ||\mathbf{a}_3 - \mathbf{a}_2|| \geq 2$, by routine computation we have $0 \leq v_1 \leq 1, \quad 0 \leq v_2 \leq 1$.

For a lattice vector

$$\mathbf{v} = z_1\mathbf{a}_1 + z_2\mathbf{a}_2 + \mathbf{a}_3 \in X(\alpha, \Lambda),$$

we have

$$||\mathbf{v}||^2 = 4z_1^2 + 4z_1v_1 + 4z_2^2 + 4z_2v_2 + v_1^2 + v_2^2 + v_3^2 \leq (2\sqrt{2})^2,$$

which means

$$z_1^2 + z_1v_1 + z_2^2 + z_2v_2 \leq 1. \tag{4.9}$$

By routine computation, a necessary condition for (4.9) is $|z_1| \leq 1, \quad |z_2| \leq 1$. For $(z_1, z_2) = (1, 0)$: $\mathbf{a}_1 + \mathbf{a}_3 \in X(\alpha, \Lambda)$ if and only if

$$v_1 = 0. \tag{4.9.1}$$

For $(z_1, z_2) = (1, -1)$: $\mathbf{a}_1 - \mathbf{a}_2 + \mathbf{a}_3 \in X(\alpha, \Lambda)$ if and only if

$$1 + v_1 - v_2 \leq 0. \tag{4.9.2}$$

For $(z_1, z_2) = (0, 1)$: $\mathbf{a}_2 + \mathbf{a}_3 \in X(\alpha, \Lambda)$ if and only if

$$v_2 = 0. \tag{4.9.3}$$

For $(z_1, z_2) = (-1, 1)$: $-\mathbf{a}_1 + \mathbf{a}_2 + \mathbf{a}_3 \in X(\alpha, \Lambda)$ if and only if

$$1 - v_1 + v_2 \leq 0. \tag{4.9.4}$$

For $(z_1, z_2) = (-1, -1)$: $-\mathbf{a}_1 - \mathbf{a}_2 + \mathbf{a}_3 \in X(\alpha, \Lambda)$ if and only if

$$1 - v_1 - v_2 \leq 0. \tag{4.9.5}$$

For $(z_1, z_2) = (1, 1)$, since $1 + v_1 + 1 + v_2 > 1$, we have

$$\mathbf{a}_1 + \mathbf{a}_2 + \mathbf{a}_3 \notin X(\alpha, \Lambda). \tag{4.9.6}$$

Obviously, (4.9.1) and (4.9.4) cannot hold simultaneously, (4.9.2) and (4.9.3) cannot hold simultaneously. Combining with (4.9.6), we have

$$\mathrm{card}\{\mathbf{v} : \ \mathbf{v} = z_1\mathbf{a}_1 + z_2\mathbf{a}_2 + \mathbf{a}_3 \in X(\alpha, \Lambda)\} \leq 6.$$

Therefore, for $\alpha = 2\sqrt{2} - 2$ we get $\mathrm{card}\{X(\alpha, \Lambda)\} \leq 20$.

To let $\mathrm{card}\{X(\alpha, \Lambda)\} = 20$, we must have: one of (4.9.1) and (4.9.4) holds, one of (4.9.2) and (4.9.3) holds, and (4.9.5) holds. By routine computation one can deduce that $\mathbf{a}_3 = (0, 1, \sqrt{3})$ or $(1, 0, \sqrt{3})$. Therefore, when $\alpha = 2\sqrt{2} - 2$, we have $\kappa^*_\alpha(B^3) = 20$, and the equality holds if and only if the corresponding lattice Λ is generated by $\mathbf{a}_1 = (2, 0, 0)$, $\mathbf{a}_2 = (0, 2, 0)$ and $\mathbf{a}_3 = (1, 0, \sqrt{3})$, up to some rotation. Lemma 4.5 is proved. □

Lemma 4.1, Lemma 4.4 and Lemma 4.5 together yields the following theorem.

Theorem 2 In $\mathbb{E}^3$, we have

$$\kappa^*_\alpha(B^3) = \begin{cases} 12, & 0 \leq \alpha < \frac{4}{3}\sqrt{3} - 2, \\ 14, & \frac{4}{3}\sqrt{3} - 2 \leq \alpha < 2\sqrt{2} - 2, \\ 20, & \alpha = 2\sqrt{2} - 2. \end{cases}$$

Remark 4.2 In fact, by repeating the calculations (4.9.1)–(4.9.6), one can deduce that for $\alpha = 2\sqrt{2}-2$, $\mathrm{card}\{X(\alpha, \Lambda)\} = 18$ if and only if Λ is generated by $\mathbf{a}_1 = (2, 0, 0)$, $\mathbf{a}_2 = (0, 2, 0)$ and $\mathbf{a}_3 = (1, 1, \sqrt{2})$, or $\mathbf{a}_1 = (2, 0, 0)$, $\mathbf{a}_2 = (0, 2, 0)$ and $\mathbf{a}_3 = (0, 0, 2)$, up to some rotation.

We end this section by a problem as following.

Problem 4.1 When $\alpha = \frac{4}{3}\sqrt{3} - 2$, is it true that $\kappa^*_\alpha(C) \geq 14$ holds for every three-dimensional centrally symmetric convex body C?

5 Kissing Numbers of Convex Bodies

Although the concept of $\kappa^*_\alpha(B^3)$ itself is interesting, it can also lead to determine the lattice kissing numbers of convex bodies which were geometrically similar to B^3. To this end, we present the following theorem:

Theorem 5.1 For a 3-dimensional centrally symmetric convex body C centered at $\mathbf{o}$, if $B^3 \subset C \subset \mathrm{int}(\frac{2}{3}\sqrt{3}B^3)$ holds, then we have $\kappa^*(C) = 12$.

Proof Let $C + \Lambda$ be a lattice packing attaining $\kappa^*(C)$ and

$$X = \{\mathbf{v}_1, \cdots, \mathbf{v}_{\kappa^*(C)}\} = \partial(2C) \cap \Lambda,$$

where $\partial(2C)$ denotes the boundary of $2C$. Since $2B^3 \subset 2C \subset \mathrm{int}(\frac{4}{3}\sqrt{3}B^3)$, we have $2 \leq ||\mathbf{v}_i|| < \frac{4}{3}\sqrt{3}$ holds for all $i = 1, 2, \cdots, \kappa^*(C)$. Since Λ is also a packing lattice of B^3, by Lemma 4.3 we get

$$\kappa^*(C) = \mathrm{card}X \leq 12.$$

On the other hand, since $\kappa^*(C) \geq 12$ holds for all 3-dimensional centrally symmetric convex body C (see [17]), we have $\kappa^*(C) = 12$.

Theorem 5.1 is proved. □

We now give several convex bodies as examples which lattice kissing numbers can be determined by Theorem 5.1.

Example 5.1 We take $\tau = \frac{\sqrt{5}+1}{2}$ and define

$$P_d = \{(v_1, v_2, v_3) : |\tau v_1| + |v_2| \leq 1,\ |\tau v_2| + |v_3| \leq 1,\ |\tau v_3| + |v_1| \leq 1\},$$

$$P_i = \Big\{(v_1, v_2, v_3) : |v_1| + |v_2| + |v_3| \leq 1,\ |\tau v_1| + |\tfrac{1}{\tau} v_3| \leq 1,\ |\tau v_2| + |\tfrac{1}{\tau} v_1| \leq 1,\ |\tau v_3| + |\tfrac{1}{\tau} v_2| \leq 1\Big\}.$$

Usually, P_d and P_i are called a dodecahedron and an icosahedron, respectively. Define

$$P_{tri} = (1+\tau)P_i \cap (4/3+\tau)P_d.$$

Usually, P_{tri} is called a truncated icosahedron.

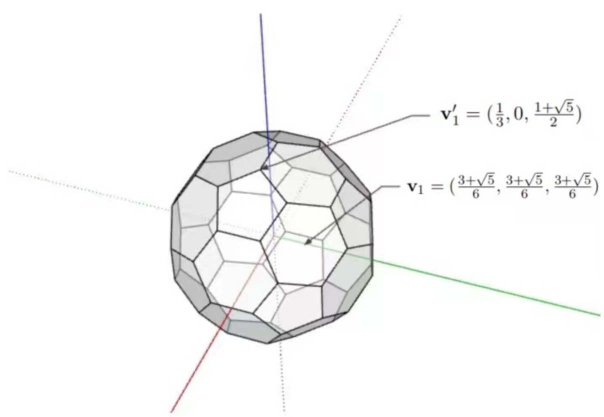

Figure 1 Truncated icosahedron P_{tri}

By routine computation, we have $||\mathbf{v}_1||B^3 \subset P_{tri} \subset ||\mathbf{v}_1'||B^3$, where $\mathbf{v}_1 = (\frac{3+\sqrt{5}}{6}, \frac{3+\sqrt{5}}{6}, \frac{3+\sqrt{5}}{6})$, $\mathbf{v}_1' = (\frac{1}{3}, 0, \frac{1+\sqrt{5}}{2})$, see Figure 1. Since

$$B^3 \subset \frac{1}{||\mathbf{v}_1||}P_{tri} \subset \frac{||\mathbf{v}_1'||}{||\mathbf{v}_1||}B^3 = 1.0929\cdots B^3 \subset \mathrm{int}\Big(\frac{4}{3}\sqrt{3}B^3\Big),$$

by Theorem 5.1 we have

$$\kappa^*(P_{tri}) = \kappa^*\Big(\frac{1}{||\mathbf{v}_1||}P_{tri}\Big) = 12.$$

Example 5.2 We define

$$P_{rtc} = \Big\{(v_1, v_2, v_3) : |\tau v_1| \leq 1, |\tau v_2| \leq 1, |\tau v_3| \leq 1, |\frac{1}{2}v_1| + |\frac{\tau}{2}v_2| + |\frac{\tau+1}{2}v_3| \leq 1,$$
$$|\frac{\tau}{2}v_1| + |\frac{\tau+1}{2}v_2| + |\frac{1}{2}v_3| \leq 1, |\frac{\tau+1}{2}v_1| + |\frac{1}{2}v_2| + |\frac{\tau}{2}v_3| \leq 1\Big\}.$$

Usually, P_{rtc} is called a rhombic triacontahedron. Define

$$P_{rid} = (3\tau+2)P_{rtc} \cap (4\tau+1)P_i \cap (3(1+\tau))P_d,$$

$$P_{trid} = (5\tau+4)P_{rtc} \cap (6\tau+3)P_i \cap (5(1+\tau))P_d.$$

Usually, P_{rid} and P_{trid} are called a rhombic icosidodecahedron and a truncated icosidodecahedron, respectively.

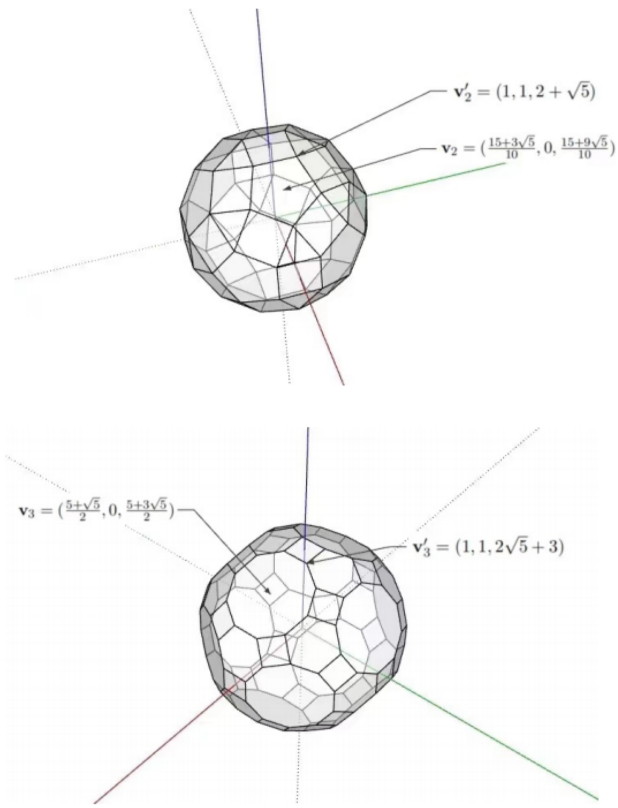

Figure 2 Rhombic icosidodecahedron P_{rid} and truncated icosidodecahedron P_{trid}

By routine computation, we have

$$||\mathbf{v}_2||B^3 \subset P_{rid} \subset ||\mathbf{v}_2'||B^3,$$

where $\mathbf{v}_2 = (\frac{15+3\sqrt{5}}{10}, 0, \frac{15+9\sqrt{5}}{10})$, $\mathbf{v}_2' = (1, 1, 2 + \sqrt{5})$, see Figure 2. Since

$$B^3 \subset \frac{1}{||\mathbf{v}_2||} P_{rid} \subset \frac{||\mathbf{v}_2'||}{||\mathbf{v}_2||} B^3 = 1.0815 \cdots B^3 \subset \mathrm{int}\Big(\frac{4}{3}\sqrt{3}B^3\Big),$$

by Theorem 5.1 we have

$$\kappa^*(P_{rid}) = \kappa^*\Big(\frac{1}{||\mathbf{v}_2||} P_{rid}\Big) = 12.$$

By routine computation, we have

$$||\mathbf{v}_3||B^3 \subset P_{trid} \subset ||\mathbf{v}_3'||B^3,$$

where $\mathbf{v}_3 = (\frac{5+\sqrt{5}}{2}, 0, \frac{5+3\sqrt{5}}{2})$, $\mathbf{v}_3' = (1, 1, 2\sqrt{5} + 3)$, see Figure 2. Since

$$B^3 \subset \frac{1}{||\mathbf{v}_3||} P_{trid} \subset \frac{||\mathbf{v}_3'||}{||\mathbf{v}_3||} B^3 = 1.1050 \cdots B^3 \subset \mathrm{int}\Big(\frac{4}{3}\sqrt{3}B^3\Big),$$

by Theorem 5.1 we have

$$\kappa^*(P_{trid}) = \kappa^*\Big(\frac{1}{||\mathbf{v}_3||} P_{trid}\Big) = 12.$$

Example 5.3 We use the configuration of snub dodecahedron given by Henk [6], denote it by P_{sd}. By routine computation, we have

$$||\mathbf{v}_4||B^3 \subset P_{sd} \subset ||\mathbf{v}_4'||B^3,$$

where

$$\mathbf{v}_4 = (-0.9661\cdots, 0, 1.5632\cdots),\ \mathbf{v}_4' = (-0.3477\cdots, -0.3069\cdots, 1.9454\cdots),$$

see Figure 3. Since

$$B^3 \subset \frac{1}{||\mathbf{v}_4||}P_{sd} \subset \frac{||\mathbf{v}_4'||}{||\mathbf{v}_4||}B^3 = 1.0883\cdots B^3 \subset \mathrm{int}\Big(\frac{4}{3}\sqrt{3}B^3\Big),$$

by Theorem 5.1 we have

$$\kappa^*(P_{sd}) = \kappa^*\Big(\frac{1}{||\mathbf{v}_4||}P_{sd}\Big) = 12.$$

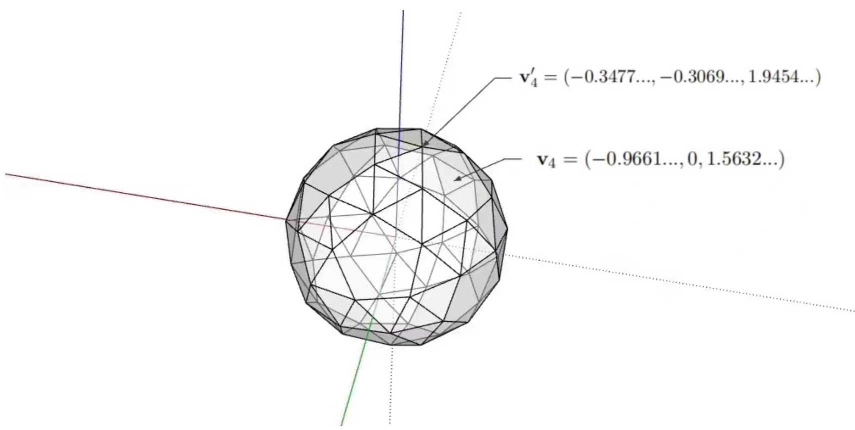

Figure 3 Snub dodecahedron P_{sd}

Example 5.4 Define

$$B_p^3 = \big\{(v_1, v_2, v_3):\ |v_1|^p + |v_2|^p + |v_3|^p \le 1\big\}.$$

Usually, B_p^3 is called a L_p unit ball in 3-dimension. When $p_1 \le p_2$, it is well known that

$$B_{p_1}^3 \subset B_{p_2}^3.$$

For $p \ge 2$ and a point $\mathbf{v} = (v_1, v_2, v_3) \in B_p^3$ where $v_1,\ v_2,\ v_3 \ge 0$, we have

$$v_1^p + v_2^p + v_3^p \le 1.$$

According to Power-Mean Inequality, we have

$$v_1^2 + v_2^2 + v_3^2 \le \Big(\Big(\frac{1}{3}\Big)^{\frac{1}{p}}\Big)^2 \times 3 = \Big(\frac{1}{3}\Big)^{\frac{2}{p}} \times 3.$$

Therefore we have

$$B^3 \subset B_p^3 \subset \sqrt{3} \times \Big(\frac{1}{3}\Big)^{\frac{1}{p}} B^3.$$

By routine computation, when $2 \le p < \frac{\ln 3}{\ln 3 - \ln 2}$ we get $\kappa^*(B_p^3) = 12$ by Theorem 5.1.

For $p < 2$, by the same deduction we obtain

$$\sqrt{3} \times \Big(\frac{1}{3}\Big)^{\frac{1}{p}} B^3 \subset B_p^3 \subset B^3,$$

which means

$$B^3 \subset \frac{1}{\sqrt{3}\times\left(\frac{1}{3}\right)^{\frac{1}{p}}} B_p^3 \subset \frac{1}{\sqrt{3}\times\left(\frac{1}{3}\right)^{\frac{1}{p}}} B^3.$$

By routine computation, when $2 > p > \frac{\ln 3}{\ln 2}$ we get $\kappa^*(B_p^3) = 12$ by Theorem 5.1.

As a conclusion of the two cases, when $\frac{\ln 3}{\ln 3 - \ln 2} > p > \frac{\ln 3}{\ln 2}$, we have $\kappa^*(B_p^3) = 12$.

Remark 5.1 Let Λ be the lattice generated by $\mathbf{a}_1 = (2,0,0), \mathbf{a}_2 = (0,2,0)$ and $\mathbf{a}_3 = (1,1,1)$, then one can verify that: when $p = \frac{\ln 3}{\ln 2}$, Λ is a packing lattice of B_p^3 and

$$\mathrm{card}\big\{\partial(2B_p^3)\cap\Lambda\big\} = 14.$$

On the other hand, since B_p^3 is a strictly convex body when $1 < p < \infty$, combining with Lemma 2.2, when $p = \frac{\ln 3}{\ln 2}$ we have $\kappa^*(B_p^3) = 14$.

6 Proof of Theorem 4

For $\alpha = 2\sqrt{2} - 2$ and a packing lattice Λ of B^n, we have the following lemma.

Lemma 6.1 One equivalent class of Λ can contain at most n pairs of vectors of $X(\alpha, \Lambda)$.

Proof Suppose

$$\pm\mathbf{v}_1, \cdots, \pm\mathbf{v}_i \in X(\alpha, \Lambda)$$

belong to the same equivalent class, $i \geq 2$. By Remark 2.1, we have $||\mathbf{v}_i|| = 2\sqrt{2}$ holds for all i and $\langle \mathbf{v}_i, \mathbf{v}_j \rangle = 0$ holds for all $i \neq j$. Therefore, one equivalent class of Λ can contain at most n pairs of vectors of $X(\alpha, \Lambda)$.

Lemma 6.1 is proved. □

Denote the numbers of equivalent classes of Λ which contain exactly i pairs of vectors of $X(\alpha, \Lambda)$ by m_i. We define a collection of sets

$$C(X(\alpha,\Lambda)) = \Big\{A = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}:\ \mathbf{v}_2 = \frac{1}{2}(\mathbf{v}_1 + \mathbf{v}_3) \text{ and } \mathbf{v}_1,\ \mathbf{v}_2,\ \mathbf{v}_3 \in X(\alpha,\Lambda)\Big\}.$$

By estimate $\mathrm{card}\{C(X(\alpha,\Lambda))\}$ in two different ways, we prove the following lemma.

Lemma 6.2

$$\sum_{i=2}^{n} 2i(i-1)m_i \leq \kappa^*(B^{n-1})\cdot m_1.$$

Proof For a set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} \in C(X(\alpha,\Lambda))$, by Remark 2.1, we have

$$||\mathbf{v}_2|| = 2,\ ||\mathbf{v}_1 - \mathbf{v}_2|| = ||\mathbf{v}_3 - \mathbf{v}_2|| = 2,\ ||\mathbf{v}_1|| = ||\mathbf{v}_3|| = 2\sqrt{2}.$$

We assume $\mathbf{v}_2 = (0,0,\cdots,0,2)$, without loss of generality. Then one can easily deduce that $\mathbf{v}_1$ and $\mathbf{v}_3$ must lie in the $(n-1)$-dimensional hyperplane

$$\pi_0:\ \big\{(v_1, v_2, \cdots, v_{n-1}, v_n):\ v_n = 2\big\}.$$

It is obvious that

$$\mathrm{card}\{\mathbf{v}:\ ||\mathbf{v} - \mathbf{v}_2|| = 2,\ \mathbf{v} \in X(\alpha,\Lambda)\cap\pi_0\} \leq \kappa^*(B^{n-1}),$$

which means

$$\mathrm{card}\{A \in C(X(\alpha,\Lambda)):\ \mathbf{v}_2 \in A\} \leq \kappa^*(B^{n-1})/2.$$

For a vector $\mathbf{v} \in X(\alpha, \Lambda)$ of length 2, by Remark 2.1, $\mathbf{v}$ cannot be equivalent to any vector of $X(\alpha, \Lambda)$, besides $\pm\mathbf{v}$. Therefore, we have

$$\mathrm{card}\{\mathbf{v} : \ ||\mathbf{v}|| = 2, \ \mathbf{v} \in X(\alpha, \Lambda)\} \leq 2m_1.$$

Consequently, we get

$$\mathrm{card}\{C(X(\alpha, \Lambda))\} \leq \kappa^*(B^{n-1}) \cdot m_1. \tag{6.1}$$

On the other hand, by the definition of $C(X(\alpha, \Lambda))$, a set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ belongs to it if and only if $\mathbf{v}_1, \mathbf{v}_3 \in X(\alpha, \Lambda)$ are equivalent and $\mathbf{v}_1 \neq \pm\mathbf{v}_3$. For an equivalent class which contains $i \geq 2$ pairs of vectors of $X(\alpha, \Lambda)$, denote it by X_{i1}. By enumeration we have

$$\mathrm{card}\big\{\{\mathbf{v}_1, \mathbf{v}_3\} : \ \{\mathbf{v}_1, \mathbf{v}_3\} \subset X_{i1}, \ \mathbf{v}_1 \neq \pm\mathbf{v}_3\big\} = 2i(i-1).$$

Therefore, we get

$$\mathrm{card}\{C(X(\alpha, \Lambda))\} = \sum_{i=2}^{n} 2i(i-1)m_i. \tag{6.2}$$

By (6.1) and (6.2), Lemma 6.2 is proved. □

Theorem 4 In $\mathbb{E}^8$, when $\alpha = 2\sqrt{2} - 2$ we have $\kappa^*_\alpha(B^8) = 2400$.

Proof For $\alpha = 2\sqrt{2} - 2$, it is well known (see [2]) that

$$\mathrm{card}\big\{X(\alpha, \sqrt{2}E_8)\big\} = 240 + 2160 = 2400,$$

where

$$E_8 = \Big\{(v_1, v_2, \cdots, v_8) : \ 2v_i \in Z; \ v_i - v_j \in Z; \ \sum v_i \in 2Z\Big\}.$$

Suppose that there is a suitable lattice Λ satisfying $\mathrm{card}\{X(\alpha, \Lambda)\} \geq 2400$, which means

$$8m_8 + 7m_7 + \cdots + 2m_2 + m_1 \geq 1200. \tag{6.3}$$

Since there are at most $2^8 - 1 = 255$ equivalent classes which can contain the vectors of $X(\alpha, \Lambda)$, we have

$$m_8 + m_7 + \cdots + m_1 \leq 255. \tag{6.4}$$

For $n = 8$ case, we restate Lemma 6.2 as

$$112m_8 + 84m_7 + 60m_6 + 40m_5 + 24m_4 + 12m_3 + 4m_2 \leq 126m_1 \tag{6.5}$$

by substituting $\kappa^*(B^7) = 126$.

By (6.3) and (6.4), we have $7m_8 + 6m_7 + \cdots + m_2 \geq 945$, multiply both sides by 34, we have

$$238m_8 + 204m_7 + 170m_6 + 136m_5 + 102m_4 + 68m_3 + 34m_2 \geq 32130. \tag{6.6}$$

By (6.4) and (6.5), one can deduce that

$$238m_8 + 210m_7 + 186m_6 + 166m_5 + 150m_4 + 138m_3 + 130m_2 \leq 32130. \tag{6.7}$$

By (6.6) and (6.7), we obtain $m_7 = m_6 = \cdots = m_2 = 0$. Combining with (6.3), (6.4) and (6.5), we have

$$\begin{cases} 8m_8 + m_1 \geq 1200, \\ m_8 + m_1 \leq 255, \\ 112m_8 \leq 126m_1. \end{cases}$$

By routine computation, one can easily deduce that $m_1 = 120,\ m_8 = 135$. Furthermore, in this case the equality in (6.1) holds, which means that

$$\mathrm{card}\{\mathbf{v}:\ ||\mathbf{v}|| = 2,\ \mathbf{v} \in X(\alpha,\Lambda)\} = 2m_1 = 240.$$

Since $\kappa^*(B^8) = 240$ and the corresponding lattice must be $\sqrt{2}E_8$, up to some rotation (see [21]), for $\alpha = 2\sqrt{2}-2$ we have $\kappa^*_\alpha(B^8) = 2400$, and the equality can be attained if and only if the corresponding lattice $\Lambda = \sqrt{2}E_8$, up to rotation and reflection.

Theorem 4 is proved. □

Based on this proof, we may make the following conjecture.

Conjecture 6.1 In $\mathbb{E}^8$, when $\alpha = 2\sqrt{2}-2$ we have $\kappa_\alpha(B^8) = 2400$.

Let Λ_{24} denote the Leech lattice (see [2]). When $\alpha = 2\sqrt{2}-2$, we have

$$\mathrm{card}\big\{X(\alpha,\Lambda_{24})\big\} = 196560 + 16773120 + 398034000 = 415003680.$$

This observation supports the following conjecture.

Conjecture 6.2 In $\mathbb{E}^{24}$, when $\alpha = 2\sqrt{2}-2$ we have $\kappa^*_\alpha(B^{24}) = 415003680$.

7 Proof of Theorem 3

Theorem 3 In $\mathbb{E}^4$, we have

$$\kappa^*_\alpha(B^4) = \begin{cases} 30, & \sqrt{6}-2 \le \alpha < 2\sqrt{2}-2, \\ 50, & \alpha = 2\sqrt{2}-2. \end{cases}$$

Proof As usual, we write

$$A_n = \Big\{(v_0, v_1, v_2, \cdots, v_n):\ v_i \in Z;\ \sum v_i = 0\Big\},$$
$$D_n = \Big\{(v_1, v_2, \cdots, v_n):\ v_i \in Z;\ \sum v_i \in 2Z\Big\}.$$

Furthermore, we denote the *dual lattice* of A_n by A_n^*, namely

$$A_n^* = \big\{\mathbf{v}:\ \langle \mathbf{v}, \mathbf{u}\rangle \in Z \text{ for all } \mathbf{u} \in A_n\big\}.$$

When $\alpha = \sqrt{6}-2$, one can verify that $\mathrm{card}\big\{X(\alpha, \sqrt{5}A_4^*)\big\} = 30$. Combining with Lemma 2.2, for $\sqrt{6}-2 \le \alpha < 2\sqrt{2}-2$ we have $\kappa^*_\alpha(B^4) = 30$.

For $\alpha = 2\sqrt{2}-2$, let Λ be the lattice generated by $\mathbf{a}_1 = (2,0,0,0)$, $\mathbf{a}_2 = (0,2,0,0)$, $\mathbf{a}_3 = (1,0,\sqrt{3},0)$, $\mathbf{a}_4 = (0,1,\frac{2}{3}\sqrt{3},\frac{\sqrt{5}}{\sqrt{3}})$. One can verify that

$$\mathrm{card}\{X(\alpha,\Lambda)\} = 50. \tag{7.1}$$

Suppose that there exists a packing lattice Λ of B^4 satisfying $\mathrm{card}\{X(\alpha,\Lambda)\} \ge 52$. We still denote the numbers of equivalent classes of Λ which contain exactly i pairs of vectors of $X(\alpha,\Lambda)$ by m_i.

If $m_4 \ne 0$, by Remark 2.1, we may assume

$$\mathbf{v}_1 = (2\sqrt{2},0,0,0),\quad \mathbf{v}_2 = (0,2\sqrt{2},0,0),\quad \mathbf{v}_3 = (0,0,2\sqrt{2},0),\quad \mathbf{v}_4 = (0,0,0,2\sqrt{2})$$

belong to $X(\alpha,\Lambda)$ and $\frac{1}{2}(\mathbf{v}_i \pm \mathbf{v}_j), i \ne j$ belong to $X(\alpha,\Lambda)$, without loss of generality. In this case, lattice Λ is generated by

$$\mathbf{a}_1 = (\sqrt{2},\sqrt{2},0,0),\ \mathbf{a}_2 = (\sqrt{2},-\sqrt{2},0,0),\ \mathbf{a}_3 = (\sqrt{2},0,\sqrt{2},0),\ \mathbf{a}_4 = (\sqrt{2},0,0,\sqrt{2}),$$

which means $\Lambda = \sqrt{2}D_4$. One can verify that

$$\mathrm{card}\big\{X(\alpha, \sqrt{2}D_4)\big\} = 24 + 24 = 48 < 52,$$

therefore $m_4 = 0$. Since $\sqrt{2}D_4$ is the unique densest packing lattice for B^4, up to rotation and reflection (see [21]), from now on we suppose

$$\det(\Lambda) > 8. \tag{7.2}$$

If $m_3 = 0$ and for every vector $\mathbf{v} \in X(\alpha, \Lambda)$ which length is 2 we have

$$\mathrm{card}\{A \in C(X(\alpha,\Lambda)) : \ \mathbf{v} \in A\} < \kappa^*(B^3)/2 = 6,$$

by restate (6.1), (6.2), (6.3) and (6.4) for $n = 4$, we obtain

$$\begin{cases} 4m_2 \leq 10m_1, \\ 2m_2 + m_1 \geq 26, \\ m_2 + m_1 \leq 15, \end{cases}$$

which admits no solution. Therefore, we have $m_3 \neq 0$ or there exist a vector $\mathbf{v} \in X(\alpha, \Lambda)$ which length is 2 satisfy

$$\mathrm{card}\{A \in C(X(\alpha,\Lambda)) : \ \mathbf{v} \in A\} = \kappa^*(B^3)/2 = 6.$$

If $m_3 \neq 0$, by Remark 2.1, we assume

$$\mathbf{v}_1 = (2, -2, 0, 0), \quad \mathbf{v}_2 = (2, 2, 0, 0), \quad \mathbf{v}_3 = (0, 0, 2\sqrt{2}, 0)$$

belong to $X(\alpha, \Lambda)$ and $\frac{1}{2}(\mathbf{v}_i \pm \mathbf{v}_j), \quad i \neq j$ belong to $X(\alpha, \Lambda)$, without loss of generality. Therefore, the basis of lattice Λ can be expanded by

$$\mathbf{a}_1 = (2, 0, 0, 0), \quad \mathbf{a}_2 = (0, 2, 0, 0), \quad \mathbf{a}_3 = (1, 1, \sqrt{2}, 0).$$

On the other hand, if there exist a vector $\mathbf{v} \in X(\alpha, \Lambda)$ which length is 2 satisfy

$$\mathrm{card}\{A \in C(X(\alpha,\Lambda)) : \ \mathbf{v} \in A\} = \kappa^*(B^3)/2 = 6,$$

then there exist a three-dimensional subspace H_0 satisfy

$$\mathrm{card}\{\mathbf{v} : \ \mathbf{v} \in H_0 \cap \Lambda, \ ||\mathbf{v}|| = 2\} = \kappa^*(B^3) = 12.$$

Therefore, we may suppose $H_0 = \{(v_1, v_2, v_3, v_4) : \ v_4 = 0\}$ and the three-dimensional lattice $H_0 \cap \Lambda$ is generated by

$$\mathbf{a}_1 = (2, 0, 0, 0), \quad \mathbf{a}_2 = (0, 2, 0, 0), \quad \mathbf{a}_3 = (1, 1, \sqrt{2}, 0),$$

without loss of generality.

As a conclusion of two cases above, we set a basis of lattice Λ by

$$\mathbf{a}_1 = (2, 0, 0, 0), \ \mathbf{a}_2 = (0, 2, 0, 0), \ \mathbf{a}_3 = (1, 1, \sqrt{2}, 0), \ \mathbf{a}_4 = (v_1, v_2, v_3, v_4)$$

and $v_1 \geq 0, v_2 \geq 0, v_3 \geq 0, v_4 \geq 0$ without loss of generality. Furthermore, by (7.2) we have $v_4 > \sqrt{2}$. Therefore, for a vector

$$\mathbf{v} = z_1\mathbf{a}_1 + z_2\mathbf{a}_2 + z_3\mathbf{a}_3 + z_4\mathbf{a}_4 \in X(\alpha, \Lambda),$$

we have $z_4 = 0$ or ± 1.

By Remark 4.2,

$$\mathrm{card}\{\mathbf{v}:\ \mathbf{v}=z_1\mathbf{a}_1+z_2\mathbf{a}_2+z_3\mathbf{a}_3\in X(\alpha,\Lambda)\}=18.$$

Since $X(\alpha,\Lambda)$ is centrally symmetric, we have

$$\mathrm{card}\{\mathbf{v}:\ \mathbf{v}=z_1\mathbf{a}_1+z_2\mathbf{a}_2+z_3\mathbf{a}_3+\mathbf{a}_4\in X(\alpha,\Lambda)\}\geq 17,$$

which means that there exist two of them is equivalent. Replace $\mathbf{a}_4$ by the mid-point of them, we may further assume

$$||\mathbf{a}_4||^2=v_1^2+v_2^2+v_3^2+v_4^2=4,$$

by Remark 2.1.

By routine computation, besides $z_1\mathbf{a}_1+z_2\mathbf{a}_2+z_3\mathbf{a}_3$, vector which belong to $X(\alpha,\Lambda)$ must be one of the following form:

$$\begin{aligned} &z_1\mathbf{a}_1+z_2\mathbf{a}_2\pm\mathbf{a}_4, && z_1\mathbf{a}_1+z_2\mathbf{a}_2\pm(\mathbf{a}_3+\mathbf{a}_4),\\ &z_1\mathbf{a}_1+z_2\mathbf{a}_2\pm(\mathbf{a}_3-\mathbf{a}_4), && z_1\mathbf{a}_1+z_2\mathbf{a}_2\pm(2\mathbf{a}_3-\mathbf{a}_4).\end{aligned}$$

To let $\mathrm{card}\{X(\alpha,\Lambda)\}\geq 52$, there exist one form above have at least ten vectors which belongs to $X(\alpha,\Lambda)$. Without loss of generality, we suppose

$$\mathrm{card}\{\mathbf{v}:\ \mathbf{v}=z_1\mathbf{a}_1+z_2\mathbf{a}_2\pm\mathbf{a}_4\in X(\alpha,\Lambda)\}\geq 10.$$

Combining with

$$\mathrm{card}\{\mathbf{v}:\ \mathbf{v}=z_1\mathbf{a}_1+z_2\mathbf{a}_2\in X(\alpha,\Lambda)\}=8,$$

by Lemma 4.4 and Remark 4.2, we may assume

$$v_1=0,\quad v_2=0,\quad v_3^2+v_4^2=4$$

or

$$v_1=0,\quad v_2=1,\quad v_3^2+v_4^2=3$$

without loss of generality.

For case $v_1=0,\ v_2=0,\ v_3^2+v_4^2=4$, by routine computation we have:

$$\begin{aligned} &\mathrm{card}\{\mathbf{v}:\ \mathbf{v}=z_1\mathbf{a}_1+z_2\mathbf{a}_2\pm\mathbf{a}_4\in X(\alpha,\Lambda)\}=10,\\ &\mathrm{card}\{\mathbf{v}:\ \mathbf{v}=z_1\mathbf{a}_1+z_2\mathbf{a}_2\pm(\mathbf{a}_3+\mathbf{a}_4)\in X(\alpha,\Lambda)\}=\begin{cases}8, & v_3=0,\\ 0, & v_3\neq 0,\end{cases}\\ &\mathrm{card}\{\mathbf{v}:\ \mathbf{v}=z_1\mathbf{a}_1+z_2\mathbf{a}_2\pm(\mathbf{a}_3-\mathbf{a}_4)\in X(\alpha,\Lambda)\}=8,\\ &\mathrm{card}\{\mathbf{v}:\ \mathbf{v}=z_1\mathbf{a}_1+z_2\mathbf{a}_2\pm(2\mathbf{a}_3-\mathbf{a}_4)\in X(\alpha,\Lambda)\}=\begin{cases}2, & v_3\geq 1/\sqrt{2},\\ 0, & v_3=0.\end{cases}\end{aligned}$$

Therefore, in this case we have $\mathrm{card}\{X(\alpha,\Lambda)\}\leq 44$.

For case $v_1=0,\ v_2=1,\ v_3^2+v_4^2=3$, since $||\mathbf{a}_3-\mathbf{a}_4||\geq 2$, we have $v_3\leq 1/\sqrt{2}$. By routine computation we have:

$$\begin{aligned} &\mathrm{card}\{\mathbf{v}:\ \mathbf{v}=z_1\mathbf{a}_1+z_2\mathbf{a}_2\pm\mathbf{a}_4\in X(\alpha,\Lambda)\}=12,\\ &\mathrm{card}\{\mathbf{v}:\ \mathbf{v}=z_1\mathbf{a}_1+z_2\mathbf{a}_2\pm(\mathbf{a}_3+\mathbf{a}_4)\in X(\alpha,\Lambda)\}=4,\end{aligned}$$

$$\mathrm{card}\{\mathbf{v}:\ \mathbf{v}=z_1\mathbf{a}_1+z_2\mathbf{a}_2\pm(\mathbf{a}_3-\mathbf{a}_4)\in X(\alpha,\Lambda)\}=\begin{cases}12, & v_3=1/\sqrt{2},\\ 4, & v_3<1/\sqrt{2},\end{cases}$$

$$\mathrm{card}\{\mathbf{v}:\ \mathbf{v}=z_1\mathbf{a}_1+z_2\mathbf{a}_2\pm(2\mathbf{a}_3-\mathbf{a}_4)\in X(\alpha,\Lambda)\}=\begin{cases}4, & v_3=1/\sqrt{2},\\ 0, & v_3<1/\sqrt{2}.\end{cases}$$

Therefore, in this case we have $\mathrm{card}\{X(\alpha,\Lambda)\}\le 50$.

As a conclusion of these two cases and (7.1), for $\alpha=2\sqrt{2}-2$ we have $\kappa^*_\alpha(B^4)=50$. Theorem 3 is proved. □

Remark 7.1 It is interesting to see that the $\sqrt{2}D_4$ lattice is not the optimal lattice in this case. Let Λ be the lattice generated by $\mathbf{a}_1=(2,0,0,0)$, $\mathbf{a}_2=(0,2,0,0)$, $\mathbf{a}_3=(1,0,\sqrt{3},0)$ and $\mathbf{a}_4=(0,1,0,\sqrt{3})$. It is easy to show that, when $\alpha=2\sqrt{2}-2$,

$$\mathrm{card}\{X(\alpha,\Lambda)\}=\mathrm{card}\big\{X(\alpha,\sqrt{2}D_4)\big\}=48.$$

8 A Link Between $\kappa^*_\alpha(B^n)$ and $\gamma^*(B^n)$

In 1964, Erdös and Rogers [3] studied the star number of the lattice covering for a convex body and proved the following result.

Theorem 8.1 Let C be an **o**-symmetric strictly convex body and Λ a covering lattice of C in $\mathbb{E}^n$. Then the star number of the covering $\{C+\mathbf{v}:\mathbf{v}\in\Lambda\}$ is at least $2^{n+1}-1$, where the star number is the numbers of the translates of C by lattice vectors, including C, which intersect the body C.

Let $\gamma^*(B^n)$ be the lattice packing-covering constant of B^n, namely

$$\gamma^*(B^n)=\min_{\Lambda}\{r:\ rB^n+\Lambda \text{ is a covering of } \mathbb{E}^n\},$$

where Λ is a lattice such that $B^n+\Lambda$ is a packing in $\mathbb{E}^n$. For more details about $\gamma^*(B^n)$, we refer to [22].

There exist a strong relation between $\kappa^*_\alpha(B^n)$ and $\gamma^*(B^n)$:

Theorem 8.2 For a given dimension n_0, suppose $\gamma^*(B^{n_0})=\sqrt{2}-\beta$ for a positive number β. Then for $\alpha\in\big[2\sqrt{2}-2\beta-2,\ 2\sqrt{2}-2\big)$ we have $\kappa^*_\alpha(B^{n_0})=2^{n_0+1}-2$. Which means that, if $\kappa^*_\alpha(B^{n_0})<2^{n_0+1}-2$ holds for $\alpha<2\sqrt{2}-2$, then we have $\gamma^*(B^{n_0})\ge\sqrt{2}$.

Proof We assume that $B^{n_0}+\Lambda$ is a lattice packing attaining $\gamma^*(B^{n_0})=\sqrt{2}-\beta$ for a positive β. For convenience, let

$$X=\big\{\mathbf{v}:\ 2\le||\mathbf{v}||\le 2\sqrt{2}-2\beta,\ \mathbf{v}\in\Lambda\big\}.$$

It is easy to see that the star number of the covering configuration $(\sqrt{2}-\beta)B^{n_0}+\Lambda$ is $\mathrm{card}X+1$. By Theorem 8.1 we have $\mathrm{card}X\ge 2^{n_0+1}-2$. Combining with Lemma 2.2, we get $\mathrm{card}X=2^{n_0+1}-2$.

Therefore, for $\alpha\in\big[2\sqrt{2}-2\beta-2,\ 2\sqrt{2}-2\big)$, we have $\kappa^*_\alpha(B^{n_0})=2^{n_0+1}-2$. Theorem 8.2 is proved. □

Remark 8.1 Notice that $\gamma^*(B^5)>\sqrt{2}$, see [22]. However, when $\alpha=2\sqrt{9/5}-2$, one can verify that

$$\mathrm{card}\big\{X(\alpha,\sqrt{24/5}A_5^*)\big\}=62.$$

Combining with Lemma 2.2, when $\alpha\in\big[2\sqrt{9/5}-2,\ 2\sqrt{2}-2\big)$ we have $\kappa^*_\alpha(B^5)=62$.

Corollary 8.1 In [15], Schürmann and Vallentin improved the former best known result (see [22]) $\gamma^*(B^6) \le \sqrt{2}$ to

$$\gamma^*(B^6) \le 2\sqrt{2\sqrt{798} - 56} = 1.411081242\cdots.$$

Therefore, by Theorem 8.2, for $\alpha \in \big[0.8222, 2\sqrt{2} - 2\big)$ we have $\kappa^*_\alpha(B^6) = 126$.

For a packing lattice Λ of B^n and an $\alpha < 2\sqrt{2} - 2$, the sufficient and necessary condition for

$$\text{card}\{X(\alpha, \Lambda)\} = 2^{n+1} - 2$$

is each equivalent class of Λ, except the class which contain $\mathbf{o}$, must contain a pair of vectors of $X(\alpha, \Lambda)$. It is reasonable to imagine that, this condition is hard to satisfy in high dimensions. If so, the following conjecture make sense.

Conjecture 8.1 There are infinity numbers of dimension n such that, when $\alpha < 2\sqrt{2} - 2$ we have $\kappa^*_\alpha(B^n) < 2^{n+1} - 2$. Especially, when $\alpha < 2\sqrt{2} - 2$ we have $\kappa^*_\alpha(B^8) < 510$ and $\kappa^*_\alpha(B^{24}) < 33554430$.

Remark 8.2 If Conjecture 8.1 is true, by Theorem 8.2, we have $\gamma^*(B^8) = \sqrt{2}$ and $\gamma^*(B^{24}) = \sqrt{2}$, which give an affirmative answer for Zong's Conjecture 3.1 in [22].

We write

$$E_7 = \Big\{\mathbf{v}:\ \mathbf{v} \in E_8;\ \sum v_i = 0\Big\}$$

and

$$E_6 = \Big\{\mathbf{v}:\ \mathbf{v} \in E_8;\ \sum v_i = v_7 + v_8 = 0\Big\}.$$

When $\alpha = 2\sqrt{2} - 2$, we have (see [2]) that

$$\text{card}\big\{X(\alpha, \sqrt{2}D_5)\big\} = 130,$$

$$\text{card}\big\{X(\alpha, \sqrt{2}E_6)\big\} = 342$$

and

$$\text{card}\big\{X(\alpha, \sqrt{2}E_7)\big\} = 882.$$

To end this article, we list some known results of $\kappa^*_\alpha(B^n)$ as the following Table 2.

Table 2

n	$\kappa^*_\alpha(B^n)$ for $\alpha < 2\sqrt{2} - 2$	$\kappa^*_\alpha(B^n)$ for $\alpha = 2\sqrt{2} - 2$
2	≤ 6 (can be attained)	$= 8$
3	≤ 14 (can be attained)	$= 20$
4	≤ 30 (can be attained)	$= 50$
5	≤ 62 (can be attained)	≥ 130
6	≤ 126 (can be attained)	≥ 342
7	≤ 254 (??)	≥ 882
8	≤ 510 (??)	$= 2400$
24	≤ 33554430 (??)	≥ 415003680

Conflict of Interest The authors declare no conflict of interest.

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