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The simultaneous lattice packing-covering constant of octahedra

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Abstract: This paper proves that the simultaneous lattice packing-covering constant of an octahedron is 7/6. In other words, 7/6 is the smallest positive number r such that for every octahedron O centered at the origin there is a lattice Λ such that $O + \Lambda$ is a packing in $\mathbb{E}^3$ and $rO + \Lambda$ is a covering of $\mathbb{E}^3$.

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1 Introduction

In 1950, C. A. Rogers [15] introduced and studied two simultaneous packing-covering constants $\gamma(C)$ and $\gamma^*(C)$ for an n-dimensional centrally symmetric convex body C centered at the origin of $\mathbb{E}^n$, namely

$$\gamma(C) = \min_X \{r : rC + X \text{ is a covering of } \mathbb{E}^n\}$$

where X is an arbitrary discrete point set such that $C + X$ is a packing in $\mathbb{E}^n$, and

$$\gamma^*(C) = \min_\Lambda \{r : rC + \Lambda \text{ is a covering of } \mathbb{E}^n\}$$

where Λ is a lattice such that $C + \Lambda$ is a packing in $\mathbb{E}^n$. By an inductive method, he proved that

$$\gamma(C) \le \gamma^*(C) \le 3$$

holds for all n-dimensional centrally symmetric convex bodies. In 1972, via mean value techniques developed by C. A. Rogers [16] and C. L. Siegel [17], G. L. Butler [4] proved that

$$\gamma^*(C) \le 2 + o(1)$$

holds for all n-dimensional centrally symmetric convex bodies.

In the 1970s, L. Fejes Tóth [6; 18] introduced and investigated two deep hole constants $\rho(C)$ and $\rho^*(C)$ for an n-dimensional centrally symmetric convex body C centered at the origin of $\mathbb{E}^n$, where $\rho(C)$ is the largest positive number r such that one can put a translate of rC into every translative packing $C + X$, and $\rho^*(C)$ is the largest positive number r^* such that one can put a translate of r^*C into every lattice packing $C + \Lambda$. Clearly, we have

$$\gamma(C) = \rho(C) + 1$$

and

$$\gamma^*(C) = \rho^*(C) + 1.$$

Let B^n denote the n-dimensional unit ball centered at the origin. Like the packing density problem and the covering density problem, to determine the values of $\gamma(B^n)$ and $\gamma^*(B^n)$ is important and interesting. The known exact results are listed in the following table:

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n	2	3	4	5
$\gamma^*(B^n)$	$2/\sqrt{3}$	$\sqrt{5/3}$	$\sqrt{2\sqrt{3}(\sqrt{3}-1)}$	$\sqrt{3/2+\sqrt{13}/6}$
$\gamma(B^n)$	$2/\sqrt{3}$	$\sqrt{5/3}$	??	??
Authors	Trivial	Böröczky [2]	Horváth [10]	Horváth [11]

In 1995, for the ball case, Rogers' reductive method was modified and his upper bound was improved by M. Henk [8] to

$$\gamma^*(B^n) \le \sqrt{21}/2 = 2.29128\cdots.$$

Clearly, this upper bound is not as good as Butler's upper bound. However, Rogers' approach has applications in lattice cryptography; see Micciancio [13]. On one hand, $\gamma^*(B^n)$ is a bridge connecting the shortest vector problem (SVP) and the closest vector problem (CVP), both fundamental in lattice cryptography. On the other hand, the reductive argument can lead to an algorithm.

In the plane, C. Zong [21; 23] proved that

$$\gamma(C) = \gamma^*(C) \le 2(2-\sqrt{2})$$

holds for all centrally symmetric convex domains and that the second equality holds if and only if C is an affinely regular octagon. It is remarkable and interesting that the maximum is not attained by circular discs! In $\mathbb{E}^3$, C. Zong [22] proved that

$$\gamma^*(C) \le 1.75$$

holds for all centrally symmetric convex bodies. It is interesting to compare with

$$\gamma(B^3) = \gamma^*(B^3) = \sqrt{5/3} = 1.29099\cdots.$$

Let O denote the regular octahedron with vertices $(1,0,0)$, $(0,1,0)$, $(0,0,1)$, $(-1,0,0)$, $(0,-1,0)$ and $(0,0,-1)$. In 1904, Minkowski [14] proved that the lattice Λ generated by $\mathbf{a}_1 = (-\frac{2}{3}, 1, \frac{1}{3})$, $\mathbf{a}_2 = (\frac{1}{3}, -\frac{2}{3}, 1)$, $\mathbf{a}_3 = (1, \frac{1}{3}, -\frac{2}{3})$ gives the optimal lattice packing density 18/19. In fact, it is easy to see that Minkowski's result shows $\gamma^*(O) \le 7/6$ and that all the uncovered spaces are regular tetrahedra, see Figure 1.

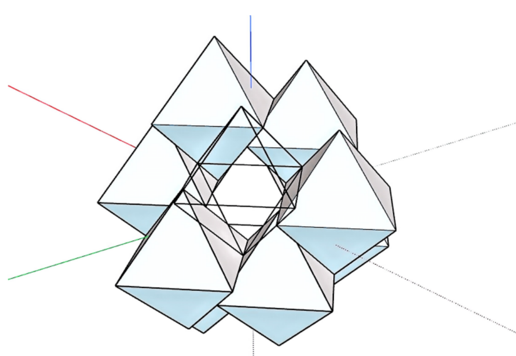

Figure 1: The optimal lattice packing configuration for O.

By studying these tetrahedral holes and all its variations, this article will prove the following theorem:

Theorem 1. *We have* $\gamma^*(O) = 7/6$.

In Section 4 we also investigate the simultaneous packing-covering constants of some other polytopes. Based on those examples, we propose the following problem:

Problem 1. Is it true that $\gamma(C) = \gamma^*(C) \le \sqrt{5/3}$ for every centrally symmetric convex body C in $\mathbb{E}^3$?

2 Technical lemmas

Let T be the regular tetrahedron with vertices (0,0,1), $(\frac{1}{3}, 0, \frac{2}{3})$, $(0, \frac{1}{3}, \frac{2}{3})$ and $(\frac{1}{3}, \frac{1}{3}, 1)$. Denote the distance function defined by O as $||\mathbf{x}||_1$ (also known as L_1 measure) and denote the distance between $\mathbf{x}$ and $\mathbf{y}$ in L_1-space as

$$||\mathbf{x}, \mathbf{y}||_1 = ||\mathbf{x} - \mathbf{y}||_1.$$

As usual, for two vector sets A and B, define the minus set and the Minkowski sum by

$$A \setminus B = \{\mathbf{x} : \mathbf{x} \in A \text{ and } \mathbf{x} \notin B\}$$

and

$$A + B = \{\mathbf{x} + \mathbf{y} : \mathbf{x} \in A \text{ and } \mathbf{y} \in B\},$$

respectively. We say a vector set A is positively homothetic to B with factor r, if $A = rB + \mathbf{x}$ with $r > 0$. We use $\operatorname{int} K$, $\operatorname{rint} K$, $\operatorname{cl} K$ and $\operatorname{conv} K$ to denote the interior of K, the relative interior of K, the closure of K and the convex hull of K as usual, and $\overline{\mathbf{xy}}$ denotes the segment with the vertices $\mathbf{x}$ and $\mathbf{y}$. We say that $\gamma(C, X) = r$, if $C + X$ is a packing and r is the minimum positive value such that $rC + X$ is a covering.

Let $\mathbf{e}_1 = (1, 0, 0)$, $\mathbf{e}_2 = (0, 1, 0)$, $\mathbf{e}_3 = (0, 0, 1)$ and denote the faces

$$\begin{gathered} \operatorname{conv}\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}, \quad \operatorname{conv}\{\mathbf{e}_1, -\mathbf{e}_2, \mathbf{e}_3\}, \quad \operatorname{conv}\{-\mathbf{e}_1, -\mathbf{e}_2, \mathbf{e}_3\}, \quad \operatorname{conv}\{-\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}, \\ \operatorname{conv}\{\mathbf{e}_1, \mathbf{e}_2, -\mathbf{e}_3\}, \quad \operatorname{conv}\{\mathbf{e}_1, -\mathbf{e}_2, -\mathbf{e}_3\}, \quad \operatorname{conv}\{-\mathbf{e}_1, -\mathbf{e}_2, -\mathbf{e}_3\}, \quad \operatorname{conv}\{-\mathbf{e}_1, \mathbf{e}_2, -\mathbf{e}_3\} \end{gathered}$$

of O by $F^{(1)}, F^{(2)}, F^{(3)}, F^{(4)}, F^{(1')}, F^{(2')}, F^{(3')}$ and $F^{(4')}$, respectively.

Observation. It is well known that, combining the regular tetrahedron with vertices $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$, $\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3$, the regular tetrahedron with vertices $-\mathbf{e}_1$, $-\mathbf{e}_2$, $-\mathbf{e}_3$, $-\mathbf{e}_1 - \mathbf{e}_2 - \mathbf{e}_3$ and O, we obtain a parallelepiped. Therefore, the sum of the dihedral angles of a regular octahedron and a regular tetrahedron is π.

On the other hand, since a regular octahedron is defined by four pairs of parallel faces, we can observe the following: if the intersections of each two of four regular octahedra O, $O + \mathbf{x}_1$, $O + \mathbf{x}_2$, $O + \mathbf{x}_3$ are two-dimensional, then the hole surrounded by them is a regular tetrahedron.

It is natural to prove the following conclusion:

Lemma 1. *If $F^{(1)} \supset O \cap (T + \mathbf{a}_1) \neq \emptyset$, then the center of gravity $\mathbf{g}$ of $T + \mathbf{a}_1$ satisfies $||\mathbf{g}||_1 \geq 7/6$.*

Proof. Define

$$X = \{\mathbf{g} : \mathbf{g} \text{ is the center of gravity of } T + \mathbf{a}_1, \text{ which satisfies } F^{(1)} \supset O \cap (T + \mathbf{a}_1) \neq \emptyset\}.$$

It is easy to see that

$$X \subset \{(x, y, z) : x + y + z = m\}, \text{ for a constant } m.$$

The center of gravity of T is $(\frac{1}{6}, \frac{1}{6}, \frac{5}{6})$, therefore $m = \frac{7}{6}$. Since $||\mathbf{o}, \mathbf{x}||_1 \geq \frac{7}{6}$ for all $\mathbf{x} \in \{(x, y, z) : x + y + z = \frac{7}{6}\}$, Lemma 1 is proved. □

Corollary 1. *Let $O + X$ be a packing such that there is a regular tetrahedron T_1 satisfying the following two conditions:*

(1) $\operatorname{int} T_1 \cap (O + X) = \emptyset$,

(2) *T_1 is positively homothetic to T with dilation factor $r \geq 1$.*

Then we have $\gamma(O, X) \geq 7/6$.

Proof. Denote the center of gravity of T_1 by $\mathbf{g}_1$. By Lemma 1, we have

$$||\mathbf{g}_1, \mathbf{x}||_1 \geq 7/6 \text{ for all } \mathbf{x} \in X.$$

Therefore $\mathbf{g}_1 \notin \operatorname{int}(\frac{7}{6}O) + X$, which means that $\gamma(O, X) \geq 7/6$. □

Suppose that $\mathbf{a}_0 = (x_0, y_0, z_0) \in \mathrm{conv}\{(0,1,1), (0, \frac{4}{3}, \frac{2}{3}), (\frac{1}{3}, 1, \frac{2}{3}), (\frac{1}{3}, \frac{4}{3}, 1)\}$. In other words, suppose that

$$\begin{aligned} x_0 + y_0 + z_0 &\geq 2, \\ -x_0 + y_0 + z_0 &\leq 2, \\ x_0 - y_0 + z_0 &\leq 0, \\ x_0 + y_0 - z_0 &\leq \frac{2}{3}. \end{aligned} \tag{1}$$

For two convex bodies C_1 and C_2, we say that C_1 is obstructed by C_2 if $C_2 \cap \mathrm{int}(C_1) \neq \emptyset$. We will prove that, for a packing $O + X$ containing O and $O + \mathbf{a}_0$, we have $\gamma(O, X) \geq 7/6$. To this end, we show that no matter how to obstruct the unpacked place by the translative of O, there must exist a regular tetrahedron positively homothetic to T with factor $r \geq 1$ which is not obstructed by $O + X$.

Define $T' = \mathrm{conv}\{\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3, \mathbf{y}_4\}$ where

$$\begin{aligned} \mathbf{y}_1 &= \Big(1 - \frac{-x_0 + y_0 + z_0}{2}, y_0 - 1, 1 - \frac{x_0 + y_0 - z_0}{2}\Big), \\ \mathbf{y}_2 &= \Big(1 - \frac{-x_0 + y_0 + z_0}{2}, \frac{-x_0 + y_0 + z_0}{2}, 0\Big), \\ \mathbf{y}_3 &= (2 - y_0, y_0 - 1, 0), \\ \mathbf{y}_4 &= \Big(2 - y_0, \frac{-x_0 + y_0 + z_0}{2}, 1 - \frac{x_0 + y_0 - z_0}{2}\Big). \end{aligned}$$

In other words, T' is a regular tetrahedron positively homothetic to T with factor r_0, formed by all the points (x, y, z) which satisfy

$$\begin{aligned} x + y + z &\geq 1, \\ -x + y + z &\leq -x_0 + y_0 + z_0 - 1, \\ x - y + z &\leq 3 - 2y_0, \\ x + y - z &\leq 1. \end{aligned} \tag{2}$$

Since $1 - \frac{x_0 + y_0 - z_0}{2} \geq \frac{2}{3}$ by (1), we have

$$r_0 \geq 2. \tag{3}$$

It is easy to see that T' contacts both O and $O + \mathbf{a}_0$ at its boundary.

Lemma 2. *If a regular octahedron $O + \mathbf{a}_1$ satisfies $O + \mathbf{a}_1 \cap \mathrm{int}(O \cup O + \mathbf{a}_0) = \emptyset$, then the following two statements are equivalent:*

(1) $O + \mathbf{a}_1 \cap \mathrm{int} T' \neq \emptyset$.

(2) $\mathbf{a}_1 \in T'' + (1, 0, 0)$ *where* $T'' = T' \setminus (\mathrm{conv}\{\mathbf{y}_1, \mathbf{y}_3, \mathbf{y}_4\} \cup \mathrm{conv}\{\mathbf{y}_2, \mathbf{y}_3, \mathbf{y}_4\})$.

Proof. Define $Y = \{\mathbf{a}_1 : (O + \mathbf{a}_1) \cap \mathrm{int} T' \neq \emptyset\}$. By routine computation we obtain

$$\begin{aligned} Y = \mathrm{int}\Big(&\mathrm{conv}\{\mathbf{y}_1 + (0,0,1), \mathbf{y}_4 + (0,0,1), \mathbf{y}_1 + (0,-1,0), \mathbf{y}_1 + (-1,0,0), \mathbf{y}_4 + (1,0,0), \mathbf{y}_4 + (0,1,0), \\ &\mathbf{y}_3 + (0,1,0), \mathbf{y}_3 + (-1,0,0), \mathbf{y}_2 + (1,0,0), \mathbf{y}_2 + (0,-1,0), \mathbf{y}_2 + (0,0,-1), \mathbf{y}_3 + (0,0,-1)\}\Big). \end{aligned}$$

With the definition

$$Y' = Y \setminus \{\mathbf{a}_1 : O + \mathbf{a}_1 \cap \mathrm{int}(O \cup O + \mathbf{a}_0) \neq \emptyset\},$$

Lemma 2 holds if and only if $Y' = T'' + (1, 0, 0)$.

On the one hand, we have

$$Y' = Y \setminus (\mathrm{int}(2O) \cup \mathrm{int}(2O + \mathbf{a}_0)).$$

Since $\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3 \in O$ and $2O$ is convex, the convex hull

$$\begin{aligned} &\mathrm{conv}\{\mathbf{y}_1 + (0,0,1), \mathbf{y}_1 + (-1,0,0), \mathbf{y}_1 + (0,-1,0), \mathbf{y}_2 + (-1,0,0), \mathbf{y}_2 + (0,1,0), \\ &\mathbf{y}_2 + (0,0,-1), \mathbf{y}_3 + (1,0,0), \mathbf{y}_3 + (0,-1,0), \mathbf{y}_3 + (0,0,-1)\} \end{aligned}$$

is a subset of $2O$; denote this convex hull by Y_1. Also, by routine computation, we have

$$\mathbf{y}_1 + (1,0,0), \mathbf{y}_1 + (0,0,1), \mathbf{y}_2 + (1,0,0), \mathbf{y}_2 + (0,1,0), \mathbf{y}_4 + (1,0,0), \mathbf{y}_4 + (0,1,0), \mathbf{y}_4 + (0,0,1) \in 2O + \mathbf{a}_0.$$

For example,

$$\begin{aligned} \|\mathbf{y}_4 + (0,0,1) - \mathbf{a}_0\|_1 &= |2 - x_0 - y_0| + \left|\frac{-x_0 - y_0 + z_0}{2}\right| + \left|2 - \frac{x_0 + y_0 + z_0}{2}\right| \\ &= 2 - x_0 - y_0 - \frac{-x_0 - y_0 + z_0}{2} + 2 - \frac{x_0 + y_0 + z_0}{2} = 4 - x_0 - y_0 - z_0 \le 2, \end{aligned}$$

by (1). Therefore, the convex hull

$$\operatorname{conv}\{\mathbf{y}_1 + (1,0,0), \mathbf{y}_1 + (0,0,1), \mathbf{y}_2 + (1,0,0), \mathbf{y}_2 + (0,1,0), \mathbf{y}_4 + (1,0,0), \mathbf{y}_4 + (0,1,0), \mathbf{y}_4 + (0,0,1)\}$$

is a subset of $2O + \mathbf{a}_0$; denote this convex hull by Y_2. It is easy to see that

$$Y \subset \operatorname{int} Y_1 \cup \operatorname{int} Y_2 \cup (T'' + (1,0,0)).$$

Therefore, we have

$$Y' \subset T'' + (1,0,0). \tag{4}$$

On the other hand, for every vector $\mathbf{w} = (w_1, w_2, w_3) \in T''$, by (1) and (2) we have

$$w_1, w_2, w_3 \ge 0, \quad w_1 + w_2 + w_3 \ge 1.$$

Thus, we have

$$\|\mathbf{w} + (1,0,0), \mathbf{o}\|_1 = w_1 + 1 + w_2 + w_3 \ge 2,$$

which means

$$O + \mathbf{w} + (1,0,0) \cap \operatorname{int} O = \emptyset.$$

Since $\mathbf{a}_0 = (x_0, y_0, z_0)$ is on the plane $-x + y + z = -x_0 + y_0 + z_0$, and $\mathbf{w} + (1,0,0)$ is in the half space $-x + y + z \le -x_0 + y_0 + z_0 - 2$, by (2), we have

$$O + \mathbf{w} + (1,0,0) \cap \operatorname{int}(O + \mathbf{a}_0) = \emptyset.$$

We argue as follows:

Case 1. $\mathbf{w} \in \operatorname{int} T''$. Combining with $\mathbf{w} \in \operatorname{int} T' \cap (O + \mathbf{w} + (1,0,0)) \ne \emptyset$, by the definition of Y' we obviously have

$$\mathbf{w} + (1,0,0) \in Y'.$$

Case 2. $\mathbf{w} \in (\operatorname{conv}\{\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_4\} \cup \operatorname{conv}\{\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3\}) \setminus (\overline{\mathbf{y}_1\mathbf{y}_3} \cup \overline{\mathbf{y}_1\mathbf{y}_4} \cup \overline{\mathbf{y}_2\mathbf{y}_3} \cup \overline{\mathbf{y}_2\mathbf{y}_4})$.
Obviously the set $\operatorname{conv}\{\mathbf{w}, \mathbf{w} + (1,-1,0), \mathbf{w} + (2,0,0), \mathbf{w} + (1,1,0)\}$, a cross section of $O + \mathbf{w} + (1,0,0)$ intersects $\operatorname{int} T'$, which means $\mathbf{w} + (1,0,0) \in Y'$.

Therefore $Y' \supset T'' + (1,0,0)$. In view of (4), we have $Y' = T'' + (1,0,0)$, and Lemma 2 is proved. □

Lemma 3. *Suppose that a regular octahedron $O + \mathbf{a}_1$ satisfies $O + \mathbf{a}_1 \cap \operatorname{int}(O \cup O + \mathbf{a}_0) = \emptyset$ and $O + \mathbf{a}_1 \cap \operatorname{int} T' \ne \emptyset$. Then for an arbitrary regular octahedron $O + \mathbf{a}_2$, the following two conditions cannot both hold:*

(a): $O + \mathbf{a}_2 \cap \operatorname{int}(O \cup (O + \mathbf{a}_0) \cup (O + \mathbf{a}_1)) = \emptyset$,

(b): $O + \mathbf{a}_2 \cap \operatorname{int} T' \ne \emptyset$.

Proof. By Lemma 2, we have $\mathbf{a}_1 \in T'' + (1,0,0)$. If (a) and (b) hold simultaneously, then $\mathbf{a}_2 \in T'' + (1,0,0)$ by Lemma 2. But for arbitrary points $\mathbf{x}, \mathbf{y} \in T''$ we have $\|\mathbf{x}, \mathbf{y}\|_1 < 2$ by (1) and (2). Therefore $O + \mathbf{a}_2 \cap \operatorname{int}(O + \mathbf{a}_1) \ne \emptyset$, which contradicts (a). Lemma 3 is proved. □

Corollary 2. *For a packing $O + X$ containing O and $O + \mathbf{a}_0$ we have $\gamma(O, X) \ge 7/6$.*

Proof. If $\operatorname{int} T' \cap (O + X) = \emptyset$, since T' is positively homothetic to T with factor $r_0 \ge 2$ by (3), the condition of Corollary 1 is satisfied.

Otherwise, if there exists $\mathbf{a}_1 \in X$ with $\operatorname{int} T' \cap (O + \mathbf{a}_1) \ne \emptyset$, then by Lemma 3, $O + \mathbf{a}_1$ is the only regular octahedron in $O + X$ which intersects $\operatorname{int} T'$. In this case, there exist a regular tetrahedron $T''' \subset T' \setminus (O + \mathbf{a}_1)$ is positively homothetic to T with factor $r_0/2 \ge 1$ and $\operatorname{int} T'' \cap (O + X) = \emptyset$, which satisfies the condition of Corollary 1. □

3 Proof of the theorem

Define

$$T_1^{(1)} = T, \quad T_2^{(1)} = T + \Big(0, \frac{1}{3}, -\frac{1}{3}\Big), \quad T_3^{(1)} = T + \Big(\frac{1}{3}, 0, -\frac{1}{3}\Big),$$
$$T_4^{(1)} = T + \Big(0, \frac{2}{3}, -\frac{2}{3}\Big), \quad T_5^{(1)} = T + \Big(\frac{1}{3}, \frac{1}{3}, -\frac{2}{3}\Big), \quad T_6^{(1)} = T + \Big(\frac{2}{3}, 0, -\frac{2}{3}\Big)$$

It is easy to see that $T_i^{(1)} \cap O \subset F^{(1)}$ for $i \in \{1, \ldots, 6\}$. Rotate $T_i^{(1)}$ anticlockwise around the z-axis by $\pi/2$ degree, π degree, $3\pi/2$ degree, and denote the results by $T_i^{(2)}, T_i^{(3)}, T_i^{(4)}$, respectively. Define

$$T_i^{(1')} = -T_i^{(3)}, \quad T_i^{(2')} = -T_i^{(4)}, \quad T_i^{(3')} = -T_i^{(1)}, \quad T_i^{(4')} = -T_i^{(2)}.$$

Similarly, we have $T_i^{(k)} \cap O \subset F^{(k)}$ for $i \in \{1, \ldots, 6\}$ and $k \in \{1, 2, 3, 4, 1', 2', 3', 4'\}$.

To generalize, we define $T_i^{(k+1)}$ by rotating $T_i^{(k)}$ anticlockwise around the z-axis by $\pi/2$ degree. By the rotation, if $k_1 \equiv k_2 \bmod 4$ then $T_i^{(k_1)} = T_i^{(k_2)}$. In the centrally symmetric condition, we suppose that $1', 2', 3', 4'$ is equivalent to $3, 4, 1, 2$, respectively.

Now we consider whether or not $T_i^{(k)}$ can be obstructed by a packing $O+X$, for all i, k. To this end, we define

$$P_i^{(k)} = \{\mathbf{x} : O + \mathbf{x} \cap \operatorname{int} T_i^{(k)} \neq \emptyset \text{ and } O + \mathbf{x} \cap \operatorname{int} O = \emptyset\} \quad \text{for all } i, k.$$

In fact, $P_i^{(k)} = \operatorname{int}(T_i^{(k)} + O) \setminus \operatorname{int}(2O)$ for all i, k. By routine computation, we have

$$\begin{aligned}
P_1^{(1)} &= \operatorname{int}\Big(\operatorname{conv}\Big\{(0,0,2), \Big(0, \frac{4}{3}, \frac{2}{3}\Big), \Big(\frac{4}{3}, 0, \frac{2}{3}\Big), \Big(\frac{1}{3}, \frac{1}{3}, 2\Big), \Big(\frac{1}{3}, \frac{4}{3}, 1\Big), \Big(\frac{4}{3}, \frac{1}{3}, 1\Big)\Big\}\Big) \\
&\quad \cup \operatorname{int}\Big(\operatorname{conv}\Big\{(0,0,2), \Big(0, \frac{4}{3}, \frac{2}{3}\Big), \Big(\frac{4}{3}, 0, \frac{2}{3}\Big)\Big\}\Big), \\
P_2^{(1)} &= P_1^{(1)} + \Big(0, \frac{1}{3}, -\frac{1}{3}\Big), \quad P_3^{(1)} = P_1^{(1)} + \Big(\frac{1}{3}, 0, -\frac{1}{3}\Big), \\
P_4^{(1)} &= P_1^{(1)} + \Big(0, \frac{2}{3}, -\frac{2}{3}\Big), \quad P_5^{(1)} = P_1^{(1)} + \Big(\frac{1}{3}, \frac{1}{3}, -\frac{2}{3}\Big), \quad P_6^{(1)} = P_1^{(1)} + \Big(\frac{2}{3}, 0, -\frac{2}{3}\Big).
\end{aligned}$$

Define $M^{(k)} = P_1^{(k)} \cup P_2^{(k)} \cup P_3^{(k)} \cup P_4^{(k)} \cup P_5^{(k)} \cup P_6^{(k)}$ for all k. It is easy to see that $M^{(k_1)} \cap M^{(k_2)} = \emptyset$ for $k_1 \neq k_2$.

We dissect $M^{(k)}$ into nineteen pieces as follows:

$$\begin{aligned}
Q_1^{(k)} &= \{\mathbf{x} : \mathbf{x} \in P_1^{(k)}, \mathbf{x} \notin (P_2^{(k)} \cup P_3^{(k)} \cup P_4^{(k)} \cup P_5^{(k)} \cup P_6^{(k)})\}, \\
Q_2^{(k)} &= \{\mathbf{x} : \mathbf{x} \in (P_1^{(k)} \cap P_2^{(k)}), \mathbf{x} \notin (P_3^{(k)} \cup P_4^{(k)} \cup P_5^{(k)} \cup P_6^{(k)})\}, \\
Q_3^{(k)} &= \{\mathbf{x} : \mathbf{x} \in (P_1^{(k)} \cap P_3^{(k)}), \mathbf{x} \notin (P_2^{(k)} \cup P_4^{(k)} \cup P_5^{(k)} \cup P_6^{(k)})\}, \\
Q_4^{(k)} &= \{\mathbf{x} : \mathbf{x} \in (P_1^{(k)} \cap P_2^{(k)} \cap P_3^{(k)} \cap P_5^{(k)}), \mathbf{x} \notin (P_4^{(k)} \cup P_6^{(k)})\}, \\
Q_5^{(k)} &= \{\mathbf{x} : \mathbf{x} \in (P_1^{(k)} \cap P_2^{(k)} \cap P_4^{(k)}), \mathbf{x} \notin (P_3^{(k)} \cup P_5^{(k)} \cup P_6^{(k)})\}, \\
Q_6^{(k)} &= \{\mathbf{x} : \mathbf{x} \in (P_1^{(k)} \cap P_2^{(k)} \cap P_3^{(k)} \cap P_4^{(k)} \cap P_5^{(k)}), \mathbf{x} \notin P_6^{(k)}\}, \\
Q_7^{(k)} &= \{\mathbf{x} : \mathbf{x} \in (P_1^{(k)} \cap P_3^{(k)} \cap P_6^{(k)}), \mathbf{x} \notin (P_2^{(k)} \cup P_4^{(k)} \cup P_5^{(k)})\}, \\
Q_8^{(k)} &= \{\mathbf{x} : \mathbf{x} \in (P_1^{(k)} \cap P_2^{(k)} \cap P_3^{(k)} \cap P_5^{(k)} \cap P_6^{(k)}), \mathbf{x} \notin P_4^{(k)}\}, \\
Q_9^{(k)} &= \{\mathbf{x} : \mathbf{x} \in P_4^{(k)}, \mathbf{x} \notin (P_1^{(k)} \cup P_2^{(k)} \cup P_3^{(k)} \cup P_5^{(k)} \cup P_6^{(k)})\}, \\
Q_{10}^{(k)} &= \{\mathbf{x} : \mathbf{x} \in (P_2^{(k)} \cap P_4^{(k)}), \mathbf{x} \notin (P_1^{(k)} \cup P_3^{(k)} \cup P_5^{(k)} \cup P_6^{(k)})\}, \\
Q_{11}^{(k)} &= \{\mathbf{x} : \mathbf{x} \in (P_4^{(k)} \cap P_5^{(k)}), \mathbf{x} \notin (P_1^{(k)} \cup P_2^{(k)} \cup P_3^{(k)} \cup P_6^{(k)})\}, \\
Q_{12}^{(k)} &= \{\mathbf{x} : \mathbf{x} \in (P_2^{(k)} \cap P_3^{(k)} \cap P_4^{(k)} \cap P_5^{(k)}), \mathbf{x} \notin (P_1^{(k)} \cup P_6^{(k)})\}, \\
Q_{13}^{(k)} &= \{\mathbf{x} : \mathbf{x} \in (P_4^{(k)} \cap P_5^{(k)} \cap P_6^{(k)}), \mathbf{x} \notin (P_1^{(k)} \cup P_2^{(k)} \cup P_3^{(k)})\},
\end{aligned}$$

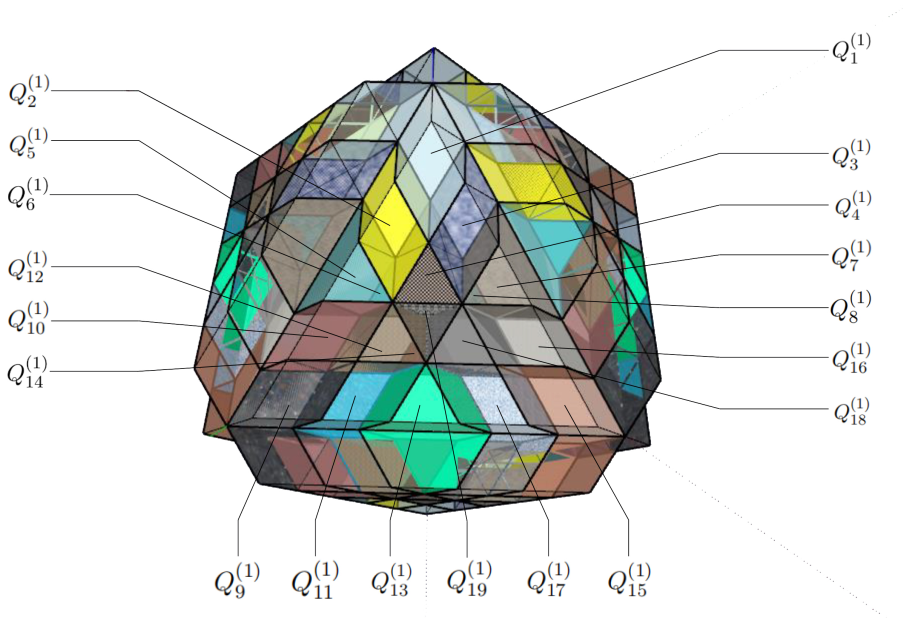

Figure 2: Dissect $M^{(1)}$ into $Q_i^{(1)}$.

$$
\begin{aligned}
Q_{14}^{(k)} &= \{\mathbf{x} : \mathbf{x} \in (P_2^{(k)} \cap P_3^{(k)} \cap P_4^{(k)} \cap P_5^{(k)} \cap P_6^{(k)}), \mathbf{x} \notin P_1^{(k)}\},\\
Q_{15}^{(k)} &= \{\mathbf{x} : \mathbf{x} \in P_6^{(k)}, \mathbf{x} \notin (P_1^{(k)} \cup P_2^{(k)} \cup P_3^{(k)} \cup P_4^{(k)} \cup P_5^{(k)})\},\\
Q_{16}^{(k)} &= \{\mathbf{x} : \mathbf{x} \in (P_3^{(k)} \cap P_6^{(k)}), \mathbf{x} \notin (P_1^{(k)} \cup P_2^{(k)} \cup P_4^{(k)} \cup P_5^{(k)})\},\\
Q_{17}^{(k)} &= \{\mathbf{x} : \mathbf{x} \in (P_5^{(k)} \cap P_6^{(k)}), \mathbf{x} \notin (P_1^{(k)} \cup P_2^{(k)} \cup P_3^{(k)} \cup P_4^{(k)})\},\\
Q_{18}^{(k)} &= \{\mathbf{x} : \mathbf{x} \in (P_2^{(k)} \cap P_3^{(k)} \cap P_5^{(k)} \cap P_6^{(k)}), \mathbf{x} \notin (P_1^{(k)} \cup P_4^{(k)})\},\\
Q_{19}^{(k)} &= \{\mathbf{x} : \mathbf{x} \in (P_2^{(k)} \cap P_3^{(k)} \cap P_5^{(k)}), \mathbf{x} \notin (P_1^{(k)} \cup P_4^{(k)} \cup P_6^{(k)})\},
\end{aligned}
$$

for all k. It is easy to see that $Q_{i_1}^{(k)} \cap Q_{i_2}^{(k)} = \emptyset$ for $i_1 \neq i_2$, see Figure 2. To show this decomposition more clearly, we give the following Figure 3.

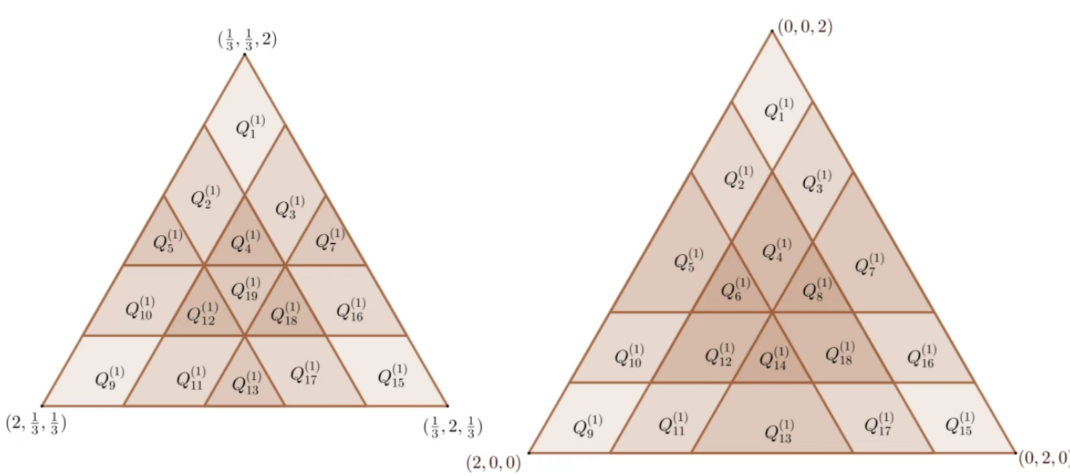

Figure 3: The intersection of $M^{(1)}$ with the planes $\{x + y + z = 8/3\}$ (left) and $\{x + y + z = 2\}$ (right).

We have

$$
\begin{aligned}
\mathrm{cl}\, Q_1^{(1)} &= \mathrm{conv}\Big\{(0,0,2), \Big(0,\frac{1}{3},\frac{5}{3}\Big), \Big(\frac{1}{3},\frac{1}{3},\frac{4}{3}\Big), \Big(\frac{1}{3},0,\frac{5}{3}\Big), \Big(\frac{1}{3},\frac{1}{3},2\Big), \Big(\frac{1}{3},\frac{2}{3},\frac{5}{3}\Big), \Big(\frac{2}{3},\frac{2}{3},\frac{4}{3}\Big), \Big(\frac{2}{3},\frac{1}{3},\frac{5}{3}\Big)\Big\},\\
\mathrm{cl}\, Q_2^{(1)} &= \mathrm{cl}\, Q_1^{(1)} + \Big(0,\frac{1}{3},-\frac{1}{3}\Big), \quad \mathrm{cl}\, Q_3^{(1)} = \mathrm{cl}\, Q_1^{(1)} + \Big(\frac{1}{3},0,-\frac{1}{3}\Big),\\
\mathrm{cl}\, Q_4^{(1)} &= \mathrm{conv}\Big\{\Big(\frac{1}{3},\frac{1}{3},\frac{4}{3}\Big), \Big(\frac{1}{3},\frac{2}{3},1\Big), \Big(\frac{2}{3},\frac{1}{3},1\Big), \Big(\frac{2}{3},\frac{2}{3},\frac{2}{3}\Big), \Big(\frac{2}{3},\frac{2}{3},\frac{4}{3}\Big), \Big(\frac{2}{3},1,1\Big), \Big(1,\frac{2}{3},1\Big)\Big\},\\
\mathrm{cl}\, Q_5^{(1)} &= \mathrm{conv}\Big\{\Big(0,\frac{2}{3},\frac{4}{3}\Big), \Big(\frac{1}{3},1,\frac{4}{3}\Big), \Big(\frac{1}{3},\frac{2}{3},1\Big), \Big(\frac{2}{3},1,1\Big), \Big(0,\frac{4}{3},\frac{2}{3}\Big), \Big(\frac{1}{3},\frac{4}{3},1\Big), \Big(\frac{1}{3},1,\frac{2}{3}\Big)\Big\},\\
\mathrm{cl}\, Q_6^{(1)} &= \mathrm{conv}\Big\{\Big(\frac{1}{3},\frac{2}{3},1\Big), \Big(\frac{2}{3},1,1\Big), \Big(\frac{1}{3},1,\frac{2}{3}\Big), \Big(\frac{2}{3},\frac{2}{3},\frac{2}{3}\Big)\Big\},\\
\mathrm{cl}\, Q_7^{(1)} &= \mathrm{conv}\Big\{\Big(\frac{2}{3},0,\frac{4}{3}\Big), \Big(1,\frac{1}{3},\frac{4}{3}\Big), \Big(\frac{2}{3},\frac{1}{3},1\Big), \Big(1,\frac{2}{3},1\Big), \Big(\frac{4}{3},\frac{1}{3},1\Big), \Big(1,\frac{1}{3},\frac{2}{3}\Big), \Big(\frac{4}{3},0,\frac{2}{3}\Big)\Big\},\\
\mathrm{cl}\, Q_8^{(1)} &= \mathrm{conv}\Big\{(\frac{2}{3},\frac{1}{3},1\Big), \Big(1,\frac{2}{3},1\Big), \Big(1,\frac{1}{3},\frac{2}{3}\Big), \Big(\frac{2}{3},\frac{2}{3},\frac{2}{3}\Big)\Big\},\\
\mathrm{cl}\, Q_9^{(1)} &= \mathrm{conv}\Big\{(0,2,0), \Big(\frac{1}{3},\frac{5}{3},0\Big), \Big(\frac{1}{3},2,\frac{1}{3}\Big), \Big(\frac{2}{3},\frac{5}{3},\frac{1}{3}\Big), \Big(0,\frac{5}{3},\frac{1}{3}\Big), \Big(\frac{1}{3},\frac{4}{3},\frac{1}{3}\Big), \Big(\frac{1}{3},\frac{5}{3},\frac{2}{3}\Big), \Big(\frac{2}{3},\frac{4}{3},\frac{2}{3}\Big)\Big\},\\
\mathrm{cl}\, Q_{10}^{(1)} &= \mathrm{cl}\, Q_9^{(1)} + \Big(0,-\frac{1}{3},\frac{1}{3}\Big), \quad \mathrm{cl}\, Q_{11}^{(1)} = \mathrm{cl}\, Q_9^{(1)} + \Big(\frac{1}{3},-\frac{1}{3},0\Big),\\
\mathrm{cl}\, Q_{12}^{(1)} &= \mathrm{conv}\Big\{\Big(\frac{1}{3},1,\frac{2}{3}\Big), \Big(\frac{2}{3},\frac{2}{3},\frac{2}{3}\Big), \Big(\frac{1}{3},\frac{4}{3},\frac{1}{3}\Big), \Big(\frac{2}{3},1,\frac{1}{3}\Big), \Big(\frac{2}{3},1,1\Big), \Big(\frac{2}{3},\frac{4}{3},\frac{2}{3}\Big), \Big(1,1,\frac{2}{3}\Big)\Big\},\\
\mathrm{cl}\, Q_{13}^{(1)} &= \mathrm{conv}\Big\{\Big(\frac{2}{3},1,\frac{1}{3}\Big), \Big(1,\frac{2}{3},\frac{1}{3}\Big), \Big(\frac{2}{3},\frac{4}{3},0\Big), \Big(\frac{4}{3},\frac{2}{3},0\Big), \Big(1,1,\frac{2}{3}\Big), \Big(1,\frac{4}{3},\frac{1}{3}\Big), \Big(\frac{4}{3},1,\frac{1}{3}\Big)\Big\},\\
\mathrm{cl}\, Q_{14}^{(1)} &= \mathrm{conv}\Big\{\Big(\frac{2}{3},\frac{2}{3},\frac{2}{3}\Big), \Big(\frac{2}{3},1,\frac{1}{3}\Big), \Big(1,\frac{2}{3},\frac{1}{3}\Big), \Big(1,1,\frac{2}{3}\Big)\Big\},\\
\mathrm{cl}\, Q_{15}^{(1)} &= \mathrm{conv}\Big\{\Big(\frac{4}{3},\frac{1}{3},\frac{1}{3}\Big), \Big(\frac{5}{3},0,\frac{1}{3}\Big), \Big(\frac{5}{3},\frac{1}{3},0\Big), (2,0,0), \Big(\frac{4}{3},\frac{2}{3},\frac{2}{3}\Big), \Big(\frac{5}{3},\frac{1}{3},\frac{2}{3}\Big), \Big(\frac{5}{3},\frac{2}{3},\frac{1}{3}\Big), \Big(2,\frac{1}{3},\frac{1}{3}\Big)\Big\},\\
\mathrm{cl}\, Q_{16}^{(1)} &= \mathrm{cl}\, Q_{15}^{(1)} + \Big(-\frac{1}{3},0,\frac{1}{3}\Big), \quad \mathrm{cl}\, Q_{17}^{(1)} = \mathrm{cl}\, Q_{15}^{(1)} + \Big(-\frac{1}{3},\frac{1}{3},0\Big),\\
\mathrm{cl}\, Q_{18}^{(1)} &= \mathrm{conv}\Big\{\Big(\frac{2}{3},\frac{2}{3},\frac{2}{3}\Big), \Big(1,\frac{1}{3},\frac{2}{3}\Big), \Big(1,\frac{2}{3},\frac{1}{3}\Big), \Big(\frac{4}{3},\frac{1}{3},\frac{1}{3}\Big), \Big(1,\frac{2}{3},1\Big), \Big(1,1,\frac{2}{3}\Big), \Big(\frac{4}{3},\frac{2}{3},\frac{2}{3}\Big)\Big\},\\
\mathrm{cl}\, Q_{19}^{(1)} &= \mathrm{conv}\Big\{\Big(\frac{2}{3},\frac{2}{3},\frac{2}{3}\Big), \Big(\frac{2}{3},1,1\Big), \Big(1,\frac{2}{3},1\Big), \Big(1,1,\frac{2}{3}\Big)\Big\}.
\end{aligned}
$$

Theorem 1. *We have* $\gamma^*(O) = 7/6$.

Proof. Suppose that $O + \Lambda$ is a lattice packing and $\gamma(O,\Lambda) < 7/6$. By Corollary 1 we have

$$(O+\Lambda) \cap \mathrm{int}\, T_i^{(k)} \neq \emptyset \text{ for all } i, k.$$

By the definition of $P_i^{(k)}$ we have

$$\Lambda \cap P_i^{(k)} \neq \emptyset \text{ for all } i, k. \tag{5}$$

Obviously, we have

$$P_1^{(k)} = Q_1^{(k)} \cup Q_2^{(k)} \cup Q_3^{(k)} \cup Q_4^{(k)} \cup Q_5^{(k)} \cup Q_6^{(k)} \cup Q_7^{(k)} \cup Q_8^{(k)}, \tag{6}$$
$$P_2^{(k)} = Q_2^{(k)} \cup Q_4^{(k)} \cup Q_5^{(k)} \cup Q_6^{(k)} \cup Q_8^{(k)} \cup Q_{10}^{(k)} \cup Q_{12}^{(k)} \cup Q_{14}^{(k)} \cup Q_{18}^{(k)} \cup Q_{19}^{(k)}, \tag{7}$$
$$P_3^{(k)} = Q_3^{(k)} \cup Q_4^{(k)} \cup Q_6^{(k)} \cup Q_7^{(k)} \cup Q_8^{(k)} \cup Q_{12}^{(k)} \cup Q_{14}^{(k)} \cup Q_{16}^{(k)} \cup Q_{18}^{(k)} \cup Q_{19}^{(k)}, \tag{8}$$
$$P_4^{(k)} = Q_5^{(k)} \cup Q_6^{(k)} \cup Q_9^{(k)} \cup Q_{10}^{(k)} \cup Q_{11}^{(k)} \cup Q_{12}^{(k)} \cup Q_{13}^{(k)} \cup Q_{14}^{(k)}, \tag{9}$$
$$P_5^{(k)} = Q_4^{(k)} \cup Q_6^{(k)} \cup Q_8^{(k)} \cup Q_{11}^{(k)} \cup Q_{12}^{(k)} \cup Q_{13}^{(k)} \cup Q_{14}^{(k)} \cup Q_{17}^{(k)} \cup Q_{18}^{(k)} \cup Q_{19}^{(k)}, \tag{10}$$
$$P_6^{(k)} = Q_7^{(k)} \cup Q_8^{(k)} \cup Q_{13}^{(k)} \cup Q_{14}^{(k)} \cup Q_{15}^{(k)} \cup Q_{16}^{(k)} \cup Q_{17}^{(k)} \cup Q_{18}^{(k)}. \tag{11}$$

Since $O + \Lambda$ is a lattice packing, by dilating the lattice a little we can achieve that $\gamma(O,\Lambda) < 7/6$ still holds and $||\mathbf{x},\mathbf{y}||_1 > 2$ for all $\mathbf{x},\mathbf{y} \in \Lambda$ with $\mathbf{x} \neq \mathbf{y}$. In other words,

$$\mathbf{y} \notin (2O + \mathbf{x}) \text{ for all } \mathbf{x},\mathbf{y} \in \Lambda \text{ with } \mathbf{x} \neq \mathbf{y}.$$

We put $\mathcal{Q}_1 = \{1, 9, 15\}$, $\mathcal{Q}_2 = \{2, 3, 10, 11, 16, 17\}$, $\mathcal{Q}_3 = \{5, 7, 13, 19\}$, $\mathcal{Q}_4 = \{4, 12, 18\}$, $\mathcal{Q}_5 = \{6, 8, 14\}$; this entails that if $m \in \mathcal{Q}_j$, then $Q_m^{(k)}$ is completely inside exactly j of the $P_i^{(k)}$. We write $\Lambda^{(k)} = \{i_1, i_2, \ldots, i_n\}$ if

$$\Lambda \cap Q_i^{(k)} \neq \emptyset \text{ for } i \in \{i_1, i_2, \ldots, i_n\} \text{ and } \Lambda \cap Q_i^{(k)} = \emptyset \text{ for } i \notin \{i_1, i_2, \ldots, i_n\}.$$

3.1 All the possible $\Lambda^{(k)}$ for a given k

To enumerate all the possible $\Lambda^{(k)}$ which satisfy (5) for a given k, we list some restricting conditions as follows:

For an arbitrary point $\mathbf{x} \in Q_2^{(k)}$, we have

$$(2O + \mathbf{x}) \supset (Q_3^{(k)} \cup Q_4^{(k)} \cup Q_5^{(k)} \cup Q_6^{(k)} \cup Q_8^{(k)}).$$

Therefore, if $2 \in \Lambda^{(k)}$, we must have

$$\Lambda^{(k)} \cap \{3, 4, 5, 6, 8\} = \emptyset. \tag{12}$$

By symmetry, we have

$$\text{if } 3 \in \Lambda^{(k)}, \text{then } \Lambda^{(k)} \cap \{2, 4, 6, 7, 8\} = \emptyset, \tag{13}$$
$$\text{if } 10 \in \Lambda^{(k)}, \text{then } \Lambda^{(k)} \cap \{5, 6, 11, 12, 14\} = \emptyset, \tag{14}$$
$$\text{if } 11 \in \Lambda^{(k)}, \text{then } \Lambda^{(k)} \cap \{6, 10, 12, 13, 14\} = \emptyset, \tag{15}$$
$$\text{if } 16 \in \Lambda^{(k)}, \text{then } \Lambda^{(k)} \cap \{7, 8, 14, 17, 18\} = \emptyset, \tag{16}$$
$$\text{if } 17 \in \Lambda^{(k)}, \text{then } \Lambda^{(k)} \cap \{8, 13, 14, 16, 18\} = \emptyset. \tag{17}$$

For an arbitrary point $\mathbf{x} \in Q_4^{(k)}$, we have

$$(2O + \mathbf{x}) \supset (Q_2^{(k)} \cup Q_3^{(k)} \cup Q_5^{(k)} \cup Q_6^{(k)} \cup Q_7^{(k)} \cup Q_8^{(k)} \cup Q_{12}^{(k)} \cup Q_{14}^{(k)} \cup Q_{18}^{(k)} \cup Q_{19}^{(k)}).$$

Therefore, if $4 \in \Lambda^{(k)}$, we must have

$$\Lambda^{(k)} \cap \{2, 3, 5, 6, 7, 8, 12, 14, 18, 19\} = \emptyset. \tag{18}$$

By symmetry, we have

$$\text{if } 12 \in \Lambda^{(k)}, \text{then } \Lambda^{(k)} \cap \{4, 5, 6, 8, 10, 11, 13, 14, 18, 19\} = \emptyset, \tag{19}$$
$$\text{if } 18 \in \Lambda^{(k)}, \text{then } \Lambda^{(k)} \cap \{4, 6, 7, 8, 12, 13, 14, 16, 17, 19\} = \emptyset. \tag{20}$$

For an arbitrary point $\mathbf{x} \in Q_5^{(k)}$, we have

$$(2O + \mathbf{x}) \supset (Q_2^{(k)} \cup Q_4^{(k)} \cup Q_6^{(k)} \cup Q_8^{(k)} \cup Q_{10}^{(k)} \cup Q_{12}^{(k)} \cup Q_{14}^{(k)} \cup Q_{19}^{(k)}).$$

Therefore, if $5 \in \Lambda^{(k)}$, we must have

$$\Lambda^{(k)} \cap \{2, 4, 6, 8, 10, 12, 14, 19\} = \emptyset. \tag{21}$$

By symmetry, we have

$$\text{if } 7 \in \Lambda^{(k)}, \text{ then } \Lambda^{(k)} \cap \{3, 4, 6, 8, 14, 16, 18, 19\} = \emptyset, \tag{22}$$
$$\text{if } 13 \in \Lambda^{(k)}, \text{ then } \Lambda^{(k)} \cap \{6, 8, 11, 12, 14, 17, 18, 19\} = \emptyset. \tag{23}$$

For an arbitrary point $\mathbf{x} \in Q_6^{(k)}$, we have

$$(2O + \mathbf{x}) \supset (Q_2^{(k)} \cup Q_3^{(k)} \cup Q_4^{(k)} \cup Q_5^{(k)} \cup Q_7^{(k)} \cup Q_8^{(k)} \cup Q_{10}^{(k)} \cup Q_{11}^{(k)} \cup Q_{12}^{(k)} \cup Q_{13}^{(k)} \cup Q_{14}^{(k)} \cup Q_{18}^{(k)} \cup Q_{19}^{(k)}).$$

Therefore, if $6 \in \Lambda^{(k)}$, we must have

$$\Lambda^{(k)} \cap \{2, 3, 4, 5, 7, 8, 10, 11, 12, 13, 14, 18, 19\} = \emptyset. \tag{24}$$

By symmetry, we have

$$\text{if } 8 \in \Lambda^{(k)}, \text{ then } \Lambda^{(k)} \cap \{2, 3, 4, 5, 6, 7, 12, 13, 14, 16, 17, 18, 19\} = \emptyset, \tag{25}$$
$$\text{if } 14 \in \Lambda^{(k)}, \text{ then } \Lambda^{(k)} \cap \{4, 5, 6, 7, 8, 10, 11, 12, 13, 16, 17, 18, 19\} = \emptyset. \tag{26}$$

For an arbitrary point $\mathbf{x} \in Q_3^{(1)}$, we have

$$(2O + \mathbf{x}) \supset \operatorname{conv}\Big\{\Big(\frac{1}{3}, \frac{2}{3}, 1\Big), \Big(0, \frac{2}{3}, \frac{4}{3}\Big), (0, 1, 1), \Big(\frac{1}{3}, 1, \frac{2}{3}\Big), \Big(\frac{1}{3}, 1, \frac{4}{3}\Big), \Big(\frac{1}{3}, \frac{4}{3}, 1\Big), \Big(\frac{2}{3}, 1, 1\Big)\Big\}.$$

Combining with Corollary 2, we have:

$$\text{if } \Lambda \cap Q_3^{(1)} \neq \emptyset, \text{ then } \Lambda \cap Q_5^{(1)} = \emptyset.$$

For an arbitrary point

$$\mathbf{x} \in \operatorname{conv}\Big\{\Big(1, \frac{1}{3}, \frac{2}{3}\Big), \Big(\frac{2}{3}, 0, \frac{4}{3}\Big), (1, 0, 1), \Big(\frac{2}{3}, \frac{1}{3}, 1\Big), \Big(1, \frac{1}{3}, \frac{4}{3}\Big), \Big(\frac{4}{3}, \frac{1}{3}, 1\Big), \Big(1, \frac{2}{3}, 1\Big)\Big\},$$

we have

$$(2O + \mathbf{x}) \supset \operatorname{conv}\Big\{\Big(\frac{1}{3}, \frac{2}{3}, 1\Big), \Big(0, \frac{2}{3}, \frac{4}{3}\Big), (0, 1, 1), \Big(\frac{1}{3}, 1, \frac{2}{3}\Big), \Big(\frac{1}{3}, 1, \frac{4}{3}\Big), \Big(\frac{1}{3}, \frac{4}{3}, 1\Big), \Big(\frac{2}{3}, 1, 1\Big)\Big\}.$$

By the symmetry of $2O$ and $M^{(1)}$, combining with Corollary 2, we have:

$$\text{if } \Lambda \cap Q_7^{(1)} \neq \emptyset, \text{ then } \Lambda \cap Q_5^{(1)} = \emptyset.$$

By symmetry, we have

$$\text{if } 5 \in \Lambda^{(k)}, \text{ then } \Lambda^{(k)} \cap \{3, 7, 11, 13\} = \emptyset, \tag{27}$$
$$\text{if } 7 \in \Lambda^{(k)}, \text{ then } \Lambda^{(k)} \cap \{2, 5, 13, 17\} = \emptyset, \tag{28}$$
$$\text{if } 13 \in \Lambda^{(k)}, \text{ then } \Lambda^{(k)} \cap \{5, 7, 10, 16\} = \emptyset. \tag{29}$$

Without loss of generality, if two different lattices Λ_1 and Λ_2 satisfy (5) and $\Lambda_1^{(k)} \subset \Lambda_2^{(k)}$, then we consider only $\Lambda_1^{(k)}$ instead of both. Suppose that

$$i_1 \in \mathcal{Q}_{j_1}, \quad i_2 \in \mathcal{Q}_{j_2}, \dots, \quad i_n \in \mathcal{Q}_{j_n};$$

to satisfy (5), a necessary condition is

$$j_1 + j_2 + \cdots + j_n \geq 6.$$

Combining with Conditions (5)–(29), we categorize all the possible $\Lambda^{(k)}$ for a given k as follows:

Category 1. $n = 2$ and $i_1 \in \mathcal{Q}_5$, $i_2 \in \mathcal{Q}_1$.

For instance, let $i_1 = 6$. By (5), (11) and (24), we have $i_2 = 15$. Therefore

$$\Lambda^{(k)} = \{14, 1\}, \{6, 15\}, \{8, 9\}$$

by the symmetry of $2O$ and $M^{(k)}$.

Category 2. $n = 2$ and $i_1 \in \mathcal{Q}_5$, $i_2 \in \mathcal{Q}_2$.

For instance, let $i_1 = 6$. By (5), (11) and (24), we have $i_2 = 16$ or 17. Therefore,

$$\Lambda^{(k)} = \{14, 2\}, \{14, 3\}, \{8, 10\}, \{8, 11\}, \{6, 16\}, \{6, 17\},$$

by the symmetry of $2O$ and $M^{(k)}$.

Category 3. $n = 2$ and $i_1 \in \mathcal{Q}_4$, $i_2 \in \mathcal{Q}_3$.

For instance, let $i_1 = 4$. By (5), (9), (11) and (18), we have $i_2 = 13$. Therefore,

$$\Lambda^{(k)} = \{4, 13\}, \{18, 5\}, \{12, 7\},$$

by the symmetry of $2O$ and $M^{(k)}$.

Category 4. $n = 3$ and $i_1 \in \mathcal{Q}_4$, $i_2, i_3 \in \mathcal{Q}_1$.

For instance, let $i_1 = 4$. By (5), (9), (11) and (18), we have $i_2 = 9$, $i_3 = 15$. Therefore,

$$\Lambda^{(k)} = \{1, 9, 18\}, \{1, 12, 15\}, \{4, 9, 15\},$$

by the symmetry of $2O$ and $M^{(k)}$.

Category 5. $n = 3$ and $i_1 \in \mathcal{Q}_4$, $i_2 \in \mathcal{Q}_2$ satisfy $\operatorname{cl} Q_{i_{j_1}}^{(k)} \cap \operatorname{cl} Q_{i_{j_2}}^{(k)} \neq \emptyset$.

For instance, let $i_1 = 4$, $i_2 = 10$. By (5), (11), (14) and (18), we have $i_3 = 15$ or16 or 17. Therefore,

$$\begin{aligned}\Lambda^{(k)} = {} & \{4, 10, 15\}, \{4, 10, 16\}, \{4, 10, 17\}, \{4, 16, 9\}, \{4, 16, 11\},\\ & \{12, 2, 15\}, \{12, 2, 16\}, \{12, 2, 17\}, \{12, 17, 1\}, \{12, 17, 3\},\\ & \{18, 11, 1\}, \{18, 11, 2\}, \{18, 11, 3\}, \{18, 3, 10\}, \{18, 3, 9\},\end{aligned}$$

by the symmetry of $2O$ and $M^{(k)}$.

Category 6. $n = 3$ and $i_1 \in \mathcal{Q}_4$, $i_2 \in \mathcal{Q}_2$, satisfies $\operatorname{cl} Q_{i_1}^{(k)} \cap \left(\operatorname{cl} Q_{i_2}^{(k)} \cup \operatorname{cl} Q_{i_3}^{(k)}\right) = \emptyset$.

For instance, let $i_1 = 4$, $i_2 = 11$. By (5), (11), (15) and (18), we have $i_3 = 15$ or 17. Therefore,

$$\Lambda^{(k)} = \{4, 11, 15\}, \{4, 11, 17\}, \{4, 17, 9\}, \{12, 3, 15\}, \{12, 3, 16\}, \{12, 16, 1\}, \{18, 10, 2\}, \{18, 10, 1\}, \{18, 2, 9\},$$

by the symmetry of $2O$ and $M^{(k)}$.

Category 7. $n = 3$ and $i_1, i_2, i_3 \in \mathcal{Q}_2$. By (5)–(17), it is easy to deduce that $\Lambda^{(k)} = \{2, 11, 16\}, \{3, 10, 17\}$.

Category 8. $n = 4$ and $i_1 = 19$, $i_2 \in \mathcal{Q}_2$.

For instance, let $i_2 = 2$. By (5), (9), (11) and (12), we have $i_3 = 9$ or 10 or 11 and $i_4 = 15$ or 16 or 17. Therefore,

$$\begin{aligned}\Lambda^{(k)} = {} & \{19, 2, 9, 15\}, \{19, 2, 9, 16\}, \{19, 2, 9, 17\}, \{19, 2, 10, 15\}, \{19, 2, 10, 16\},\\ & \{19, 2, 10, 17\}, \{19, 2, 11, 15\}, \{19, 2, 11, 17\}, \{19, 3, 9, 15\}, \{19, 3, 9, 16\},\\ & \{19, 3, 9, 17\}, \{19, 3, 10, 15\}, \{19, 3, 10, 16\}, \{19, 3, 11, 15\}, \{19, 3, 11, 16\},\\ & \{19, 3, 11, 17\}, \{19, 16, 9, 1\}, \{19, 16, 10, 1\}, \{19, 16, 11, 1\}, \{19, 17, 9, 1\},\\ & \{19, 17, 10, 1\}, \{19, 17, 11, 1\}, \{19, 11, 15, 1\}, \{19, 10, 15, 1\},\end{aligned}$$

by the symmetry of $2O$ and $M^{(k)}$.

Category 9. $n = 4$ and $i_1 = 19$, $i_2, i_3, i_4 \in \mathcal{Q}_1$. Then $\Lambda^{(k)} = \{1, 9, 15, 19\}$, obviously.

3.2 The restriction between $\Lambda^{(k_1)}$ and $\Lambda^{(k_2)}$ for $k_1 \neq k_2$

By routine computation, we obtain some restricting conditions between different faces as follows:

For an arbitrary point $\mathbf{x} \in Q_1^{(k)}$, we have

$$(2O + \mathbf{x}) \supset \left(Q_1^{(k+1)} \cup Q_1^{(k+3)}\right).$$

Therefore, if $1 \in \Lambda^{(k)}$, we must have

$$1 \notin \Lambda^{(k+1)}, \quad 1 \notin \Lambda^{(k+3)}. \tag{30}$$

For an arbitrary point $\mathbf{x} \in Q_2^{(k)}$, we have

$$(2O + \mathbf{x}) \supset \left(Q_3^{(k+3)} \cup Q_4^{(k+3)} \cup Q_7^{(k+3)} \cup Q_8^{(k+3)}\right).$$

Therefore, if $2 \in \Lambda^{(k)}$, we must have

$$\Lambda^{(k+3)} \cap \{3, 4, 7, 8\} = \emptyset. \tag{31}$$

For an arbitrary point $\mathbf{x} \in Q_4^{(k)}$, we have

$$(2O + \mathbf{x}) \supset (Q_2^{(k+1)} \cup Q_4^{(k+1)} \cup Q_5^{(k+1)} \cup Q_6^{(k+1)} \cup Q_3^{(k+3)} \cup Q_4^{(k+3)} \cup Q_7^{(k+3)} \cup Q_8^{(k+3)}).$$

Therefore, if $4 \in \Lambda^{(k)}$, we must have

$$\Lambda^{(k+1)} \cap \{2, 4, 5, 6\} = \emptyset, \quad \Lambda^{(k+3)} \cap \{3, 4, 7, 8\} = \emptyset. \tag{32}$$

For an arbitrary point $\mathbf{x} \in (Q_5^{(k)} \cup Q_6^{(k)})$, we have

$$(2O + \mathbf{x}) \supset (Q_3^{(k+3)} \cup Q_4^{(k+3)} \cup Q_7^{(k+3)} \cup Q_8^{(k+3)} \cup Q_{16}^{(k+3)} \cup Q_{18}^{(k+3)} \cup Q_{19}^{(k+3)}).$$

Therefore, if $\Lambda^{(k)} \cap \{5, 6\} \neq \emptyset$, we must have

$$\Lambda^{(k+3)} \cap \{3, 4, 7, 8, 16, 18, 19\} = \emptyset. \tag{33}$$

For an arbitrary point $\mathbf{x} \in Q_{19}^{(k)}$, we have

$$(2O + \mathbf{x}) \supset (Q_5^{(k+1)} \cup Q_6^{(k+1)} \cup Q_{13}^{(k')} \cup Q_{14}^{(k')} \cup Q_7^{(k+3)} \cup Q_8^{(k+3)}).$$

Therefore, if $19 \in \Lambda^{(k)}$, we must have

$$\Lambda^{(k+1)} \cap \{5, 6\} = \emptyset, \quad \Lambda^{(k+2)} \cap \{13, 14\} = \emptyset, \quad \Lambda^{(k+3)} \cap \{7, 8\} = \emptyset. \tag{34}$$

By symmetry, we have

$$\text{if } 3 \in \Lambda^{(k)}, \text{ then } \Lambda^{(k+1)} \cap \{2, 4, 5, 6\} = \emptyset, \tag{35}$$
$$\text{if } \Lambda^{(k)} \cap \{7, 8\} \neq \emptyset, \text{ then } \Lambda^{(k+1)} \cap \{2, 4, 5, 6, 10, 12, 19\} = \emptyset, \tag{36}$$
$$\text{if } 9 \in \Lambda^{(k)}, \text{ then } 9 \notin \Lambda^{(k+2)}, \ 15 \notin \Lambda^{(k+3)}, \tag{37}$$
$$\text{if } 10 \in \Lambda^{(k)}, \text{ then } \Lambda^{(k+3)} \cap \{7, 8, 16, 18\} = \emptyset, \tag{38}$$
$$\text{if } 11 \in \Lambda^{(k)}, \text{ then } \Lambda^{(k+2)} \cap \{11, 12, 13, 14\} = \emptyset, \tag{39}$$
$$\text{if } 12 \in \Lambda^{(k)}, \text{ then } \Lambda^{(k+2)} \cap \{11, 12, 13, 14\} = \emptyset, \ \Lambda^{(k+3)} \cap \{7, 8, 16, 18\} = \emptyset, \tag{40}$$
$$\text{if } \Lambda^{(k)} \cap \{13, 14\} \neq \emptyset, \text{ then } \Lambda^{(k+2)} \cap \{11, 12, 13, 14, 17, 18, 19\} = \emptyset, \tag{41}$$
$$\text{if } 15 \in \Lambda^{(k)}, \text{ then } 9 \notin \Lambda^{(k+1)}, \ 15 \notin \Lambda^{(k+2)}, \tag{42}$$
$$\text{if } 16 \in \Lambda^{(k)}, \text{ then } \Lambda^{(k+1)} \cap \{5, 6, 10, 12\} = \emptyset, \tag{43}$$
$$\text{if } 17 \in \Lambda^{(k)}, \text{ then } \Lambda^{(k+2)} \cap \{13, 14, 17, 18\} = \emptyset, \tag{44}$$
$$\text{if } 18 \in \Lambda^{(k)}, \text{ then } \Lambda^{(k+1)} \cap \{5, 6, 10, 12\} = \emptyset, \ \Lambda^{(k+2)} \cap \{13, 14, 17, 18\} = \emptyset. \tag{45}$$

For an arbitrary point $\mathbf{x} \in (Q_4^{(1)} \cup Q_6^{(1)} \cup Q_{19}^{(1)})$, we have

$$(2O + \mathbf{x}) \supset \operatorname{conv}\left\{\left(1, \frac{1}{3}, \frac{2}{3}\right), \left(\frac{4}{3}, 0, \frac{2}{3}\right), \left(\frac{4}{3}, \frac{1}{3}, \frac{1}{3}\right), \left(1, \frac{2}{3}, 1\right), \left(\frac{4}{3}, \frac{1}{3}, 1\right), \left(\frac{4}{3}, \frac{2}{3}, \frac{2}{3}\right), \left(\frac{5}{3}, \frac{1}{3}, \frac{2}{3}\right)\right\}.$$

For an arbitrary point

$$\mathbf{y} \in \left(Q_{16}^{(1)} \setminus \operatorname{conv}\left\{\left(1, \frac{1}{3}, \frac{2}{3}\right), \left(\frac{4}{3}, 0, \frac{2}{3}\right), \left(\frac{4}{3}, \frac{1}{3}, \frac{1}{3}\right), \left(1, \frac{2}{3}, 1\right), \left(\frac{4}{3}, \frac{1}{3}, 1\right), \left(\frac{4}{3}, \frac{2}{3}, \frac{2}{3}\right), \left(\frac{5}{3}, \frac{1}{3}, \frac{2}{3}\right)\right\}\right),$$

we have

$$(2O + \mathbf{y}) \supset (Q_9^{(2)} \cup Q_{11}^{(2)} \cup Q_{15}^{(1')}).$$

Therefore, if $\Lambda^{(1)} \cap \{4, 6, 19\} \neq \emptyset$ and $16 \in \Lambda^{(1)}$, we must have

$$\Lambda^{(2)} \cap \{9, 11\} = \emptyset, \quad 15 \notin \Lambda^{(3)}.$$

By symmetry, we have

$$\text{if } \Lambda^{(k)} \cap \{4, 6, 19\} \neq \emptyset \text{ and } 16 \in \Lambda^{(k)}, \text{ then } \Lambda^{(k+1)} \cap \{9, 11\} = \emptyset,\ 15 \notin \Lambda^{(k+2)}, \tag{46}$$
$$\text{if } \Lambda^{(k)} \cap \{6, 12, 19\} \neq \emptyset \text{ and } 17 \in \Lambda^{(k)}, \text{ then } \Lambda^{(k+2)} \cap \{15, 16\} = \emptyset,\ 9 \notin \Lambda^{(k+1)}, \tag{47}$$
$$\text{if } \Lambda^{(k)} \cap \{8, 18, 19\} \neq \emptyset \text{ and } 11 \in \Lambda^{(k)}, \text{ then } \Lambda^{(k+2)} \cap \{9, 10\} = \emptyset,\ 15 \notin \Lambda^{(k+3)}, \tag{48}$$
$$\text{if } \Lambda^{(k)} \cap \{12, 14, 19\} \neq \emptyset \text{ and } 2 \in \Lambda^{(k)}, \text{ then } \Lambda^{(k+3)} \cap \{1, 2\} = \emptyset,\ 1 \notin \Lambda^{(k+1)}, \tag{49}$$
$$\text{if } \Lambda^{(k)} \cap \{14, 18, 19\} \neq \emptyset \text{ and } 3 \in \Lambda^{(k)}, \text{ then } \Lambda^{(k+1)} \cap \{1, 3\} = \emptyset,\ 1 \notin \Lambda^{(k+3)}, \tag{50}$$
$$\text{if } \Lambda^{(k)} \cap \{4, 8, 19\} \neq \emptyset \text{ and } 10 \in \Lambda^{(k)}, \text{ then } \Lambda^{(k+3)} \cap \{15, 17\} = \emptyset,\ 9 \notin \Lambda^{(k+2)}. \tag{51}$$

For an arbitrary point $\mathbf{x} \in Q_{13}^{(1)}$, we have

$$(2O + \mathbf{x}) \supset \operatorname{conv}\Big\{\Big(\frac{1}{3}, \frac{2}{3}, 1\Big), \Big(\frac{2}{3}, \frac{1}{3}, 1\Big), \Big(\frac{2}{3}, \frac{2}{3}, \frac{2}{3}\Big), \Big(\frac{2}{3}, \frac{2}{3}, \frac{4}{3}\Big), \Big(\frac{2}{3}, 1, 1\Big), \Big(1, \frac{2}{3}, 1\Big)\Big\}.$$

For an arbitrary point

$$\mathbf{y} \in \Big(Q_4^{(1)} \setminus \operatorname{conv}\Big\{\Big(\frac{1}{3}, \frac{2}{3}, 1\Big), \Big(\frac{2}{3}, \frac{1}{3}, 1\Big), \Big(\frac{2}{3}, \frac{2}{3}, \frac{2}{3}\Big), \Big(\frac{2}{3}, \frac{2}{3}, \frac{4}{3}\Big), \Big(\frac{2}{3}, 1, 1\Big), \Big(1, \frac{2}{3}, 1\Big)\Big\}\Big),$$

we have

$$(2O + \mathbf{y}) \supset (Q_1^{(2)} \cup Q_3^{(2)} \cup Q_1^{(4)} \cup Q_2^{(4)}).$$

Therefore, if $\{4, 13\} \subset \Lambda^{(1)}$, we must have

$$\Lambda^{(2)} \cap \{1, 3\} = \emptyset,\ \Lambda^{(4)} \cap \{1, 2\} = \emptyset.$$

By symmetry, we have

$$\text{if } \{4, 13\} \subset \Lambda^{(k)}, \text{ then } \Lambda^{(k+1)} \cap \{1, 3\} = \emptyset,\ \Lambda^{(k+3)} \cap \{1, 2\} = \emptyset, \tag{52}$$
$$\text{if } \{7, 12\} \subset \Lambda^{(k)}, \text{ then } \Lambda^{(k+2)} \cap \{9, 10\} = \emptyset,\ \Lambda^{(k+3)} \cap \{15, 17\} = \emptyset, \tag{53}$$
$$\text{if } \{5, 18\} \subset \Lambda^{(k)}, \text{ then } \Lambda^{(k+1)} \cap \{9, 11\} = \emptyset,\ \Lambda^{(k+2)} \cap \{15, 16\} = \emptyset. \tag{54}$$

For an arbitrary point $\mathbf{x} \in (Q_1^{(2)} \cup Q_1^{(4)})$, we have

$$\begin{aligned}(2O + \mathbf{x}) \supset \operatorname{conv}\Big\{&\Big(0, \frac{1}{3}, \frac{5}{3}\Big), \Big(\frac{1}{3}, 0, \frac{5}{3}\Big), \Big(0, \frac{2}{3}, \frac{4}{3}\Big), \Big(\frac{2}{3}, 0, \frac{4}{3}\Big), \Big(\frac{1}{3}, \frac{2}{3}, 1\Big),\\ &\Big(\frac{2}{3}, \frac{1}{3}, 1\Big), \Big(\frac{1}{3}, \frac{2}{3}, \frac{5}{3}\Big), \Big(\frac{2}{3}, \frac{1}{3}, \frac{5}{3}\Big), \Big(\frac{1}{3}, 1, \frac{4}{3}\Big), \Big(1, \frac{1}{3}, \frac{4}{3}\Big)\Big\}.\end{aligned}$$

For an arbitrary point $\mathbf{y} \in (Q_{15}^{(4)} \cup Q_9^{(1')})$, we have

$$\begin{aligned}(2O + \mathbf{y}) \supset \operatorname{conv}\Big\{&\Big(\frac{1}{3}, \frac{5}{3}, 0\Big), \Big(0, \frac{5}{3}, \frac{1}{3}\Big), \Big(\frac{2}{3}, \frac{4}{3}, 0\Big), \Big(0, \frac{4}{3}, \frac{2}{3}\Big), \Big(\frac{2}{3}, 1, \frac{1}{3}\Big),\\ &\Big(\frac{1}{3}, 1, \frac{2}{3}\Big), \Big(\frac{2}{3}, \frac{5}{3}, \frac{1}{3}\Big), \Big(\frac{1}{3}, \frac{5}{3}, \frac{2}{3}\Big), \Big(1, \frac{4}{3}, \frac{1}{3}\Big), \Big(\frac{1}{3}, \frac{4}{3}, 1\Big)\Big\}.\end{aligned}$$

For an arbitrary point

$$\begin{aligned}\mathbf{z} \in (Q_{10}^{(1)} \cup Q_{11}^{(1)} \cup Q_{12}^{(1)}) \setminus \operatorname{conv}\Big\{&\Big(\frac{1}{3}, \frac{5}{3}, 0\Big), \Big(0, \frac{5}{3}, \frac{1}{3}\Big), \Big(\frac{2}{3}, \frac{4}{3}, 0\Big), \Big(0, \frac{4}{3}, \frac{2}{3}\Big), \Big(\frac{2}{3}, 1, \frac{1}{3}\Big),\\ &\Big(\frac{1}{3}, 1, \frac{2}{3}\Big), \Big(\frac{2}{3}, \frac{5}{3}, \frac{1}{3}\Big), \Big(\frac{1}{3}, \frac{5}{3}, \frac{2}{3}\Big), \Big(1, \frac{4}{3}, \frac{1}{3}\Big), \Big(\frac{1}{3}, \frac{4}{3}, 1\Big)\Big\},\end{aligned}$$

we have

$$(2O+\mathbf{z})\supset(Q_2^{(1)}\cup Q_3^{(1)}\cup Q_4^{(1)})\setminus\operatorname{conv}\Big\{\Big(0,\frac{1}{3},\frac{5}{3}\Big),\Big(\frac{1}{3},0,\frac{5}{3}\Big),\Big(0,\frac{2}{3},\frac{4}{3}\Big),\Big(\frac{2}{3},0,\frac{4}{3}\Big),\Big(\frac{1}{3},\frac{2}{3},1\Big),\Big(\frac{2}{3},\frac{1}{3},1\Big),\Big(\frac{1}{3},\frac{2}{3},\frac{5}{3}\Big),\Big(\frac{2}{3},\frac{1}{3},\frac{5}{3}\Big),\Big(\frac{1}{3},1,\frac{4}{3}\Big),\Big(1,\frac{1}{3},\frac{4}{3}\Big)\Big\}.$$

Therefore, if $1\in\Lambda^{(2)}$ or $1\in\Lambda^{(4)}$, and $9\in\Lambda^{(3)}$ or $15\in\Lambda^{(4)}$, combining with (12)–(15), we have

$$\operatorname{card}\{\Lambda^{(1)}\cap\{2,3,4,10,11,12\}\}\le 1.$$

By symmetry, we have:
If $1\in\Lambda^{(k+1)}$ or $1\in\Lambda^{(k+3)}$, and $9\in\Lambda^{(k+2)}$ or $15\in\Lambda^{(k+3)}$, then

$$\operatorname{card}\{\Lambda^{(k)}\cap\{2,3,4,10,11,12\}\}\le 1. \tag{55}$$

If $1\in\Lambda^{(k+1)}$ or $1\in\Lambda^{(k+3)}$, and $9\in\Lambda^{(k+1)}$ or $15\in\Lambda^{(k+2)}$, then

$$\operatorname{card}\{\Lambda^{(k)}\cap\{2,3,4,16,17,18\}\}\le 1. \tag{56}$$

If $9\in\Lambda^{(k+1)}$ or $15\in\Lambda^{(k+2)}$, and $9\in\Lambda^{(k+2)}$ or $15\in\Lambda^{(k+3)}$, then

$$\operatorname{card}\{\Lambda^{(k)}\cap\{10,11,12,16,17,18\}\}\le 1. \tag{57}$$

For an arbitrary point $\mathbf{x}\in\operatorname{conv}\{(\frac{1}{3},\frac{1}{3},\frac{4}{3}),(\frac{1}{3},\frac{2}{3},1),(\frac{2}{3},\frac{1}{3},1),(\frac{2}{3},\frac{2}{3},\frac{4}{3})\}$, we have

$$(2O+\mathbf{x})\supset(Q_1^{(2)}\cup Q_3^{(2)}\cup Q_1^{(4)}\cup Q_2^{(4)}).$$

For an arbitrary point $\mathbf{y}\in\operatorname{conv}\{(\frac{4}{3},\frac{1}{3},\frac{1}{3}),(\frac{5}{3},\frac{2}{3},\frac{1}{3}),(\frac{4}{3},\frac{2}{3},0),(\frac{5}{3},\frac{1}{3},0)\}$, we have

$$(2O+\mathbf{y})\supset(Q_9^{(2)}\cup Q_{15}^{(1')}\cup Q_{16}^{(1')}).$$

For an arbitrary point $\mathbf{z}\in(Q_4^{(1)}\setminus\operatorname{conv}\{(\frac{1}{3},\frac{1}{3},\frac{4}{3}),(\frac{1}{3},\frac{2}{3},1),(\frac{2}{3},\frac{1}{3},1),(\frac{2}{3},\frac{2}{3},\frac{4}{3})\})$, we have

$$(2O+\mathbf{z})\supset\Big(Q_{17}^{(1)}\setminus\operatorname{conv}\Big\{\Big(\frac{4}{3},\frac{1}{3},\frac{1}{3}\Big),\Big(\frac{5}{3},\frac{2}{3},\frac{1}{3}\Big),\Big(\frac{4}{3},\frac{2}{3},0\Big),\Big(\frac{5}{3},\frac{1}{3},0\Big)\Big\}\Big).$$

Therefore, if $\{4,17\}\subset\Lambda^{(1)}$, we have

$$\Lambda^{(2)}\cap\{1,3\}=\emptyset,\quad \Lambda^{(4)}\cap\{1,2\}=\emptyset,$$

or

$$9\notin\Lambda^{(2)},\quad \Lambda^{(3)}\cap\{15,16\}=\emptyset. \tag{58}$$

By symmetry, we have: If $\{4,11\}\subset\Lambda^{(1)}$, then

$$\Lambda^{(2)}\cap\{1,3\}=\emptyset,\quad \Lambda^{(4)}\cap\{1,2\}=\emptyset,$$

or

$$15\notin\Lambda^{(4)},\quad \Lambda^{(3)}\cap\{9,10\}=\emptyset. \tag{59}$$

Now we show that a combination of $\Lambda^{(k)}$, with $k=1,2,3,4$ as categorized before, cannot satisfy Conditions (30)–(59).

Case 1. Category 3 is used. Without loss of generality, we suppose

$$\Lambda^{(1)}=\{4,13\}. \tag{1.1}$$

Since $\Lambda\cap P_1^{(2)}\neq\emptyset$ and $\Lambda\cap P_1^{(4)}\neq\emptyset$, by (5), combining (1.1) with (6), (32) and (52), we have

$$\Lambda^{(2)}\cap\{7,8\}\neq\emptyset \tag{1.2}$$

and

$$\Lambda^{(4)} \cap \{5, 6\} \neq \emptyset. \tag{1.3}$$

Combining (1.1), (1.2), (1.3) with (7), (33), (36) and (41), we have $\Lambda \cap P_2^{(3)} = \emptyset$, which contradicts (5). Therefore, Category 3 cannot be used, and

$$\Lambda^{(k)} \cap \{5, 7, 13\} = \emptyset$$

holds for all k.

Case 2. Category 5 is used. Without loss of generality, we suppose

$$\Lambda^{(1)} \supset \{4, 16\}, \quad \Lambda^{(1)} \cap \{9, 10, 11\} \neq \emptyset. \tag{2.1}$$

Since $\Lambda \cap P_4^{(2)} \neq \emptyset$ by (5), combining (2.1) with (9), (43), (46) and the conclusion of Case 1, we have

$$14 \in \Lambda^{(2)} \tag{2.2}$$

and

$$\Lambda^{(2)} \cap \{1, 2, 3\} \neq \emptyset, \tag{2.3}$$

by the categorization before.

Since $\Lambda \cap P_5^{(4)} \neq \emptyset$ by (5), combining (2.1), (2.2) with (10), (32) and (41), we have

$$6 \in \Lambda^{(4)} \tag{2.4}$$

and

$$\Lambda^{(4)} \cap \{15, 16\} \neq \emptyset, \tag{2.5}$$

by the categorization before.

Suppose

$$16 \in \Lambda^{(4)}, \tag{2.5.1}$$

combining with (2.4), (43) and (46), we have $\Lambda^{(1)} \cap \{9, 10, 11\} = \emptyset$, which contradicts (2.1). Therefore, we have

$$15 \in \Lambda^{(4)}, \tag{2.5.2}$$

combining with (2.1) and (42), we have

$$\Lambda^{(1)} \cap \{10, 11\} \neq \emptyset. \tag{2.6}$$

If

$$1 \in \Lambda^{(2)}, \tag{2.3.1}$$

combining with (2.5.2) and (55), we have $\mathrm{card}\{\Lambda^{(1)} \cap \{4, 10, 11\}\} \leq 1$, which contradicts (2.1) and (2.6).

If

$$2 \in \Lambda^{(2)}, \tag{2.3.2}$$

by (31), we have $4 \notin \Lambda^{(1)}$, which contradicts (2.1). Therefore, we have

$$3 \in \Lambda^{(2)}. \tag{2.3.3}$$

By (2.3.3), (2.2), (6), (35), (50) and the conclusion of Case 1, we have $8 \in \Lambda^{(3)}$, which contradicts (2.4) and (33). Therefore, Category 5 cannot be used.

Case 3. Category 6 is used. Without loss of generality, we suppose

$$\Lambda^{(1)} \supset \{4, 17\}, \quad \Lambda^{(1)} \cap \{9, 11\} \neq \emptyset. \tag{3.1}$$

Suppose

$$\Lambda^{(2)} \cap \{1, 3\} = \emptyset, \quad \Lambda^{(4)} \cap \{1, 2\} = \emptyset. \tag{3.2.1}$$

Since $\Lambda \cap P_1^{(2)} \neq \emptyset$ and $\Lambda \cap P_1^{(4)} \neq \emptyset$ by (5), combining with (3.1), (6), (32) and the conclusion of Case 1, we have

$$8 \in \Lambda^{(2)} \tag{3.2.1.1}$$

and

$$6 \in \Lambda^{(4)}. \tag{3.2.1.2}$$

Since $\Lambda \cap P_2^{(3)} \neq \emptyset$ by (5), combining with (7), (3.2.1.1), (3.2.1.2), (33) and (36), we have

$$14 \in \Lambda^{(3)}. \tag{3.2.1.3}$$

By (41), we have $17 \notin \Lambda^{(1)}$, which contradicts (3.1).

Suppose that (3.2.1) does not hold, then by (3.1) and (58) we have

$$9 \notin \Lambda^{(2)}, \quad \Lambda^{(3)} \cap \{15, 16\} = \emptyset. \tag{3.2.2}$$

Since $\Lambda \cap P_6^{(3)} \neq \emptyset$ by (5), combining with (3.1), (11), (44) and the conclusion of Case 1, we have

$$8 \in \Lambda^{(3)} \tag{3.2.2.1}$$

and

$$\Lambda^{(3)} \cap \{9, 10, 11\} \neq \emptyset, \tag{3.2.2.2}$$

by the categorization before. By (3.2.2.1), (3.2.2.2), (37), (48) and (51), we have

$$9 \notin \Lambda^{(1)}, \tag{3.2.2.3}$$

combining with (3.1), we have

$$11 \in \Lambda^{(1)}. \tag{3.2.2.4}$$

By (3.1), (3.2.2.4), (39) and (59), we have $\Lambda^{(3)} \cap \{9, 10, 11\} = \emptyset$, which contradicts (3.2.2.2). Therefore, Category 6 cannot be used.

Case 4. Category 2 is used. Without loss of generality, we suppose

$$\Lambda^{(1)} = \{6, 16\}. \tag{4.1}$$

Since $\Lambda \cap P_4^{(2)} \neq \emptyset$ by (5), combining with (4.1), (9), (43), (46) and the conclusion of Case 1, we have

$$14 \in \Lambda^{(2)} \tag{4.2}$$

and

$$\Lambda^{(2)} \cap \{1, 2, 3\} \neq \emptyset, \tag{4.3}$$

by the categorization before.

By (4.2), (4.3), (30), (49) and (50), we have

$$1 \notin \Lambda^{(3)}. \tag{4.4}$$

Since $\Lambda \cap P_3^{(4)} \neq \emptyset$ by (5), combining with (4.1), (4.2), (8), (33) and (41), we have

$$6 \in \Lambda^{(4)}. \tag{4.5}$$

Since $\Lambda \cap P_1^{(3)} \neq \emptyset$ by (5), combining with (4.4), (4.5), (6), (33) and the conclusion of Case 1, we have

$$\Lambda^{(3)} \cap \{2, 6\} \neq \emptyset. \tag{4.6}$$

If

$$2 \in \Lambda^{(3)}, \tag{4.6.1}$$

since $\Lambda \cap P_3^{(3)} \neq \emptyset$ by (5), combining with (4.5), (8), (12) and (33), we have $\Lambda^{(3)} \cap \{12, 14\} \neq \emptyset$. By the conclusion of Case 2, we have $12 \notin \Lambda^{(3)}$. Therefore, we have $14 \in \Lambda^{(3)}$. Combining with (4.6.1), (31) and (49), we have $\Lambda^{(2)} \cap \{1, 2, 3\} = \emptyset$, which contradicts (4.3).

If

$$6 \in \Lambda^{(3)}, \tag{4.6.2}$$

then we have

$$\Lambda^{(3)} \cap \{15, 16, 17\} \neq \emptyset, \tag{4.6.2.1}$$

by the categorization before. Combining with (4.1), (4.5), (33) and (46), we have

$$17 \in \Lambda^{(3)}, \tag{4.6.2.2}$$

combining with (4.6.2) and (47), we have $16 \notin \Lambda^{(1)}$, which contradicts (4.1). Therefore, Category 2 cannot be used.

Case 5. Category 8 is used. Without loss of generality, we suppose

$$\Lambda^{(1)} \supset \{16, 19\}, \quad \Lambda^{(1)} \cap \{1, 2, 3\} \neq \emptyset, \quad \Lambda^{(1)} \cap \{9, 10, 11\} \neq \emptyset. \tag{5.1}$$

Since $\Lambda \cap P_4^{(2)} \neq \emptyset$, combining with (5.1), (9), (43), (46) and the conclusion of Case 1, we have

$$14 \in \Lambda^{(2)} \tag{5.2}$$

and

$$\Lambda^{(2)} \cap \{1, 2, 3\} \neq \emptyset, \tag{5.3}$$

by the categorization before. By (5.2), (5.3), (30), (49) and (50), we have

$$1 \notin \Lambda^{(1)}. \tag{5.4}$$

Suppose that

$$3 \in \Lambda^{(1)}; \tag{5.1.1}$$

combining with (5.1), (35) and (50), we have $\Lambda^{(2)} \cap \{1, 2, 3\} = \emptyset$, which contradicts (5.3). Therefore, we have

$$3 \notin \Lambda^{(1)}, \tag{5.1.2}$$

and combining with (5.1) and (5.4) we have

$$2 \in \Lambda^{(1)}. \tag{5.5}$$

Since $\Lambda \cap P_1^{(4)} \neq \emptyset$ by (5), combining with (5.1), (5.5), (6), (31), (49) and the conclusion of Case 1, we have

$$6 \in \Lambda^{(4)} \tag{5.6}$$

and

$$\Lambda^{(4)} \cap \{15, 16, 17\} \neq \emptyset, \tag{5.7}$$

by the categorization before. By (5.1), (37), (48) and (51), we have

$$15 \notin \Lambda^{(4)}. \tag{5.8}$$

If

$$16 \in \Lambda^{(4)}, \tag{5.9}$$

combining with (5.6), (43) and (46), we have $\Lambda^{(1)} \cap \{9, 10, 11\} = \emptyset$, which contradicts (5.1). Therefore $16 \notin \Lambda^{(4)}$. Combining with (5.7) and (5.8), we have

$$17 \in \Lambda^{(4)}. \tag{5.10}$$

By (5.10) and (44), we have $14 \notin \Lambda^{(2)}$, which contradicts (5.2). Therefore, Category 8 cannot be used.

Case 6. Category 9 is used. Without loss of generality, we suppose

$$\Lambda^{(1)} = \{1, 9, 15, 19\}. \tag{6.1}$$

By (6.1), (30), (34) and (42), we have

$$\Lambda^{(2)} \cap \{1, 5, 6, 9\} = \emptyset. \tag{6.2}$$

Therefore, $\Lambda^{(2)}$ cannot use Category 1, 4 and 9, $\Lambda^{(2)}$ must use Category 7. In this case, card$\{\Lambda^{(2)} \cap \{2, 3, 10, 11\}\} \geq 2$, which contradicts (6.1) and (55). Therefore, Category 9 cannot be used.

Case 7. Category 7 is used. Without loss of generality, we suppose

$$\Lambda^{(1)} = \{2, 11, 16\}. \tag{7.1}$$

Since $\Lambda \cap P_4^{(2)} \neq \emptyset$ by (5), combining with (7.1), (9), (43) and the conclusion of Case 1, we have

$$\Lambda^{(2)} \cap \{9, 11, 14\} \neq \emptyset. \tag{7.2}$$

Case 7.1.

$$9 \in \Lambda^{(2)}. \tag{7.2.1}$$

Since only Category 1, 4 and 7 is still available, we have only three options for $F^{(2)}$:

Case 7.1.1. If

$$\Lambda^{(2)} = \{8, 9\}, \tag{7.2.1.1}$$

since $\Lambda \cap P_4^{(3)} \neq \emptyset$ by (5), combining with (7.1), (9), (36) and (39) we have $9 \in \Lambda^{(3)}$. Combining with (7.2.1) and (57), we have card$\{\Lambda^{(1)} \cap \{10, 11, 12, 16, 17, 18\}\} \leq 1$, which contradicts (7.1).

Case 7.1.2. If

$$\Lambda^{(2)} = \{1, 9, 18\}, \tag{7.2.1.2}$$

by (56) we have card$\{\Lambda^{(1)} \cap \{2, 3, 4, 16, 17, 18\}\} \leq 1$, which contradicts (7.1).

Case 7.1.3. If

$$\Lambda^{(2)} = \{4, 9, 15\}, \tag{7.2.1.3}$$

since $\Lambda \cap P_4^{(3)} \neq \emptyset$ by (5), combining with (9), (7.1), (32), (39) and (42) we have $10 \in \Lambda^{(3)}$. Since only Category 1, 4 and 7 is still available, by the categorization before we have $\Lambda^{(3)} = \{3, 10, 17\}$. Since $\Lambda \cap P_1^{(4)} \neq \emptyset$ by (5), combining with (6), (7.1), (31) and (35), we have $1 \in \Lambda^{(4)}$. Combining with (7.2.1.3) and (56), we have card$\{\Lambda^{(1)} \cap \{2, 3, 4, 16, 17, 18\}\} \leq 1$, which contradicts (7.1).

Case 7.2.

$$11 \in \Lambda^{(2)}. \tag{7.2.2}$$

Since only Category 1, 4 and 7 is still available, by the categorization before, we have

$$\Lambda^{(2)} = \{2, 11, 16\}. \tag{7.2.2.1}$$

Since $\Lambda \cap P_4^{(3)} \neq \emptyset$ by (5), combining with (7.1), (9), (39) and (43) we have $9 \in \Lambda^{(3)}$, which is the same as Case 7.1, up to symmetry.

Case 7.3.

$$14 \in \Lambda^{(2)}. \tag{7.2.3}$$

In this case, we have

$$\Lambda^{(2)} = \{1, 14\}, \tag{7.2.3.1}$$

by the categorization before. Since $\Lambda \cap P_5^{(4)} \neq \emptyset$ by (5), combining with (7.1), (7.2.3), (10), (31) and (41) we have $6 \in \Lambda^{(4)}$. Therefore, we have

$$\Lambda^{(4)} = \{6, 15\}, \tag{7.2.3.2}$$

by the categorization before. By (7.2.3.1), (7.2.3.2) and (55), we have card$\{\Lambda^{(1)} \cap \{2, 3, 4, 10, 11, 12\}\} \leq 1$, which contradicts (7.1).

As a conclusion, Category 7 cannot be used.

Case 8. Category 4 is used. Without loss of generality, we suppose

$$\Lambda^{(1)} = \{1, 9, 18\}. \tag{8.1}$$

Since only Category 1 and 4 is still available, combining with (8.1), (30) and (37), we have

$$\Lambda^{(4)} = \{8, 9\}. \tag{8.2}$$

By (8.1), (8.2) and (37), we have

$$\Lambda^{(3)} = \{1, 14\}. \tag{8.3}$$

By (8.3) and (41), we have $18 \notin \Lambda^{(1)}$, which contradicts (8.1). Therefore, Category 4 cannot be used.

Case 9. Category 1 is used by all the faces. Without loss of generality, we suppose

$$\Lambda^{(1)} = \{1, 14\}. \tag{9.1}$$

Since $\Lambda \cap P_5^{(3)} \neq \emptyset$ by (5), combining with (9.1), (10) and (41), we have

$$\Lambda^{(3)} \cap \{6, 8\} \neq \emptyset. \tag{9.2}$$

Without loss of generality, we suppose

$$\Lambda^{(3)} = \{6, 15\}. \tag{9.3}$$

By (9.1), (9.3), (30) and (33), we have $\Lambda^{(2)} \cap \{1, 8\} = \emptyset$. Therefore, we have

$$\Lambda^{(2)} = \{6, 15\}, \tag{9.4}$$

by the categorization before. By (9.1), (9.3), (9.4), (30) and (42), we have $\Lambda^{(4)} \cap \{1, 9, 15\} = \emptyset$, which is a contradiction, since Category 1 cannot be used in $\Lambda^{(4)}$.

As a conclusion, for lattice packings $O + \Lambda$ the condition (5) cannot hold, which means that $\gamma(O, \Lambda) \geq 7/6$ holds for all lattice packings. In particular, since the lattice Λ generated by $\mathbf{a}_1 = (-\frac{2}{3}, 1, \frac{1}{3})$, $\mathbf{a}_2 = (\frac{1}{3}, -\frac{2}{3}, 1)$, $\mathbf{a}_3 = (1, \frac{1}{3}, -\frac{2}{3})$ given in [14], see also [1], can be easily verified to satisfy $\gamma(O, \Lambda) = 7/6$, we obtain

$$\gamma^*(O) = 7/6,$$

and Theorem 1 is proved. □

4 Several examples about Problem 1

In $\mathbb{E}^2$, it is known that the density of the densest lattice packing of a smoothed octagon is smaller than the density of the densest packing of a circular disk (see [7]) and the simultaneous packing-covering constant of a regular octagon is bigger than the simultaneous packing-covering constant of a circular disk (see [21; 23]). However, in $\mathbb{E}^3$, some evidence supports Ulam's conjecture (see [5]) which claims that the density of the densest packing of a convex body attains its minimum at balls. In this section, we present some examples about the simultaneous packing-covering analogy of Ulam's conjecture.

Suppose that C is a centrally symmetric convex body in $\mathbb{E}^3$. Let Λ be a lattice generated by $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$, let V denote the set $\{\mathbf{o}, \mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_1 + \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_3 - \mathbf{a}_2\}$, and let P denote the convex hull of V. Then we have the following criterion:

Lemma 4 (Zong [22]). *If $P \subset C + V$, then $C + \Lambda$ is a lattice covering.*

In 2000, Betke and Henk [1] discovered an algorithm by which one can determine the density of the densest lattice packing of any given three-dimensional convex polytope. As an application they calculated densest lattice packings of all regular and Archimedean polytopes. Applying this criterion to Betke and Henk's discoveries, we obtain the following examples:

Example 1. Let C_1 be the octahedron defined by $\{(x_1, x_2, x_3) : |x_1| + |x_2| + |x_3| \le 1\}$ and let Λ_1 denote the lattice with the basis $\mathbf{a}_1 = (2/3, 1, 1/3)$, $\mathbf{a}_2 = (-1/3, -2/3, 1)$ and $\mathbf{a}_3 = (-1, 1/3, -2/3)$. One can prove that $C_1 + \Lambda_1$ is a packing in $\mathbb{E}^3$ and that $\frac{7}{6}C_1 + \Lambda_1$ is a covering of $\mathbb{E}^3$. Therefore we have

$$\gamma^*(C_1) \le \frac{7}{6} < \sqrt{5/3} = \gamma^*(B^3).$$

Example 2. We take $\tau = \frac{1}{2}(\sqrt{5} + 1)$ and define

$$\begin{aligned}
C_2 &= \{(x_1, x_2, x_3) : |\tau x_1| + |x_2| \le 1, |\tau x_2| + |x_3| \le 1, |\tau x_3| + |x_1| \le 1\},\\
C_5 &= \Big\{(x_1, x_2, x_3) : |x_1| + |x_2| + |x_3| \le 1, |\tau x_1| + \Big|\frac{1}{\tau}x_3\Big| \le 1, |\tau x_2| + \Big|\frac{1}{\tau}x_1\Big| \le 1, |\tau x_3| + \Big|\frac{1}{\tau}x_2\Big| \le 1\Big\},\\
C_3 &= C_2 \cap C_5, \quad \text{and}\\
C_4 &= C_2 \cap \Big(\frac{7 + 12\tau}{(3 + 4\tau)(1 + \tau)}C_5\Big).
\end{aligned}$$

Usually, C_2, C_5, C_3 and C_4 are called a dodecahedron, an icosahedron, an icosidodecahedron and a truncated dodecahedron. Let $\Lambda_2 = \Lambda_3 = \Lambda_4$ be the lattice with the basis $\mathbf{a}_1 = (0, \frac{2}{1+\tau}, \frac{2}{1+\tau})$, $\mathbf{a}_2 = (\frac{2}{1+\tau}, 0, \frac{2}{1+\tau})$ and $\mathbf{a}_3 = (\frac{2}{1+\tau}, \frac{2}{1+\tau}, 0)$. One can prove that $C_i + \Lambda_i$ is a packing in $\mathbb{E}^3$ and $(\sqrt{5} - 1)C_i + \Lambda_i$ is a covering of $\mathbb{E}^3$ for $i = 2, 3, 4$. Therefore we have

$$\gamma^*(C_i) \le \sqrt{5} - 1 < \sqrt{5/3} = \gamma^*(B^3) \qquad \text{for } i = 2, 3, 4.$$

Example 3. We continue to use the notation of Example 2 and define

$$C_6 = C_5 \cap \Big(\frac{\frac{4}{3} + \tau}{1 + \tau}C_2\Big).$$

Usually, C_6 is called a truncated icosahedron. Let $\Lambda_5 = \Lambda_6$ be the lattice with the basis $\mathbf{a}_1 = (\frac{4}{3}, 0, 0)$, $\mathbf{a}_2 = (0, \frac{4}{3}, 0)$ and $\mathbf{a}_3 = (\frac{2}{3}, \frac{2}{3}, \frac{2}{3})$. One can prove that $C_i + \Lambda_i$ is a packing in $\mathbb{E}^3$ and $\sqrt{5/3}C_i + \Lambda_i$ is a covering of $\mathbb{E}^3$ for $i = 5$ and 6. Therefore we have

$$\gamma^*(C_i) \le \sqrt{5/3} = \gamma^*(B^3) \qquad \text{for } i = 5, 6.$$

Remark 1. It is interesting that for an icosahedron the optimal lattices for the packing density are no longer optimal for the simultaneous packing-covering constant. It is well-known (see [5; 20]) that for the unit ball B^3, the optimal lattices for the packing density are different from the optimal lattices for the covering density which are identical with the optimal lattices for the simultaneous packing-covering constant.

Example 4. Let C_0 denote the cube $\{(x_1, x_2, x_3) : |x_1|, |x_2|, |x_3| \le 1\}$. We define

$$C_7 = C_0 \cap (2C_1).$$

Usually C_7 is called a cubeoctahedron. Let Λ_7 denote the lattice with the basis $\mathbf{a}_1 = (2, -\frac{1}{3}, -\frac{1}{3})$, $\mathbf{a}_2 = (-\frac{1}{3}, 2, -\frac{1}{3})$ and $\mathbf{a}_3 = (-\frac{1}{3}, -\frac{1}{3}, 2)$. One can prove that $C_7 + \Lambda_7$ is a packing in $\mathbb{E}^3$ and $\frac{7}{6}C_7 + \Lambda_7$ is a covering of $\mathbb{E}^3$. Therefore we have

$$\gamma^*(C_7) \le \frac{7}{6} < \sqrt{5/3} = \gamma^*(B^3).$$

Example 5. We define

$$C_8 = \{(x_1, x_2, x_3) : |x_1| + |x_2| \le 2 + 3\sqrt{2}, |x_1| + |x_3| \le 2 + 3\sqrt{2}, |x_2| + |x_3| \le 2 + 3\sqrt{2}\} \cap (2\sqrt{2} + 1)C_0 \cap (3\sqrt{2} + 3)C_1.$$

Usually C_8 is called a truncated cubeoctahedron. Let Λ_8 denote the lattice with a basis $\mathbf{a}_1 = (7.6568\cdots, -2.0339\cdots, 2.0339\cdots)$, $\mathbf{a}_2 = (1.5185\cdots, 0.6901\cdots, 7.6568\cdots)$ and $\mathbf{a}_3 = (6.1383\cdots, 5.6228\cdots, 2.7241\cdots)$. One can prove that $C_8 + \Lambda_8$ is a packing in $\mathbb{E}^3$ and $\sqrt{5/3}C_8 + \Lambda_8$ is a covering of $\mathbb{E}^3$. Therefore we have

$$\gamma^*(C_8) \le \sqrt{5/3} = \gamma^*(B^3).$$

Example 6. We define

$$C_9 = \{(x_1, x_2, x_3) : |x_1| + |x_2| \le 2, |x_1| + |x_3| \le 2, |x_2| + |x_3| \le 2\} \cap \sqrt{2}C_0 \cap (4 - \sqrt{2})C_1.$$

Usually C_9 is called a rhombic cubeoctahedron. Let Λ_9 denote the lattice generated by $\mathbf{a}_1 = (0, 2, 2)$, $\mathbf{a}_2 = (2, 0, 2)$ and $\mathbf{a}_3 = (2, 2, 0)$. It was shown in [1] that the density of the densest lattice packing of C_9 is attained at $C_9 + \Lambda_9$. However, the simultaneous packing-covering constant of $C_9 + \Lambda_9$ is $\sqrt{2}$, which is much bigger than $\sqrt{5/3}$. On the other hand, let Λ_9^* denote the body cubic center lattice generated by $\mathbf{a}_1 = (\frac{4(4-\sqrt{2})}{3}, 0, 0)$, $\mathbf{a}_2 = (0, \frac{4(4-\sqrt{2})}{3}, 0)$ and $\mathbf{a}_3 = (\frac{2(4-\sqrt{2})}{3}, \frac{2(4-\sqrt{2})}{3}, \frac{2(4-\sqrt{2})}{3})$. It can be verified that the simultaneous packing-covering constant of $C_9 + \Lambda_9^*$ is between $\sqrt{5/3}$ and $\sqrt{2}$.

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