NON-LOCAL OPERATORS WITH LOW SINGULARITY KERNELS: REGULARITY ESTIMATES AND MARTINGALE PROBLEM

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ABSTRACT. We consider the linear non-local operator $\mathcal{L}$ denoted by

$$\mathcal{L}u(x) = \int_{\mathbb{R}^d} \left(u(x+z) - u(x)\right) a(x,z) J(z) \, \mathrm{d}z.$$

Here $a(x,z)$ is bounded and $J(z)$ is the jump kernel of a Lévy process, which only has a low-order singularity near the origin and does not allow for standard scaling. The aim of this work is twofold. Firstly, we introduce generalized Orlicz-Besov spaces tailored to accommodate the analysis of elliptic equations associated with $\mathcal{L}$, and establish regularity results for the solutions of such equations in these spaces. Secondly, we investigate the martingale problem associated with $\mathcal{L}$. By utilizing analytic results, we prove the well-posedness of the martingale problem under mild conditions. Finally, we obtain a new Krylov-type estimate for the martingale solution through the use of a Morrey-type inequality for generalized Orlicz-Besov spaces.

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Keywords: Non-local operator, Orlicz space, Besov space, Morrey inequality, Subordinate Brownian motion, Martingale problem, Krylov estimate

1. Introduction

Over the past two decades, extensive research efforts have been dedicated to investigating a class of non-local operators related to jump Markov processes, called α stable-like operators, which can be formulated as:

$$\mathscr{L}u(x) = \int_{\mathbb{R}^d} (u(x+z) - u(x) - \nabla u(x) \cdot z \mathbf{1}_{B_1}(z)) \frac{a(x,z)}{|z|^{d+\alpha}} \, \mathrm{d}z, \quad \alpha \in (0,2) \qquad (1.1)$$

(cf. [Bas88]). Here $a : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ is a bounded measurable function, and $u \in C_b^\infty$. When a is smooth, it is known that there exists a pure jump Markov process X generated by $\mathscr{L}$, and $a(x,z)|z|^{-d-\alpha}$ is the intensity of jumps from x to $x+z$ (cf. [Jac05]).

The exploration of regularity properties for linear and non-linear equations corresponding to the above non-local operators has been the subject of extensive investigation. In particular, Schauder estimates for linear equations were investigated in [BK15, Bas09, DK13, LZ22, MP14], while the non-linear cases were studied in [DJZ18, DZ18]; L^p-estimates for non-local equations are available in [DJK23, DK12, MP92, Zha13]; Harnack inequalities for positive $\mathscr{L}$-harmonic functions and Hölder continuity of $\mathscr{L}$-harmonic functions were first established in [BL02] under the assumption that the function a satisfies certain boundedness conditions. For more related results, readers can refer to [CCV11, CS09, CK03, DK20, Kas09] and the references therein.

In the context of jump Markov processes, several studies have focused on the martingale problem for $\mathscr{L}$ given by (1.1), including [AK09, BT09, CZ16, MP14], among others. Related problems involving stochastic differential equations driven by rectilinear α-stable processes were also explored in [BC06] and [CZZ21].

In this paper, we consider a class of non-local operators including the following one:

$$Lu(x) = \int_{\mathbb{R}^d} (u(x+z) - u(x)) \, a(x,z) \frac{\mathbf{1}_{B_1}(z)}{|z|^d} \, \mathrm{d}z. \qquad (1.2)$$

Here $a : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ is a non-negative bounded measurable function and B_1 is the unit ball in $\mathbb{R}^d$.

Comparing (1.1) and (1.2), one can roughly think of (1.2) as being about the "$\alpha = 0$" case, which turns out to be a more challenging situation. Although there is limited research on this type of operators compared to the stable case, we have noted that several significant papers have examined issues related to operators like (1.2) from various perspectives, highlighting their important applications in different fields. Notably, Šikić-Song-Vondraček [ŠSV06] first studied the potential theory of subordinate Brownian motions with geometric stable subordinators, whose generator is given by $\log(I + (-\Delta)^{\alpha/2})$ (below we will see that $\log(I + (-\Delta)^{\alpha/2})$ is a variant of L). They established the asymptotic behaviors of the Green function and the Lévy density of these processes. Among the PDE articles, Chen-Weth [CW19] demonstrated

that

$$\frac{\mathrm{d}}{\mathrm{d}s}(-\Delta)^s u(x)\Big|_{s=0} = \log(-\Delta)u(x)$$

$$= -c_d \int_{\mathbb{R}^d} (u(x+z) - u(x))\frac{\mathbf{1}_{B_1}(z)}{|z|^d} \,\mathrm{d}z - c_d \int_{|z|\geqslant 1} \frac{u(x+z)}{|z|^d} \,\mathrm{d}z + \rho_d u(x),$$

where c_d and ρ_d are constants only depending on d. So the dominating term of $\frac{\mathrm{d}}{\mathrm{d}s}(-\Delta)^s\big|_{s=0}$ is $-c_d L$, as defined in (1.2) with $a = 1$. In the same work, the authors studied the eigenvalue of Dirichlet problem for the logarithmic Laplacian. Later, Jarohs-Saldaña-Weth [JSnW20] offered a novel look at the fractional Poisson problem via the logarithmic Laplacian given above. We also mention that there is some literature revealing the importance of logarithmic regularization operator $T := \log^{-1}(eI - \Delta)$ (T^{-1} is also a variant of L). For instance, T is widely used in the studies of the local (global) well-posedness of the regularized inviscid models, such as the regularized 2D Euler equation, the surface quasi-geostrophic equation, and the Boussinesq equations (see [CW12] and the references therein).

Our work is partially inspired by Kassmann-Mimica[KM17], where Kassmann and Mimica considered a much larger class of non-local operators including (1.1) and (1.2). To overcome the difficulties arising from the lack of the scaling invariant property of L, they introduced a new intrinsic scale, which can be used to establish a growth lemma for L and a modification of hitting probability estimate for Markov process corresponding to L. As an application, $\log^{-\beta}(1/r)$-order (for some $0 < \beta < 1$) regularity estimate for solutions to non-local elliptic equation $Lu = f$ was obtained under the assumption that the coefficient a only satisfies some boundedness conditions (see also [Mim14] for the case $a = 1$). In parallel, the study conducted by Chen-Zhang [CZ14] delves into parabolic equations with drift terms. However, an unresolved question remains: can the continuity modulus of u reach $\log^{-1}(1/r)$ when we impose continuity conditions on a, along with the condition that $Lu \in L^\infty$?

On the other hand, while the well-posedness of the martingale problem for the α stable-like operator $\mathscr{L}$ ($0 < \alpha < 2$) as defined in equation (1.1) has been extensively studied in various literature, to the best of our knowledge, there is a notable absence of research on the martingale problem for L. However, the existence and uniqueness of martingale solutions associated with L appear as premise assumptions in some works, such as [CZ14] and [KM17]. So it is necessary to provide sufficient conditions for the establishment of these assumptions.

Therefore, in this paper, we aim to address the following two questions:

($\mathbf{Q}_1$) Does the solution u to the equation $Lu = f$ exhibit additional smoothness under certain continuity conditions satisfied by a?

($\mathbf{Q}_2$) Is it possible to identify a unique strong Markov process X, for which the infinitesimal generator matches L when acting upon smooth functions, even when a only meets some minimal conditions?

Recall that, for a second order, uniformly elliptic operator $Au = \sum_{i,j} a_{ij}\partial_{ij}u$, the standard Schauder estimates and L^p-estimates are as follows:

(a) For any $\alpha \in (0,1)$, there is a constant C such that for each $u \in C^{2,\alpha}$,

$$\|u\|_{C^{2+\alpha}} \leqslant C\left(\|Au\|_{C^\alpha} + \|u\|_\infty\right),$$

provided that $a \in C^\alpha$;

(b) For any $p \in (1,\infty)$, there is a constant C such that for each $u \in W_p^2$,

$$\|u\|_{W_p^2} \leqslant C\left(\|Au\|_p + \|u\|_p\right),$$

provided that a is uniformly continuous (see [HL11]). Both of these results are useful in the study of diffusion processes. In particular, (b) can be directly used to prove the uniqueness of the martingale problem corresponding to A when a is merely continuous (see Stroock and Varadhan [SV07]).

Following the path of Stroock and Varadhan, in response to $(\mathbf{Q}_1)$ and $(\mathbf{Q}_2)$, in this work, we

- offer a series of analytical results for the elliptic equation

$$\lambda u - Lu = f$$

  encompassing Schauder-type estimates (see Theorem 1.1 (a) below).
- construct the strong Markov process associated with L (see Theorem 1.4).
- provide an a priori estimate for the solutions to the corresponding elliptic equation within a generalized Orlicz-Besov space (see Lemma 4.4), facilitating the derivation of a Krylov-type estimate (or occupation time estimate) for the jump processes (see Theorem 1.1 (b)).

1.1. **Main results and examples.** Although an important example of an operator we have in mind is (1.2), we will consider a more general class of non-local operators that naturally arise from jump Markov processes. To state our main theorems, we must first introduce some fundamental concepts from probability theory. Specifically, let $S = (S_t)_{t \geqslant 0}$ denote a driftless subordinator, which is a Lévy process with non-decreasing paths and non-negative jumps. Its behavior is fully characterized by its Lévy measure Π, defined on $(0, \infty)$, satisfying the condition $\int_0^\infty (1 \wedge t)\,\Pi(\mathrm{d}t) < \infty$. We assume that the Laplace exponent for S is given by

$$\phi(s) = -\log \mathbf{E}\mathrm{e}^{-sS_1} = \int_0^\infty (1 - \mathrm{e}^{-st})\Pi(\mathrm{d}t) = s\int_0^\infty \mathrm{e}^{-st}\Pi((t,\infty))\mathrm{d}t. \tag{1.3}$$

Let B be a d-dimensional Brownian motion independent of S. Set

$$Z_t = \sqrt{2}B_{S_t},$$

which is a pure jump Lévy process known as the subordinate Brownian motion corresponding to S. Its jump kernel $J(z)$ is rotationally symmetric, meaning that there is a function $j : (0, \infty) \to (0, \infty)$ such that

$$J(z) = j(|z|) = \int_0^\infty (4\pi t)^{-\frac{d}{2}}\mathrm{e}^{-\frac{|z|^2}{4t}}\,\Pi(\mathrm{d}t). \tag{1.4}$$

The Lévy exponent of Z is given by the formula

$$\psi(\xi) := -\log \mathbf{E}\mathrm{e}^{i\xi \cdot Z_1} = \phi(|\xi|^2).$$

With a slight abuse of notation, the function ψ can be viewed as a function from $\mathbb{R}_+$ to $\mathbb{R}_+$. Given ψ, we put

$$\rho(r) = \rho_\psi(r) := \frac{1}{\psi(r^{-1})}. \tag{1.5}$$

We study the following spatial inhomogeneous non-local operator:

$$\mathcal{L}u = \int_{\mathbb{R}^d} \left(u(x + z) - u(x)\right) a(x, z)J(z)\,\mathrm{d}z. \tag{1.6}$$

Our assumptions on ϕ (or Π) and a are

Assumption 1. ($\boldsymbol{\Phi}$) : ϕ *is a slowly varying function at infinity (see Definition 2.7) and* $\lim_{s\to\infty}\phi(s)=\infty$;

($\mathbf{A_1}$) : *There are positive real numbers* $\rho_0, c_0 \in (0,1)$ *such that*

$$a(x,z) \geqslant c_0, \quad \text{for every } x \in \mathbb{R}^d, z \in B_{\rho_0} = \{x \in \mathbb{R}^d : |x| < \rho_0\}; \tag{L_a}$$

($\mathbf{A_2}$) : *There are real numbers* $\alpha > 0, c_0 \in (0,1)$ *such that*

$$\|a(\cdot, z)\|_{C_\rho^\alpha} \leqslant c_0^{-1}, \quad \text{for every } z \in \mathbb{R}^d, \tag{H_a}$$

where ρ is given by (1.5) *and C_ρ^α is the generalized Hölder-type space defined in Section 3 (Definition 3.5).*

We emphasize that many important examples satisfy the above conditions, such as the core examples in this paper: $\log(I - \Delta)$ and L in (1.2).

We now can formulate our first main result:

Theorem 1.1 (Schauder-type estimates). *Suppose that ϕ satisfies* ($\boldsymbol{\Phi}$), *and a satisfies* ($\mathbf{A_1}$)-($\mathbf{A_2}$). *Then there exists a constant $\lambda_0 > 0$ such that for each $\lambda \geqslant 2\lambda_0$ and $f \in C_\rho^\alpha$, the following Poisson equation*

$$\lambda u - \mathcal{L}u = f \tag{PE}$$

admits a unique solution in $C_\rho^{1+\alpha}$. Moreover, for any $\beta \in [0, \alpha]$,

$$\lambda \|u\|_{C_\rho^\beta} + \|u\|_{C_\rho^{1+\beta}} \leqslant C\|f\|_{C_\rho^\beta},$$

where C only depends on $d, \psi, \rho_0, c_0, \alpha$ and β.

Remark 1.2. *(i) The method employed in this article is quite robust. With some modifications, most of the results can be readily extended to the case where ϕ is a general regularly varying function (see Definition 2.7). However, for the sake of readability of the article, we only consider the case where ϕ is slowly varying at infinity.*

(ii) $\rho(|x-y|)$ can be regarded as the "intrinsic" distance between x and y, corresponding to the operator $\mathcal{L}$. Consider the example where $\phi(s) = \sqrt{s}$, then $\mathcal{L} = -\sqrt{-\Delta}$ and $\rho(|x-y|)$ represents the usual Euclidean distance. In this context, the result analogous to Theorem 1.1 is: for any $\alpha \in (0,1)$,

$$\|u\|_{C^{1+\alpha}} \leqslant C\left(\|\sqrt{-\Delta}u\|_{C^\alpha} + \|u\|_{L^\infty}\right),$$

which was proved by Bass in [Bas09].

(iii) In this paper, we also establish some estimates for (PE) *in Orlicz-Besov spaces, considering cases where a is spatially homogeneous or satisfies a small oscillation condition. Refer to Theorem 4.1 and Lemma 4.4 for details.*

Theorem 1.1 implies that

Corollary 1.3. *Let $c_0 \in (0,1)$, $\alpha > 0$ and $\rho(r) = 1/\log(1 + \frac{1}{r})$. Suppose that $c_0 \leqslant a \leqslant c_0^{-1}$ and $|a(x,z) - a(x',z)| \leqslant c_0^{-1}\rho^\alpha(|x - x'|)$, then for each $\beta \in [0, \alpha]$, it holds that*

$$\|u\|_{C_\rho^{1+\beta}} \leqslant C\left(\|Lu\|_{C_\rho^\beta} + \|u\|_\infty\right), \quad \forall u \in C_\rho^{1+\beta}.$$

Here L is the operator defined in (1.2), *and C only depends on d, ρ_0, c_0, α and β.*

In Section 2.3, we will verify that the following key examples satisfy our assumptions.

Example 1. *One notable example of S is the gamma subordinator, whose Lévy measure, density and Laplace exponent are given by*

$$\Pi(\mathrm{d}t) = \mathrm{e}^{-t}t^{-1}\mathrm{d}t, \ \ \mathbf{P}(S_t = u) = \frac{1}{\Gamma(t)}u^{t-1}\mathrm{e}^{-u} \ \ and \ \ \phi(s) = \log(1+s),$$

respectively. In this case, the process $Z = \sqrt{2}B_{S_t}$ is called the variance gamma process whose infinitesimal generator is

$$\mathcal{L}u(x) = -\log(I - \Delta)u(x) = \int_{\mathbb{R}^d} (u(x+z) - u(x)) \frac{\mathrm{d}z}{|z|^d \ell(|z|)}.$$

Here $\ell : (0, \infty) \to (0, \infty)$ satisfies $\ell(r) \asymp 1$ $(r \to 0^+)$ (see more details after Proposition 2.10).

Example 2. *Another example is the infinitesimal generator of the solution to stochastic differential equation (SDE) driven by* variance gamma process $Z_t = \sqrt{2}B_{S_t}$*:*

$$\mathrm{d}X_t = \sigma(X_{t-})\,\mathrm{d}Z_t, \quad X_0 = x. \tag{1.7}$$

Assume that $\sigma : \mathbb{R}^d \to \mathbb{R}^{d \times d}$ is invertible. Then the infinitesimal generator $\mathcal{L}_\sigma$ of X is formulated by

$$\begin{aligned}
\mathcal{L}_\sigma u(x) &= \int_{\mathbb{R}^d} (u(x + \sigma(x)z) - u(x)) \underbrace{\frac{1}{|z|^d \ell(|z|)}}_{J(z)} \, \mathrm{d}z \\
&= \int_{\mathbb{R}^d} (u(x+z) - u(x)) \underbrace{\frac{|z|^d \ell(|z|)}{|\det \sigma(x)| \cdot |\sigma^{-1}(x)z|^d \ell(|\sigma^{-1}(x)z|)}}_{a(x,z)} J(z)\mathrm{d}z.
\end{aligned} \tag{1.8}$$

When σ is ρ^α-continuous and $c_0|\xi| \leqslant |\sigma\xi| \leqslant c_0^{-1}|\xi|$, for all $\xi \in \mathbb{R}^d$, then $a(x,z)$ satisfies $(\mathbf{A_1})$-$(\mathbf{A_2})$ (see the discussion after Proposition 2.10).

Although we consider general non-local operators with low singularity kernels in our study, it is worth mentioning that one of examples we have in mind is the operator L defined in (1.2).

Example 3. *We can rewrite L in (1.2) as*

$$Lu(x) = \int_{\mathbb{R}^d} (u(x+z) - u(x)) \underbrace{a(x,z)\mathbf{1}_{B_1}(z)\ell(|z|)}_{\widetilde{a}(x,z)} \underbrace{\frac{1}{|z|^d \ell(|z|)}}_{J(z)} \, \mathrm{d}z,$$

where J is the jump kernel of the variance gamma process given in Example 1. $\widetilde{a}$ satisfies $(\mathbf{A_1})$-$(\mathbf{A_2})$ if a satisfies the same conditions, due to the fact that $\ell(r)$ is bounded from below and above (see also the discussion after Proposition 2.10).

Our second main result concerns the weak well-posedness of equation (1.7) and relies on the martingale problem associated with non-local operators of the form (1.6), which includes $\mathcal{L}_\sigma$.

Theorem 1.4. *Suppose $(\mathbf{\Phi})$ and $(\mathbf{A_1})$-$(\mathbf{A_2})$ are satisfied. Then*

(a) *for each $x \in \mathbb{R}^d$, the martingale problem $(\mathcal{L}, \delta_x)$ has a unique solution $\mathbb{P}_x$, and the family $(\mathbb{P}_x, X)_{x \in \mathbb{R}^d}$ forms a strong Markov process on $\mathbb{R}^d$;*

(b) *for any N-function A (see Definition 2.1) satisfying $A(t) \gtrsim_\varepsilon [\psi^{-1}(t^{1+\varepsilon})]^d$ $(\forall \varepsilon > 0)$, the following Krylov-type estimate is valid*

$$\mathbb{E}_x \int_0^\infty e^{-\lambda t} f(X_t) \, dt \leqslant C \|f\|_A / \lambda, \quad \lambda > 0 \text{ and } f \in L_A. \tag{1.9}$$

Here $\|f\|_A$ is the Luxemburg norm of f with respect to A (see Definition 2.3), and C does not depend on f.

Corollary 1.5. *Let Z be the variance gamma process. Let $c_0 \in (0,1)$, $\alpha > 0$ and $\rho(r) = 1/\log(1 + r^{-1})$. Suppose $c_0|\xi| \leqslant |\sigma\xi| \leqslant c_0^{-1}|\xi|$, for all $\xi \in \mathbb{R}^d$, and*

$$|\sigma(x) - \sigma(x')| \leqslant c_0^{-1} \rho^\alpha(|x - x'|).$$

Then SDE (1.7) has a unique weak solution X. Moreover, for any $\lambda > 0$ and $\beta > 1$, X satisfies

$$\mathbf{E} \int_0^\infty e^{-\lambda t} f(X_t) \, dt \leqslant \frac{C}{\lambda} \inf \left\{ \lambda > 0 : \int_{\mathbb{R}^d} \left(\exp |f(x)/\lambda|^\beta - 1 \right) dx \leqslant 1 \right\}, \quad f \in L_A. \tag{1.10}$$

Here the constant C does not depend on f.

Remark 1.6. *(i) When we seek to compare the classical results of Stroock-Varadhan for diffusion processes (see [SV07]) with the aforementioned findings, a natural question emerges: can we relax the assumption $(\mathbf{A_2})$ in Theorem 1.4 to the point where $x \mapsto a(x, z)$ is only uniformly continuous? We believe the answer is yes, but currently, we do not have a solution to this problem. One obstacle is that the singular integrals may not be bounded in general Orlicz spaces (see Remark 5.2 below for further discussion);*

(ii) The inequalities (1.9) and (1.10) are commonly referred to as Krylov-type estimates. For nondegenerate Itô processes, the dominant term on the right-hand side of the inequality is $C \|f\|_{L^d} / \lambda$, which can be derived from the ABP maximal principle (see [Kry09]). On the other hand, for the Markov process associated with $\mathcal{L}$, the "diffusion" rate of the process is exceedingly sluggish, causing the occupation time $\mathbb{E}_0 \int_0^1 \mathbf{1}_{B_r}(X_t) \, dt$ to decay more slowly than r^s, for any $s > 0$, as $r \to 0$. Consequently, we require a stronger integrability condition for f in this case.

(iii) By leveraging the transformations outlined in Example 2, and using Theorem 1.4 we can address the weak well-posedness of SDE (1.7) driven by general subordinate Brownian. However, this approach requires verifying that $\ell(r)$ is Hölder continuous near $r = 0$, a condition that is generally non-trivial to validate. In some specific cases, asymptotic analysis of $\ell(r)$ near the origin offers a practical pathway to address this requirement (see more discussion after Proposition 2.10).

1.2. **Approach.** As mentioned before, by comparing (1.1) and (1.2), one can roughly interpret (1.2) as dealing with the "$\alpha = 0$" case, which turns out to be more challenging. Two of the reasons for this are:

(1) Considering the nearly integrable nature of the kernel $|z|^{-d} \mathbf{1}_{B_1}(z)$ (indeed $z \mapsto |z|^{-d} \mathbf{1}_{B_1}(z)$ is in weak L^1 space), it can be intuitively predicted that the regularity improvement of L^{-1} will be rather weak. As a result, it seems difficult to deal with such operator using standard elliptic theory;

(2) Although when $a = 1$, operator L corresponds to a nice Lévy process Y, it is essential to highlight that Y does not exhibit scaling invariance, a distinctive characteristic that sets it apart from α-stable processes. The lack of scaling invariance introduces a substantial challenge in the study of these Markov operators, as it renders the conventional scaling techniques, highly effective for α stable-like operators, but ineffective in this context.

In this work, we mainly use a modified Littlewood-Paley type decomposition and some tools from probability theory to deal with our two issues. In order to demonstrate the robustness of our approach, in Appendix A, we apply the classic Littlewood-Paley theory and scaling techniques to reprove and extend the main result of [Bas09] for stable-like operators. As can be seen from our proof, it is convenient to consider Schauder-type estimates in the Hölder-Zygmund space $\mathscr{C}^s$. Here $\mathscr{C}^s$ is defined using Δ_j, which is called the non-homgeneous dyadic blocks (cf. [BCD11]). When $s > 0$ with $s \notin \mathbb{N}$, it is well known that $\mathscr{C}^s$ coincides with the usual Hölder space C^s.

Naturally, we attempt to extend the above mentioned idea to the study of $\mathcal{L}$, but immediately face difficulties. The usual dyadic decomposition is no longer applicable to the situation we are concerned with. For example, for the operator $\mathcal{L} = -\log(I - \Delta)$, a naive substitute of $\mathscr{C}^s$ is the function space $\mathscr{X}^s$ defined by

$$\mathscr{X}^s = \left\{ u \in \mathscr{S}'(\mathbb{R}^d) : \sup_{j \geqslant -1} (2 + j)^s \|\Delta_j f\|_\infty < \infty \right\} \tag{1.11}$$

(see [CW12]). However, as we demonstrate in Appendix B, $\mathscr{X}^1$ contains unbounded discontinuous functions. Thus, classical decomposition theory is no longer applicable to the problems we are concerned with.

To overcome the aforementioned obstacle, we leverage the concept introduced in [KM17] and propose a novel decomposition for distributions, which we refer to as the "intrinsic dyadic decomposition" or "ψ-decomposition" (where ψ denotes any positive, increasing function, though we ultimately choose it as the Lévy exponent of Z for our purposes). This new decomposition replaces the role of Δ_j with the ψ-dyadic block Π_j^ψ, which is defined in Section 3.1. Additionally, the generalized Orlicz-Besov space $B_A^{\psi,s}$ (as presented in Section 2.1) supplants the position of the Besov space $B_{p,\infty}^s$ in classical theory. One of the key observations in proving Schauder-type estimates is Theorem 3.6, which serves as an analogy to Theorem A.2 and establishes that the generalized Hölder space C_ρ^s and $B_\infty^{\psi,s}$ are consistent.

The second issue at hand pertains to the inadequacy of scaling methods in proving Schauder estimates for the very low-order operators (further details are provided in Appendix A). Consequently, we begin by focusing on a specific subset of non-local operators, namely the infinitesimal generators of subordinate Brownian motions with slowly varying symbols. Using the favorable analytical properties of the subordinator S and the Gaussian kernel, we establish the desired a priori estimates in Orlicz-Besov spaces (as stated in Theorem 4.1). Subsequently, we gradually expand these estimates to encompass the general case using classical techniques developed for differential operators.

Regarding the martingale problem associated with $\mathcal{L}$, following [SV07], one can see that well-posedness is a consequence of the solvability of (PE) in generalized Hölder spaces. However, certain crucial properties of the Markov process corresponding to $\mathcal{L}$ cannot be inferred from estimates in Hölder-type spaces. For example, such estimates do not inform us whether

the occupation time $\int_0^T \mathbf{1}_B(X_s)\mathrm{d}s$ is nicely upper or lower bounded. To address this, we turn to L^∞-estimates for (PE) under the assumption that f belongs to some Orlicz space. The L^∞ bounds for solutions to (PE) are derived via a Morrey-type inequality (3.24) and an a priori estimate for solutions in Orlicz-Besov spaces (as described in Lemma 4.4). We note that although L^p-estimates can also be established (see for instance [KP25]), they are inadequate for obtaining the probabilistic results we require since the following embedding result fails for any $p < \infty$: $[I + \log(I - \Delta)]^{-1} L^p \hookrightarrow L^\infty$.

Organization: The paper is organized into several sections. Section 2 provides a review of Orlicz spaces, slowly varying functions, and some important properties of subordinators with slowly varying Laplace exponents. In Section 3, we introduce the ψ-decomposition, generalized Orlicz-Besov spaces, and present the proof of Theorem 3.6. This section also includes crucial Morrey-type inequalities. In Section 4, we present the proof of Theorem 1.1 and Corollary 1.3. Section 5 investigates the martingale problem associated with $\mathcal{L}$. Finally, to illustrate the technical difficulties we mentioned above and the need to introduce the new dyadic decomposition, we offer an appendix including a brief discussion on a theorem by Bass [Bas09] and a remark on the space $\mathscr{X}^s$.

Notations. Reciprocal of f is denoted by $\frac{1}{f}$ and f^{-1} denote the inverse function of f. The letter c or C with or without subscripts stands for an unimportant constant, whose value may change in different places. We use $a \asymp b$ to denote that a and b are comparable up to a constant, and use $a \lesssim b$ ($a \gtrsim b$) to denote $a \leqslant Cb$ ($a \geqslant Cb$) for some constant C.

2. Preliminaries

2.1. **Orlicz space.** In this section, we review some basic facts about Orlicz space. Most of the results can be found in [AF03, Chapter VIII].

The notion of Orlicz space extends the usual notion of L^p space. The function t^p entering the definition of L^p is replaced by a more general convex function $A(t)$, which is called an N-function.

Definition 2.1. *Let $\alpha(t)$ be a real valued function defined on $[0, \infty)$ and having the following properties:*
(a) $\alpha(0) = 0$, $\alpha(t) > 0$ if $t > 0$, $\lim_{t \to \infty} \alpha(t) = \infty$;
(b) $\alpha(t)$ is nondecreasing;
(c) $\alpha(t)$ is right continuous.
Then the real valued function A defined on $[0, \infty)$ by

$$A(t) = \int_0^t \alpha(s)\,\mathrm{d}s, \quad t \geqslant 0$$

is called an N-function.

Typical examples of N-functions include $A(t) = t^p$ for $1 < p < \infty$ and $A(t) = \mathrm{e}^{t^\beta} - 1$ for $\beta \geqslant 1$.

The Legendre transform of a convex function A on $[0, \infty)$ is given by

$$A_*(s) := \sup_{t \geqslant 0}[st - A(t)]. \tag{2.1}$$

By this, one can verify that $A_* : [0, \infty) \to [0, \infty)$ is also an N-function (cf. [AF03]). By A^{-1} we denote the inverse of A:

$$A^{-1}(s) = \inf \{t : A(t) > s\}, \quad s \geqslant 0.$$

The following result can be found in [AF03, page 265, equation (4)].

Proposition 2.2. *Assume that $A : [0, \infty) \to [0, \infty)$ is an N-function, then*

$$s \leqslant A_*^{-1}(s) \cdot A^{-1}(s) \leqslant 2s. \tag{2.2}$$

Given an N-function A, for each measurable function f on $\mathbb{R}^d$, we define

$$I_A(f) := \int_{\mathbb{R}^d} A(|f|(x)) \, dx.$$

Definition 2.3 (Luxemburg norm)**.** *Let A be an N-function. For any measurable function f, define*

$$\|f\|_A := \inf \{\lambda > 0 : I_A(f/\lambda) \leqslant 1\} \in [0, \infty].$$

The Orlicz space L_A is the collection of all functions with finite Luxemburg norm. L_A is a Banach space. In general, the space L_A is not necessarily separable unless A satisfies the Δ_2-condition (see [AF03, 8.6, page 266]). However, one can consider a subspace of L_A, denoted by E_A, which is defined as the closure in L_A of $L_A \cap C_0$, where C_0 denotes the space of continuous functions vanishing at infinity.

Orlicz spaces have good duality and interpolation properties, making them useful in many areas of mathematics, including probability theory, functional analysis, and PDEs. The following two propositions can also be found in [AF03, 8.11, page 269 and Lemma 8.17, page 272].

Proposition 2.4. *For all $f \in L_A$ and $g \in L_{A^*}$, it holds that*

$$\int |fg| \leqslant 2\|f\|_A \|g\|_{A^*}. \tag{2.3}$$

Proposition 2.5.

$$\|f\|_A \asymp \sup \left\{ \|fg\|_1 : \int_{\mathbb{R}^d} A_*(g(x)) \, dx \leqslant 1 \right\}.$$

The following Young's inequality is crucial.

Proposition 2.6 ([O'N65, Theorem 2.5])**.** *Assume that A, B and C are N-functions and that $A^{-1}(t)B^{-1}(t) \leqslant tC^{-1}(t)$ ($\forall t \geqslant 0$), then*

$$\|f * g\|_C \leqslant 2\|f\|_A \|g\|_B. \tag{2.4}$$

2.2. **Regular variation.**

Definition 2.7. *A measurable function $f : (0, \infty) \to (0, \infty)$ is regularly varying at infinity with index $\alpha \in \mathbb{R}$ (denoted by $f \in \mathcal{R}_\alpha(\infty)$), if for all $b > 0$,*

$$\lim_{x \to \infty} \frac{f(bx)}{f(x)} = b^\alpha.$$

If $\alpha = 0$, then we say that f is slowly varying. Regular (slow) variation at zero can be defined similarly, by changing $x \to \infty$ to $x \to 0^+$.

The following two Karamata's theorems (cf. [BGT87, Theorem 1.5.11]) will be used frequently.

Lemma 2.8 (Karamata's representation theorem)**.** *A function f is slowly varying if and only if there exists $M > 0$ such that for all $x \geqslant M$ the function can be written in the form*

$$f(x) = c(x) \exp\left(\int_M^x \frac{\varepsilon(t)}{t} dt\right)$$

where $c(x)$ is a bounded measurable function of a real variable converging to a finite number as x goes to infinity, and $\varepsilon(x)$ is a bounded measurable function of a real variable converging to zero as x goes to infinity.

Lemma 2.9 (Karamata's integral theorem)**.** *Let f be a slowly varying function at infinity. Then*

(i) for any $\sigma \geqslant -1$ and $M > 0$,

$$\lim_{s \to \infty} \frac{s^{\sigma+1} f(s)}{\int_M^s t^\sigma f(t)\, dt} = \sigma + 1;$$

(ii) for any $\sigma < -1$,

$$\lim_{s \to \infty} \frac{s^{\sigma+1} f(s)}{\int_s^\infty t^\sigma f(t)\, dt} = -(\sigma + 1).$$

2.3. **Subordinators and jump kernels.** Recall that S is a subordinator without drift. ϕ, the Laplace exponent of S is given by (1.3). Since $(st)e^{-st} \leqslant 1 - e^{-st}$ for each $s, t > 0$, by (1.3), we have

$$s\phi'(s) \leqslant \phi(s). \tag{2.5}$$

Moreover, (1.3) also implies

$$\phi(u^{-1}) \geqslant u^{-1} \int_0^u e^{-t/u}\Pi((t,\infty))dt \geqslant e^{-1}\Pi((u,\infty)), \quad u > 0. \tag{2.6}$$

Recall also that B is a d-dimensional Brownian motion independent of S, $Z_t = \sqrt{2}B_{S_t}$, and that

$$\psi(\xi) = -\log \mathbf{E}e^{i\xi \cdot Z_1} = \phi(|\xi|^2)$$

is the Lévy exponent of the subordinate Brownian motion Z. $J(z)$, the jump kernel of Z, is given by (1.4).

For any $\lambda \geqslant 0$, let

$$U_\lambda(du) = \int_0^\infty e^{-\lambda t}\, \mathbf{P}(S_t \in du)\, dt$$

be the λ-potential measure of S. By Fubini's theorem,

$$\begin{aligned}
\int_0^\infty e^{-\tau u} U_\lambda(du) &= \int_0^\infty e^{-\tau u} \int_0^\infty e^{-\lambda t}\mathbf{P}(S_t \in du)\, dt \\
&= \int_0^\infty e^{-\lambda t}\, dt \int_0^\infty e^{-\tau u}\mathbf{P}(S_t \in du) \\
&= \int_0^\infty e^{-(\lambda + \phi(\tau))t}\, dt = \frac{1}{\lambda + \phi(\tau)}.
\end{aligned}$$

This implies

$$\frac{1}{\lambda + \phi(\tau^{-1})} \geqslant \int_0^\tau e^{-\frac{u}{\tau}} U_\lambda(\mathrm{d}u) \geqslant e^{-1} U_\lambda((0,\tau)), \quad \tau > 0. \tag{2.7}$$

The following proposition will be used in the proof for our main results.

Proposition 2.10. *Suppose ϕ satisfies ($\mathbf{\Phi}$), then for any $r \ll 1$, it holds that*

$$\int_{B_r} |z| J(z) \, \mathrm{d}z \lesssim r\psi(r^{-1}), \tag{2.8}$$

and

$$\int_{B_r^c} J(z) \, \mathrm{d}z \lesssim \psi(r^{-1}). \tag{2.9}$$

Proof. For (2.8), in virtue of Fubini's theorem,(1.4) and (2.6), we have

$$\int_{B_r} |z| J(z) \, \mathrm{d}z \asymp \int_0^r j(s) s^d \, \mathrm{d}s \lesssim \int_0^r s^d \mathrm{d}s \int_0^\infty t^{-\frac{d}{2}} e^{-\frac{s^2}{4t}} \Pi(\mathrm{d}t)$$

$$= \int_0^\infty t^{-\frac{d}{2}} \Pi(\mathrm{d}t) \int_0^r s^d e^{-\frac{s^2}{4t}} \mathrm{d}s \asymp \int_0^\infty t^{\frac{1}{2}} \Pi(\mathrm{d}t) \int_0^{\frac{r^2}{4t}} u^{\frac{d-1}{2}} e^{-u} \mathrm{d}u$$

$$\asymp \int_0^{r^2} t^{\frac{1}{2}} \Pi(\mathrm{d}t) \int_0^{\frac{r^2}{4t}} u^{\frac{d-1}{2}} e^{-u} \mathrm{d}u + \int_{r^2}^\infty t^{\frac{1}{2}} \Pi(\mathrm{d}t) \int_0^{\frac{r^2}{4t}} u^{\frac{d-1}{2}} e^{-u} \mathrm{d}u$$

$$\asymp \int_0^{r^2} t^{\frac{1}{2}} \Pi(\mathrm{d}t) + r^{d+1} \int_{r^2}^\infty t^{-\frac{d}{2}} \Pi(\mathrm{d}t)$$

$$\lesssim \int_0^\infty \int_0^\infty \mathbf{1}_{\{s < t \leqslant r^2\}} s^{-\frac{1}{2}} \Pi(\mathrm{d}t) \, \mathrm{d}s + r\Pi((r^2, \infty))$$

$$\lesssim \int_0^{r^2} s^{-\frac{1}{2}} \Pi((s, \infty)) \mathrm{d}s + r\Pi((r^2, \infty))$$

$$\overset{(2.6)}{\lesssim} \int_0^{r^2} s^{-\frac{1}{2}} \phi(s^{-1}) \mathrm{d}s + r\phi(r^{-2}) \lesssim r\psi(r^{-1}), \quad 0 < r \ll 1.$$

Here we used Lemma 2.9 (ii) with $\sigma = -\frac{3}{2}$ and $s = r^{-2}$ in the last inequality.

For (2.9), again by Fubini's theorem, (2.6) and Lemma 2.9 (ii) with $\sigma = -2$ and $s = r^{-2}$, we have

$$\int_{B_r^c} J(z) \, \mathrm{d}z \asymp \int_r^\infty s^{d-1} j(s) \mathrm{d}s = \int_r^\infty s^{d-1} \mathrm{d}s \int_0^\infty t^{-\frac{d}{2}} e^{-\frac{s^2}{4t}} \Pi(\mathrm{d}t)$$

$$= \int_0^\infty t^{-\frac{d}{2}} \Pi(\mathrm{d}t) \int_r^\infty s^{d-1} e^{-\frac{s^2}{4t}} \mathrm{d}s \asymp \int_0^\infty \Pi(\mathrm{d}t) \int_{\frac{r^2}{4t}}^\infty u^{\frac{d}{2}-1} e^{-u} \mathrm{d}u$$

$$= \int_0^\infty u^{\frac{d}{2}-1} e^{-u} \mathrm{d}u \int_{\frac{r^2}{4u}}^\infty \Pi(\mathrm{d}t) \overset{(2.6)}{\lesssim} \int_0^\infty u^{\frac{d}{2}-1} e^{-u} \phi(4u/r^2) \, \mathrm{d}u$$

$$\lesssim \phi(r^{-2}) \int_0^{\frac{1}{4}} u^{\frac{d}{2}-1} e^{-u} \mathrm{d}u + \int_{\frac{1}{4}}^\infty u^{-2} \phi(4u/r^2) \mathrm{d}u$$

$$\lesssim \phi(r^{-2}) + r^{-2} \int_{r^{-2}}^\infty s^{-2} \phi(s) \mathrm{d}s \lesssim \psi(r^{-1}).$$

$\square$

To verify that Example 2 and Example 3 satisfy our assumptions, we need to conduct a more detailed analysis. The specific calculations are as follows.

For the gamma subordinator, recall that $\Pi(dt) = e^{-t}t^{-1}dt$. By (1.4),

$$j(r) = (4\pi)^{-\frac{d}{2}} \int_0^\infty t^{-\frac{d}{2}-1}e^{-\frac{r^2}{4t}-t}dt = \frac{1}{r^d\ell(r)},$$

which yields that

$$1/\ell(r) = (4\pi)^{-\frac{d}{2}}r^d \int_0^\infty t^{-\frac{d}{2}-1}e^{-\frac{r^2}{4t}-t}dt$$

and

$$(1/\ell(r))' = (4\pi)^{-\frac{d}{2}}d\ r^{d-1} \int_0^\infty t^{-\frac{d}{2}-1}e^{-\frac{r^2}{4t}-t}dt - (4\pi)^{-\frac{d}{2}}\frac{r^{d+1}}{2}\int_0^\infty t^{-\frac{d}{2}-2}e^{-\frac{r^2}{4t}-t}dt$$

For any $\gamma \in \mathbb{R}$, set

$$I_\gamma(r) := \int_0^\infty t^\gamma e^{-\frac{r^2}{4t}-t}dt.$$

By integration by parts formula, we get

$$\begin{aligned}I_\gamma(r) =& \frac{1}{\gamma+1}\int_0^\infty t^{\gamma+1}e^{-\frac{r^2}{4t}-t}(1-r^2/(4t^2))dt \\ =& \frac{I_{\gamma+1}(r)}{\gamma+1} - \frac{r^2}{4(\gamma+1)}I_{\gamma-1}(r), \quad \gamma \neq -1.\end{aligned}$$

Therefore,

$$1/\ell(r) = 2^{-d}\pi^{-\frac{d}{2}}r^d I_{-\frac{d}{2}-1}(r)$$

and

$$(1/\ell(r))' = (4\pi)^{-\frac{d}{2}}dr^{d-1}\left(I_{-\frac{d}{2}-1}(r) - \frac{r^2}{2d}I_{-\frac{d}{2}-2}(r)\right) = -2^{1-d}\pi^{-\frac{d}{2}}r^{d-1}I_{-\frac{d}{2}}(r).$$

Noting that

$$\begin{aligned}I_\gamma(r) = \int_0^\infty t^\gamma e^{-\frac{r^2}{4t}-t}dt = \int_0^{r^2/4} t^\gamma e^{-\frac{r^2}{4t}-t}dt + \int_{r^2/4}^\infty t^\gamma e^{-\frac{r^2}{4t}-t}dt \\ \asymp \begin{cases} r^{2\gamma+2}, & \gamma < -1 \\ -\log(r), & \gamma = -1 \quad\quad 0 < r \ll 1. \\ 1, & \gamma > -1, \end{cases}\end{aligned}$$

and $I_\gamma(r) \lesssim e^{-r/3}$, $r \gg 1$, we obtain

$$j(r) = 1/(r^d\ell(r)) \asymp I_{-\frac{d}{2}-1}(r) \asymp r^{-d}, \ r \to 0^+$$

and

$$j(r) = 1/(r^d\ell(r)) \lesssim r^{-d}e^{-r/3}, \ r > 0.$$

Moreover, we have

$$|\ell'(r)| \lesssim r^{d-1}I_{-\frac{d}{2}}(r) \lesssim 1, \ r > 0,$$

which implies that the coefficient a in (1.8) satisfies $(\mathbf{A_1})$ and $(\mathbf{A_2})$, provided that σ is ρ^α-continuous and $c_0|\xi| \leqslant |\sigma\xi| \leqslant c_0^{-1}|\xi|$ for all $\xi \in \mathbb{R}^d$.

3. Intrinsic dyadic decomposition, function spaces and embedding theorems

The contents of this section are purely analytical. Generalized Orlicz-Besov spaces, which is defined by a refined Littlewood-Paley decomposition called ψ-decomposition, are introduced. Such function space is a natural extension of Besov space $B^s_{p,\infty}$ (see [BCD11] for its definition). Crucial Morrey-type inequalities are also proved in this section.

3.1. **Intrinsic dyadic decomposition.** Let $\psi : (0,\infty) \to (0,\infty)$ be a strictly increasing function such that $\lim_{R\to\infty} \psi(R) = \infty$. We point out that in this section, ψ needs not be the Lévy exponent of a subordinate Brownian motion. Set

$$\psi^{-1}(r) := \inf\left\{s > 0 : \psi(s) > r\right\}, \ r > 0.$$

Let χ be a rotationally symmetric, nonnegative and smooth function with compact support such that

$$\chi(\xi) = \chi(|\xi|) = \begin{cases} 1 & \text{when } |\xi| \leqslant 3/4, \\ 0 & \text{when } |\xi| \geqslant 1. \end{cases}$$

Let $\varrho := \mathcal{F}^{-1}(\chi)$. We define

$$\bar{\chi}(\xi) = \chi(2\xi) \ \text{ and } \ \widetilde{\chi}(\xi) = \chi(\xi/2); \quad \bar{\varrho} = \mathcal{F}^{-1}(\bar{\chi}) \ \text{ and } \ \widetilde{\varrho} = \mathcal{F}^{-1}(\widetilde{\chi}).$$

Given $j \geqslant -1$, set

$$\chi_j(\xi) := \chi\left(\frac{\xi}{\psi^{-1}(2^j)}\right), \quad \varrho_j(x) := \mathcal{F}^{-1}(\chi_j)(x) = \left(\psi^{-1}(2^j)\right)^d \varrho\left(\psi^{-1}(2^j)x\right).$$

Similarly, we define

$$\bar{\chi}_j = \bar{\chi}\left(\frac{\xi}{\psi^{-1}(2^j)}\right) = \chi\left(\frac{2\xi}{\psi^{-1}(2^j)}\right), \quad \bar{\varrho}_j(x) = \left(\tfrac{1}{2}\psi^{-1}(2^j)\right)^d \varrho\left(\tfrac{1}{2}\psi^{-1}(2^j)x\right),$$

and

$$\widetilde{\chi}_j = \widetilde{\chi}\left(\frac{\xi}{\psi^{-1}(2^j)}\right) = \chi\left(\frac{\xi}{2\psi^{-1}(2^j)}\right), \quad \widetilde{\varrho}_j(x) = \left(2\psi^{-1}(2^j)\right)^d \varrho\left(2\psi^{-1}(2^j)x\right).$$

The ψ-dyadic block Π^ψ_j is formulated by

$$\Pi^\psi_j f := \mathcal{F}^{-1}((\chi_{j+1} - \chi_j)\mathcal{F}(f)) = (\varrho_{j+1} - \varrho_j) * f, \ j \geqslant 0$$

and

$$\Pi^\psi_{-1} f := \mathcal{F}^{-1}\left(\chi\left(\frac{\xi}{\psi^{-1}(1)}\right)\mathcal{F}(f)(\xi)\right) = \varrho_0 * f.$$

Similarly, for each $j \geqslant 0$, we define

$$\widetilde{\Pi}^\psi_j f := \mathcal{F}^{-1}((\widetilde{\chi}_{j+1} - \bar{\chi}_j)\mathcal{F}(f)).$$

Obviously, for each $j \geqslant 0$, it holds that

$$\widetilde{\Pi}^\psi_j \Pi^\psi_j = \Pi^\psi_j \widetilde{\Pi}^\psi_j = \Pi^\psi_j.$$

The following simple result will be used several times later.

Lemma 3.1. *Let A be an N-function or $A = \infty$. It holds that*

$$\|\nabla \Pi^\psi_j f\|_A \leqslant C\psi^{-1}(2^{j+1}) \|\Pi^\psi_j f\|_A, \quad j \geqslant -1. \tag{3.1}$$

Proof. Noting that $\Pi_j^\psi f = \widetilde{\varrho}_{j+1} * (\Pi_j^\psi f)$, by Young's inequality, we get

$$\|\nabla \Pi_j^\psi f\|_A \leqslant \|\nabla \widetilde{\varrho}_{j+1} * (\Pi_j^\psi f)\|_A \overset{(2.4)}{\leqslant} \|\nabla \widetilde{\varrho}_{j+1}\|_1 \|\Pi_j^\psi f\|_A$$
$$\leqslant C\psi^{-1}(2^{j+1}) \|\Pi_j^\psi f\|_A.$$

$\square$

3.2. **Generalized Orlicz-Besov and Orlicz-Bessel potential spaces.**

Definition 3.2. *Let $s \in \mathbb{R}$ and A be an N-function. The $\| \cdot \|_{\psi,s;A}$-norm of a distribution f is given by*

$$\|f\|_{\psi,s;A} := \sup_{j \geqslant -1} 2^{js} \|\Pi_j^\psi f\|_A,$$

and the collection of all distributions with finite $\| \cdot \|_{\psi,s;A}$-norm is denoted by $B_A^{\psi,s}$. If $A = \infty$, we define the space $B_\infty^{\psi,s}$ in the same way, and also denote it by $\mathscr{C}_\psi^s$. For simplicity, we denote $B_A^{\psi,1}$ ($\mathscr{C}_\psi^1$) as B_A^ψ ($\mathscr{C}_\psi$).

By the definition of $B_A^{\psi,s}$, it is easy to get the following interpolation result:

Lemma 3.3. *Suppose $s, s_0, s_1 \in \mathbb{R}$, $\theta \in (0,1)$ and $s = \theta s_0 + (1-\theta)s_1$. Then*

$$\|f\|_{B_A^{\psi,s}} \leqslant \|f\|_{B_A^{\psi,s_0}}^\theta \|f\|_{B_A^{\psi,s_1}}^{1-\theta}.$$

We also introduce the definition of Orlicz-Bessel potential spaces. Recall that E_A is defined in Definition 2.3.

Definition 3.4. *Let A be an N-function. The space*

$$H_A^\psi := \left\{ f \in \mathscr{S}'(\mathbb{R}^d) : \mathcal{F}^{-1}[(1 + \psi(\cdot))\mathcal{F}(f)] \in E_A \right\}$$

with the norm

$$\|f\|_{H_A^\psi} := \|(1 + \psi(\sqrt{-\Delta}))f\|_A$$

is called generalized Orlicz-Bessel potential space.

We remark that the space H_A^ψ is the collection of all functions $f \in E_A$ such that $\psi(\sqrt{-\Delta})f \in E_A$. Here we use the restricted space E_A instead of L_A for the following reasons. In fact, we will use the space H_A^ψ in the proof of Theorem 3.10, and we first prove Theorem 3.10 for 'smooth' functions (which are dense in E_A, but may not be dense in L_A) and then for general functions by the standard approximation arguments. On the other hand, E_A already contains enough 'good' functions in many cases (for instance in the case of the hypothesis of Corollary 3.12).

Definition 3.5. *Let $s > 0$. Let $\omega : (0,\infty) \to (0,\infty)$ be a strictly increasing function satisfying $\lim_{r \downarrow 0} \omega(r) = 0$. Assume $\omega^s(t)/t \to \infty$ as $t \to 0$. The $\| \cdot \|_{C_\omega^s}$-norm of a measurable function f is given by*

$$\|f\|_{C_\omega^s} := \|f\|_\infty + \sup_{x \neq y} \frac{|f(x) - f(y)|}{\omega^s(|x-y|)}.$$

The collection of all functions with finite $\| \cdot \|_{C_\omega^s}$-norm is denoted by C_ω^s, which is referred to as the generalized Hölder space.

Recall that

$$\rho(r) = \rho_\psi(r) := \frac{1}{\psi(r^{-1})}.$$

The following result gives a characterization of generalized Hölder space C_ρ^s in terms of ψ-decomposition.

Theorem 3.6. *Let $s > 0$. Let ρ be the function defined by (1.5). Assume*

$$r \mapsto \frac{r}{\psi^s(r)} \quad \text{is an increasing function and} \quad \int_\varepsilon^R \frac{\mathrm{d}r}{\psi^s(r)} \lesssim_\varepsilon \frac{R}{\psi^s(R)} \quad \text{for all } \varepsilon \ll 1 \ll R. \quad (3.2)$$

Then there is a constant $c \in (0,1)$ independent of f such that

$$c\|f\|_{\mathscr{C}_\psi^s} \leqslant \|f\|_{C_\rho^s} \leqslant c^{-1}\|f\|_{\mathscr{C}_\psi^s}. \quad (3.3)$$

Proof. We only prove the case that $s = 1$ (the proof for the general case is similar). Assume that $\|f\|_{\mathscr{C}_\psi} < \infty$. For any $|x - y| \ll 1$,

$$|f(x) - f(y)| \leqslant \sum_{j \geqslant -1} |\Pi_j^\psi f(x) - \Pi_j^\psi f(y)|$$

$$\lesssim \sum_{j \geqslant -1} \|\Pi_j^\psi f\|_\infty \wedge |x - y|\|\nabla \Pi_j^\psi f\|_\infty.$$

Thanks to (3.1), for each $j \geqslant -1$,

$$\|\nabla \Pi_j^\psi f\|_\infty \lesssim \psi^{-1}(2^{j+1})\|\Pi_j^\psi f\|_\infty.$$

So for each $K \in \mathbb{N}$, it holds that

$$|f(x) - f(y)| \lesssim \|f\|_{\mathscr{C}_\psi} \sum_{j \geqslant -1} 2^{-j} \wedge |x - y|\psi^{-1}(2^{j+1})2^{-j}$$

$$\lesssim \|f\|_{\mathscr{C}_\psi} \sum_{K+1}^\infty 2^{-j} + \|f\|_{\mathscr{C}_\psi}|x - y| \sum_{j=-1}^K \psi^{-1}(2^{j+1})2^{-j} \quad (3.4)$$

$$\lesssim \|f\|_{\mathscr{C}_\psi} \left(2^{-K} + |x - y| \int_0^{K+1} \psi^{-1}(2^t)2^{-t}\,\mathrm{d}t \right).$$

For $R \gg 1$, by integration by parts and (3.2), we have

$$\int_0^R \psi^{-1}(2^t)2^{-t}\mathrm{d}t \lesssim \int_1^{2^R} \psi^{-1}(r)r^{-2}\mathrm{d}r = -\int_{\psi^{-1}(1)}^{\psi^{-1}(2^R)} u\,\mathrm{d}\left(\frac{1}{\psi(u)}\right)$$

$$\sim \frac{u}{\psi(u)}\Big|_{\psi^{-1}(R)}^{\psi^{-1}(1)} + \int_{\psi^{-1}(1)}^{\psi^{-1}(2^R)} \frac{\mathrm{d}u}{\psi(u)} \lesssim \psi^{-1}(2^R)2^{-R}.$$

Choosing $R = K + 1 = [\log_2 \psi(|x - y|^{-1})]$, one sees that

$$2^{-K} + |x - y| \int_0^{K+1} \psi^{-1}(2^t)2^{-t}\mathrm{d}t \lesssim 2^{-K} \lesssim \frac{1}{\psi(|x - y|^{-1})}. \quad (3.5)$$

Plugging (3.5) into (3.4), we see that for any $|x - y| \ll 1$,

$$\frac{|f(x) - f(y)|}{\rho(|x - y|)} = |f(x) - f(y)|\psi(|x - y|^{-1}) \lesssim \|f\|_{\mathscr{C}_\psi}.$$

This together with the fact that $\|f\|_\infty \lesssim \|f\|_{\mathscr{C}_\psi}$ implies

$$\|f\|_{C_\rho^1} = \|f\|_\infty + \sup_{x \neq y} \frac{|f(x) - f(y)|}{\rho(|x - y|)} \lesssim \|f\|_{\mathscr{C}_\psi}.$$

On the other hand, suppose that

$$M = \sup_{x \neq y} \frac{|f(x) - f(y)|}{\rho(|x - y|)} + \|f\|_\infty < \infty.$$

By the definition of Π_j^ψ and the fact that $\int \varrho_j = \chi_j(0) = 1 \, (j \geqslant 0)$, we have

$$
\begin{aligned}
|\Pi_j^\psi f(x)| &= \left| \int_{\mathbb{R}^d} (\varrho_{j+1} - \varrho_j)(z) f(x + z) \, \mathrm{d}z \right| \\
&= \left| \int_{\mathbb{R}^d} (\varrho_{j+1} - \varrho_j)(z)(f(x + z) - f(x)) \, \mathrm{d}z \right| \\
&\leqslant \int_{\mathbb{R}^d} |\varrho(z)| \left[\left| f\left(x + \frac{z}{\psi^{-1}(2^j)}\right) - f(x) \right| + \left| f\left(x + \frac{z}{\psi^{-1}(2^{j+1})}\right) - f(x) \right| \right] \, \mathrm{d}z \\
&\leqslant M \int_{\mathbb{R}^d} |\varrho(z)| \left[1 \wedge \frac{1}{\psi\left(\frac{\psi^{-1}(2^j)}{|z|}\right)} \right] \, \mathrm{d}z + M \int_{\mathbb{R}^d} |\varrho(z)| \left[1 \wedge \frac{1}{\psi\left(\frac{\psi^{-1}(2^{j+1})}{|z|}\right)} \right] \, \mathrm{d}z \\
&\lesssim M \int_0^\infty \frac{\varrho(r) r^{d-1}}{1 \vee \psi\left(\frac{\psi^{-1}(2^j)}{r}\right)} \, \mathrm{d}r + M \int_0^\infty \frac{\varrho(r) r^{d-1}}{1 \vee \psi\left(\frac{\psi^{-1}(2^{j+1})}{r}\right)} \, \mathrm{d}r \quad (j \geqslant 0).
\end{aligned}
\tag{3.6}
$$

So we need to dominate the integration $\int_0^\infty \frac{\varrho(r) r^{d-1}}{1 \vee \psi\left(\frac{s}{r}\right)} \, \mathrm{d}r$. Using the fact that $|\varrho(t)| \lesssim_k (1 \wedge t^{-k}) \, (\forall k > 0)$, one sees that for each $s > 0$,

$$
\begin{aligned}
\int_0^\infty \frac{\varrho(r) r^{d-1}}{1 \vee \psi(s/r)} \, \mathrm{d}r &= s^d \int_0^\infty \frac{\varrho(s/t)}{t^{d+1}[1 \vee \psi(t)]} \, \mathrm{d}t \\
&\lesssim s^{d-k} \int_0^s \frac{t^k}{t^{d+1}[1 \vee \psi(t)]} \, \mathrm{d}t + s^d \int_s^\infty \frac{1}{t^{d+1}[1 \vee \psi(t)]} \, \mathrm{d}t \\
&= I_1 + I_2.
\end{aligned}
$$

Choosing $k = d + 2$ and using (3.2), we get

$$I_1 = s^{-2} \int_0^s \frac{t}{1 \vee \psi(t)} \, \mathrm{d}t \lesssim s^{-1} \int_0^s \frac{\mathrm{d}t}{1 \vee \psi(t)} \overset{(3.2)}{\lesssim} \frac{1}{1 \vee \psi(s)}.$$

Moreover,

$$I_2 = s^d \int_s^\infty \frac{1}{t^{d+1}[1 \vee \psi(t)]} \mathrm{d}t \leqslant \frac{1}{1 \vee \psi(s)} s^d \int_s^\infty t^{-d-1} \mathrm{d}t \lesssim \frac{1}{1 \vee \psi(s)}.$$

Thus, for each $s > 0$,

$$\int_0^\infty \frac{\varrho(r) r^{d-1}}{1 \vee \psi(s/r)} \, \mathrm{d}r \lesssim \frac{1}{1 \vee \psi(s)}. \tag{3.7}$$

Combining (3.6) and (3.7), we arrive

$$\|\Pi_j^\psi f\|_\infty \lesssim M 2^{-j}, \quad \text{for all } j \geqslant -1.$$

So we complete our proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

3.3. **Morrey-type inequalities.** In this section, we prove two Morrey-type inequalities, (3.11) and (3.24). The novelty of them is that both regularity and integrability of u are characterized by monotone functions ψ and A, respectively, and ψ can be a slowly varying function.

We need to make some necessary preparations. Let S be a subordinator with Laplace exponent ϕ. Recall Z is the subordinate Brownian motion corresponding to S. Here and below, the function ψ is given by the Lévy exponent of Z, i.e.,

$$\psi(\xi) = -\log \mathbf{E} e^{i\xi \cdot Z_1} = \phi(|\xi|^2).$$

In this section, we assume S satisfies

$(\mathbf{\Phi'_1})$: $\phi \in \mathcal{R}_{\frac{\alpha}{2}}(\infty)$ with $\alpha \in [0,1)$, i.e. $\psi \in \mathcal{R}_\alpha(\infty)$ with $\alpha \in [0,1)$.

Lemma 3.7. *Assume that ϕ satisfies $(\mathbf{\Phi'_1})$. Then for any $R \geqslant 1$, it holds that*

$$\int_1^R \frac{\mathrm{d}r}{\psi(r)} \lesssim \frac{R}{\psi(R)} \tag{3.8}$$

and

$$\frac{r}{\psi(r)} \lesssim \frac{R}{\psi(R)}, \quad r \in [1,R]. \tag{3.9}$$

Proof. If $\psi \in \mathcal{R}_\alpha(\infty)$ with $\alpha \in [0,1)$, then there exists a function $\ell \in \mathcal{R}_0(\infty)$ such that $\psi(r) = r^\alpha \ell(r)$. For $R \geqslant 1$, using Lemma 2.9 (i), we have

$$\int_1^R \frac{\mathrm{d}r}{\psi(r)} \lesssim \int_1^R \frac{\mathrm{d}r}{r^\alpha \ell(r)} \lesssim \frac{R^{1-\alpha}}{\ell(R)} \lesssim \frac{R}{\psi(R)}.$$

For (3.9). By Lemma 2.8, $\psi(r) = r^\alpha c(r) \exp(\int_1^r \varepsilon(t)/t \mathrm{d}t)$, with $c(r) \to c \in \mathbb{R}$ and $\varepsilon(r) \to 0$ as $r \to \infty$. Therefore,

$$\frac{r\psi(R)}{R\psi(r)} \lesssim \exp\left(\int_r^R \frac{\varepsilon(t) - (1-\alpha)}{t} \mathrm{d}t\right) \lesssim 1,$$

which implies our desired assertion. $\qquad\qquad\square$

3.3.1. *Morrey-type inequality for Orlicz-Bessel potential spaces.* The following Morrey-type inequality for Orlicz-Sobolev space W_A^1 can be found in [AF03].

Proposition 3.8. *Let A be an N-function. Assume that $\int_1^\infty \frac{A^{-1}(s^d)}{s^2} \mathrm{d}t < \infty$, then*

$$\|f\|_{C_\rho} \lesssim \|f\|_A + \|\nabla f\|_A,$$

where W_A^1 is the Orlicz-Sobolev space and

$$\rho(r) = \left(\int_{\frac{1}{r}}^\infty \frac{A^{-1}(s^d)}{s^2} \mathrm{d}s\right)^{-1}.$$

Remark 3.9. *As far as the authors are aware, there is no result guaranteeing that the Riesz transform is bounded in the Orlicz space L_A, so in general H_A^1 does not coincide with W_A^1. Therefore, $H_A^1 \hookrightarrow C_\rho$ cannot be derived from Proposition 3.8.*

The following result is about Morrey inequalities for generalized Orlicz-Bessel potential spaces.

Theorem 3.10. *Assume that ϕ satisfies $(\mathbf{\Phi_1'})$. Let A be an N-function satisfying*

$$\int_1^\infty \frac{\mathrm{d}A^{-1}(t^d)}{\psi(t)}\,\mathrm{d}t < \infty.$$

Let

$$\Psi(R) := \left(\int_R^\infty \frac{\mathrm{d}A^{-1}(t^d)}{\psi(t)}\,\mathrm{d}t\right)^{-1}. \tag{3.10}$$

Then it holds that

$$\|u\|_{\mathscr{C}_\Psi} \lesssim \|u\|_{H_A^\psi}. \tag{3.11}$$

Remark 3.11. *From a perspective of theoretical completeness, exploring Sobolev-type inequalities for generalized Orlicz-Bessel potential spaces can provide a more comprehensive understanding of these function spaces. However, while this inequality is of interest, it is not directly utilized in the proof of our main results. Therefore, we have chosen to defer the investigation of this topic to a future paper.*

Our result implies the classical Morrey's inequality for Bessel potential space:

Corollary 3.12. (a) *Let $\alpha \in (0,1)$. Assume $\psi(R) = \phi(R^2) = R^\alpha$ and $A(t) = t^p$ with $p > d/\alpha$. Then*

$$\|u\|_{C^{\alpha - \frac{d}{p}}} \lesssim \|u\|_{H_A^\psi} = \|u\|_{H_p^\alpha}.$$

(b) *Let $\psi(R) = \log(1 + R^2)$ and $A(t) = e^{t^\beta} - 1$ with $\beta > 1$. It holds that*

$$\|u\|_\infty + \sup_{|x-y| < \frac{1}{2}} |u(x) - u(y)| \cdot (-\log|x - y|)^{1 - \frac{1}{\beta}}$$
$$\lesssim \|u\|_{H_A^\psi} = \|u + \log(I - \Delta)u\|_A.$$

Proof. (a) Let $\phi(r) = r^{\frac{\alpha}{2}}$. Since

$$\rho_\Psi(r) = \frac{1}{\Psi(\frac{1}{r})} = \int_{\frac{1}{r}}^\infty \frac{\mathrm{d}A^{-1}(t^d)}{\psi(t)} \asymp \int_{\frac{1}{r}}^\infty t^{\frac{d}{p} - \alpha - 1}\,\mathrm{d}s \asymp r^{\alpha - \frac{d}{p}},$$

the conclusion follows by Theorem 3.10 and Theorem 3.6.

(b) By definition,

$$A^{-1}(t) = \log^{\frac{1}{\beta}}(1 + t).$$

For any $r \ll 1$,

$$\rho_\Psi(r) = \frac{1}{\Psi(\frac{1}{r})} = \int_{\frac{1}{r}}^\infty \frac{\mathrm{d}A^{-1}(t^d)}{\psi(t)} \asymp \int_{\frac{1}{r}}^\infty \frac{t^{d-1}\log^{\frac{1}{\beta} - 1}(1 + t^d)}{(1 + t^d)\log(1 + t^2)}\,\mathrm{d}t$$
$$\asymp \int_{r^{-d}}^\infty t^{-1}\log^{\frac{1}{\beta} - 2} t\,\mathrm{d}t \asymp \left(\log\frac{1}{r}\right)^{\frac{1}{\beta} - 1}.$$

Again by Theorem 3.10 and Theorem 3.6, we get the desired result. $\qquad\square$

Now we give the proof for Theorem 3.10 below.

Proof of Theorem 3.10. Recall that S is a subordinator with Laplace exponent ϕ, and that Z is the subordinate Brownian motion corresponding to S. For any $\lambda > 0$ and $f \in C_0(\mathbb{R}^d)$, set

$$P_t f(x) = \mathbf{E} f(x + Z_t) \quad \text{and} \quad G_\lambda f(x) = \int_0^\infty e^{-\lambda t} P_t f(x)\, \mathrm{d}t.$$

Then

$$(\lambda + \phi(-\Delta)) G_\lambda f = f, \quad f \in C_0(\mathbb{R}^d) \tag{3.12}$$

and

$$G_\lambda(\lambda + \phi(-\Delta)) u = u \text{ with } u = G_\lambda f, \quad f \in C_0(\mathbb{R}^d). \tag{3.13}$$

Since $f \in C_0 \cap L_A$ is dense in E_A, the operator G_λ then can be extended to a mapping from E_A to H_A^ψ. Moreover, (3.12) still holds for all $f \in E_A$, and $H_A^\psi = G_\lambda E_A$. Thus,

$$G_\lambda(\lambda + \phi(-\Delta)) u = u, \quad u \in H_A^\psi.$$

Let $f = u + \phi(-\Delta) u$. Then

$$u = G_1 f = \int_0^\infty e^{-t} P_t f\, \mathrm{d}t.$$

We need to prove

$$\|\Pi_j^\Psi u\|_\infty = \left\| \Pi_j^\Psi \int_0^\infty e^{-t} P_t f\, \mathrm{d}t \right\|_\infty \leqslant C 2^{-j} \|f\|_A. \tag{3.14}$$

Denote by h_t and p_t the transition probability density functions of Brownian motion $\sqrt{2} B_t$ and Z_t, respectively. Let $s_t(\mathrm{d}u)$ be the distribution of S_t. Let $\zeta_j(x) = (\Psi^{-1}(2^j))^d \varrho(\Psi^{-1}(2^j) x)$, where ϱ is the smooth function in Section 3.1. Thanks to Hölder's inequality (2.3), we have

$$\|\Pi_j^\Psi u\|_\infty \leqslant \left\| \Pi_j^\Psi \int_0^\infty e^{-t} P_t f\, \mathrm{d}t \right\|_\infty \leqslant \left\| \int_0^\infty e^{-t} p_t * (\zeta_{j+1} - \zeta_j) * f\, \mathrm{d}t \right\|_\infty \tag{3.15}$$
$$\lesssim \|f\|_A \int_0^\infty e^{-t} \|p_t * (\zeta_{j+1} - \zeta_j)\|_{A_*}\, \mathrm{d}t.$$

Thus, to prove (3.14), we need to estimate $\|p_t * (\zeta_{j+1} - \zeta_j)\|_{A_*}$.

Let $R' > R \gg 1$ and $\varrho^R(x) = R^d \varrho(Rx)$. Using the fact that

$$p_t(x) = \int_0^\infty h_u(x) s_t(\mathrm{d}u),$$

we get

$$p_t * (\varrho^{R'} - \varrho^R)(x)$$
$$= \int_{\mathbb{R}^d} p_t(x + z) R'^d \varrho(R'z)\, \mathrm{d}z - \int_{\mathbb{R}^d} p_t(x + z) R^d \varrho(Rz)\, \mathrm{d}z$$
$$= \int_{\mathbb{R}^d} \left[p_t\left(x + \frac{z}{R'}\right) - p_t\left(x + \frac{z}{R}\right) \right] \varrho(z)\, \mathrm{d}z \tag{3.16}$$
$$= \int_0^\infty \int_{\mathbb{R}^d} \left| h_u\left(x + \frac{z}{R'}\right) - h_u\left(x + \frac{z}{R}\right) \right| \varrho(z)\, s_t(\mathrm{d}u)\, \mathrm{d}z$$
$$= \int_0^\infty \int_{\mathbb{R}^d} u^{-\frac{d}{2}} \left| h_1\left(\frac{x}{\sqrt{u}} + \frac{z}{\sqrt{u} R'}\right) - h_1\left(\frac{x}{\sqrt{u}} + \frac{z}{\sqrt{u} R}\right) \right| s_t(\mathrm{d}u) \varrho(z)\, \mathrm{d}z.$$

Here we use the fact that $h_u(x) = u^{-\frac{d}{2}} h_1(x/\sqrt{u})$. By mean value theorem, one sees that

$$\left| h_1\left(\frac{x}{\sqrt{u}} + \frac{z}{\sqrt{u} R'}\right) - h_1\left(\frac{x}{\sqrt{u}} + \frac{z}{\sqrt{u} R}\right) \right|$$

$$\leqslant \left(\left| h_1\left(\frac{x}{\sqrt{u}} + \frac{z}{\sqrt{u}R'} \right) \right| + \left| h_1\left(\frac{x}{\sqrt{u}} + \frac{z}{\sqrt{u}R} \right) \right| \right) \wedge$$
$$\left| \frac{z}{\sqrt{u}R'} - \frac{z}{\sqrt{u}R} \right| \int_0^1 \left| \nabla h_1\left(\frac{x}{\sqrt{u}} + (1-\theta)\frac{z}{\sqrt{u}R'} + \theta\frac{z}{\sqrt{u}R} \right) \right| \, \mathrm{d}\theta.$$

This implies

$$u^{-\frac{d}{2}} \left\| h_1\left(\frac{\cdot}{\sqrt{u}} + \frac{z}{\sqrt{u}R'} \right) - h_1\left(\frac{\cdot}{\sqrt{u}} + \frac{z}{\sqrt{u}R} \right) \right\|_{A_*}$$
$$\lesssim \left\| u^{-\frac{d}{2}} h_1\left(\frac{\cdot}{\sqrt{u}} \right) \right\|_{A_*} \wedge \left| \frac{z}{\sqrt{u}R'} - \frac{z}{\sqrt{u}R} \right| \left\| u^{-\frac{d}{2}} \nabla h_1\left(\frac{\cdot}{\sqrt{u}} \right) \right\|_{A_*} \qquad (3.17)$$
$$\lesssim \left(1 \wedge \frac{z}{\sqrt{u}R} \right) \left(\|h_u\|_{A_*} + \left\| u^{-\frac{d}{2}} \nabla h_1\left(\frac{\cdot}{\sqrt{u}} \right) \right\|_{A_*} \right).$$

Setting

$$f_{A_*}(u) = \|h_u\|_{A_*} + \left\| u^{-\frac{d}{2}} \nabla h_1\left(\frac{\cdot}{\sqrt{u}} \right) \right\|_{A_*}, \qquad (3.18)$$

and combining (3.16) and (3.17), we get

$$\|p_t * (\varrho^{R'} - \varrho^R)\|_{A_*}$$
$$\lesssim \int_0^\infty \int_{\mathbb{R}^d} \left(1 \wedge \frac{z}{\sqrt{u}R} \right) \varrho(z) f_{A_*}(u) \, \mathrm{d}z \, s_t(\mathrm{d}u) \qquad (3.19)$$
$$\lesssim \int_0^\infty \left(\frac{1}{\sqrt{u}R} \int_0^{\sqrt{u}R} |\varrho(r)| r^d \, \mathrm{d}r + \int_{\sqrt{u}R}^\infty |\varrho(r)| r^{d-1} \, \mathrm{d}r \right) f_{A_*}(u) \, s_t(\mathrm{d}u).$$

Noting that $|\varrho(r)| \lesssim 1 \wedge r^{-k} \ (\forall k > 0)$, we have

$$\frac{1}{N} \int_0^N |\varrho(r)| r^d \, \mathrm{d}r \lesssim 1 \wedge \frac{1}{N} \ \text{ and } \ \int_N^\infty |\varrho(r)| r^{d-1} \, \mathrm{d}r \lesssim 1 \wedge \frac{1}{N}, \ \forall N > 0. \qquad (3.20)$$

Plugging estimate (3.20) into (3.19), we get

$$\|P_t(\varrho^{R'} - \varrho^R)\|_{A_*} \lesssim \int_0^\infty \left(1 \wedge \frac{1}{\sqrt{u}R} \right) f_{A_*}(u) \, s_t(\mathrm{d}u). \qquad (3.21)$$

So the main problem comes down to the estimation of $f_{A_*}(u)$. Let us estimate the first term on the right-hand side of (3.18). By the scaling property of h_u, one sees that

$$I_{A_*}(h_u/\lambda) \leqslant 1 \iff \int_{\mathbb{R}^d} A_*\left(\frac{h_1\left(x/\sqrt{u} \right)}{(\sqrt{u})^d \lambda} \right) \, \mathrm{d}x \leqslant 1$$
$$\iff \int_{\mathbb{R}^d} A_*\left(\frac{h_1\left(x \right)}{(\sqrt{u})^d \lambda} \right) \, \mathrm{d}x \leqslant u^{-\frac{d}{2}}. \qquad (3.22)$$

Let $K = (4\pi u)^{-d/2}$. Basic calculation yields

$$\int_{\mathbb{R}^d} A_* \left(\frac{h_1(x)}{(\sqrt{u})^d \lambda} \right) \mathrm{d}x = \int_{\mathbb{R}^d} A_* \left(\frac{\mathrm{e}^{-\frac{|x|^2}{4}}}{(\sqrt{4\pi u})^d \lambda} \right) \mathrm{d}x$$

$$= c_d \int_0^\infty A_* \left(\frac{\mathrm{e}^{-r^2/4}}{(\sqrt{4\pi u})^d \lambda} \right) r^{d-1} \, \mathrm{d}r$$

$$= c_d \int_0^{\frac{K}{\lambda}} A_*(r) \left(\log \frac{K}{\lambda} - \log r \right)^{\frac{d}{2}-1} r^{-1} \, \mathrm{d}r$$

$$= c_d \int_0^{\frac{K}{\lambda}} \left(\log \frac{K}{\lambda} - \log r \right)^{\frac{d}{2}-1} r^{-1} \, \mathrm{d}r \int_0^r \mathrm{d}A_*(s)$$

$$= c_d \int_0^{\frac{K}{\lambda}} \mathrm{d}A_*(s) \int_s^{\frac{K}{\lambda}} \left(\log \frac{K}{\lambda} - \log r \right)^{\frac{d}{2}-1} r^{-1} \, \mathrm{d}r$$

$$= c_d \int_0^{\frac{K}{\lambda}} \left(\log \frac{K}{\lambda} - \log s \right)^{\frac{d}{2}} D_- A_*(s) \, \mathrm{d}s,$$

where $D_- f$ is the left derivative of f. Noting that $D_- A_*$ is increasing and $D_- A_*(x)(y-x) \leqslant A_*(y) - A_*(x)$, we get

$$\int_{\mathbb{R}^d} A_* \left(\frac{h_1(x)}{(\sqrt{u})^d \lambda} \right) \mathrm{d}x \leqslant c_d \int_0^\infty u^{\frac{d}{2}} \mathrm{e}^{-u} \, \mathrm{d}u \cdot D_- A_* \left(\frac{K}{\lambda} \right) \frac{K}{\lambda}$$

$$\leqslant c_d \left[A_* \left(\frac{2K}{\lambda} \right) - A_* \left(\frac{K}{\lambda} \right) \right] \leqslant c_d A_* \left(\frac{2K}{\lambda} \right).$$

Using (3.22), we have

$$I_{A_*}(h_u/\lambda) \leqslant 1 \Longleftarrow A_* \left(\frac{2K}{\lambda} \right) \leqslant CK.$$

This yields

$$\|h_u\|_{A_*} = \inf \left\{ \lambda : I_{A_*}(h_u/\lambda) \leqslant 1 \right\} \leqslant \frac{2K}{A_*^{-1}(CK)} \lesssim \frac{u^{-d/2}}{A_*^{-1}(cu^{-d/2})}, \quad c \geqslant 1.$$

Similarly, we also have

$$\left\| u^{-\frac{d}{2}} \nabla h_1 \left(\frac{\cdot}{\sqrt{u}} \right) \right\|_{A_*} \lesssim \frac{u^{-d/2}}{A_*^{-1}(cu^{-d/2})}.$$

Thus, there exists a constant $c > 0$ such that

$$f_{A_*}(u) \lesssim \frac{u^{-d/2}}{A_*^{-1}(cu^{-d/2})}.$$

This together with (3.21) and (2.2) yields

$$\int_0^\infty \mathrm{e}^{-t} \| p_t * (\varrho^{R'} - \varrho^R) \|_{A_*} \, \mathrm{d}t$$

$$\lesssim \int_0^\infty \mathrm{e}^{-t} \int_0^\infty \frac{1 \wedge \frac{1}{\sqrt{u}R}}{u^{d/2} A_*^{-1}(cu^{-d/2})} \, s_t(\mathrm{d}u) \, \mathrm{d}t$$

$$\lesssim \int_0^{R^{-2}} A^{-1}(cu^{-d/2})U_1(\mathrm{d}u) + \frac{1}{R}\int_{R^{-2}}^\infty \frac{1}{\sqrt{u}}A^{-1}(cu^{-d/2})U_1(\mathrm{d}u)$$

$$=: I_1 + I_2.$$

where U_1 is the 1- potential measure of S_t.

Recall that $A(t) = \int_0^t \alpha(s)\mathrm{d}s$, we have $A^{-1}(t) = \int_0^t \beta(s)\mathrm{d}s$ with $\beta(s) = \frac{1}{\alpha(A^{-1}(s))}$.

For I_1, by Fubini theorem and (2.7), we have

$$I_1 \lesssim \int_0^{R^{-2}}\int_0^{cu^{-\frac{d}{2}}} \beta(s)\ \mathrm{d}s\ U_0(\mathrm{d}u)$$

$$= \int_0^\infty\int_0^\infty \mathbf{1}_{\{u<(s/c)^{-2/d}\wedge R^{-2}\}}\beta(s)\ U_0(\mathrm{d}u)\mathrm{d}s$$

$$= \int_0^\infty \beta(s)U_0((0,(s/c)^{-2/d}\wedge R^{-2}))\mathrm{d}s$$

$$= A^{-1}(cR^d)U_0((0,R^{-2})) + \int_{cR^d}^\infty U_0((0,(s/c)^{-2/d}))\mathrm{d}A^{-1}(s)$$

$$\overset{(2.7)}{\lesssim} \frac{A^{-1}(cR^d)}{\psi(R)} + \int_{cR^d}^\infty \frac{\mathrm{d}A^{-1}(s)}{\psi((s/c)^{1/d})} \lesssim \frac{A^{-1}(cR^d)}{\psi(R)} + \int_R^\infty \frac{\mathrm{d}A^{-1}(ct^d)}{\psi(t)}.$$

For I_2, again by Fubini theorem, (2.7), (3.8) and (3.9), we get

$$I_2 = \frac{1}{R}\int_{R^{-2}}^\infty \frac{1}{\sqrt{u}}A^{-1}(cu^{-d/2})U_1(\mathrm{d}u)$$

$$= \frac{1}{R}\int_0^\infty\int_0^\infty \mathbf{1}_{\{R^{-2}<u<(s/c)^{-2/d}\}}\frac{\beta(s)}{\sqrt{u}}\ U_1(\mathrm{d}u)\mathrm{d}s$$

$$\asymp \frac{1}{R}\int_{(0,\infty)^3} \mathbf{1}_{\{R^{-2}<u<(s/c)^{-2/d}\wedge t\}}\beta(s)t^{-\frac{3}{2}}U_1(\mathrm{d}u)\mathrm{d}s\mathrm{d}t$$

$$\lesssim \frac{1}{R}\int_{(0,\infty)^2} \mathbf{1}_{\{t>R^{-2}\}}\mathbf{1}_{\{s<cR^d\}}U_1(0,(s/c)^{-2/d}\wedge t)\beta(s)t^{-\frac{3}{2}}\mathrm{d}s\mathrm{d}t$$

$$\overset{(2.7)}{\lesssim} \frac{1}{R}\int_{(0,\infty)^2} \mathbf{1}_{\{t>R^{-2}\}}\mathbf{1}_{\{s<cR^d\}}\frac{1}{1+\phi((s/c)^{2/d}\vee t^{-1})}t^{-\frac{3}{2}}\beta(s)\ \mathrm{d}s\mathrm{d}t$$

$$\lesssim \frac{1}{R}\int_{R^{-2}}^\infty t^{-\frac{3}{2}}\left[\int_0^{ct^{-\frac{d}{2}}} \frac{\beta(s)}{1+\phi(t^{-1})}\mathrm{d}s + \int_{ct^{-\frac{d}{2}}}^{cR^d} \frac{\beta(s)}{1+\phi((s/c)^{2/d})}\mathrm{d}s\right]\mathrm{d}t$$

$$\lesssim \frac{1}{R}\int_0^R \frac{A^{-1}(cu^d)}{1+\psi(u)}\mathrm{d}u + \frac{1}{R}\int_0^{cR^d} \frac{\beta(s)}{1+\psi((s/c)^{1/d})}\int_{(\frac{s}{c})^{-\frac{2}{d}}}^\infty t^{-\frac{3}{2}}\ \mathrm{d}t\mathrm{d}s$$

$$\lesssim A^{-1}(cR^d)\frac{1}{R}\int_0^R \frac{1}{1+\psi(u)}\mathrm{d}u + \frac{1}{R}\int_0^{cR^d} \frac{s^{\frac{1}{d}}\mathrm{d}A^{-1}(s)}{1+\psi((s/c)^{1/d})}$$

$$\overset{(3.8),(3.9)}{\lesssim} \frac{A^{-1}(cR^d)}{\psi(R)}.$$

Noting that

$$\int_R^\infty \frac{\mathrm{d}A^{-1}(ct^d)}{\psi(t)} \gtrsim \int_R^{2R} \frac{\mathrm{d}A^{-1}(ct^d)}{\psi(t)} \gtrsim \frac{A^{-1}(cR^d)}{\psi(2R)} \gtrsim \frac{A^{-1}(cR^d)}{\psi(R)},$$

Therefore, there is a constant $R_0 > 1$ such that for all $R' > R \geqslant R_0$, it holds that

$$
\begin{aligned}
\int_0^\infty e^{-t} \| P_t(\varrho^{R'} - \varrho^R) \|_{A_*} \, dt &\lesssim \int_R^\infty \frac{d A^{-1}(ct^d)}{\psi(t)} \lesssim \int_R^\infty \frac{d A^{-1}(ct^d)}{\psi(t)} \\
&\lesssim \int_R^\infty \frac{d A^{-1}(t^d)}{\psi(t)} \overset{(3.10)}{=} \frac{1}{\Psi(R)}.
\end{aligned}
\tag{3.23}
$$

When $j \gtrsim \log_2 R_0$, by letting $R' = \Psi^{-1}(2^{j+1}) \gg 1$ and $R = \Psi^{-1}(2^j) \gg 1$, and plugging (3.23) to (3.15), we obtain

$$
\| \Pi_j^\Psi u \|_\infty \lesssim \frac{\|f\|_A}{\Psi\left(\Psi^{-1}(2^j)\right)} \lesssim 2^{-j} \|f\|_A.
$$

When $j \lesssim \log_2 R_0$, in view of (3.15), we have

$$
\| \Pi_j^\Psi u \|_\infty \lesssim \|f\|_A \sup_{j \lesssim \log_2 R_0} \|\zeta_j\|_A \lesssim \|f\|_A \lesssim 2^{-j} \|f\|_A.
$$

So we obtain (3.14), and complete our proof. $\qquad \square$

3.3.2. *Morrey-type inequality for generalized Orlicz-Besov spaces.* We also prove a Morrey inequality for generalized Orlicz-Besov spaces, which will be used in the proofs for our main results.

Theorem 3.13. *Assume S satisfies (Φ_1'). If A is an N-function satisfies $A(t) \geqslant [\psi^{-1}(ct^{1+\varepsilon})]^d$ for some $\varepsilon > 0$, then*

$$
\|u\|_{\mathscr{C}_\psi^{\frac{\varepsilon}{1+\varepsilon}}} \lesssim \|u\|_{B_A^\psi}.
\tag{3.24}
$$

Proof. For any $u \in B_A^\psi$ and $j \geqslant -1$, put $h_j = \varrho_{j+1} - \varrho_j \in L_{A_*}$ and $\widetilde{u}_j = \widetilde{\Pi}_j u \in L_A$. Note that for any $|x| \geqslant 2R \gg 1$,

$$
\begin{aligned}
|\Pi_j^\psi u(x)| &= \left| \int_{|y| \leqslant R} \widetilde{u}_j(y) h_j(x-y) \right| + \left| \int_{|y| > R} \widetilde{u}_j(y) h_j(x-y) \right| \\
&\lesssim \|\widetilde{u}_j\|_A \| h_j \mathbf{1}_{B_R^c} \|_{A_*} + \|\widetilde{u}_j \mathbf{1}_{B_R^c}\|_A \|h_j\|_{A_*} \to 0, \quad R \to \infty.
\end{aligned}
$$

Therefore, $\Pi_j^\psi u \in C_0 \cap L_A$, for any $u \in B_A^\psi$. Similarly, one can verify that $(1 + \phi(-\Delta)) \Pi_j^\psi u = (1 + \phi(-\Delta)) \Pi_j^\psi u_j \in C_0 \cap L_A$. Thus,

$$
\Pi_j^\psi u = G_1(1 + \phi(-\Delta)) \Pi_j^\psi u = G_1 \Pi_j^\psi (1 + \phi(-\Delta)) u, \quad u \in B_A^\psi.
$$

Put $f = (1 + \phi(-\Delta)) u$. Following the proof for (3.15), one can verify that

$$
\begin{aligned}
\| \Pi_j^\psi u \|_\infty &\leqslant \left\| \int_0^\infty e^{-t} P_t \, \widetilde{\Pi}_j^\psi \, (\Pi_j^\psi f) \, dt \right\|_\infty \\
&\leqslant \left\| \int_0^\infty e^{-t} p_t * (\widetilde{\varrho}_{j+1} - \bar{\varrho}_j) * (\Pi_j^\psi f) \, dt \right\|_\infty \\
&\lesssim \| \Pi_j^\psi f \|_A \int_0^\infty e^{-t} \| p_t * (\widetilde{\varrho}_{j+1} - \bar{\varrho}_j) \|_{A_*} \, dt.
\end{aligned}
$$

By our assumption $A(t) \geqslant [\psi^{-1}(ct^{1+\varepsilon})]^d$, we have

$$
A^{-1}(s) \leqslant C \psi^{\frac{1}{1+\varepsilon}}(s^{\frac{1}{d}}).
$$

Using this and following the proof for (3.23), we get

$$\int_0^\infty e^{-t}\|p_t * (\widetilde{\varrho}_{j+1} - \bar{\varrho}_j)\|_{A_*}\,dt$$

$$\lesssim \int_{\frac{1}{2}\psi^{-1}(2^j)}^\infty \frac{dA^{-1}(s^d)}{\psi(s)} \lesssim \int_{\frac{1}{2}\psi^{-1}(2^j)}^\infty \frac{\psi^{\frac{1}{1+\varepsilon}}(s)\psi'(s)}{\psi^2(s)}\,ds$$

$$\lesssim \frac{1}{\psi^{\frac{\varepsilon}{1+\varepsilon}}(\frac{1}{2}\psi^{-1}(2^j))} \lesssim 2^{-\frac{\varepsilon j}{1+\varepsilon}}.$$

Here we used the fact that for each $b > 0$,

$$\lim_{R\to\infty} \frac{\psi\big(b\psi^{-1}(R)\big)}{R} = \lim_{R\to\infty} \frac{\psi\big(b\psi^{-1}(R)\big)}{\psi\big(\psi^{-1}(R)\big)} = b^\alpha.$$

Thus,

$$2^{\frac{\varepsilon j}{1+\varepsilon}}\|\Pi_j^\psi u\|_\infty \lesssim \|\Pi_j^\psi f\|_A \lesssim \|u\|_{B_A^{\psi,1}}.$$

So we complete the proof for (3.24). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

4. Studies of Poisson equations

In this section, we study the Poisson equation (PE) in the generalized Orlicz-Besov space. Our analysis is divided into two parts: In the first part, we explore the case where the coefficient a is independent of x. Subsequently, we extended our findings to the more general case by using the method of frozen coefficients.

4.1. **Spatially homogeneous case.**

Theorem 4.1. *Let A be an N-function or $A = \infty$. Assume (Φ) holds. Suppose that $a : \mathbb{R}^d \to [0, \infty)$ is measurable,*

$$\int_{\mathbb{R}^d}(1 \wedge |z|^2)\, a(z)J(z)\,dz < \infty, \tag{4.1}$$

and there are positive numbers $\rho_0, c_0 \in (0, 1)$ such that

$$a(z) \geqslant c_0, \quad \text{for all } z \in B_{\rho_0}. \tag{4.2}$$

Then there exists a constant $\lambda_0 > 0$ such that for any $\lambda \geqslant \lambda_0$ and any $\alpha \in \mathbb{R}$,

$$\lambda\|u\|_{B_A^{\psi,\alpha}} + \|u\|_{B_A^{\psi,1+\alpha}} \leqslant C\|\lambda u - \mathcal{L}u\|_{B_A^{\psi,\alpha}}, \tag{4.3}$$

provided that $u \in B_A^{\psi,1+\alpha}$. Here C is a constant only depending on d, ψ, ρ_0, c_0 and α.

Remark 4.2. *We would like to emphasize that Theorem 4.1 does not require the assumption that $\alpha \geqslant 0$.*

Proof. Without loss of generality, we can assume that $c_0 = 1$ to simplify the analysis. By utilizing the condition in (4.1), we can identify a Lévy process Z whose infinitesimal generator corresponds to the operator $\mathcal{L}$. As presented in the proof for Theorem 3.13, for any $u \in B_A^\psi$, we have

$$\Pi_j^\psi u = G_\lambda(\lambda - \mathcal{L})\Pi_j^\psi u = G_\lambda\Pi_j^\psi(\lambda - \mathcal{L})u.$$

Using this and letting $f = \lambda u - \mathcal{L}u$, we obtain

$$
\begin{aligned}
\|\Pi_j^\psi u\|_A = \left\|\Pi_j^\psi G_\lambda f\right\|_A &= \left\|\Pi_j^\psi \int_0^\infty \mathrm{e}^{-\lambda t}(P_t f)\mathrm{d}t\right\|_A \\
&\leqslant \int_0^\infty \mathrm{e}^{-\lambda t}\|P_t \Pi_j^\psi f\|_A \,\mathrm{d}t \leqslant \lambda^{-1}\|\Pi_j^\psi f\|_A.
\end{aligned}
\tag{4.4}
$$

This yields

$$
\lambda\|u\|_{B_A^{\psi,\alpha}} \leqslant \|f\|_{B_A^{\psi,\alpha}} = \|\lambda u - \mathcal{L}u\|_{B_A^{\psi,\alpha}}.
\tag{4.5}
$$

Thus, it remains to show that the second term on the left-hand side of (4.3) can be dominated by $C\|\lambda u - \mathcal{L}u\|_{B_A^{\psi,\alpha}}$.

We divide the proof into several steps.

Step 1. We first consider the case that $a \equiv 1$. In this case, $\mathcal{L}$ is the infinitesimal generator of a subordinate Brownian motion $Z_t = \sqrt{2}B_{S_t}$. It is enough to show

$$
\|\Pi_j^\psi u\|_A = \left\|\Pi_j^\psi \int_0^\infty \mathrm{e}^{-\lambda t}(P_t f)\mathrm{d}t\right\|_A \leqslant C2^{-j}\|\Pi_j^\psi f\|_A, \quad j \geqslant 0.
\tag{4.6}
$$

Using the fact that $\widetilde{\Pi}_j^\psi \Pi_j^\psi = \Pi_j^\psi$ ($j \geqslant 0$), we see that

$$
\Pi_j^\psi(P_t f) = P_t(\Pi_j^\psi f) = P_t(\widetilde{\Pi}_j^\psi \Pi_j^\psi f).
$$

Thus,

$$
\|\Pi_j^\psi(P_t f)\|_A \leqslant \|P_t(\widetilde{\varrho}_{j+1} - \bar{\varrho}_j)\|_1 \, \|\Pi_j^\psi f\|_A,
\tag{4.7}
$$

due to Young's inequality (2.4). So our problem boils down to estimating $\|P_t(\widetilde{\varrho}_{j+1} - \bar{\varrho}_j)\|_1$ ($j \geqslant 0$). For any $R' > R \gg 1$, put

$$
\varrho^R(x) = R^d \varrho(Rx), \quad \varrho^{R'}(x) = (R')^d \varrho(R'x).
$$

Employing the procedure deducing (3.19), for any $t > 0$, we have

$$
\begin{aligned}
\|P_t(\varrho^{R'} - \varrho^R)\|_1 &\lesssim \int_0^\infty \int_{\mathbb{R}^d} \left(1 \wedge \frac{|z|}{\sqrt{u}R}\right) \, |\varrho(z)|\mathrm{d}z \, s_t(\mathrm{d}u) \\
&\lesssim \int_0^\infty \left(\frac{1}{\sqrt{u}R} \int_0^{\sqrt{u}R} |\varrho(r)|r^d \,\mathrm{d}r + \int_{\sqrt{u}R}^\infty |\varrho(r)|r^{d-1} \,\mathrm{d}r\right) s_t(\mathrm{d}u).
\end{aligned}
\tag{4.8}
$$

Noting that $|\varrho(r)| \lesssim 1 \wedge r^{-k}$ ($\forall k > 0$), one sees that

$$
\frac{1}{N}\int_0^N |\varrho(r)|r^d \,\mathrm{d}r \lesssim 1 \wedge \frac{1}{N} \text{ and } \int_N^\infty |\varrho(r)|r^{d-1} \,\mathrm{d}r \lesssim 1 \wedge \frac{1}{N}, \ \forall N > 0.
\tag{4.9}
$$

Combining (4.8) and (4.9), we obtain

$$
\begin{aligned}
\|P_t(\varrho^{R'} - \varrho^R)\|_1 &\lesssim \int_0^\infty \left(1 \wedge \frac{1}{\sqrt{u}R}\right) s_t(\mathrm{d}u) \\
&\leqslant \mathbf{P}\left(S_t \leqslant R^{-2}\right) + \frac{1}{R}\mathbf{E}\left(S_t^{-\frac{1}{2}} \mathbf{1}_{\{S_t \geqslant R^{-2}\}}\right) \\
&=: I_1(t, R) + I_2(t, R).
\end{aligned}
\tag{4.10}
$$

For I_1, noting that $\mathrm{e}^{-R^2 S_t} \geqslant \mathrm{e}^{-1}\mathbf{1}_{\{S_t \leqslant R^{-2}\}}$, we have

$$
I_1(t, R) = \mathbf{P}\left(S_t \leqslant R^{-2}\right) \leqslant \mathbf{E}\exp(1 - R^2 S_t) \lesssim \mathrm{e}^{-t\psi(R)}.
$$

Thus,

$$\int_0^\infty e^{-\lambda t} I_1(t, R)\, dt \lesssim \frac{1}{\lambda + \psi(R)}. \tag{4.11}$$

For I_2, in virtue of Lemma 2.9 (i) with $\sigma = -\frac{1}{2}$, $s = R^2$ and $f = 1/(\lambda + \phi)$, and (2.7), we have

$$\begin{aligned}
\int_0^\infty e^{-\lambda t} I_2(t, R)\, dt &= \frac{1}{R} \int_0^\infty e^{-\lambda t} \mathbf{E}\left(S_t^{-\frac{1}{2}} \mathbf{1}_{S_t \geqslant R^{-2}}\right) dt \\
&= \frac{1}{R} \int_{R^{-2}}^\infty u^{-\frac{1}{2}} \int_0^\infty e^{-\lambda t}\, \mathbf{P}(S_t \in du)\, dt = \frac{1}{R} \int_{R^{-2}}^\infty u^{-\frac{1}{2}} U_\lambda(du) \\
&= \frac{1}{R} \int_{(0,\infty)^2} \mathbf{1}_{\{R^{-2} < u < t\}} t^{-\frac{3}{2}} U_\lambda(du) dt \\
&\lesssim \frac{1}{R} \int_{R^{-2}}^\infty \frac{t^{-\frac{3}{2}}}{\lambda + \phi(t^{-1})} dt \lesssim \frac{1}{\lambda + \psi(R)}.
\end{aligned} \tag{4.12}$$

Here we use Karamata's theorem in the last inequality. By (4.10)-(4.12), we get

$$\int_0^\infty e^{-\lambda t} \|P_t(\varrho^{R'} - \varrho^R)\|_1\, dt \lesssim \frac{1}{\psi(R)}, \quad R' \geqslant R \gg 1. \tag{4.13}$$

In view of (4.7) and (4.13), there is a constant $j_0 \gg 1$ such that for any $j \geqslant j_0$, it holds that

$$\|\Pi_j^\psi u\|_A = \left\|\Pi_j^\psi \int_0^\infty e^{-\lambda t} (P_t f) dt\right\|_A \overset{(4.7)}{\leqslant} \|\Pi_j^\psi f\|_A \int_0^\infty e^{-\lambda t} \|P_t(\widetilde{\varrho}_{j+1} - \bar{\varrho}_j)\|_1\, dt$$

$$\overset{(4.13)}{\lesssim} \frac{\|\Pi_j^\psi f\|_A}{\psi\left(\frac{1}{2}\psi^{-1}(2^j)\right)} \lesssim 2^{-j} \|\Pi_j^\psi f\|_A.$$

Here we choose $R' = 2\psi^{-1}(2^{j+1})$ and $R = \frac{1}{2}\psi^{-1}(2^j)$ in (4.13), and use the fact that for each $b > 0$,

$$\lim_{R \to \infty} \frac{\psi(b\psi^{-1}(R))}{R} = \lim_{R \to \infty} \frac{\psi(b\psi^{-1}(R))}{\psi(\psi^{-1}(R))} = 1.$$

For $0 \leqslant j < j_0$, in view of (4.4), we have

$$\|\Pi_j^\psi u\|_A \lesssim \|\Pi_j^\psi f\|_A \lesssim 2^{-j_0} \|\Pi_j^\psi f\|_A \lesssim 2^{-j} \|\Pi_j^\psi f\|_A.$$

So we complete the proof for (4.6) when $a(x, z) = a(z) = 1$.

Step 2. Now assume that a satisfies (4.2) with $c_0 = 1$ and $\rho_0 = \infty$. Let

$$\bar{\nu}(dz) = J(z)dz \text{ and } \widetilde{\nu}(dz) = \underbrace{(a(z) - 1)}_{\geqslant 0} J(z)dz.$$

Since $\bar{\nu}$ and $\widetilde{\nu}$ are two Lévy measures, there are two pure jump Lévy processes $\bar{Z}$ and $\widetilde{Z}$ associated with $\bar{\nu}$ and $\widetilde{\nu}$, respectively. Let $\bar{P}_t$ and $\widetilde{P}_t$ be the semigroups corresponding to $\bar{Z}$ and $\widetilde{Z}$, respectively. Noting that $\Pi_j^\psi P_t f = \Pi_j^\psi \bar{P}_t \widetilde{P}_t f$, by *Step 1*, one sees that

$$\left\|\Pi_j^\psi \int_0^\infty e^{-\lambda t} (P_t f) dt\right\|_A \overset{(4.7)}{\leqslant} \|\Pi_j^\psi f\|_A \int_0^\infty e^{-\lambda t} \|P_t(\widetilde{\varrho}_{j+1} - \bar{\varrho}_j)\|_1\, dt \lesssim 2^{-j} \|\Pi_j^\psi f\|_A.$$

Thus, we get (4.6) in the case that a is bounded below by $c_0 > 0$.

Step 3. Now suppose a only satisfies (4.2) with $c_0 = 1$. Set

$$\bar{\mathcal{L}} u := \int_{B_{\rho_0}} (u(x+z) - u(x)) \left(a(z) \mathbf{1}_{B_{\rho_0}}(z) + \mathbf{1}_{B_{\rho_0}^c}(z)\right) J(z)\, dz$$

and

$$Ru := \int_{\mathbb{R}^d} (u(x+z) - u(x)) \left((a(z)-1)\mathbf{1}_{B_{\rho_0}^c} \right) J(z)\,\mathrm{d}z.$$

Then $\bar{\mathcal{L}}$ satisfies (4.2) with $c_0 = 1$ and $\rho_0 = \infty$. By *Step 2* and noting that $\lambda u - \bar{\mathcal{L}}u = \lambda u - \mathcal{L}u + Ru$, we get

$$\|u\|_{B_A^{\psi,1+\alpha}} \lesssim \|\lambda u - \bar{\mathcal{L}}u\|_{B_A^{\psi,\alpha}} \lesssim \|\lambda u - \mathcal{L}u\|_{B_A^{\psi,\alpha}} + \|Ru\|_{B_A^{\psi,\alpha}}. \tag{4.14}$$

Since

$$\|Rf\|_A \lesssim \|f\|_A \int_{|z|\geqslant \rho_0} (1+|a(z)|)J(z)\,\mathrm{d}z \lesssim \|f\|_A,$$

one sees that

$$\begin{aligned}
\|Ru\|_{B_A^{\psi,\alpha}} &= \sup_{j\geqslant -1} 2^{j\alpha}\|\Pi_j^\psi Ru\|_A = \sup_{j\geqslant -1} 2^{j\alpha}\|R\Pi_j^\psi u\|_A \\
&\lesssim \sup_{j\geqslant -1} 2^{j\alpha}\|\Pi_j^\psi u\|_A = \|u\|_{B_A^{\psi,\alpha}}.
\end{aligned} \tag{4.15}$$

Combining (4.5), (4.14) and (4.15), we obtain

$$\lambda\|u\|_{B_A^{\psi,\alpha}} + \|u\|_{B_A^{\psi,1+\alpha}} \leqslant C \left(\|\lambda u - \mathcal{L}u\|_{B_A^{\psi,\alpha}} + \|u\|_{B_A^{\psi,\alpha}} \right).$$

Choosing $\lambda_0 = 2C$, we obtain (4.3) for all $\lambda \geqslant \lambda_0$. This completes our proof. $\qquad\square$

4.2. Spatially inhomogeneous case.
Before proving our main results, we need to make some necessary preparations. Under assumption ($\boldsymbol{\Phi}$), one can verify that there is an integer $N \geqslant 1$ such that for all $s \geqslant 1$,

$$\psi^{-1}(2^{1-N}s) \leqslant \frac{3}{8}\psi^{-1}(s). \tag{4.16}$$

Define

$$S_k^\psi f = \sum_{l\prec k}\Pi_l^\psi f, \quad T_f^\psi g = \sum_k S_k^\psi f \cdot \Pi_k^\psi g, \quad R^\psi(f,g) = \sum_{k\sim l}\Pi_k^\psi f \cdot \Pi_l^\psi g.$$

Here $l \prec k$ means $l < k - N$ and $k \sim l$ means $|k-l| \leqslant N$. Thus,

$$f \cdot g = \sum_{k,l}\Pi_k^\psi f \cdot \Pi_l^\psi g = T_f^\psi g + T_g^\psi f + R^\psi(f,g).$$

For $R_1, R_2 \geqslant 0$ with $R_1 < R_2$, denote $D_{R_1,R_2} := \{x \in \mathbb{R}^d : R_1 \leqslant |x| \leqslant R_2\}$. The following simple fact will be used frequently: For any two functions $\hat{f}$ and $\hat{g}$ whose supports are in B_{R_0} and D_{R_1,R_2}, respectively, then

$$\operatorname{supp}[\hat{f} * \hat{g}] \subseteq D_{(R_1-R_0)^+,R_2+R_0}. \tag{4.17}$$

Noting that $\operatorname{supp}[\mathcal{F}(S_k^\psi f)] \subseteq B_{\psi^{-1}(2^{k-N})}$ and $\operatorname{supp}[\mathcal{F}(\Pi_k^\psi g)] \subseteq D_{\frac{3}{4}\psi^{-1}(2^k),\psi^{-1}(2^{k+1})}$, in view of (4.17) and (4.16), we have

$$\begin{aligned}
\operatorname{supp}\left[\mathcal{F}\left(S_k^\psi f \cdot \Pi_k^\psi g\right)\right] &= \operatorname{supp}\left[\mathcal{F}\left(S_k^\psi f\right) * \mathcal{F}\left(\Pi_k^\psi g\right)\right] \subseteq D_{\frac{3}{8}\psi^{-1}(2^k),\ 2\psi^{-1}(2^{k+1})} \\
&\subseteq D_{\psi^{-1}(2^{k-N}),\ \frac{3}{4}\psi^{-1}(2^{k+N})}.
\end{aligned}$$

This yields that

$$\Pi_j^\psi(S_k^\psi f \cdot \Pi_k^\psi g) = 0, \text{ if } |j-k| \geqslant N. \tag{4.18}$$

Similarly, one can also verify that

$$\Pi_j^\psi \left(\sum_{k:k\sim l} \Pi_k^\psi f \cdot \Pi_l^\psi g \right) = 0, \text{ if } j \geqslant k + 2N. \tag{4.19}$$

Lemma 4.3. *Let A be an N-function or $A = \infty$. Let $a : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ be a bounded measurable function (need not be positive). Assume the linear operator $\mathcal{L}$ is given by (1.6), and (Φ) are satisfied.*

(a) For any $\theta > 0$, it holds that

$$\|\mathcal{L}u\|_{B_A^{\psi,0}} \leqslant C\|u\|_{B_A^{\psi,1}} \cdot \sup_{z\in\mathbb{R}^d} \|a(\cdot,z)\|_{\mathscr{C}_\psi^\theta}, \tag{4.20}$$

where C only depends on d, c_0, ψ and θ.

(b) For any $\alpha > 0$ and $\theta \in (0, \alpha)$, it holds that

$$\|\mathcal{L}u\|_{B_A^{\psi,\alpha}} \leqslant C \left(\|u\|_{B_A^{\psi,1+\alpha}} \cdot \sup_{z\in\mathbb{R}^d} \|a(\cdot,z)\|_{\mathscr{C}_\psi^\theta} + \|u\|_{B_A^{\psi,1+\theta}} \cdot \sup_{z\in\mathbb{R}^d} \|a(\cdot,z)\|_{\mathscr{C}_\psi^\alpha} \right), \tag{4.21}$$

where the constant C only depends on d, c_0, ψ, α and θ.

Proof. Set

$$\delta_z u(x) = u(x+z) - u(x), \quad a_z(x) = a(x,z).$$

By definition,

$$\begin{aligned}
\Pi_j^\psi \mathcal{L}u(x) =& \Pi_j^\psi \left(\int_{\mathbb{R}^d} \delta_z u(\cdot) \, a(\cdot,z) J(z) \, \mathrm{d}z \right) \\
=& \int_{\mathbb{R}^d} \left(\Pi_j^\psi \sum_{k,l\geqslant -1} \Pi_k^\psi(\delta_z u) \cdot \Pi_l^\psi a_z \right)(x) \, J(z)\, \mathrm{d}z =: \int_{\mathbb{R}^d} I_j(x,z) \, J(z)\, \mathrm{d}z.
\end{aligned} \tag{4.22}$$

We drop the variable x below for simplicity. By (4.18) and (4.19),

$$\begin{aligned}
|I_j(z)| \quad = & \quad \left| \Pi_j^\psi \sum_{k,l\geqslant -1} \left[(\delta_z \Pi_k^\psi u) \cdot \Pi_l^\psi a_z \right] \right| \\
= & \quad \left| \Pi_j^\psi \left(\sum_{k\prec l} \delta_z \Pi_k^\psi u \cdot \Pi_l^\psi a_z + \sum_{k\succ l} \delta_z \Pi_k^\psi u \cdot \Pi_l^\psi a_z + \sum_{k\sim l} \delta_z \Pi_k^\psi u \cdot \Pi_l^\psi a_z \right) \right| \\
\overset{(4.18),(4.19)}{\leqslant} & \quad \sum_{k\prec l\sim j} \left| \delta_z \Pi_k^\psi u \cdot \Pi_l^\psi a_z \right| + \sum_{l\prec k\sim j} \left| \delta_z \Pi_k^\psi u \cdot \Pi_l^\psi a_z \right| + \sum_{l\sim k; k>j-2N} \left| \delta_z \Pi_k^\psi u \cdot \Pi_l^\psi a_z \right| \\
=: & \quad I_j^{(1)}(z) + I_j^{(2)}(z) + I_j^{(3)}(z).
\end{aligned}$$

By mean value theorem, we get

$$|\delta_z \Pi_k^\psi u(x)| = \left| \int_0^1 z \cdot \nabla \Pi_k^\psi u(x+tz) \mathrm{d}t \right| \leqslant |z| \int_0^1 \left| \nabla \Pi_k^\psi u(x+tz) \right| \, \mathrm{d}t.$$

Thus,

$$|\delta_z \Pi_k^\psi u(x)| \leqslant \left(|\Pi_k^\psi u(x)| + |\Pi_k^\psi u(x+z)| \right) \wedge \left(|z| \int_0^1 \left| \nabla \Pi_k^\psi u(x+tz) \right| \, \mathrm{d}t \right).$$

This and (3.1) imply that

$$
\begin{aligned}
\|\delta_z \Pi_k^\psi u \cdot \Pi_l^\psi a_z\|_A &\leqslant \|\delta_z \Pi_k^\psi u\|_A \ \sup_z \|\Pi_l^\psi a_z\|_\infty \\
&\leqslant \sup_z \|\Pi_l^\psi a_z\|_\infty \ \|\Pi_k^\psi u\|_A \left(1 \wedge |z|\psi^{-1}(2^{k+1})\right).
\end{aligned}
\tag{4.23}
$$

(i) If $\alpha = 0$. Combining inequalities (4.23), (2.8) and (2.9), we get

$$
\begin{aligned}
\int_{\mathbb{R}^d} \|I_j^{(1)}(z)\|_A J(z)\,\mathrm{d}z &\leqslant \sum_{k \prec l \sim j} \int_{\mathbb{R}^d} \left\|\delta_z \Pi_k^\psi u \cdot \Pi_l^\psi a_z\right\|_A J(z)\,\mathrm{d}z \\
&\overset{(4.23)}{\leqslant} 2^{-\theta j} \sup_z \|a_z\|_{\mathscr{C}_\psi^\theta} \|u\|_{B_A^{\psi,1}} \sum_{k \prec l \sim j} 2^{-k} \\
&\quad \left(\psi^{-1}(2^{k+1}) \int_{|z| \leqslant \frac{1}{\psi^{-1}(2^{k+1})}} |z| J(z)\,\mathrm{d}z + \int_{|z| > \frac{1}{\psi^{-1}(2^{k+1})}} J(z)\,\mathrm{d}z\right) \\
&\overset{(2.8),(2.9)}{\lesssim} \|u\|_{B_A^{\psi,1}} \sup_z \|a_z\|_{\mathscr{C}_\psi^\theta} 2^{-\theta j} \sum_{k \prec l \sim j} 2^{-k} \left(\frac{2^k}{\psi^{-1}(2^{k+1})} + 2^k\right) \\
&\lesssim \|u\|_{B_A^{\psi,1}} \sup_z \|a_z\|_{\mathscr{C}_\psi^\theta} j 2^{-\theta j}.
\end{aligned}
\tag{4.24}
$$

Similarly, noting that $\theta > 0$, we have

$$
\begin{aligned}
\int_{\mathbb{R}^d} \|I_j^{(2)}(z)\|_A J(z)\,\mathrm{d}z &\leqslant \sum_{l \prec k \sim j} \int_{\mathbb{R}^d} \left\|\delta_z \Pi_k^\psi u \cdot \Pi_l^\psi a_z\right\|_A J(z)\,\mathrm{d}z \\
&\lesssim \|u\|_{B_A^{\psi,1}} \sup_z \|a_z\|_{\mathscr{C}_\psi^\theta} \sum_{l \prec k \sim j} 2^{-\theta l} 2^{-k} \left(\frac{2^k}{\psi^{-1}(2^{k+1})} + 2^k\right) \\
&\lesssim \|u\|_{B_A^{\psi,1}} \sup_z \|a_z\|_{\mathscr{C}_\psi^\theta},
\end{aligned}
\tag{4.25}
$$

and

$$
\begin{aligned}
\int_{\mathbb{R}^d} \|I_j^{(3)}(z)\|_A J(z)\,\mathrm{d}z &\lesssim \|u\|_{B_A^{\psi,1}} \sup_z \|a_z\|_{\mathscr{C}_\psi^\theta} \sum_{l \sim k; k > j - 2N} 2^{-\theta l} 2^{-k} \left(\frac{2^k}{\psi^{-1}(2^{k+1})} + 2^k\right) \\
&\lesssim \|u\|_{B_A^{\psi,1}} \sup_z \|a_z\|_{\mathscr{C}_\psi^\theta} 2^{-\theta j}.
\end{aligned}
\tag{4.26}
$$

Combining (4.22), (4.24), (4.25) and (4.26), we obtain (4.20).

(ii) When $\alpha > 0$, by (4.23) and similar discussions in (i), one can see that

$$
\begin{aligned}
\int_{\mathbb{R}^d} \|I_j^{(1)}(z)\|_A J(z)\,\mathrm{d}z &\lesssim 2^{-\alpha j} \sup_z \|a_z\|_{\mathscr{C}_\psi^\alpha} \|u\|_{B_A^{\psi,1+\theta}} \sum_{k \prec l \sim j} 2^{-(1+\theta)k} \left(\frac{2^k}{\psi^{-1}(2^{k+1})} + 2^k\right) \\
&\lesssim 2^{-\alpha j} \|u\|_{B_A^{\psi,1+\theta}} \sup_z \|a_z\|_{\mathscr{C}_\psi^\alpha},
\end{aligned}
$$

$$
\begin{aligned}
\int_{\mathbb{R}^d} \|I_j^{(2)}(z)\|_A J(z)\,\mathrm{d}z &\lesssim 2^{-\alpha j} \|u\|_{B_A^{\psi,1+\alpha}} \sup_z \|a_z\|_{\mathscr{C}_\psi^\theta} \sum_{l \prec k \sim j} 2^{-\theta l} 2^{-k} \left(\frac{2^k}{\psi^{-1}(2^{k+1})} + 2^k\right) \\
&\lesssim 2^{-\alpha j} \|u\|_{B_A^{\psi,1+\alpha}} \sup_z \|a_z\|_{\mathscr{C}_\psi^\theta},
\end{aligned}
$$

and

$$\int_{\mathbb{R}^d} \|I_j^{(3)}(z)\|_A J(z)\, \mathrm{d}z \lesssim \|u\|_{B_A^{\psi,1+\alpha}} \sum_{l\sim k;k>j-2N} 2^{-(1+\alpha)k} \left(\frac{2^k}{\psi^{-1}(2^{k+1})} + 2^k\right) \sup_z \|\Pi_l^\psi a_z\|_\infty$$

$$\lesssim 2^{-\alpha j}\|u\|_{B_A^{\psi,1+\alpha}} \sup_{x,z} |a(x,z)|.$$

Combining the above estimates, we get (4.21). $\qquad\square$

Lemma 4.4. *Let A be an N-function or $A = \infty$. Suppose that (Φ) and (A_1)-(A_2) are satisfied, then there are universal constants $\varepsilon_1 > 0$ and $\lambda_0 > 0$ such that for each $\lambda \geqslant 2\lambda_0$ and $f \in B_A^{\psi,\beta}$ with $\beta \in [0,\alpha]$, the Poisson equation (PE) has a unique solution $u \in B_A^{\psi,1+\beta}$, provided that $|a(x,z) - a(0,z)| \leqslant \varepsilon_1$ and $\lambda \geqslant \lambda_0$. Moreover,*

$$\lambda\|u\|_{B_A^{\psi,\beta}} + \|u\|_{B_A^{\psi,1+\beta}} \leqslant C\|f\|_{B_A^{\psi,\beta}}, \quad \text{for each } \beta \in [0,\alpha]. \tag{4.27}$$

In particular, it holds that

$$\lambda\|u\|_{C_\rho^\alpha} + \|u\|_{C_\rho^{1+\alpha}} \leqslant C\|f\|_{C_\rho^\alpha}, \tag{4.28}$$

if $|a(x,z) - a(0,z)| \leqslant \varepsilon_1$ and $\lambda \geqslant \lambda_0$, and $f \in C_\rho^\alpha$.

Remark 4.5. *Although the above analytic result relies on a special assumption regarding the oscillation of a, it is sufficient for us to establish our Theorem 1.4. In the case where $A(t) = \infty$ and $s > 0$, the space $B_A^{\psi,s}$ can be identified as the generalized Hölder space C_ρ^s, which exhibits the following localization property:*

$$\|f\|_{C_\rho^s} \asymp_\varepsilon \sup_{y\in\mathbb{R}^d} \|f\eta((\cdot - y))\|_{C_\rho^s},$$

where η is some smooth cut-off function. The method of frozen coefficients, combined with the aforementioned result, can be employed to eliminate the requirement that the oscillation of a is small. We will present a detailed proof of this approach for Theorem 1.1. However, it should be noted that the localization property can not extend to $B_{p,\infty}^s$ (see [Tri92]). As a result, it is unclear whether the assumption on the oscillation of a can be entirely eliminated.

Proof of Lemma 4.4. When $a \equiv 1$, by the proof of Theorem 4.1, we see that if $f \in B_A^{\psi,\beta}$, then $G_\lambda f \in B_A^{\psi,1+\beta}$ is a solution to (PE) ($\forall \lambda > 0$), and estimate (4.27) holds true. Through the continuity method, it is enough to prove (4.27), under the premise that $u \in B_A^{\psi,1+\beta}$ is a solution to equation (PE). By interpolation theorem, we only need to show (4.27) for $\beta = 0$ and $\beta = \alpha$.

Assume $u \in B_A^{\psi,1+\alpha}$. Set $f = \lambda u - \mathcal{L}u$,

$$a_0(z) = a(0,z) \quad \text{and} \quad \mathcal{L}_0 u(x) = \int_{\mathbb{R}^d} (u(x+z) - u(x))a_0(z)J(z)\, \mathrm{d}z.$$

Thus,

$$\lambda u - \mathcal{L}_0 u = (\mathcal{L} - \mathcal{L}_0)u + f.$$

Using Theorem 4.1, we have

$$\lambda\|u\|_{B_A^{\psi,\alpha}} + \|u\|_{B_A^{\psi,1+\alpha}} \leqslant C\left(\|(\mathcal{L} - \mathcal{L}_0)u\|_{B_A^{\psi,\alpha}} + \|f\|_{B_A^{\psi,\alpha}}\right). \tag{4.29}$$

Choosing $\theta \in (0,\alpha)$, we have

$$\|a(\cdot,z) - a_0(\cdot,z)\|_{\mathscr{C}_\psi^\theta} \leqslant \|a(\cdot,z) - a_0(\cdot,z)\|_{\mathscr{C}_\psi^0}^{1-\frac{\theta}{\alpha}} \|a(\cdot,z) - a_0(\cdot,z)\|_{\mathscr{C}_\psi^\alpha}^{\frac{\theta}{\alpha}} \leqslant c_0^{-1}\varepsilon_1^{1-\frac{\theta}{\alpha}}.$$

By (4.21) and interpolation, we have

$$
\begin{aligned}
\|(\mathcal{L} - \mathcal{L}_0)u\|_{B_A^{\psi,\alpha}} &\leqslant C\varepsilon_1^{1-\frac{\theta}{\alpha}}\|u\|_{B_A^{\psi,1+\alpha}} + C\|u\|_{B_A^{\psi,1+\theta}} \\
&\leqslant C\varepsilon_1^{1-\frac{\theta}{\alpha}}\|u\|_{B_A^{\psi,1+\alpha}} + C\|u\|_{B_A^{\psi,\alpha}}.
\end{aligned}
\tag{4.30}
$$

Combining (4.29) and (4.30), and choosing $\lambda \gg 1$, we get

$$
\lambda\|u\|_{B_A^{\psi,\alpha}} + \|u\|_{B_A^{\psi,1+\alpha}} \leqslant C\varepsilon_1^{1-\frac{\theta}{\alpha}}\|u\|_{B_A^{\psi,1+\alpha}} + C\|f\|_{B_A^{\psi,\alpha}}.
$$

Choosing ε_1 sufficiently small, so that $C\varepsilon_1^{1-\frac{\theta}{\alpha}} \leqslant 1/2$, then we obtain (4.27) for $\beta = \alpha$.

The case $\beta = 0$ can be proved by using (4.20) and following the same procedure above. So we complete our proof. $\qquad\square$

Now we are at the point of proving Theorem 1.1.

Proof of Theorem 1.1. Let χ be the nonnegative smooth function with compact support defined in Section 3.1. For fixed $x_0 \in \mathbb{R}^d$, define

$$
\chi_\varepsilon^{x_0}(x) := \chi\Big(\frac{x - x_0}{\varepsilon}\Big) \text{ and } a_\varepsilon^{x_0}(x,z) := [a(x,z) - a(x_0,z)]\chi_\varepsilon(x).
$$

By definition, $a_\varepsilon^{x_0}$ satisfies that

$$
|a_\varepsilon^{x_0}(x,z)| \lesssim 1
\tag{4.31}
$$

and for every $z \in \mathbb{R}^d$ and $|x - x'| < 1$,

$$
|a_\varepsilon^{x_0}(x,z) - a_\varepsilon^{x_0}(x',z)| \leqslant C_\varepsilon \rho^\alpha(|x - x'|).
\tag{4.32}
$$

For simplicity, we omit the superscript x_0 below. Let $v = u\chi_\varepsilon$,

$$
\delta_z f(x) = f(x + z) - f(x) \text{ and } \mathcal{L}_0 u(x) = \int_{\mathbb{R}^d} \delta_z f(x)\, a(x_0,z)J(z)\,\mathrm{d}z.
$$

We have

$$
\begin{aligned}
&\lambda v - \mathcal{L}_0 v \\
={}&[f\chi_\varepsilon - u\mathcal{L}_0\chi_\varepsilon] + (\mathcal{L}u - \mathcal{L}_0 u)\chi_\varepsilon - [\mathcal{L}_0(u\chi_\varepsilon) - (\mathcal{L}_0 u)\chi_\varepsilon - u(\mathcal{L}_0\chi_\varepsilon)] \\
=:{}&\sum_{i=1}^3 I_\varepsilon^{(i)}.
\end{aligned}
\tag{4.33}
$$

Obviously,

$$
\|I_\varepsilon^{(1)}\|_{C_\rho^\alpha} \leqslant \|f\chi_\varepsilon\|_{C_\rho^\alpha} + \|u\mathcal{L}_0\chi_\varepsilon\|_{C_\rho^\alpha} \lesssim_\varepsilon \|f\|_{C_\rho^\alpha} + \|u\|_{C_\rho^\alpha}.
\tag{4.34}
$$

Noting that

$$
I_\varepsilon^{(2)}(x) = (\mathcal{L}_0 u(x) - \mathcal{L}u(x))\chi_\varepsilon(x) = -\int_{\mathbb{R}^d} \delta_z u(x) a_\varepsilon(x,z)\, J(z)\,\mathrm{d}z,
$$

then by (4.31), (4.32) and (4.21), for any $\theta \in (0,\alpha)$

$$
\|I_\varepsilon^{(2)}\|_{C_\rho^\alpha} \leqslant c_1\rho^\alpha(\varepsilon)\|u\|_{C_\rho^{1+\alpha}} + C\|u\|_{C_\rho^{1+\theta}},
\tag{4.35}
$$

where the constant c_1 is independent with ε. For $I_\varepsilon^{(3)}$, by definition

$$
I_\varepsilon^{(3)}(x) = \int_{\mathbb{R}^d} \delta_z\chi_\varepsilon(x)\, \delta_z u(x)\, a(x_0,z)J(z)\,\mathrm{d}z
\tag{4.36}
$$

and

$$I_\varepsilon^{(3)}(x) - I_\varepsilon^{(3)}(y) = \int_{\mathbb{R}^d} \delta_z \chi_\varepsilon(x) \big[\delta_z u(x) - \delta_z u(y) \big] \, a(x_0, z) J(z) \, \mathrm{d}z$$
$$+ \int_{\mathbb{R}^d} \big[\delta_z \chi_\varepsilon(x) - \delta_z \chi_\varepsilon(y) \big] \delta_z u(y) \, a(x_0, z) J(z) \, \mathrm{d}z. \tag{4.37}$$

By (4.36), we have

$$\big| I_\varepsilon^{(3)}(x) \big| \leqslant \|u\|_\infty \left(\int_{|z| \leqslant 1} \|\nabla \chi_\varepsilon\|_\infty |z| a(x_0, z) J(z) \, \mathrm{d}z + 2 \int_{|z| > 1} \|\chi_\varepsilon\|_\infty a(x_0, z) J(z) \, \mathrm{d}z \right)$$
$$\lesssim_\varepsilon \|u\|_\infty.$$

By (4.37), one can see that

$$\big| I_\varepsilon^{(3)}(x) - I_\varepsilon^{(3)}(y) \big|$$
$$\lesssim_\varepsilon \rho^\alpha(|x - y|) \, \|u\|_{C_\rho^\alpha} \left(\|\nabla \chi_\varepsilon\|_\infty \int_{|z| \leqslant 1} |z| J(z) \, \mathrm{d}z + \|\chi_\varepsilon\|_\infty \int_{|z| > 1} J(z) \, \mathrm{d}z \right)$$
$$+ |x - y| \left(\|\nabla^2 \chi_\varepsilon\|_\infty \|u\|_\infty \int_{|z| \leqslant 1} |z| J(z) \, \mathrm{d}z + \|\nabla \chi_\varepsilon\|_\infty \|u\|_\infty \int_{|z| > 1} J(z) \, \mathrm{d}z \right)$$
$$\lesssim_\varepsilon \rho^\alpha(|x - y|) \, \|u\|_{C_\rho^\alpha}.$$

Therefore,

$$\|I_\varepsilon^{(3)}\|_{C_\rho^\alpha} \lesssim_\varepsilon \|u\|_{C_\rho^\alpha}. \tag{4.38}$$

Using Theorem 4.1, and combining (4.34), we obtain that (4.35) and (4.38),

$$\lambda \|u\|_{C_\rho^\alpha(B_{\varepsilon/2}(x_0))} + \|u\|_{C_\rho^{1+\alpha}(B_{\varepsilon/2}(x_0))}$$
$$\leqslant C\|v\|_{C_\rho^{1+\alpha}} \leqslant C\|v\|_{C_\rho^\alpha} + C\|\mathcal{L}_0 v\|_{C_\rho^\alpha}$$
$$\leqslant c_2 \rho^\alpha(\varepsilon) \|u\|_{C_\rho^{1+\alpha}} + C\|u\|_{C_\rho^{1+\theta}} + C\|u\|_{C_\rho^\alpha} + C\|f\|_{C_\rho^\alpha},$$

for all $\lambda \geqslant \lambda_0'(d, \alpha, \psi, c_0, \rho_0) > 0$. Using the fact that

$$\|u\|_{C_\rho^{1+\alpha}} \leqslant \sup_{x_0 \in \mathbb{R}^d} \|u\|_{C_\rho^{1+\alpha}(B_{\varepsilon/2}(x_0))} + C_\varepsilon \|u\|_\infty,$$

we obtain

$$\|u\|_{C_\rho^{1+\alpha}(B_{\varepsilon/2}(x_0))} \leqslant c_2 \rho^\alpha(\varepsilon) \sup_{x_0 \in \mathbb{R}^d} \|u\|_{C_\rho^{1+\alpha}(B_{\varepsilon/2}(x_0))} + C \left(\|u\|_{C_\rho^{1+\theta}} + \|u\|_{C_\rho^\alpha} + \|f\|_{C_\rho^\alpha} \right).$$

We then fix ε_0 sufficiently small, such that $c_2 \rho^\alpha(\varepsilon) \leqslant 1/2$, so we arrive

$$\sup_{x_0 \in \mathbb{R}^d} \left(\lambda \|u\|_{C_\rho^\alpha(B_{\varepsilon_0/2}(x_0))} + \|u\|_{C_\rho^{1+\alpha}(B_{\varepsilon_0/2}(x_0))} \right) \leqslant C \left(\|u\|_{C_\rho^{1+\theta}} + \|u\|_{C_\rho^\alpha} + \|f\|_{C_\rho^\alpha} \right).$$

This yields

$$\lambda \|u\|_{C_\rho^\alpha} + \|u\|_{C_\rho^{1+\alpha}} \leqslant C_{\varepsilon_0} \sup_{x_0 \in \mathbb{R}^d} \left(\lambda \|u\|_{C_\rho^\alpha(B_{\varepsilon_0/2}(x_0))} + \|u\|_{C_\rho^{1+\alpha}(B_{\varepsilon_0/2}(x_0))} \right)$$
$$\lesssim \left(\|u\|_{C_\rho^{1+\theta}} + \|u\|_{C_\rho^\alpha} + \|f\|_{C_\rho^\alpha} \right).$$

By interpolation, one sees that

$$\lambda \|u\|_{C_\rho^\alpha} + \|u\|_{C_\rho^{1+\alpha}} \lesssim \|f\|_{C_\rho^\alpha},$$

for all $\lambda \geqslant \lambda_0(d, \alpha, \psi, c_0, \rho_0) > 0$. $\square$

Proof of Corollary 1.3. Let S be the gamma subordinator, whose Laplace exponent is given by $\phi(s) = \log(1 + s)$. By Proposition 2.10, one sees $j(r) \asymp r^{-d}, r \to 0$. Let

$$a'(x, z) = |z|^{-d} j^{-1}(|z|) \mathbf{1}_{B_1}(z) \, a(x, z).$$

Then $a'(x, z) \geqslant c_0$, and $\|a'(\cdot, z)\|_{C_\rho^\alpha} \lesssim \|a(\cdot, z)\|_{C_\rho^\alpha} \leqslant C$, where $\rho(r) = \frac{1}{\psi(r^{-1})} = \frac{1}{\log(1 + r^{-2})}$. Noting that

$$Lu(x) = \int_{B_1} (u(x + z) - u(x)) \frac{a(x, z)}{|z|^d} \, \mathrm{d}z = \int_{\mathbb{R}^d} (u(x + z) - u(x)) \, a'(x, z) J(z) \, \mathrm{d}z,$$

by Theorem 1.1, we obtain our assertion. $\square$

5. Martingale problem

In this section, we study the martingale problem associated with $\mathcal{L}$. We need to introduce some necessary terminology.

Let $D = D([0, \infty), \mathbb{R}^d)$ be the Skorokhod space of càdlàg $\mathbb{R}^d$-valued trajectories and let $X_t = X_t(\omega) = \omega_t$ $(\omega \in D)$, be the canonical process on it. Set

$$\mathcal{D}_t = \bigcap_{\varepsilon > 0} \sigma(X_s, 0 \leqslant s \leqslant t + \varepsilon), \quad \mathcal{D} = \sigma(\mathcal{D}_t, t \geqslant 0).$$

Definition 5.1 (Martingale problem). *Let μ be a probability measure on $\mathbb{R}^d$. A probability measure $\mathbb{P}$ on $(D, \mathcal{D})$ is said to be a solution to the martingale problem for $(\mathcal{L}, \mu)$ if $\mathbb{P} \circ X_0^{-1} = \mu$ and for each $f \in C_b^\infty$,*

$$M_t^f := f(X_t) - f(x) - \int_0^t \mathcal{L}f(X_s) \, \mathrm{d}s \;\; \text{is a } \mathcal{D}_t\text{-martingale under } \mathbb{P}.$$

Below we briefly introduce the SDE corresponding to the martingale problem $(\mathcal{L}, \mu)$. Let $(\Omega, \mathbf{P}, \mathcal{F})$ be a probability space and $N(\mathrm{d}r, \mathrm{d}z, \mathrm{d}s)$ be a Poisson random measure on $\mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}_+$ with intensity measure $\mathrm{d}r \, J(z) \mathrm{d}z \, \mathrm{d}s$. Consider the following SDE driven by Poisson random measure N:

$$X_t = x + \int_0^t \int_{\mathbb{R}^d} \int_0^\infty z \mathbf{1}_{[0, a(X_{s-}, z))}(r) N(\mathrm{d}r, \mathrm{d}z, \mathrm{d}s). \tag{5.1}$$

If $|a(x, z) - a(x, z)| \leqslant C|x - y|$ for any $x, y \in \mathbb{R}^d$, then

$$\int_{\mathbb{R}^d} \int_0^\infty |z|^2 \left(\mathbf{1}_{[0, a(x, z))}(r) - \mathbf{1}_{[0, a(y, z))}(r) \right)^2 \mathrm{d}r \, J(z) \mathrm{d}z \leqslant C|x - y|^2.$$

This implies (5.1) admits a unique strong solution (see [App09]). By Itô's formula, we see that

$$f(X_t) - f(x) = \int_0^t \int_{\mathbb{R}^d} \int_0^\infty \left(f\left(X_{s-} + z\mathbf{1}_{[0, a(X_{s-}, z))}(r)\right) - f(X_{s-}) \right) N(\mathrm{d}r, \mathrm{d}z, \mathrm{d}s)$$

$$= \int_0^t \int_{\mathbb{R}^d} (f(X_{s-} + z) - f(X_{s-})) \, a(X_{s-}, z) \, J(z) \mathrm{d}z \, \mathrm{d}s$$

$$+ \int_0^t \int_{\mathbb{R}^d} \int_0^\infty (f(X_{s-} + z) - f(X_{s-})) \, \mathbf{1}_{[0, a(X_{s-}, z)]}(r) \widetilde{N}(\mathrm{d}r, \mathrm{d}z, \mathrm{d}s),$$

$$= \int_0^t \mathcal{L}f(X_{s-})\mathrm{d}s + M_t^f,$$

where $\widetilde{N}(\mathrm{d}r,\mathrm{d}z,\mathrm{d}t) = N(\mathrm{d}r,\mathrm{d}z,\mathrm{d}t) - \mathrm{d}r J(z)\mathrm{d}z\mathrm{d}t$ and M_t^f is a martingale. Therefore, when $x \mapsto a(x,z)$ is Lipschitz continuous (uniformly in z), $\mathbf{P} \circ X^{-1}$ is a martingale solution to $(\mathcal{L}, \delta_x)$.

Now we are at the point of proving Theorem 1.4.

Proof of Theorem 1.4. **Existence:** The proof for the existence is standard, for the convenience of the reader, we give the details here. Let $\eta \in C_c^\infty(B_1)$ satisfying $\int \eta = 1$. Set $\eta^n(x) = n^d\eta(nx)$ and $a^n(x,z) = \int a(x-y,z)\eta_n(y)\,\mathrm{d}y$. Let X_t^n be the unique strong solution to (5.1) (a is replaced by a^n), and $\mathbb{P}_x^n := \mathbf{P} \circ (X^n)^{-1}$. By the discussion above, $\mathbb{P}_x^n$ is a martingale solution associated with $(\mathcal{L}^n, \delta_x)$, and

$$\mathcal{L}^n f(x) = \int_{\mathbb{R}^d} (f(x+z) - f(x))\, a^n(x,z)J(z)\,\mathrm{d}z.$$

We claim that $\{\mathbb{P}_x^n\}$ is tight in $\mathscr{P}(D)$. Then upon taking a subsequence, still denoted by n, we can assume that $\mathbb{P}_x^n \Rightarrow \mathbb{P}_x$. For any $0 \leqslant s_1 \leqslant \cdots \leqslant s_k \leqslant s \leqslant t$, $f \in C_c^\infty(\mathbb{R}^d)$ and $h_i \in C_c^\infty(\mathbb{R}^d)$, $i \in \{1,2\cdots,k\}$, we have

$$\mathbf{E}_{\mathbb{P}_x}\left\{\left[f(X_t) - f(X_s) - \int_s^t \mathcal{L}f(X_u)\,\mathrm{d}u\right]\prod_{i=1}^k h_i(X_{t_i})\right\}$$

$$= \lim_{n\to\infty} \mathbf{E}\left\{\left[f(X_t^n) - f(X_s^n) - \int_s^t \mathcal{L}f(X_u^n)\,\mathrm{d}u\right]\prod_{i=1}^k h_i(X_{t_i}^n)\right\}$$

$$= \lim_{n\to\infty} \mathbf{E}\left\{\left[f(X_t^n) - f(X_s^n) - \int_s^t \mathcal{L}^n f(X_u^n)\,\mathrm{d}u\right]\prod_{i=1}^k h_i(X_{t_i}^n)\right\}$$

$$+ \lim_{n\to\infty} \mathbf{E}\left\{\left[\int_s^t (\mathcal{L}^n - \mathcal{L})f(X_u^n)\,\mathrm{d}u\right]\prod_{i=1}^k h_i(X_{t_i}^n)\right\} = 0.$$

Here we use the fact that $\mathbf{P} \circ (X^n)^{-1}$ is a solution to $(\mathcal{L}^n, \delta_x)$, and $(\mathcal{L}^n - \mathcal{L})f \to 0$ uniformly (since $a^n \to a$ uniformly). Thus, $\mathbb{P}_x$ is a martingale solution to $(\mathcal{L}, \delta_x)$.

It remains to show $\{\mathbb{P}_x^n\}$ is tight. Given a bounded stopping time τ, we define

$$N_\tau(\mathrm{d}z) = \int_\tau^{\tau+1}\int_0^{c_0^{-1}} N(\mathrm{d}r,\mathrm{d}z,\mathrm{d}s).$$

$N_\tau(\mathrm{d}z)$ is a Poisson random measure on $\mathbb{R}^d$ with intensity measure $c_0^{-1}J(z)\mathrm{d}z$.

$$\mathbf{P}(|X_{\tau+\delta}^n - X_\tau^n| > \varepsilon)$$

$$\leqslant \mathbf{P}(|X_{\tau+\delta}^n - X_\tau^n| > \varepsilon; N_\tau(B_M^c) = 0) + \mathbf{P}(N_\tau(B_M^c) \geqslant 1)$$

$$\leqslant \varepsilon^{-1}\mathbf{E}\left|\int_\tau^{\tau+\delta}\int_{B_M}\int_0^{a(X_{s-},z)} zN(\mathrm{d}r,\mathrm{d}z,\mathrm{d}s)\right| + \mathbf{P}(N_\tau(B_M^c) \geqslant 1)$$

$$\leqslant \frac{\delta}{c_0\varepsilon}\int_{|z|\leqslant M} |z|J(z)\mathrm{d}z + c_0^{-1}\int_{|z|>M} J(z)\mathrm{d}z.$$

Letting $\delta \to 0$ and then $M \to 0$, we get $\sup_{n,\tau} \mathbf{P}(|X^n_{\tau+\delta} - X^n_\tau| > \varepsilon) \to 0$ ($\delta \to 0$). Similarly, one can also see that $\sup_n \mathbf{P}(|X^n| > M) \to 0$ ($M \to \infty$). By Aldous tightness criterion, we get the tightness of $\{\mathbb{P}^n_x\}$.

Uniqueness: For any $f \in C^\infty_b$, consider equation (PE) in $C^{1+\alpha}_\rho$. By Theorem 1.1, equation (PE) admits a unique solution $u \in C^{1+\alpha}_\rho$. Suppose that $(\mathbb{P}^i_x, X_t)$ $(i = 1, 2)$ are two solutions to martingale problem $(\mathcal{L}, \delta_x)$, one can easily verify that

$$M^u_t = u(X_t) - u(x) - \int_0^t \mathcal{L}u(X_t)\,\mathrm{d}t$$

is a martingale under $\mathbb{P}^i_x$. This implies

$$\begin{aligned}
\mathrm{d}(\mathrm{e}^{-\lambda t}u(X_t)) &= -\mathrm{e}^{-\lambda t}(\lambda - \mathcal{L})u(X_t)\,\mathrm{d}t + \mathrm{e}^{-\lambda t}\,\mathrm{d}M^u_t \\
&= -\mathrm{e}^{-\lambda t}f(X_t)\,\mathrm{d}t + \mathrm{e}^{-\lambda t}\mathrm{d}M^u_t.
\end{aligned}$$

Taking expectation, one sees

$$u(x) = \mathbb{E}^i_x \int_0^\infty \mathrm{e}^{-\lambda t}f(X_t)\,\mathrm{d}t, \quad i = 1, 2.$$

Thanks to [EK09, Theorem 4.2], $\mathbb{P}^1_x = \mathbb{P}^2_x$.

Krylov-type estimate: Let $\delta \in (0, 1 \wedge \rho_0/10)$ such that

$$\rho^\alpha(2\delta) < \varepsilon_1,$$

where ε_1 is the same constant in Lemma 4.4. For each $y \in \mathbb{R}^d$, let

$$a^y(x, z) = \begin{cases} a(x, z) & \text{if } x \in B_\delta(y), \\ a\left(y + \dfrac{\delta^2(x - y)}{|x - y|^2}, z\right) & \text{if } x \in B^c_\delta(y). \end{cases}$$

Then a^y satisfies $(\mathbf{L}_a)$, $(\mathbf{H}_a)$ and

$$|a^y(x, z) - a^y(y, z)| < \varepsilon_1.$$

Let

$$\mathcal{L}^y u(x) = \int_{\mathbb{R}^d} (u(x + z) - u(x))\ a^y(x, z)J(z)\,\mathrm{d}z.$$

Since a^y also satisfies conditions $(\mathbf{A_1})$ and $(\mathbf{A_2})$, by Theorem 1.4 (a), for each $y \in \mathbb{R}^d$, the martingale problem $(\mathcal{L}^y, \delta_y)$ admits a unique solution $(\mathbb{Q}_y, X_t)$. Moreover, for each $y \in \mathbb{R}^d$ and $f \in C^\infty_c$, by the proof for Theorem 1.4 (a), we have

$$\mathbf{E}_{\mathbb{Q}_y} \int_0^\infty \mathrm{e}^{-\lambda t}f(X_t)\,\mathrm{d}t = u_{a^y}(y), \tag{5.2}$$

where u_{a^y} is the solution to (PE) with $\mathcal{L}$ replaced by $\mathcal{L}^y$. Noting that a^y meets all the assumptions in Lemma 4.4, we have

$$\|u_{a^y}\|_{B^{\psi,1}_A} \leqslant C\|f\|_{B^{\psi,0}_A} \leqslant C\|f\|_A. \tag{5.3}$$

Combining and (5.2) and (5.3) and using Theorem 3.13, we get

$$\mathbf{E}_{\mathbb{Q}_y} \int_0^\infty \mathrm{e}^{-\lambda t}f(X_t)\,\mathrm{d}t \overset{(5.2)}{\leqslant} C\|u_{a^y}\|_{\mathscr{C}^{\frac{\varepsilon}{1+\varepsilon}}_\psi} \overset{(3.24)}{\leqslant} C\|u_{a^y}\|_{B^{\psi,1}_A} \overset{(5.3)}{\leqslant} C\|f\|_A. \tag{5.4}$$

Now let $\tau_1 = \inf\{t > 0 : |X_t - X_0| > \delta\}$ and $\tau_{k+1} = \tau_k + \tau_1 \circ \theta_{\tau_k}$. It holds that

$$\mathbb{Q}_y|_{\mathscr{F}_{\tau_1}} = \mathbb{P}_y|_{\mathscr{F}_{\tau_1}},$$

due to the fact that $a^y(\cdot, z)|_{B_\delta(y)} = a(\cdot, z)|_{B_\delta(y)}$ and the uniqueness of martingale problem $(\mathcal{L}, \delta_y)$. Suppose that $(\mathbb{P}_\mu, X_t)$ is the unique martingale solution to $(\mathcal{L}, \mu)$, then for any non-negative function $f \in C_c^\infty$,

$$\mathbb{E}_\mu \int_0^{\tau_1} \mathrm{e}^{-\lambda t} f(X_t) \, \mathrm{d}t = \int_{\mathbb{R}^d} \mu(\mathrm{d}y) \, \mathbb{E}_y \int_0^{\tau_1} \mathrm{e}^{-\lambda t} f(X_t) \, \mathrm{d}t$$

$$= \int_{\mathbb{R}^d} \mu(\mathrm{d}y) \, \mathbf{E}_{\mathbb{Q}_y} \int_0^{\tau_1} \mathrm{e}^{-\lambda t} f(X_t) \, \mathrm{d}t \leqslant \sup_y \mathbf{E}_{\mathbb{Q}_y} \int_0^\infty \mathrm{e}^{-\lambda t} f(X_t) \, \mathrm{d}t \lesssim \|f\|_A.$$

By strong Markov property,

$$\mathbb{E}_\mu \int_{\tau_k}^{\tau_{k+1}} \mathrm{e}^{-\lambda t} f(X_t) \, \mathrm{d}t = \mathbb{E}_\mu \int_{\tau_k}^{\tau_{k+1}} \mathrm{e}^{-\lambda t} f(X_t) \, \mathrm{d}t = \mathbb{E}_\mu \int_0^{\tau_1 \circ \theta_{\tau_k}} \mathrm{e}^{-\lambda(\tau_k + t)} f(X_t \circ \theta_{\tau_k}) \, \mathrm{d}t$$

$$= \mathbb{E}_\mu \mathrm{e}^{-\lambda \tau_k} \left(\int_0^{\tau_1} \mathrm{e}^{-\lambda t} f(X_t) \, \mathrm{d}t \right) \circ \theta_{\tau_k}$$

$$= \mathbb{E}_\mu \left(\mathrm{e}^{-\lambda \tau_k} \mathbb{E}_{X_{\tau_k}} \int_0^{\tau_1} \mathrm{e}^{-\lambda t} f(X_t) \, \mathrm{d}t \right)$$

$$\lesssim \|f\|_A \mathbb{E}_\mu \mathrm{e}^{-\lambda \tau_k} \leqslant \|f\|_A \sup_y \mathbb{E}_y \mathrm{e}^{-\lambda \tau_k}.$$

To estimate $\sup_y \mathbb{E}_y \mathrm{e}^{-\lambda \tau_k}$, using strong Markov property again, we have

$$\mathbb{E}_x \mathrm{e}^{-\lambda \tau_{k+1}} = \mathbb{E}_x \mathrm{e}^{-\lambda(\tau_k + \tau_1 \circ \theta_{\tau_k})} = \mathbb{E}_x \left[\mathrm{e}^{-\lambda \tau_k} \mathbb{E}_x \left(\mathrm{e}^{-\lambda \tau_1} \circ \theta_{\tau_k} | \mathscr{F}_{\tau_k} \right) \right]$$

$$= \mathbb{E}_x \left[\mathrm{e}^{-\lambda \tau_k} \mathbb{E}_{X_{\tau_k}} \mathrm{e}^{-\lambda \tau_1} \right] \leqslant \left(\sup_y \mathbb{E}_y \mathrm{e}^{-\lambda \tau_1} \right) \mathbb{E}_x \mathrm{e}^{-\lambda \tau_k}$$

$$\leqslant \cdots \leqslant \left(\sup_y \mathbb{E}_y \mathrm{e}^{-\lambda \tau_1} \right)^{k+1}.$$

Thus,

$$\mathbb{E}_\mu \int_0^\infty \mathrm{e}^{-\lambda t} f(X_t) \, \mathrm{d}t \lesssim \sum_{k=0}^\infty \mathbb{E}_\mu \int_{\tau_k}^{\tau_{k+1}} \mathrm{e}^{-\lambda t} f(X_t) \, \mathrm{d}t$$

$$\lesssim \|f\|_A \sum_{k=1}^\infty \left(\sup_y \mathbb{E}_y \mathrm{e}^{-\lambda \tau_k} \right) \leqslant \|f\|_A \sum_{k=1}^\infty \left(\sup_y \mathbb{E}_y \mathrm{e}^{-\lambda \tau_1} \right)^k.$$

So the desired conclusion follows if $\mathbb{E}_y \mathrm{e}^{-\lambda \tau_1} \leqslant 1/2$ for all y. To achieve this, we choose $g \in C_b^\infty$, which satisfies $g(x) = 0$ if $x \in B_{\delta/2}$, $g(x) = 1$ if $x \in B_\delta$ and $\|\nabla g\|_\infty \leqslant C\delta^{-1}$. Let $g_y(x) = g(x - y)$. It is easy to verify that there exists a constant $K_\delta < \infty$ such that

$$\mathcal{L} g_y(x) - \lambda g_y(x) \leqslant \|\mathcal{L} g_y\|_\infty \leqslant K_\delta.$$

Note that

$$\mathbb{E}_y \left[\mathrm{e}^{-\lambda \tau_1} g_y(X_{\tau_1}) \right] - g_y(y) = \mathbb{E}_y \int_0^\infty \mathrm{e}^{-\lambda s} (\mathcal{L} g_y - \lambda g_y)(X_s) \, \mathrm{d}s.$$

Since $g_y(y) = g(0) = 0$ and $g_y(X_{\tau_1}) = 1$, $\mathbb{P}_y$-a.s., we have

$$\mathbb{E}_y \mathrm{e}^{-\lambda \tau_1} \leqslant K_\delta/\lambda.$$

Choosing $\lambda \geqslant 2K \vee \lambda_0$, we have $\sup_y \mathbb{E}_y \mathrm{e}^{-\lambda \tau_1} \leqslant 1/2$. Therefore,

$$\mathbb{E}_\mu \int_0^\infty \mathrm{e}^{-\lambda t} f(X_t)\,\mathrm{d}t \lesssim \|f\|_A \sum_{k=1}^\infty \frac{1}{2^k} \lesssim \|f\|_A, \quad \text{for all } \lambda \geqslant 2K \vee \lambda_0.$$

For any $\lambda > 0$ and $\mu \in \mathcal{P}(\mathbb{R}^d)$,

$$\mathbb{E}_\mu \int_0^\infty \mathrm{e}^{-\lambda t} f(X_t)\,\mathrm{d}t \leqslant \sup_y \mathbb{E}_y \int_0^1 f(X_t)\,\mathrm{d}t \cdot \sum_{k=0}^\infty \mathrm{e}^{-\lambda k}$$

$$\lesssim \lambda^{-1} \sup_y \mathbb{E}_y \int_0^\infty \mathrm{e}^{-(2K \vee \lambda_0)t} f(X_t)\,\mathrm{d}t \lesssim \|f\|_A/\lambda.$$

So we complete our proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Remark 5.2. *For diffusion operators, Stroock and Varadhan showed that if the diffusion coefficients are uniformly elliptic and continuous, then the corresponding martingale problem is well-posed (see for instance, [SV07]). Two key ingredients proving the uniqueness are:*

(1) $(I - \Delta)^{-1} L^p \hookrightarrow L^\infty$, $p > d/2$;
(2) the L^p boundedness of the Riesz transforms (singular integrals).

Following the approach of Stroock and Varadhan to obtain a similar result for $\mathcal{L}$ when the coefficient a is merely uniformly continuous in x, one would need to establish the boundedness of singular integrals in the Orlicz space L_A, which satisfies $(I + \phi(-\Delta))^{-1} L_A \hookrightarrow L^\infty$. Unfortunately, the following example demonstrates that if A grows rapidly, even the Hilbert transform (the archetypal singular integral operator) may not be bounded in L_A: let $d = 1$ and H be the Hilbert transform. Set $I = (0,1)$ and $A(t) = \mathrm{e}^{t^2} - 1$. Then

$$H\chi_I(x) = \log\left|\frac{x}{(x-1)}\right|.$$

For each $\lambda > 0$, we have

$$\int A(|H\chi_I|/\lambda) = \int_0^\infty A'(t)\,|\{x : H\chi_I(x) > \lambda t\}|\,\mathrm{d}t \geqslant \int_1^\infty \frac{\mathrm{e}^{t^2}}{(\mathrm{e}^{\lambda t} - 1)}\,\mathrm{d}t = \infty.$$

Thus, it appears that we may need to explore new frameworks to address this problem when a does not satisfies $(\mathbf{A_2})$.

We conclude this section by providing the proof of Corollary 1.5.

Proof of Corollary 1.5. The existence of a weak solution to (1.7) follows from standard arguments under the assumption that the coefficient is continuous and bounded.

For the uniqueness, let S be the gamma subordinator, whose Laplace exponent is given by $\phi(s) = \log(1 + s)$. Under the assumption that σ is non-degenerate, as presented in Example 2, the generator operator $\mathcal{L}_\sigma$ associated with (1.7) is given by (1.8). In Section 2.3, we have verified that the jump kernel of the variance gamma process and the function a satisfy conditions $(\mathbf{\Phi})$ and $(\mathbf{A_1})$-$(\mathbf{A_2})$, respectively. Since any weak solution to (1.7) is also a solution to martingale problem $(\mathcal{L}_\sigma, \delta_x)$, in the light of Theorem 1.4, (1.7) admits a unique solution.

For (1.10), set $A(t) = e^{t^\beta} - 1$ with $\beta > 1$. Then $[\psi^{-1}(t^{1+\varepsilon})]^d = (e^{t^{1+\varepsilon}} - 1)^{d/2} \lesssim A(t)$, for all $t \gg 1$ and some $\varepsilon \in (0, \beta - 1)$. Thanks to Theorem 1.4 (inequality (1.9)), we obtain that

$$
\begin{aligned}
\mathbb{E}_x \int_0^\infty e^{-\lambda t} f(X_t)\, dt &\leqslant \frac{C}{\lambda} \inf \left\{ \lambda > 0 : \int_{\mathbb{R}^d} A\left(f(x)/\lambda\right) dx \leqslant 1 \right\} \\
&= \frac{C}{\lambda} \inf \left\{ \lambda > 0 : \int_{\mathbb{R}^d} \left(\exp\left[(f(x)/\lambda)^\beta \right] - 1 \right) dx \leqslant 1 \right\}.
\end{aligned}
$$

$\square$

Appendix A. Remarks on a theorem by R. Bass

This section extends the main results of [Bas09] on stable-like operators using Littlewood-Paley theory and scaling techniques, and emphasizes that it is more natural to study a priori estimates in the $\mathscr{C}^s$ space. We also observe that the function space X mentioned in Section 1.2 (aimed to replace $\mathscr{C}^s$) contains unbounded and discontinuous functions, so it is not suitable for the problems we care about. We will also show that techniques that have proven to be successful for stable-like operators can not yield satisfactory results for our main problem of concern.

To begin, we review some fundamental concepts from classical Littlewood-Paley theory. Let χ be the same smooth function defined in Section 3.

$$
\varphi(\xi) := \chi(\xi) - \chi(2\xi).
$$

It is easy to see that $\varphi \geqslant 0$,

$$
\operatorname{supp} \varphi \subset \mathcal{C} := B_1 \setminus B_{\frac{3}{8}} \quad \text{and} \quad \varphi(x) = 1 \text{ if } x \in B_{\frac{3}{4}} \setminus B_{\frac{1}{2}}. \tag{A.1}
$$

Operators Δ_j is defined by

$$
\Delta_j f := \begin{cases} \mathcal{F}^{-1}(\chi(2\cdot)\mathcal{F}f), & j = -1, \\ \mathcal{F}^{-1}(\varphi(2^{-j}\cdot)\mathcal{F}f), & j \geqslant 0. \end{cases}
$$

Definition A.1 (Hölder-Zygmund space)**.** *Assume that $s \in \mathbb{R}$, let $\mathscr{C}^s$ denote the collection of all distribution f satisfying*

$$
\|f\|_{\mathscr{C}^s} := \sup_{j \geqslant -1} 2^{js} \|\Delta_j f\|_\infty < \infty.
$$

The following result is well-known.

Theorem A.2 ([BCD11, Theorem 2.36])**.** *Assume $s > 0$ and $s \notin \mathbb{N}$, then $\mathscr{C}^s$ is the usual Hölder space and*

$$
\|f\|_{\mathscr{C}^s} \asymp \|f\|_\infty + \sup_{x \neq y} \frac{|\nabla^{[s]} f(x) - \nabla^{[s]} f(y)|}{|x - y|^{s - [s]}}.
$$

Let ν be a non-degenerate α-stable measure, i.e., for each $\xi \in \mathbb{R}^d \setminus \{0\}$,

$$
\int_{\mathbb{S}^{d-1}} |\sigma \cdot \xi|^2 \Sigma(d\sigma) > 0.
$$

Here Σ is the spectral measure of ν, i.e.

$$
\nu(dz) = \frac{dr}{r^{1+\alpha}} \Sigma(d\sigma), \quad z = r\sigma.
$$

Let a is a bounded positive function on $\mathbb{R}^d \times \mathbb{R}^d$. The operator $\mathscr{L}_a$ is defined by

$$\mathscr{L}_a u(x) = \int_{\mathbb{R}^d} \left(u(x+z) - u(x) - \nabla u(x) \cdot z \mathbf{1}_{\alpha \in (1,2)} \right) \ a(x,z) \nu(\mathrm{d}z)$$

when $\alpha \in (0,2)$ and $\alpha \neq 1$. If $\alpha = 1$, we always assume $\int_{\mathbb{S}^{d-1}} \sigma \ a(x, r\sigma) \Sigma(\mathrm{d}\sigma) = 0$ and define

$$\mathscr{L}_a u(x) = \int_{\mathbb{R}^d} \left(u(x+z) - u(x) - \nabla u(x) \cdot z \mathbf{1}_{B_1}(z) \right) \ a(x,z) \nu(\mathrm{d}z).$$

The following result is a generalization of Proposition 4.2 and Proposition 4.3 in [Bas09]. Note that here we do not need to assume ν is absolutely continuous with respect to Lebesgue measure.

Lemma A.3. *Let $c_0 \in (0,1)$. Suppose ν is a non-degenerate α-stable measure. Assume that a only depends on z and $c_0 \leqslant a \leqslant c_0^{-1}$, then for each $\lambda \geqslant 1, \beta \in \mathbb{R}$, there is a constant $C = C(d, \nu, \alpha, \beta, c_0)$ such that*

$$\lambda \|u\|_{\mathscr{C}^\beta} + \|u\|_{\mathscr{C}^{\alpha+\beta}} \leqslant C \|\lambda u - \mathscr{L}_a u\|_{\mathscr{C}^\beta}. \tag{A.2}$$

Proof. We only prove the case $\alpha \neq 1$ here.
Step 1. Let

$$\mathcal{A} = \left\{ a \in L^\infty(\mathbb{R}^d) : c_0 \leqslant a(z) \leqslant c_0^{-1}, z \in \mathbb{R}^d \right\}.$$

We first prove that for any $v \in \mathscr{S}'(\mathbb{R}^d)$ satisfying supp $\hat{v} \subseteq \mathcal{C} = B_1 \setminus B_{\frac{3}{8}}$, it holds that

$$\|v\|_\infty \leqslant C \inf_{a \in \mathcal{A}} \|\mathscr{L}_a v\|_\infty. \tag{A.3}$$

Here C is a constant independent of on v.

Assume (A.3) does not hold. Then, there is a sequence v_n such that $\widehat{v_n}$ is supported on $\mathcal{C}$, and a sequence $a_n \in \mathcal{A}$ such that

$$1 = \|v_n\|_\infty \geqslant n \|\mathscr{L}_{a_n} v_n\|_\infty. \tag{A.4}$$

Let $h = \mathscr{F}^{-1}(\chi(\cdot/2) - \chi(4\cdot))$, where χ is the same function in Section 3. Noting that $(\chi(\cdot/2) - \chi(4\cdot))\widehat{v_n} = \widehat{v_n}$, we have

$$v_n(x) = \int_{\mathbb{R}^d} h(x-y) v_n(y) \mathrm{d}y.$$

So for any $k \in \mathbb{N}$,

$$\|\nabla^k v_n\|_\infty = \|\nabla^k h * v_n\|_\infty \leqslant \|\nabla^k h\|_1 \|v_n\|_\infty \leqslant C_k. \tag{A.5}$$

By Ascoli-Azela's lemma and diagonal argument, there is a subsequence of $\{v_n\}$ (still denoted by v_n for simplicity) and $v \in C_b^\infty$ such that $\nabla^k v_n$ converges to $\nabla^k v$ uniformly on any compact set. Let $\chi_R(\cdot) = \chi(\cdot/R)$. For any $\phi \in \mathscr{S}(\mathbb{R}^d)$,

$$\left| \int \phi(v_n - v) \right| \leqslant \int |\phi \chi_R \cdot (v_n - v)| + \int |\phi(1 - \chi_R)(v_n - v)|$$

$$\leqslant \|\phi\|_{L^1} \|v_n - v\|_{L^\infty(B_{3R/2})} + 2 \sup_{|x| > R} |\phi(x)|.$$

Letting $n \to \infty$ and then $R \to \infty$, we get

$$\langle \phi, v_n \rangle \to \langle \phi, u \rangle, \quad \forall \phi \in \mathscr{S}(\mathbb{R}^d).$$

Here $\langle,\rangle$ denotes the dual pair of $\mathscr{S}(\mathbb{R}^d)$ and $\mathscr{S}'(\mathbb{R}^d)$. That is to say $v_n \to v$ in $\mathscr{S}'(\mathbb{R}^d)$ and consequently, $\hat{v_n} \to \widehat{v}$ in $\mathscr{S}'(\mathbb{R}^d)$. For any $\phi \in \mathscr{S}(\mathbb{R}^d)$ supported on $\mathbb{R}^d\backslash\mathcal{C}$, we have

$$\langle\phi,\hat{v}\rangle = \lim_{n\to\infty}\langle\phi,\widehat{v_n}\rangle = 0,$$

which means $\hat{v}$ is also supported on $\mathcal{C}$.

On the other hand, note that $a_n\nu$ is a sequence of Radon measures on $\mathbb{R}^d\backslash\{0\}$, and $\sup_n \int_K a_n(z)\nu(\mathrm{d}z) \leqslant c_0^{-1}\nu(K) < \infty$ for each compact subset K in $\mathbb{R}^d\backslash\{0\}$. Thus, there is a subsequence of a_n (still denoted by a_n for simplicity) such that

$$a_n\nu \xrightarrow{v} \nu_0 \quad \text{and} \quad a_0 := \mathrm{d}\nu_0/\mathrm{d}\nu \in [c_0, c_0^{-1}]. \tag{A.6}$$

Using (A.4)-(A.6) and the fact that $\nabla^k v_n \to \nabla^k v$ uniformly on each compact subset, we have

$$\mathscr{L}_{a_0}v(x) = \int_{\mathbb{R}^d} \left(v(x+z) - v(x) - \nabla v(x)\cdot z\mathbf{1}_{\alpha\in(1,2)}\right) a(z)\nu(\mathrm{d}z)$$

$$= \lim_{n\to\infty} \left|\int_{\mathbb{R}^d} \left(v_n(x+z) - v_n(x) - \nabla v_n(x)\cdot z\mathbf{1}_{\alpha\in(1,2)}\right) a_n(z)\nu(\mathrm{d}z)\right|$$

$$\overset{\text{(A.4)}}{\leqslant} \lim_{n\to\infty} \frac{\|v_n\|_\infty}{n} = 0.$$

This implies $0 = \widehat{\mathscr{L}_{a_0}v} = \psi_{a_0}\hat{v}$. Since for $\xi \neq 0$,

$$\mathrm{Re}(\psi_{a_0}(\xi)) = \int_{\mathbb{R}^d}(1 - \cos(z\cdot\xi))\, a_0(z)\nu(\mathrm{d}z) \gtrsim \int_{|z|\leqslant\frac{\pi}{4|\xi|}} |z\cdot\xi|^2\nu(\mathrm{d}z)$$

$$= \int_0^{\frac{\pi}{4|\xi|}} r^{1-\alpha}\mathrm{d}r \int_{\mathbb{R}^{d-1}} |\sigma\cdot\xi|^2\Sigma(\mathrm{d}\sigma) > 0,$$

we get that the support of $\hat{v}$ is the origin. This contradicts the previous conclusion that supp $\hat{v} \subseteq \mathcal{C}$. So we complete the proof for (A.3).

Step 2. Suppose $u \in \mathscr{S}'(\mathbb{R}^d)$ and supp $\hat{u} \subseteq \lambda\mathcal{C}$. Let $v(x) := u(\lambda^{-1}x)$ and $a^\lambda(z) = a(\lambda^{-1}z)$. By the scaling property of ν, we get

$$\mathscr{L}_{a^\lambda}v(x) = \int_{\mathbb{R}^d} \left(u(\lambda^{-1}x + \lambda^{-1}z) - u(\lambda^{-1}x) - \nabla u(\lambda^{-1}(x))\cdot(\lambda^{-1}z)\mathbf{1}_{\alpha\in(1,2)}\right) a^\lambda(z)\nu(\mathrm{d}z)$$

$$= \int_{\mathbb{R}^d} \left(u(\lambda^{-1}x + z') - u(\lambda^{-1}x) - \nabla u(\lambda^{-1}(x))\cdot z'\mathbf{1}_{\alpha\in(1,2)}\right) a(z')\lambda^{-\alpha}\nu(\mathrm{d}z')$$

$$= \lambda^{-\alpha}(\mathscr{L}_a u)(\lambda^{-1}x).$$

Thus, by (A.3), we get

$$\|u\|_\infty = \|v\|_\infty \leqslant C\|\mathscr{L}_{a^\lambda}v\|_\infty = C\lambda^{-\alpha}\|\mathscr{L}_a u\|_\infty, \tag{A.7}$$

provided that supp $\hat{u} \in \lambda\mathcal{C}$.

Step 3. Like the proof for Theorem 4.1, it is easy to see that for each $j \geqslant -1$,

$$\lambda\|\Delta_j u\|_\infty \leqslant \|\lambda\Delta_j u - \mathscr{L}_a\Delta_j u\|_\infty.$$

By (A.7), for any $j \geqslant 0$,

$$2^{j\alpha}\|\Delta_j u\|_\infty \leqslant C\|\mathscr{L}_a\Delta_j u\|_\infty \leqslant C\lambda\|\Delta_j u\|_\infty + C\|\lambda\Delta_j u - \mathscr{L}_a\Delta_j u\|_\infty$$

$$\leqslant \|\lambda\Delta_j u - \mathscr{L}_a\Delta_j u\|_\infty.$$

Thus, for all $\lambda \geqslant 1$ and $\beta \in \mathbb{R}$, we get (A.2). $\qquad\qquad\square$

Theorem A.4. *Let ν be a non-degenerate α-stable measure with $\alpha \in (0,2)$. Suppose that there exist constants c_0 and $\gamma \in (0,1)$, such that $c_0 \leqslant a \leqslant c_0^{-1}$ and $\sup_{z \in \mathbb{R}^d} \|a(\cdot,z)\|_{\mathscr{C}^\gamma} \leqslant c_0^{-1}$, then for any $\lambda > 0$ and $\beta \in [0,\gamma]$, there is a constant $C = C(d,\nu,\alpha,\beta,\gamma,c_0)$ such that*

$$\lambda \|u\|_{\mathscr{C}^\beta} + \|u\|_{\mathscr{C}^{\alpha+\beta}} \leqslant C \|\lambda u - \mathscr{L}_a u\|_{\mathscr{C}^\beta}.$$

With Lemma A.3 at hand, utilizing the method of frozen coefficients as before, we can prove the case that a is a positive function on $\mathbb{R}^d \times \mathbb{R}^d$. Since the proof for this result is just a repetition of the proof of Theorem 1.1, we omit it here.

We point out that the approach used in the proof of Lemma A.3 cannot be directly extended to the case of $\nu(\mathrm{d}z) = \frac{\mathbf{1}_{B_1}(z)}{|z|^d} \mathrm{d}z$, and it may not be possible. One significant hurdle arises from the need to obtain the desired regularity estimate, which requires the use of the ψ-decomposition with $\psi(R) \asymp \log R$ ($R \to \infty$). However, when attempting to repeat the same procedure, it becomes apparent that *Step 2* in the proof for Lemma A.3 is invalid in the current situation.

Appendix B. A remark on space $\mathscr{X}^s$

In this section, we show that $\mathscr{X}^s$ given by (1.11) may contains unbounded discontinuous functions, even if $s > 0$.

Proposition B.1. *Let $\mathscr{X}^1$ be the function space given by (1.11). Then $\mathscr{X}^1 \nsubseteq L^\infty$.*

Proof. We provide an example to demonstrate that $\mathscr{X}^1$ encompasses unbounded, discontinuous functions.

Let $\phi \in \mathscr{S}(\mathbb{R}^d)$ such that $\mathcal{F}(\phi) \in C_c^\infty(B_{\frac{3}{4}} \setminus B_{\frac{1}{2}})$, $\mathcal{F}(\phi) \in [0,1]$ and $\mathcal{F}(\phi) = 1$ on $B_{\frac{11}{16}} \setminus B_{\frac{9}{16}}$. Define

$$f(x) = \begin{cases} \displaystyle\sum_{j \geqslant 1} j^{-1} \phi(2^j x), & x \neq 0 \\ 0, & x = 0. \end{cases}$$

Due to (A.1), $\mathcal{F}(\phi)\varphi = \mathcal{F}(\phi)$. Thus,

$$\begin{aligned} \mathcal{F}(f)(\xi)\varphi(2^{-j}\xi) &= j^{-1} 2^{-j} (\mathcal{F}(\phi)\varphi)(2^{-j}\xi) \\ &= j^{-1} 2^{-j} \mathcal{F}(\phi)(2^{-j}\xi). \end{aligned}$$

This yields

$$\sup_{j \geqslant -1} \max\{1,j\} \|\Delta_j f\|_\infty = \sup_{j \geqslant 1} \|\phi(2^j \cdot)\| = \|\phi\|_\infty < \infty.$$

Therefore, $f \in \mathscr{X}^1$.

On the other hand, since $\phi \in \mathscr{S}(\mathbb{R}^d)$ and

$$|\phi(x)| = \left| \int_{\mathbb{R}^d} \mathcal{F}(\phi)(\xi) e^{i2\pi x \cdot \xi} \mathrm{d}\xi \right| \leqslant \int_{\mathbb{R}^d} \mathcal{F}(\phi)(\xi) \mathrm{d}\xi = \phi(0) > 0,$$

there are constants $\varepsilon_0, \delta_0 \in (0, 2^{-10})$ such that $\phi(x) \geqslant 2\varepsilon_0$, when $|x| \leqslant \delta_0$. Now put $\delta := \delta_0^{10}$. For any $x \in B_\delta \setminus \{0\}$, set $j_x := [-\log_2 |x|]$. It holds that

$$C \log j_x \geqslant \phi(0) \sum_{1 \leqslant j \leqslant j_x} j^{-1} \geqslant \sum_{1 \leqslant j \leqslant j_x + \log_2 \delta_0} j^{-1} \phi(2^j x) \geqslant \varepsilon_0 \log j_x. \tag{B.1}$$

Noting that $|\phi(2^j x)| \leqslant C(2^j|x|)^{-N}$ if $j \geqslant j_x$, we get

$$\sum_{j>j_x} |j^{-1}\phi(2^j x)| \leqslant C \sum_{j>j_x} j^{-1} 2^{-jN}|x|^{-N} \leqslant C j_x^{-1} \ll 1. \tag{B.2}$$

Moreover,

$$\sum_{j_x+\log_2 \delta_0 < j \leqslant j_x} |j^{-1}\phi(2^j x)| \leqslant -C\log_2 \delta_0. \tag{B.3}$$

Combining (B.1)-(B.3), one sees that for any $x \in B_\delta \backslash \{0\}$,

$$f(x) \geqslant \varepsilon_0 \log j_x - C \geqslant \frac{\varepsilon_0}{2} \log\log_2 |x|^{-1} - C$$

and

$$|f(x)| \leqslant C \log\log_2 |x|^{-1} + C.$$

This implies $f \notin L^\infty$ and $f \in L^1(B_\delta)$.

To sum up, we see that $f \in (\mathscr{X}^1 \cap L_{\mathrm{loc}}^1)\backslash L^\infty$. $\qquad\square$

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