DIRICHLET HEAT KERNEL ESTIMATES FOR RECTILINEAR STABLE PROCESSES

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Abstract. Let $d \geq 2$, $\alpha \in (0,2)$, and X be the rectilinear α-stable process on $\mathbb{R}^d$. We first present a geometric characterization of open subset $D \subset \mathbb{R}^d$ so that the part process X^D of X in D is irreducible. We then study the properties of the transition density functions of X^D, including the strict positivity property as well as their sharp two-sided bounds in $C^{1,1}$ domains in $\mathbb{R}^d$. Our bounds are shown to be sharp for a class of $C^{1,1}$ domains.

Contents

1. Introduction and main results

Dirichlet heat kernels for non-local operators are a fundamental subject both in analysis and in probability theory. Sharp two-sided Dirichlet heat kernel estimates for fractional Laplacian $\Delta^{\alpha/2} := -(-\Delta)^{\alpha/2}$ in $C^{1,1}$ open subsets of $\mathbb{R}^d$ with $\alpha \in (0,2)$ have first been obtained in [8]. Since then, there are many works in extending it to certain classes of symmetric Markov processes and their lower order perturbations as well as to more general open sets. In many of these works, the jump measures of the Markov processes are absolutely continuous with respect to the Lebesgue measure.

Let $d \geq 2$ and $\alpha \in (0,2)$. The purpose of this paper is to study the Dirichlet heat kernels for

$$\mathcal{L} := -\sum_{k=1}^{d} \left(-\frac{\partial^2}{(\partial x^{(k)})^2} \right)^{\alpha/2}$$

in open subsets of $\mathbb{R}^d$. Here $x^{(k)}$ is the k^{th}-coordinate of a point $x = (x^{(1)}, x^{(2)}, \cdots, x^{(d)}) \in \mathbb{R}^d$. We call $\mathcal{L}$ a rectilinear fractional Laplace operator, which is more singular than the usual isotropic fractional Laplacian $\Delta^{\alpha/2}$. The rectilinear fractional Laplacian $\mathcal{L}$ is the infinitesimal generator of the rectilinear α-stable process

$$X = \left\{ X_t = \left(X_t^{(1)}, X_t^{(2)}, \cdots, X_t^{(d)} \right); t \geq 0 \right\}$$

on $\mathbb{R}^d$, where $X^{(1)}$, $X^{(2)}, \cdots, X^{(d)}$ are independent one-dimensional symmetric α-stable processes. The process X is a Lévy process on $\mathbb{R}^d$ whose Lévy exponent is $\Psi(\xi) = \sum_{j=1}^{d} |\xi^{(j)}|^\alpha$ for $\xi = (\xi^{(1)}, \xi^{(2)}, \cdots, \xi^{(d)}) \in$

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$\mathbb{R}^d$; that is,

$$\mathbb{E}e^{i\xi\cdot(X_t-X_0)}=e^{-t\sum_{j=1}^d|\xi^{(j)}|^\alpha}\quad\text{for } t>0 \text{ and } \xi\in\mathbb{R}^d.$$

The Lévy measure of X is singular with respect to the Lebesgue measure on $\mathbb{R}^d$; see (1.7).

Unlike the isotropic (or, rotationally symmetric) α-stable process Z on $\mathbb{R}^d$, the distribution of the increments of the rectilinear α-stable process X is not rotationally invariant. The isotropic α-stable process Z is a Lévy process on $\mathbb{R}^d$ having infinitesimal generator $\Delta^{\alpha/2}$ and Lévy exponent $|\xi|^\alpha:=\big(\sum_{j=1}^d|\xi^{(j)}|^2\big)^{\alpha/2}$. For $f\in C_c^2(\mathbb{R}^d)$,

$$\Delta^{\alpha/2}f(x)=\int_{\mathbb{R}^d\setminus\{0\}}\big(f(x+z)-f(x)-\nabla f(x)\cdot z\mathbb{1}_{\{|z|\le1\}}\big)\frac{\mathcal{C}_{d,\alpha}}{|z|^{d+\alpha}}dz, \tag{1.1}$$

while

$$\mathcal{L}f(x)=\sum_{j=1}^d\int_{\mathbb{R}\setminus\{0\}}\left(f(x+we_j)-f(x)-w\mathbb{1}_{\{|w|\le1\}}\frac{\partial f(x)}{\partial x^{(j)}}\right)\frac{\mathcal{C}_{1,\alpha}}{|w|^{1+\alpha}}dw,$$

where e_j is the unit vector in the positive $x^{(j)}$-direction and

$$\mathcal{C}_{d,\alpha}=\frac{\alpha2^{\alpha-1}\Gamma((d+\alpha)/2)}{\pi^{d/2}\Gamma(1-\alpha/2)}. \tag{1.2}$$

Here Γ is the usual Gamma function defined by $\Gamma(\lambda):=\int_0^\infty t^{\lambda-1}e^{-t}dt$ for $\lambda>0$.

It is well known (see [3, Theorem 2.1] via stable scaling) that the isotropic α-stable process Z on $\mathbb{R}^d$ has a smooth density function $p^{(d,\alpha)}(t,x,y)$ with respect to the Lebesgue measure on $\mathbb{R}^d$ and there are positive constants $c_2>c_1>0$ that depend only on d and α so that

$$c_1\left(t^{-d/\alpha}\wedge\frac{t}{|x-y|^{d+\alpha}}\right)\le p^{(d,\alpha)}(t,x,y)\le c_2\left(t^{-d/\alpha}\wedge\frac{t}{|x-y|^{d+\alpha}}\right)\quad\text{for } t>0,\ x,y\in\mathbb{R}^d. \tag{1.3}$$

In this paper, we will use $:=$ as a way of definition. For $a,b\in\mathbb{R}$, $a\wedge b:=\min\{a,b\}$ and $a\vee b:=\max\{a,b\}$. Since for $a>0$ and $b>0$,

$$\frac{ab}{a+b}\le a\wedge b\le\frac{2ab}{a+b},$$

we can rewrite the estimates in (1.3) by

$$\frac{c_3\,t}{(t^{1/\alpha}+|x-y|)^{d+\alpha}}\le p^{(d,\alpha)}(t,x,y)\le\frac{c_4\,t}{(t^{1/\alpha}+|x-y|)^{d+\alpha}}\quad\text{for } t>0 \text{ and } x,y\in\mathbb{R}^d, \tag{1.4}$$

where constants $c_4>c_3>0$ depend only on d and α.

By the independence between its coordinate processes, the rectilinear α-stable process X on $\mathbb{R}^d$ has a smooth transition density function

$$p(t,x,y)=\prod_{k=1}^d p^{(1,\alpha)}(t,x^{(k)},y^{(k)})\quad\text{for } t>0 \text{ and } x=(x^{(k)}),\ y=(y^{(k)})\in\mathbb{R}^d, \tag{1.5}$$

with respect to the Lebesgue measure on $\mathbb{R}^d$. By (1.3), there is a constant $C_1=C_1(d,\alpha)>1$ so that

$$C_1^{-1}\prod_{k=1}^d\left(t^{-1/\alpha}\wedge\frac{t}{|x^{(k)}-y^{(k)}|^{1+\alpha}}\right)\le p(t,x,y)\le C_1\prod_{k=1}^d\left(t^{-1/\alpha}\wedge\frac{t}{|x^{(k)}-y^{(k)}|^{1+\alpha}}\right) \tag{1.6}$$

for all $t>0$, and $x=(x^{(k)})$, $y=(y^{(k)})\in\mathbb{R}^d$. This is clearly quite different from the estimates for $p^{(d,\alpha)}(t,x,y)$ of the isotropic or rotationally symmetric α-stable process Z on $\mathbb{R}^d$.

Though the rectilinear α-stable process X and the isotropic α-stable process Z are both Lévy processes that are invariant under the α-stable scaling, they have many fundamentally different properties, which will be further revealed in this paper. For instance, it is shown in [1] that Harnack inequality fails for rectilinear stable processes, while scale invariant Harnack inequality holds for isotropic stable processes. The root of these differences lies in the fact that the isotropic α-stable process Z can jump in any direction uniformly, while the rectilinear α-stable process X can only jump along the directions of coordinate axes, one at a time, and thus is much singular. The Lévy measure of X is

$$\mu(dz) = \sum_{j=1}^{d} \frac{\mathcal{C}_{1,\alpha}}{|z^{(j)}|^{1+\alpha}} dz^{(j)} \otimes \prod_{\substack{k=1\\k\neq j}}^{d} \delta_{\{0\}}(dz^{(k)}), \tag{1.7}$$

where $\delta_{\{0\}}$ denotes the Dirac measure concentrated at 0 and $z = (z^{(1)}, z^{(2)}, \cdots, z^{(d)}) \in \mathbb{R}^d$. The Lévy measure μ describes how the rectilinear α-stable process X jumps. For any non-negative measurable function f on $\mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d$ with $f(s, x, x) = 0$ for any $s \geq 0$ and $x \in \mathbb{R}^d$ and for any stopping time S with respect to the minimum augmented filtration generated by X, we have

$$\mathbb{E}_x \left[\sum_{s\leq S} f(s, X_{s-}, X_s) \right] = \mathbb{E}_x \left[\int_0^S \int_{\mathbb{R}^d} f(s, X_s, X_s + z)\mu(dz)ds \right]. \tag{1.8}$$

See, e.g., [12, proof of Lemma 4.7] and [13, Appendix A]. We mention that recently it is shown in [19] that the transition density functions of the symmetric pure jump processes whose jumping measure $J(dx, dy)$ is comparable to $dx\mu(dy - x)$, where μ is the Lévy measure of (1.7), have the two-sided estimates (1.6). This result has been further extended to more general rectilinear Lévy processes in [20].

For any non-empty open subset $D \subset \mathbb{R}^d$, let $\tau_D(\omega) = \inf\{t > 0, X_t(\omega) \notin D\}$ denote the first exit time from D by X. Taking $f(x, y) = \mathbb{1}_D(x)\mathbb{1}_{D^c}(y)\varphi(y)$ and $S = \tau_D$ in (1.8), where φ is a bounded function defined on D^c, yields

$$\mathbb{E}_x \left[\varphi(X_{\tau_D}); X_{\tau_D -} \neq X_{\tau_D}\right] = \mathbb{E}_x \left[\int_0^{\tau_D} \int_{D^c} \varphi(z)\mu(dz - X_s)ds \right]. \tag{1.9}$$

The subprocess X^D of X killed upon leaving D is defined as

$$X_t^D(\omega) = \begin{cases} X_t(\omega) & \text{for } t < \tau_D(\omega) \\ \partial & \text{for } t \geq \tau_D(\omega) \end{cases},$$

where ∂ is a cemetery state. The subprocesses of other Markov processes in an open set can be defined in a similar way. Denote by $\mathcal{L}^D$ the infinitesimal generator of X^D, which is the non-local operator $\mathcal{L}$ in D satisfying the zero exterior condition.

Let $\{P_t; t \geq 0\}$ be the transition semigroup of the rectilinear α-stable process X; that is, for $t > 0$, $x \in \mathbb{R}^d$ and $f \geq 0$ on $\mathbb{R}^d$,

$$P_t f(x) := \mathbb{E}_x [f(X_t)] = \int_{\mathbb{R}^d} p(t, x, y)f(y)dy.$$

By (1.6), $\{P_t; t \geq 0\}$ is a strongly continuous semigroup in the Banach space $C_\infty(\mathbb{R}^d)$ of continuous functions that vanish at infinity equipped with the uniform norm $\|f\|_\infty := \sup_{x\in\mathbb{R}^d} |f(x)|$. Moreover, since $p(t, x, y)$ is jointly continuous and has estimates (1.6), $\{P_t; t \geq 0\}$ has strong Feller property in the sense that for every $t > 0$ and any bounded function f on $\mathbb{R}^d$, $P_t f$ is a bounded continuous function on $\mathbb{R}^d$. Thus the Lévy process X is a Feller process having strong Feller property. By the proof of [15, Theorem on p. 68], the semigroup $\{P_t^D; t \geq 0\}$ of X^D has strong Feller property for any non-empty open subset $D \subset \mathbb{R}^d$. (Observe that the proof of the strong Feller property of $\{P_t^D; t \geq 0\}$ in [15] does not need regular assumption on D.) In this paper, we will show that X^D has a jointly locally Hölder continuous transition density $p_D(t, x, y)$ with respect to the Lebesgue measure. Furthermore, we will investigate the strict positivity property and the two-sided estimates of $p_D(t, x, y)$ for a class of open subsets $D \subset \mathbb{R}^d$.

Theorem 1.1. *For any non-empty open set $D \subset \mathbb{R}^d$, the subprocess X^D has a jointly (locally) Hölder continuous density function $p_D(t, x, y)$ on $(0, \infty) \times D \times D$; that is, for any $x \in D$ and any non-negative Borel measurable function φ on D,*

$$\mathbb{E}_x \left[\varphi(X_t^D)\right] = \int_D p_D(t, x, y)\varphi(y)dy. \tag{1.10}$$

Throughout this paper, we use the convention that any function φ defined on D is extended to ∂ by setting $\varphi(\partial) = 0$. We also call $p_D(t, x, y)$ the heat kernel of X^D (or, equivalently, of $\mathcal{L}^D$), or the Dirichlet heat kernel of X (or, equivalently, of $\mathcal{L}$) in D.

Unlike the rotationally symmetric α-stable process Z, the behavior of X^D and $p_D(t,x,y)$ are strongly dependent on the shape of the domain D due to the special structure of the Lévy measure of the rectilinear α-stable process X. For example, X^D can be reducible for some smooth bounded open sets D.

Definition 1.2. *We say a Markov process $\{Y,\mathbb{P}_x\}$ on a topological state space E is irreducible if for any non-empty open subset $U\subset E$,*

$$\mathbb{P}_x(\sigma_U<\infty)>0 \quad \text{for every } x\in E,$$

where $\sigma_U:=\inf\{t>0:Y_t\in U\}$. Otherwise, we say the process $\{Y,\mathbb{P}_x\}$ is reducible.

The next result gives a geometric criterion on D for the irreducibility of the subprocess X^D in D.

Theorem 1.3. *Let $D\subset\mathbb{R}^d$ be a non-empty open set. The subprocess X^D is irreducible if and only if*

$$\begin{aligned}&\text{for every } x,y\in D,\text{ there is } N\geq 1 \text{ and some } \{x_i\}_{i=0}^N\subset D \text{ with } x_0=x \text{ and } x_N=y \text{ so}\\ &\text{that each consecutive pair } (x_{i-1},x_i),\ 1\leq i\leq N, \text{ has only one different coordinate.}\end{aligned} \tag{1.11}$$

Moreover, X^D is irreducible if and only if $p_D(t,x,y)>0$ for every $t>0$ and $x,y\in D$.

Theorem 1.3 together with Theorem 1.1 in particular implies that for any connected open set D, X^D is irreducible and has a strictly positive continuous transition density function $p_D(t,x,y)$.

Corollary 1.4. *Suppose that $D\subset\mathbb{R}^d$ is a non-empty open set, and D_1 and D_2 are two disjoint connected components of D. Then*

- (i) *$p_D(t,x,y)>0$ for every $t>0$ and $x,y\in D_1$.*
- (ii) *Either $p_D(t,x,y)>0$ for every $(t,x,y)\in(0,\infty)\times D_1\times D_2$ or $p_D(t,x,y)=0$ for every $(t,x,y)\in(0,\infty)\times D_1\times D_2$. The former happens if and only if there exists a finite sequence $\{x_i\}_{i=0}^N\subset D$ with $x_0\in D_1$ and $x_N\in D_2$ so that each consecutive pair (x_{i-1},x_i), $1\leq i\leq N$, has only one different coordinate.*

See Theorems 5.2-5.3 for further information on the positivity of $p_D(t,x,y)$. To obtain more precise bounds (for example the two-sided estimate) on $p_D(t,x,y)$, we need certain smoothness of D and some additional geometric condition on D beyond (1.11) (or equivalently, the irreducibility of X^D).

Recall that an open set $D\subset\mathbb{R}^d$ is said to be $C^{1,1}$ with characteristics (R,Λ) for some $R,\Lambda>0$, if for every $z\in\partial D$, there is a $C^{1,1}$-function $\phi=\phi_z:\mathbb{R}^{d-1}\to\mathbb{R}$ satisfying $\phi(0)=0$, $\nabla\phi(0)=0$, $|\nabla\phi(\tilde{x})-\nabla\phi(\tilde{y})|\leq\Lambda|\tilde{x}-\tilde{y}|$, $\tilde{x},\tilde{y}\in\mathbb{R}^{d-1}$, and an orthogonal coordinate system $CS_z: y=(y^{(1)},\cdots,y^{(d-1)},y^{(d)})=:(\tilde{y},y^{(d)})$ with its origin at z such that

$$B(z,R)\cap D=\left\{y=(\tilde{y},y^{(d)})\in B(0,R) \text{ in } CS_z: y^{(d)}>\phi(\tilde{y})\right\}.$$

The pair (R,Λ) is called the characteristics of the $C^{1,1}$ open set D. Note that the $C^{1,1}$ open set D may be disconnected and may have infinite number of components. However, the distances between any two distinct connected components of D are bounded from below uniformly by a positive constant.

For an open set $D\subset\mathbb{R}^d$ and $x\in D$, let $\delta_D(x)$ be the Euclidean distance between x and D^c. We say D satisfies the uniform interior ball condition with radius $R_1>0$, if, for every $x\in D$ with $\delta_D(x)\leq R_1$, there is $z_x\in\partial D$ such that $|x-z_x|=\delta_D(x)$ and $B(x_0,R_1)\subset D$ for $x_0:=z_x+R_1(x-z_x)/|x-z_x|$; see [9, 14]. Similarly, we can define the uniform exterior ball condition.

It is well known that D being a $C^{1,1}$ open set is equivalent to that D satisfies both the uniform interior and exterior ball conditions. Thus without loss of generality, in this paper, for a $C^{1,1}$ open set D, we always assume its $C^{1,1}$ characteristics (R,Λ) have the property that $R\leq 1$, $\Lambda\geq 1$ and it satisfies the uniform interior and exterior ball conditions with radius R.

For $u=(u^{(1)},u^{(2)},\cdots,u^{(d)})\in\mathbb{R}^d$, $a\in\mathbb{R}$ and $1\leq i\leq d$, let

$$[u]_a^i:=(u^{(1)},\cdots,u^{(i-1)},a,u^{(i+1)},\cdots,u^{(d)});$$

that is, $[u]_a^i$ is the point in $\mathbb{R}^d$ by changing its i^{th}-coordinate to a. For $x,y\in\mathbb{R}^d$ and a permutation $\{i_1,i_2,\cdots,i_d\}$ of $\{1,2,\cdots,d\}$, let

$$\overline{xy}_1:=[x]^{i_1}_{y^{(i_1)}},\quad \overline{xy}_2:=[\overline{xy}_1]^{i_2}_{y^{(i_2)}},\quad \overline{xy}_3:=[\overline{xy}_2]^{i_3}_{y^{(i_3)}},\quad \cdots,\quad \overline{xy}_d:=[\overline{xy}_{d-1}]^{i_d}_{y^{(i_d)}}=y.$$

That is, $\overline{xy}_j$ is the point obtained by swapping consecutively the i_k^{th}-coordinate of x with that of y for $k = 1, 2, \cdots, j$.

We consider the following geometric condition on an open set D. Let $\gamma \in (0, 1]$.

($\mathbf{H}_\gamma$): An open set $D \subset \mathbb{R}^d$ is said to satisfy condition ($\mathbf{H}_\gamma$) if for any $x, y \in D$ with $\delta_D(x) \wedge \delta_D(y) \geq r > 0$, there exists a permutation $\{i_1, i_2, \cdots, i_d\}$ of $\{1, 2, \cdots, d\}$ so that $B(\overline{xy}_k, \gamma r) \subset D,\ k = 1, 2, \cdots, d$.

Clearly, for any $0 < \gamma_1 < \gamma_2 \leq 1$, condition ($\mathbf{H}_{\gamma_2}$) implies ($\mathbf{H}_{\gamma_1}$) and any ($\mathbf{H}_\gamma$) with $\gamma \in (0, 1]$ implies the irreducibility condition (1.11). Many open sets in $\mathbb{R}^d$ satisfy condition ($\mathbf{H}_\gamma$). For example, all balls, complements of closed balls, and the open sets shown in Figure 1 satisfy ($\mathbf{H_1}$). But there are also many open sets which do not satisfy condition ($\mathbf{H}_\gamma$); see Section 6 for some examples.

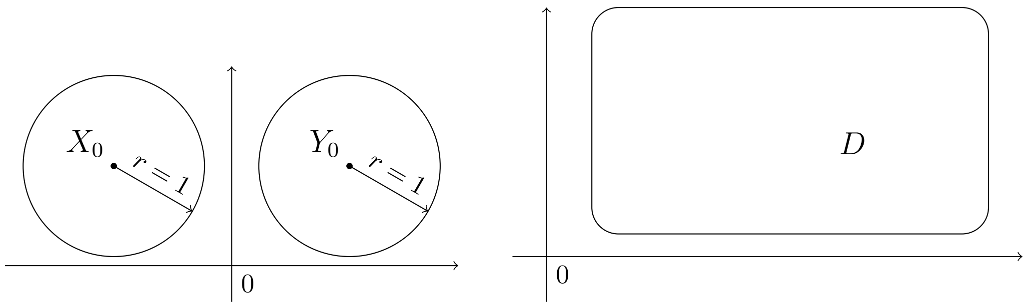

FIGURE 1. The set $D := B(X_0, r) \cup B(Y_0, r)$ with $r = 1$, and the set $D :=$ the cubes with round corners in $\mathbb{R}^2$

Recall that $p(t, x, y)$ is the transition density function (also called the heat kernel) of the rectilinear stable process X, and, for any open set $D \subset \mathbb{R}^d$, $p_D(t, x, y)$ is the heat kernel of X^D. Recall also that $\mathcal{L}^D$ is the infinitesimal generator of X^D.

Theorem 1.5. *Let $D \subset \mathbb{R}^d$ be a $C^{1,1}$ open set with characteristics (R, Λ).*

(i) *For any $T > 0$, there exists $C_2 = C_2(d, \alpha, R, \Lambda, T) > 0$ such that for all $t \in (0, T]$ and $x, y \in D$,*

$$p_D(t, x, y) \leq C_2 \left(1 \wedge \frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}}\right) \left(1 \wedge \frac{\delta_D(y)^{\alpha/2}}{\sqrt{t}}\right) p(t, x, y). \tag{1.12}$$

(ii) *Assume in addition that D satisfies ($\mathbf{H}_\gamma$) for some $\gamma \in (0, 1]$. Then, for any $T > 0$, there exists $C_3 = C_3(d, \alpha, R, \Lambda, \gamma, T) > 0$ such that for all $t \in (0, T]$ and $x, y \in D$,*

$$p_D(t, x, y) \geq C_3 \left(1 \wedge \frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}}\right) \left(1 \wedge \frac{\delta_D(y)^{\alpha/2}}{\sqrt{t}}\right) p(t, x, y). \tag{1.13}$$

(iii) *Assume in addition that D is bounded and satisfies ($\mathbf{H}_\gamma$) for some $\gamma \in (0, 1]$. Denote by $\lambda_1(D)$ the first eigenvalue of $-\mathcal{L}^{\mathcal{D}}$. Then, there exists $C_4 = C_4(d, \alpha, R, \Lambda, \gamma, \mathrm{diam}(D)) > 1$ such that*

$$C_4^{-1} \leq \lambda_1(D) \leq C_4, \tag{1.14}$$

and, for any $T > 0$, there exists $C_5 = C_5(d, \alpha, R, \Lambda, \gamma, T, \mathrm{diam}(D)) > 0$ such that for all $t \in [T, \infty)$ and $x, y \in D$,

$$C_5^{-1} e^{-\lambda_1(D)t} \delta_D(x)^{\alpha/2} \delta_D(y)^{\alpha/2} \leq p_D(t, x, y) \leq C_5 e^{-\lambda_1(D)t} \delta_D(x)^{\alpha/2} \delta_D(y)^{\alpha/2}. \tag{1.15}$$

For the comparable lower bound estimate (1.13) to hold, certain geometric condition beyond smoothness of the bounded open set D is needed. We show by Examples 6.1 and 6.2 in Section 6 that there are smooth connected bounded domains, even some smooth convex domains, that do not satisfy condition ($\mathbf{H}_\gamma$) for any $\gamma \in (0, 1]$ and the lower bound of Dirichlet heat kernel estimate (1.13) fails. In Example 6.3, a bounded smooth open set D is given that does not satisfy the irreducibility condition (1.11) and thus X^D is not irreducible. These facts are in strong contrast with that for the rotationally symmetric stable processes in $\mathbb{R}^d$, whose subprocesses in open sets are always irreducible and the comparable two-sided Dirichlet heat kernel estimates, obtained in [8], are known to hold for any $C^{1,1}$ smooth open sets. We next present a bounded smooth open set D that satisfies the irreducibility condition (1.11) but does not satisfy condition

($\mathbf{H}_\gamma$) for any $\gamma \in (0,1]$ and the lower bound (1.13) fails for $p_D(t,x,y)$, nevertheless for which we can still derive comparable upper and lower bound of Dirichlet heat kernel estimates.

Let $O_i \in \mathbb{R}^2$, $i=1,\cdots,4$ be four points such that, the line through O_1 and O_2 is paralleled with x-axis, the line through O_2 and O_3 is paralleled with y-axis, the line through O_3 and O_4 is paralleled with x-axis, and $|O_i - O_{i+1}| = 3$ for $i=1,2,3$; see Figure 2. Let A_i, $i=1,\cdots,4$ be four squares with round corners and with edge-length 2 centered at O_i respectively. Consider the open set $D := \cup_{i=1}^4 A_i$.

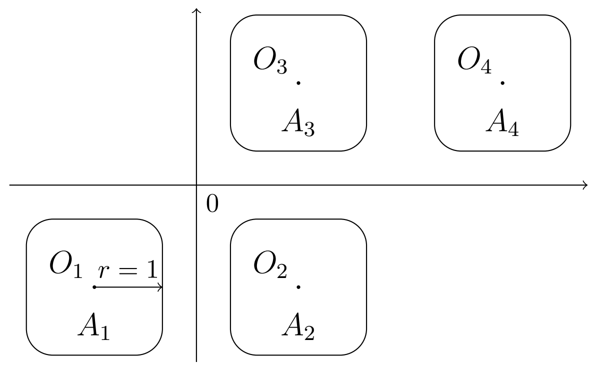

FIGURE 2. The set D is the union of four squares with round corners in $\mathbb{R}^2$

Note that $D \subset \mathbb{R}^2$ is a bounded smooth open set that satisfies the irreducibility condition (1.11) but does not satisfy condition ($\mathbf{H}_\gamma$) for any $\gamma \in (0,1]$ as for any $x \in A_1$ and $y \in A_4$, swapping any coordinate of x by that of y results a point falling outside D.

Theorem 1.6. *Let $D \subset \mathbb{R}^2$ be the above smooth open set as shown in Figure 2 and $T>0$.*

(i) *There exists $C_6 = C_6(\alpha, T) > 0$ such that for all $t \in (0,T]$, $x \in A_i$ and $y \in A_j$ with $|i-j| \le 2$,*

$$p_D(t,x,y) \overset{C_6}{\asymp} \left(1 \wedge \frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}}\right)\left(1 \wedge \frac{\delta_D(y)^{\alpha/2}}{\sqrt{t}}\right) p(t,x,y). \tag{1.16}$$

(ii) *There exist $C_7 = C_7(\alpha,T) > 0$ and $C_8 = C_8(\alpha,T) > 0$ such that for all $t \in (0,T]$, $x \in A_1$ and $y \in A_4$,*

$$p_D(t,x,y) \overset{C_7}{\asymp} t^3 \left(1 \wedge \frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}}\right)\left(1 \wedge \frac{\delta_D(y)^{\alpha/2}}{\sqrt{t}}\right) \tag{1.17}$$

$$\overset{C_8}{\asymp} t \left(1 \wedge \frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}}\right)\left(1 \wedge \frac{\delta_D(y)^{\alpha/2}}{\sqrt{t}}\right) p(t,x,y). \tag{1.18}$$

Here and in the sequel, for two functions f, g and a positive constant C, the notation $f \overset{C}{\asymp} g$ means that $C^{-1} f \le g \le Cf$ holds true on their common domains. Theorem 1.6(ii) shows that, for the smooth open set D in Figure 2, the lower bound (1.13) fails for $p_D(t,x,y)$.

For an open set $D \subset \mathbb{R}^d$, we call $G_D(x,y) := \int_0^\infty p_D(t,x,y)dt$ the Green function of X in D. It follows from (1.10) that for any $x \in D$ and any non-negative Borel measurable function φ on D,

$$\mathbb{E}_x \int_0^{\tau_D} \varphi(X_s)ds = \int_D G_D(x,y)\varphi(y)dy.$$

From the Dirichlet heat kernel estimates in Theorem 1.5 for $p_D(t,x,y)$, one can clearly derive the Green function estimates for $G_D(x,y)$.

Finally, we mention that boundary regularity of solutions to the Dirichlet problem for the generator of isotropic stable processes is studied in [21]. These regularity results were later extended to more general stable operators in [23, 24], and further to nonlinear nonlocal equations in [22]. However, for Dirichlet heat kernel estimates, the singular nature of rectilinear stable processes poses significant challenges. For instance, as mentioned above, Harnack inequality fails for rectilinear stable processes. Thus we can not use the approach developed in [8] for the study of Dirichlet heat kernel estimates for rotationally symmetric stable processes directly. To see this through a concrete case, we invite the interested reader to pause for a few minutes and think about possible ways to establish the joint local Hölder continuity of the transition density $p_D(t,x,y)$ of X^D in any open subset $D\subset\mathbb{R}^d$ before reading the proof of Theorem 2.5. Some new ideas and methods are needed for the study of rectilinear stable processes. We employ a combination of the probabilistic and analytic methods in our investigation.

The rest of the paper is organized as follows. In Section 2, we show that the part process X^D in any open subset $D\subset\mathbb{R}^d$ has a locally Hölder continuous transition density function. Boundary properties of the harmonic measures of $\mathcal{L}$, or equivalently, the exit distributions of X, in $C^{1,1}$ open sets are investigated in Section 3, using testing function methods developed in [5, 11]. Various Dirichlet heat kernel estimates are obtained in Section 4, and the proof of Theorem 1.5 is given in Subsection 4.3. For the upper bound estimates of $p_D(t,x,y)$, we use the exit time estimates, strong Markov property and the Lévy system of the rectilinear stable process X. For the lower bound estimates of $p_D(t,x,y)$, we first obtain its near diagonal interior estimate in Lemma 4.7, and then the interior estimates under the condition $(\mathbf{H}_\gamma)$ for some $\gamma\in(0,1]$ in Lemma 4.9 using the Chapman-Kolmogorov equation, a chaining argument and a delicate probability lower bound estimate for X^U_t taking values in suitable cubes. The sharp lower bound estimates for $p_D(t,x,y)$ over some bounded time interval $(0,t_*]$ in any $C^{1,1}$ open set D satisfying the condition $(\mathbf{H}_\gamma)$ for some $\gamma\in(0,1]$ is established in Lemmas 4.11-4.12 though a careful probabilistic argument that boils down to the exit time estimates for X. The proof of Theorem 1.5 is given Subsection 4.3, where in particular the lower bound estimate in Lemma 4.12 over *some* bounded time interval is shown to hold over *any* bounded time interval through a chaining argument. For any two fixed distinct points $x,y\in D$, a geometric condition for the positivity of $p_D(t,x,y)$ is given in Theorems 5.2 and 5.3, whose proof uses some of the lower bound estimates derived in Section 4. From these, we give in Section 5 a geometric criterion on D for the irreducibility of X^D as well as the strict positivity property of $p_D(t,x,y)$ (see Theorem 1.3). In addition to the proof of Theorem 1.6, three additional examples of bounded smooth open sets are given in Section 6, two of them are connected open sets, that do not satisfy the condition $(\mathbf{H}_\gamma)$ for any $\gamma\in(0,1]$, for which the lower bound estimate (1.13) is shown to fail.

In this paper, for $x=(x^{(1)},x^{(2)},\cdots,x^{(d)})\in\mathbb{R}^d$ and $r>0$, we will use $Q(x,r)$ to denote the cube centered at x with edge-length $2r$, that is,

$$Q(x,r):=\left\{y=(y^{(1)},y^{(2)},\cdots,y^{(d)})\in\mathbb{R}^d:|x^{(i)}-y^{(i)}|<r,\ i=1,2,\cdots,d\right\}.$$

For an open set $U\subset\mathbb{R}^d$ and $\lambda>0$, unless otherwise stated, we define

$$\lambda U:=\{\lambda y:y\in U\}.$$

For a measurable set $A\subset\mathbb{R}^d$, we use $|A|$ to denote its Lebesgue measure.

There is a more detailed arXiv version [7] of this paper, where the reader can find additional details of some calculations.

2. Hölder regularity of Dirichlet heat kernel

In this section, we fix a non-empty open set $D\subset\mathbb{R}^d$. Recall that $\tau_D(\omega)=\inf\{t>0,X_t(\omega)\notin D\}$ is the first exit time from D by the rectilinear α-stable process X. Since X_t has a continuous transition density function with respect to the Lebesgue measure, we have the following property by the same proof as that in [16, Proposition 1.20].

Proposition 2.1. *For every $t>0$ and $x\in\mathbb{R}^d$, $\mathbb{P}_x(\tau_D=t)=0$.*

For $t>0$, $x,y\in\mathbb{R}^d$, set $r_D(t,x,y):=\mathbb{E}_x\left[p(t-\tau_D,X_{\tau_D},y);\tau_D<t\right]$, and

$$p_D(t,x,y):=p(t,x,y)-r_D(t,x,y). \tag{2.1}$$

Note that the function $p_D(t,x,y)$ is pointwise well-defined. One can follow the proof of [16, Theorem 2.4, p. 33] to prove the following lemma. Let $\mathcal{B}(\mathbb{R}^d)$ denote the collection of all Borel measurable sets in $\mathbb{R}^d$.

Lemma 2.2. *For any $t>0$, $x\in\mathbb{R}^d$ and $A\in\mathcal{B}(\mathbb{R}^d)$,*

$$\mathbb{P}_x\left(X_t\in A,\ t<\tau_D\right)=\int_A p_D(t,x,y)dy. \tag{2.2}$$

The function $p_D(t,x,y)$ is almost surely symmetric on $\mathbb{R}^d\times\mathbb{R}^d$: for all $t>0$,

$$p_D(t,x,y)=p_D(t,y,x) \quad \text{for a.a. } (x,y)\in D\times D. \tag{2.3}$$

Moreover, for any $s,t>0$ and $x\in\mathbb{R}^d$, we have

$$p_D(t+s,x,y)=\int_{\mathbb{R}^d} p_D(t,x,z)p_D(s,z,y)dz \quad \text{for a.a. } y\in D. \tag{2.4}$$

Unfortunately, we can not use the approach in [16, Theorem 2.4 on p.33] to establish the joint continuity of $p_D(t,\cdot,\cdot)$ on $(\mathbb{R}^d\setminus\partial D)\times(\mathbb{R}^d\setminus\partial D)$, and hence to improve the identities in (2.3) and (2.4) from almost every point to every point. The main issue is that unlike Brownian motion or rotationally symmetric stable processes case, in our setting, the function $\mathbb{1}_{\{\tau_D<t\}}p(t-\tau_D,X_{\tau_D},y)$ is unbounded. However, we can apply the result in [1] and the ideas in [6, Proposition 2.5, p.1603] to establish the joint Hölder continuity of $p_D(t,\cdot,\cdot)$ on $D\times D$ in Theorem 2.5 (see also [2] or [18] for another approach). In order to make the proof as self-contained as possible, we show all the details.

The rectilinear α-stable process X has the following scaling property: for any $\lambda>0$, the processes $\{\lambda X_{\lambda^{-\alpha}t};t\geq 0\}$ conditioned on $X_0=x$ has the same distribution as $\{X_t;t\geq 0\}$ conditioned on $X_0=\lambda x$. Consequently, since the heat kernel $p(t,x,y)$ is continuous, it has the following scaling property: for any $\lambda>0$,

$$p(t,x,y)=\lambda^{-d}p(\lambda^{-\alpha}t,\lambda^{-1}x,\lambda^{-1}y), \qquad t>0,\ x,y\in\mathbb{R}^d. \tag{2.5}$$

Moreover, $\{\lambda X^D_{\lambda^{-\alpha}t};t\geq 0\}$ conditioned on $X_0=x\in D$ has the same distribution as $\{X^{\lambda D}_t;t\geq 0\}$ conditioned on $X_0=\lambda x$. It follows that for every $\lambda>0$, $t>0$ and $x\in D$,

$$p_D(t,x,y)=\lambda^d p_{\lambda D}(\lambda^\alpha t,\lambda x,\lambda y) \quad \text{for a.a. } y\in D. \tag{2.6}$$

The following lemma gives the exit time estimates for X from balls.

Lemma 2.3. *There exist positive constants $c_i:=c_i(d,\alpha)>0$, $i=1,2$, such that for any $x_0\in\mathbb{R}^d$ and $r>0$,*

(i) $\mathbb{E}_x\left[\tau_{B(x_0,r)}\right]\leq c_1r^\alpha$ *for all $x\in B(x_0,r)$;*
(ii) $\mathbb{E}_x\left[\tau_{B(x_0,r)}\right]\geq c_2r^\alpha$ *for all $x\in B(x_0,r/2)$.*

Proof. By the Lévy property of X we may assume that $x_0=0$. When $r=1$, this lemma follows directly from [1, Proposition 2.1, p. 492] with the matrix A being the $d\times d$ identity matrix. For general $r>0$, the desired property follows from the scaling property of $X^{B(0,r)}$ that $\mathbb{E}_x\left[\tau_{B(0,r)}\right]=r^\alpha\mathbb{E}_{x/r}\left[\tau_{B(0,1)}\right]$. □

Definition 2.4. *A bounded function h on $\mathbb{R}^d$ is said to be harmonic (with respect to X) in a ball $B\subset\mathbb{R}^d$ if*

$$h(x)=\mathbb{E}_x\left[h(X_{\tau_B})\right] \qquad \text{for all } x\in B.$$

For a non-negative function f on D, let

$$P^D_tf(x):=\int_D p_D(t,x,z)f(z)dz=\mathbb{E}_x[f(X^D_t)], \quad t>0,\ x\in\mathbb{R}^d, \tag{2.7}$$

where the second equality is due to (2.2). Thus $\{P^D_t,t>0\}$ is the transition semigroup of X^D. It follows from (2.3) that $\{P^D_t,t>0\}$ is a strongly continuous symmetric contractive semigroup on $L^2(D;dx)$. Moreover, by (2.7) and the Markov property of X^D, we have for any $s,t>0$

$$P^D_{t+s}f(x)=P^D_tP^D_sf(x), \quad x\in\mathbb{R}^d\setminus\partial D. \tag{2.8}$$

We further define the 1-potential

$$G^D_1f(x):=\mathbb{E}_x\left[\int_0^\infty e^{-t}f(X^D_t)dt\right]=\int_0^\infty e^{-t}P^D_tf(x)dt, \quad x\in\mathbb{R}^d\setminus\partial D.$$

Let $\{\theta_t;t\geq 0\}$ be the time-shifting operators on the canonical probability sample space Ω for the Lévy process X; that is, $X_r(\theta_t\omega)=X_{r+t}(\omega)$ for $\omega\in\Omega$ and $t,r\geq 0$.

Theorem 2.5. *There is a jointly (locally) Hölder continuous function $q(t,x,y)$ on $D\times D$ so that*

(i) *for any $t>0$ and $x\in D$,*

$$q(t,x,z)=p_D(t,x,z) \quad \text{for a.a. } z\in D;$$

(ii) *for any $t>0$, $q(t,x,y)$ is symmetric on $D\times D$;*

(iii) *for any* $s,t>0$ *and* $x,y\in D$,

$$q(t+s,x,y)=\int_D q(t,x,z)q(s,z,y)dz.$$

Proof. The proof uses a result from [1] and the ideas from [6, Proposition 2.5, p. 1603]. For the reader's convenience, we spell out all the details. We divide the proof into five steps.

Step 1. Let $r\in(0,1]$ and $B:=B(x_0,r)\subset D$. Suppose that h is a bounded function on $\mathbb{R}^d$ and is harmonic with respect to X in B. By the Hölder regularity obtained in [1, Theorem 2.9, p. 499], taking the matrix A there the $d\times d$ identity matrix, there exist positive constants c_1 and β depending only on d and α such that

$$|h(x)-h(y)|\le c_1\left(\frac{|x-y|}{r}\right)^\beta \sup_{z\in\mathbb{R}^d}|h(z)| \quad \text{for all } x,y\in B(x_0,r/2). \tag{2.9}$$

Step 2. Let $f\in L^\infty(D)\cap L^2(D)$. By the strong Markov property, we obtain that for any $x\in B$,

$$\begin{aligned}
G_1^D f(x) &= \mathbb{E}_x\left[\int_0^{\tau_B} e^{-t}f(X_t^D)dt\right]+\mathbb{E}_x\left[\int_{\tau_B}^\infty e^{-t}f(X_t^D)dt\right]\\
&= \mathbb{E}_x\left[\int_0^{\tau_B} e^{-t}f(X_t^D)dt\right]+\mathbb{E}_x\left[e^{-\tau_B}\left(\int_0^\infty e^{-t}f(X_t^D)dt\right)\circ\theta_{\tau_B}\right]\\
&= \mathbb{E}_x\left[\int_0^{\tau_B} e^{-t}f(X_t^D)dt\right]+\mathbb{E}_x\left[e^{-\tau_B}G_1^D f(X_{\tau_B}^D)\right]\\
&= \mathbb{E}_x\left[\int_0^{\tau_B} e^{-t}f(X_t^D)dt\right]+\mathbb{E}_x\left[(e^{-\tau_B}-1)G_1^D f(X_{\tau_B}^D)\right]+\mathbb{E}_x\left[G_1^D f(X_{\tau_B}^D)\right]\\
&= I_1(x)+I_2(x)+I_3(x).
\end{aligned}$$

By Lemma 2.3(i) and the elementary inequality that $1-e^{-a}\le a$ for $a\ge 0$, we have

$$|I_1(x)|\le \|f\|_{L^\infty(D)}\mathbb{E}_x[\tau_B]\le c_2 r^\alpha\|f\|_{L^\infty(D)},$$

and

$$|I_2(x)|\le \|G_1^D f\|_{L^\infty(D)}\mathbb{E}_x[\tau_B]\le c_2 r^\alpha\|f\|_{L^\infty(D)}.$$

Since $z\mapsto I_3(z)=\mathbb{E}_z\left[G_1^D f(X_{\tau_B}^D)\right]$ is bounded and harmonic in B, we have by (2.9) that for any $x,y\in B(x_0,r/2)\subset D$,

$$|I_3(x)-I_3(y)|\le c_1\left(\frac{|x-y|}{r}\right)^\beta\sup_{z\in\mathbb{R}^d}|I_3(z)|\le c_1\left(\frac{|x-y|}{r}\right)^\beta\|G_1^D f\|_{L^\infty(D)}\le c_1\left(\frac{|x-y|}{r}\right)^\beta\|f\|_{L^\infty(D)}.$$

Combining the above four formulas, we obtain that for all $x,y\in B(x_0,r/2)\subset D$,

$$|G_1^D f(x)-G_1^D f(y)|\le(4c_2+c_1)\left(r^\alpha+\frac{|x-y|^\beta}{r^\beta}\right)\|f\|_{L^\infty(D)}. \tag{2.10}$$

Step 3. Recall that $\mathcal{L}^D$ is the generator of the heat semigroup $\{P_t^D,t>0\}$ on $L^2(D)$. Note that $\mathcal{L}^D$ is *negative* definite self-adjoint. By general theory of heat semigroup, we have that for any $s,s'>0$ and $f\in L^2(D)$,

$$P_s^D\mathcal{L}^D P_{s'}^D f=P_{s'}^D\mathcal{L}^D P_s^D f \quad \text{a.e.}. \tag{2.11}$$

For a fixed $f\in L^\infty(D)\cap L^2(D)$, set

$$h_t:=P_t^D f-\mathcal{L}^D P_t^D f\in L^2(D),\quad t>0.$$

By spectral theory, there exists a spectral family $\{E_\lambda,\lambda\in\mathbb{R}\}$ such that

$$\mathcal{L}^D=-\int_0^\infty\lambda dE_\lambda,\quad f=\int_0^\infty dE_\lambda f \quad\text{and}\quad P_t^D f\overset{\text{a.e.}}{=}\int_0^\infty e^{-\lambda t}dE_\lambda f. \tag{2.12}$$

Consequently,

$$(1-\mathcal{L}^D)G_1^D f\overset{\text{a.e.}}{=}f \quad\text{and}\quad h_t\overset{\text{a.e.}}{=}\int_0^\infty(1+\lambda)e^{-\lambda t}dE_\lambda f. \tag{2.13}$$

For any $g \in L^1(D)$, by (1.6), we have for any $t > 0$

$$\|P_t^D g\|_{L^\infty(D)} = \left\| \int_D p_D(t,\cdot,z) g(z) dz \right\|_{L^\infty(D)} \le \left\| \int_D |p(t,\cdot,z)||g(z)| dz \right\|_{L^\infty(D)} \le C_1 t^{-d/\alpha} \|g\|_{L^1(D)},$$

and then,

$$\|P_t^D g\|_{L^2(D)} \le \sqrt{\|P_t^D g\|_{L^1(D)} \|P_t^D g\|_{L^\infty(D)}} \le \sqrt{\|g\|_{L^1(D)} C_1 t^{-d/\alpha} \|g\|_{L^1(D)}} = \sqrt{C_1} t^{-\frac{d}{2\alpha}} \|g\|_{L^1(D)}. \quad (2.14)$$

Using the above inequality, Cauchy-Schwarz and the facts that

$$\sup_{\lambda > 0} (1+\lambda) e^{-\lambda t} \le (t \wedge 1)^{-1} < \infty \quad \text{and} \quad \sup_{\lambda > 0} (1+\lambda) e^{-\lambda t/2} \le 2(t \wedge 1)^{-1} < \infty,$$

we obtain

$$\begin{aligned}
(h_t, g)_{L^2(D)} &= \int_0^\infty (1+\lambda) e^{-\lambda t} d(E_\lambda f, g)_{L^2(D)} \\
&\le \left(\int_0^\infty (1+\lambda) e^{-\lambda t} d(E_\lambda f, f)_{L^2(D)} \right)^{1/2} \left(\int_0^\infty (1+\lambda) e^{-\lambda t} d(E_\lambda g, g)_{L^2(D)} \right)^{1/2} \\
&\le 2(t \wedge 1)^{-1} \left(\int_0^\infty d(E_\lambda f, f)_{L^2(D)} \right)^{1/2} \left(\int_0^\infty e^{-\lambda t/2} d(E_\lambda g, g)_{L^2(D)} \right)^{1/2} \\
&= 2(t \wedge 1)^{-1} \|f\|_{L^2(D)} \|P_{t/2}^D g\|_{L^2(D)} \\
&\le 2\sqrt{C_1} (t \wedge 1)^{-1} t^{-\frac{d}{2\alpha}} \|f\|_{L^2(D)} \|g\|_{L^1(D)}.
\end{aligned}$$

Since $g \in L^1(D)$ is arbitrary, we obtain

$$\|h_t\|_{L^\infty} \le 2\sqrt{C_1} (t \wedge 1)^{-1} t^{-\frac{d}{2\alpha}} \|f\|_{L^2(D)}.$$

On the other hand, we have by (2.11) and (2.13) that a.e. on D,

$$G_1^D h_t = G_1^D P_t^D f - G_1^D \mathcal{L}^D P_t^D f = P_t^D \left(G_1^D f - \mathcal{L}^D G_1^D f \right) = P_t^D f.$$

As noted earlier, $P_t^D f$ and $P_t^D h_t$ are continuous functions on D by the strong Feller property of X^D. By the dominated convergence theorem, $G_1^D h_t(x) = \int_0^\infty e^{-s} P_s^D h_t(x) ds$ is a bounded continuous function on D. Hence we have

$$P_t^D f(x) = G_1^D h_t(x) \quad \text{for every } x \in D.$$

This together with (2.10) yields that for all $t > 0$ and $x, y \in B(x_0, r/2) \subset D$,

$$\begin{aligned}
|P_t^D f(x) - P_t^D f(y)| &= |G_1^D h_t(x) - G_1^D h_t(y)| \\
&\le (4c_2 + c_1) \left(r^\alpha + \frac{|x-y|^\beta}{r^\beta} \right) \|h_t\|_{L^\infty(D)} \\
&\le 2\sqrt{C_1} (4c_2 + c_1) \left(r^\alpha + \frac{|x-y|^\beta}{r^\beta} \right) (t \wedge 1)^{-1} t^{-\frac{d}{2\alpha}} \|f\|_{L^2(D)}.
\end{aligned} \quad (2.15)$$

Step 4. For any fixed compact set $K \subset D$ and $x, y \in K$, let $\delta_K = \frac{1}{4}(\text{dist}(K, \partial D) \wedge \text{dist}(K, \partial D)^2 \wedge 1)$ and $x_0 = x$.

Case 1: $|x - y| < \delta_K$. Setting $r := |x-y|^{1/2}$, we have

$$\text{dist}(K, \partial D) > \sqrt{\delta_K} > |x-y|^{1/2} = r > 2|x-y|.$$

Applying (2.15) for this r and $x_0 = x$, we have

$$\begin{aligned}
|P_t^D f(x) - P_t^D f(y)| &\le 2\sqrt{C_1} (4c_2 + c_1) \left(r^\alpha + \frac{|x-y|^\beta}{r^\beta} \right) (t \wedge 1)^{-1} t^{-\frac{d}{2\alpha}} \|f\|_{L^2(D)} \\
&\le 4\sqrt{C_1} (4c_2 + c_1) |x-y|^{(\alpha \wedge \beta)/2} (t \wedge 1)^{-1} t^{-\frac{d}{2\alpha}} \|f\|_{L^2(D)}.
\end{aligned}$$

Case 2: $|x - y| \ge \delta_K$. By the definition of $P_t^D f$ and Cauchy-Schwarz inequality, we have for all $t > 0$ and $z \in D$,

$$|P_t^D f(z)| \le \sqrt{\int_D p_D(t,z,y)^2 dy} \, \|f\|_{L^2(D)} \le \sqrt{p(2t,z,z)} \|f\|_{L^2(D)} \le \sqrt{C_1} (2t)^{-d/(2\alpha)} \|f\|_{L^2(D)}.$$

This implies that

$$\begin{aligned}|P_t^D f(x) - P_t^D f(y)| &\le |P_t^D f(x)| + |P_t^D f(y)| \le 2\sqrt{C_1} t^{-\frac{d}{2\alpha}} \|f\|_{L^2(D)} \\ &\le 2\sqrt{C_1} \left(\frac{|x-y|}{\delta_K} \right)^{(\alpha\wedge\beta)/2} t^{-\frac{d}{2\alpha}} \|f\|_{L^2(D)}.\end{aligned}$$

Combining the above two cases, we obtain for any $x, y \in K$,

$$|P_t^D f(x) - P_t^D f(y)| \le c_3(d, \alpha, \mathrm{dist}(K, \partial D)) |x-y|^{(\alpha\wedge\beta)/2} (t\wedge 1)^{-1} t^{-\frac{d}{2\alpha}} \|f\|_{L^2(D)}. \tag{2.16}$$

Step 5. For any $0 < s < t$, we define

$$q_s(t,x,y) := \int_D p_D(t-s,x,z) p_D(s,y,z) dz = P_{t-s}^D p_D(s,y,\cdot)(x) = P_s^D p_D(t-s,x,\cdot)(y), \quad x,y \in D.$$

By (2.3) and (2.4), we obtain for any $0 < s < t$ and $x \in D$,

$$q_s(t,x,y) = p_D(t,x,y), \quad \text{a.a. } y \in D. \tag{2.17}$$

On the other hand, let K be any compact subset of D as in Step 4, $t > 0$ and $x, y, x', y' \in K$. Replacing t by $t-s$ and f by $p_D(s,y,\cdot)$ in (2.16) and using (1.6), we obtain

$$\begin{aligned}|q_s(t,x,y) - q_s(t,x',y)| &= |P_{t-s}^D f(x) - P_{t-s}^D f(x')| \\ &\le c_3(d,\alpha,\mathrm{dist}(K,\partial D)) |x-x'|^{(\alpha\wedge\beta)/2} ((t-s)\wedge 1)^{-1} (t-s)^{-\frac{d}{2\alpha}} \|p_D(s,y,\cdot)\|_{L^2(D)} \\ &\le c_3 |x-x'|^{(\alpha\wedge\beta)/2} ((t-s)\wedge 1)^{-1} (t-s)^{-\frac{d}{2\alpha}} \sqrt{p(2s,y,y)} \\ &\le c_3\sqrt{C_1} |x-x'|^{(\alpha\wedge\beta)/2} ((t-s)\wedge 1)^{-1} (t-s)^{-\frac{d}{2\alpha}} (2s)^{-\frac{d}{2\alpha}}.\end{aligned}$$

Similarly, replacing t by s and f by $p_D(t-s,x',\cdot)$ in (2.16) and using (1.6), we obtain

$$\begin{aligned}|q_s(t,x',y) - q_s(t,x',y')| &= |P_{t-s}^D f(y) - P_{t-s}^D f(y')| \\ &\le c_3\sqrt{C_1} |y-y'|^{(\alpha\wedge\beta)/2} (s\wedge 1)^{-1} s^{-\frac{d}{2\alpha}} (2(t-s))^{-\frac{d}{2\alpha}}.\end{aligned}$$

Adding up the above two inequalities, we have that $q_s(t,x,y)$ is jointly Hölder continuous on $K \times K$, and hence, is jointly (locally) Hölder continuous on $D \times D$. Moreover, the (locally) Hölder continuity of $q_s(t,x,y)$ on $D\times D$ and (2.17) imply that q_s does not depend on the choice of s, that is, for any $0 < s, s' < t$,

$$q_s(t,x,y) = q_{s'}(t,x,y), \qquad \text{for all } x, y \in D.$$

Hence, we can define

$$q(t,x,y) = q_s(t,x,y), \quad t > 0, \ x,y \in D.$$

This together with (2.17) yields (i). By the definition of q_s and (2.17), we obtain for all $t > 0$ and $x, y \in D$,

$$q(t,x,y) = q_{t/2}(t,x,y) = \int_D q(t/2,x,z) q(t/2,y,z) dz = \int_D q(t/2,y,z) q(t/2,x,z) dz = q_{t/2}(t,y,x) = q(t,y,x),$$

which is (ii). By the definition of q_s, (2.17) and the symmetry of q, we obtain for all $0 < s < t$ and $x, y \in D$,

$$q(t,x,y) = q_s(t,x,y) = \int_D p_D(t-s,x,z) p_D(s,y,z) dz = \int_D q(t-s,x,z) q(s,z,y) dz,$$

which is (iii). The proof is complete. □

Proof of Theorem 1.1. Theorem 1.1 follows directly from Lemma 2.2 and Theorem 2.5. Indeed, we need only to rename the heat kernel $q(t,x,y)$ from Theorem 2.5 to $p_D(t,x,y)$. □

In the sequel, for any open set $D \subset \mathbb{R}^d$, we always use $p_D(t,x,y)$ to denote the (locally) Hölder continuous version of the heat kernel obtained in Theorem 1.1.

3. Harmonic measures

Let Π be a hyperplane described by the function $\Phi(x) = (a, x - x_0)$, $x \in \mathbb{R}^d$, where $x_0 \in \mathbb{R}^d$, $0 \neq a = (a^{(1)}, a^{(2)}, \cdots, a^{(d)}) \in \mathbb{R}^d$ and $(\cdot, \cdot)$ is the inner product in $\mathbb{R}^d$. That is, $\Pi = \{y : \Phi(y) = 0\}$. We define $\delta_\Pi(y) = (\Phi(y) \vee 0)|a|^{-1}$ to be the distance from y to the lower half space separated by the hyperplane Π.

The following estimates are given in [22, Lemma 2.3]. See also the arXiv version of this paper, [7, Lemma 3.2], for another proof which adopts the approach from [5, Lemmas 4.1 and 5.1] for censored stable processes in upper half space and [11, Lemma 2.1] for the process that is the independent sum of Brownian motion and isotropic stable processes.

Lemma 3.1. *Let $p > 0$ and suppose that Π and δ_Π are defined as above. Then, there are two constants $C_i = C_i(d, \alpha, p) > 0, i = 9, 10$ such that for every $x \in \mathbb{R}^d$ with $\Phi(x) > 0$,*

$$C_{10}\delta_\Pi(x)^{p-\alpha} \leq \left(\mathcal{L}\delta_\Pi^p\right)(x) \leq C_9\delta_\Pi(x)^{p-\alpha}, \qquad \text{if } p \in (\alpha/2, \alpha), \tag{3.1}$$

$$\left(\mathcal{L}\delta_\Pi^p\right)(x) = 0, \qquad \text{if } p = \alpha/2. \tag{3.2}$$

Let $D \subset \mathbb{R}^d$ be a $C^{1,1}$ open set with characteristics (R, Λ). For $Q \in \partial D$ and the coordinate system CS_Q, we define $\rho_Q(y) := y^{(d)} - \phi_Q(\tilde{y})$ for $y := (\tilde{y}, y^{(d)}) \in CS_Q$. Note that for every $Q \in \partial D$ and $y \in B(Q, R) \cap D$, we have

$$\frac{\rho_Q(y)}{\sqrt{1 + \Lambda^2}} \leq \delta_D(y) \leq \rho_Q(y).$$

Set

$$R_0 := R_0(R, \Lambda) = \frac{R}{\sqrt{1 + \Lambda^2}} \quad \text{and} \quad r_0 := r_0(R, \Lambda) = \frac{R}{4(1 + \Lambda^2)}. \tag{3.3}$$

For simplicity, we denote $-(-\partial^2_{x^{(k)}x^{(k)}})^{\alpha/2}$ by $\Delta_k^{\alpha/2}$ for $1 \leq k \leq d$. That is, by (1.1),

$$\Delta_k^{\alpha/2} f(x) = \lim_{\varepsilon \to 0+} \mathcal{C}_{1,\alpha} \int_{|t| > \varepsilon} \frac{(f(x + te_k) - f(x))}{|t|^{1+\alpha}} dt. \tag{3.4}$$

Recall that e_k is the unit vector in the positive $x^{(k)}$-direction.

Lemma 3.2. *Let $Q \in \partial D$ and fix the coordinate system CS_Q so that*

$$B(Q, R) \cap D = \{y = (\tilde{y}, y^{(d)}) \in B(0, R) \text{ in } CS_Q : y^{(d)} > \phi_Q(\tilde{y})\}.$$

For $p \in [\alpha/2, \alpha)$, we define

$$h_p(y) = (\rho_Q(y))^p \, \mathbb{1}_{D \cap B(Q, R_0)}(y), \qquad y \in \mathbb{R}^d.$$

Then, there exist $C_i = C_i(d, \alpha, R, \Lambda, p) > 0$, $i = 11, 12, 13$ such that for all $x \in D$ with $\rho_Q(x) < r_0$ and $|\tilde{x}| < r_0$,

(1) *if $p = \alpha/2$, then, we have*

$$|\mathcal{L}h_p(x)| \leq C_{11}|\ln \rho_Q(x)|, \tag{3.5}$$

(2) *if $\alpha/2 < p < \alpha$, then, we have*

$$C_{13}\left(\rho_Q(x)\right)^{p-\alpha} \leq \mathcal{L}h_p(x) \leq C_{12}\left(\rho_Q(x)\right)^{p-\alpha}. \tag{3.6}$$

Proof. Note that any $C^{1,1}$ open set is locally very close to the upper half space. We will use this property and apply Lemma 3.1 to prove this lemma.

Fix $x = (\tilde{x}, x^{(d)}) \in CS_Q$ with $\rho_Q(x) < r_0$ and $|\tilde{x}| < r_0$, and choose $x_0 \in \partial D$ with $\tilde{x} = \tilde{x}_0$. See Figure 3 for a special case.

Denote by Π the hyperplane tangent to ∂D at the point x_0. Then, the function $\Gamma^* : \mathbb{R}^{d-1} \mapsto \mathbb{R}$ defined by $\Gamma^*(\tilde{y}) := \phi_Q(\tilde{x}_0) + \nabla\phi_Q(\tilde{x}_0)(\tilde{y} - \tilde{x})$ describes the plane Π. We use the following items:

$$\overline{h}(y) := \overline{h}_x(y) := (y^{(d)} - \Gamma^*(\tilde{y})) \vee 0,$$

$$D_{\Gamma^*} = \{y \in \mathbb{R}^d : y^{(d)} > \Gamma^*(\tilde{y})\},$$

$$\delta_\Pi(y) = \operatorname{dist}(y, \Pi)\mathbb{1}_{D_{\Gamma^*}}(y), \quad y \in \mathbb{R}^d,$$

$$b_x := \sqrt{1 + |\nabla\phi_Q(\tilde{x})|^2} \quad \text{and} \quad h_{x,p}(y) := (\overline{h}(y))^p \text{ for } \alpha/2 \leq p < \alpha.$$

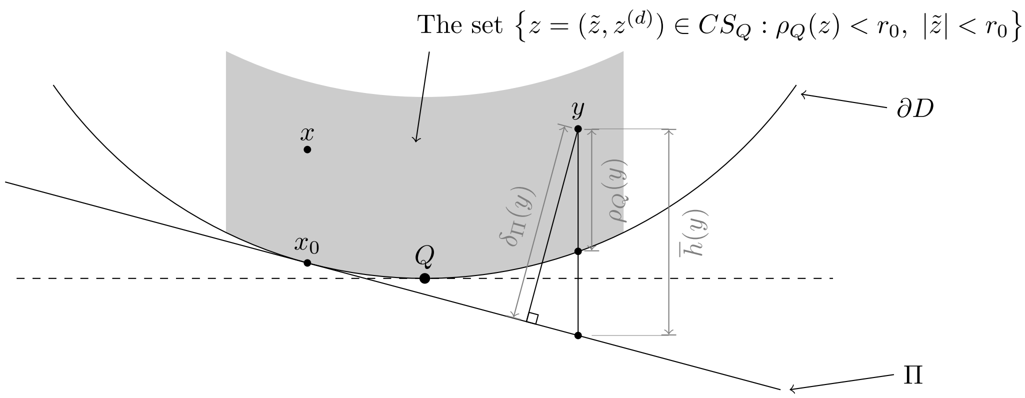

FIGURE 3. The points Q, x and x_0, etc

Note that $1 \le b_x \le \sqrt{1+\Lambda^2}$ and $h_{x,p}(x) = h_p(x)$. Since $\overline{h}(y) = b_x \delta_\Pi(y)$, by (3.1) and (3.2), we have for $y \in D_{\Gamma^*}$,

$$\mathcal{L}h_{x,p}(y) = b_x^p \mathcal{L}\delta_\Pi^p(y) = 0, \quad p = \alpha/2, \tag{3.7}$$

and

$$C_{10} b_x^p (\delta_\Pi(y))^{p-\alpha} \le \mathcal{L}h_{x,p}(y) = b_x^p \mathcal{L}\delta_\Pi^p(y) \le C_9 b_x^p (\delta_\Pi(y))^{p-\alpha}, \quad \alpha/2 < p < \alpha.$$

Note that $b_x \delta_\Pi(x) = \rho_Q(x)$. By the last inequality, we have for $\alpha/2 < p < \alpha$,

$$\begin{aligned} C_{10}(\rho_Q(x))^{p-\alpha} \le C_{10} b_x^\alpha (b_x \delta_\Pi(x))^{p-\alpha} &\le \mathcal{L}h_{x,p}(x) \\ &\le C_9 b_x^\alpha (b_x \delta_\Pi(x)))^{p-\alpha} \le C_9(1+\Lambda^2)^{\alpha/2} (\rho_Q(x))^{p-\alpha}. \end{aligned} \tag{3.8}$$

We claim that,

$$|\mathcal{L}(h_p - h_{x,p})(x)| \le \begin{cases} c_1 < \infty, & p \in (\frac{\alpha}{2}, \alpha), \\ c_2 |\ln \rho_Q(x)|, & p = \frac{\alpha}{2}, \end{cases} \tag{3.9}$$

for some constant $c_1 = c_1(d, \alpha, p, R, \Lambda) > 0$ and $c_2 = c_2(d, \alpha, R, \Lambda) > 0$, which together with (3.7) and (3.8) will establish this lemma.

Let

$$\begin{aligned} A &:= \{y : \Gamma^*(\tilde{y}) < y^{(d)} < \phi_Q(\tilde{y}) \text{ and } |\tilde{y} - \tilde{x}| < r_0\} \cup \{y : \Gamma^*(\tilde{y}) > y^{(d)} > \phi_Q(\tilde{y}) \text{ and } |\tilde{y} - \tilde{x}| < r_0\}, \\ E &:= \{y \in D \backslash A : |\tilde{y} - \tilde{x}| < r_0 \text{ and } \rho_Q(y) < r_0(2+\Lambda)\}. \end{aligned}$$

Note that, if $y \in D \cap B(x, r_0)$, then

$$\rho_Q(y) = y^{(d)} - \phi_Q(\tilde{y}) \le |y^{(d)} - x^{(d)}| + |x^{(d)} - \phi_Q(\tilde{x})| + |\phi_Q(\tilde{x}) - \phi_Q(\tilde{y})| < r_0(2+\Lambda).$$

If $|\tilde{y} - \tilde{x}| < r_0$ and $\rho_Q(y) < r_0(2+\Lambda)$, then

$$\begin{aligned} |y - Q|^2 &= |\tilde{y}|^2 + |y^{(d)}|^2 \\ &\le (|\tilde{y} - \tilde{x}| + |\tilde{x}|)^2 + (|y^{(d)} - \phi_Q(\tilde{y})| + |\phi_Q(\tilde{y})|)^2 \\ &\le (2r_0)^2 + (r_0(2+\Lambda) + |\phi_Q(\tilde{y})|)^2 \\ &\le (2r_0)^2 + (r_0(2+\Lambda) + \Lambda|\tilde{y}|)^2 < R_0^2. \end{aligned}$$

Consequently, we have

$$D \cap B(x, r_0) \subset D \cap \{y : |\tilde{y} - \tilde{x}| < r_0 \text{ and } \rho_Q(y) < r_0(2+\Lambda)\} \subset D \cap B(Q, R_0),$$

and then,

$$A^c \cap D \cap B(x, r_0) \subset E \subset D \cap B(Q, R_0). \tag{3.10}$$

Hence, by (3.10) and the fact that $B(x, r_0) \subset F$, we have

$$B(x, r_0) \cap \{z : h_p - h_{x,p} \neq 0\} \subset A \cup E; \tag{3.11}$$

see [7, p. 16] for details. We first consider the case for $p \in (\alpha/2, \alpha)$. Let $e_k = (0, \cdots, 0, 1, 0, \cdots, 0) \in \mathbb{R}^d$ be the unit vector along the k^{th} axis for $1 \le k \le d$. Using (3.11), we have by (3.4), for $1 \le k \le d$,

$$
\begin{aligned}
&(\mathcal{C}_{1,\alpha})^{-1} \left| \Delta_k^{\alpha/2}(h_p - h_{x,p})(x) \right| \\
\le& \left| \int_{|t| \ge r_0} (h_p(x + te_k) - h_{x,p}(x + te_k)) \frac{dt}{|t|^{1+\alpha}} \right| + \lim_{\varepsilon \to 0} \int_{r_0 > |t| > \varepsilon} |h_p(x + te_k) - h_{x,p}(x + te_k)| \frac{dt}{|t|^{1+\alpha}} \\
\le& \int_{|t| \ge r_0} |h_p(x + te_k) - h_{x,p}(x + te_k)| \frac{dt}{|t|^{1+\alpha}} + \varliminf_{\varepsilon \to 0} \int_{\{r_0 > |t| > \varepsilon\} \cap \{t : x + te_k \in A\}} |h_p(x + te_k) - h_{x,p}(x + te_k)| \frac{dt}{|t|^{1+\alpha}} \\
&+ \varliminf_{\varepsilon \to 0} \int_{\{r_0 > |t| > \varepsilon\} \cap \{t : x + te_k \in E\}} |h_p(x + te_k) - h_{x,p}(x + te_k)| \frac{dt}{|t|^{1+\alpha}} \quad \text{(by (3.11))} \\
=:& I_1 + I_2 + I_3. \qquad (3.12)
\end{aligned}
$$

We estimate I_1, I_2, I_3 separately. Note that for any $y \in \mathbb{R}^d$,

$$
0 \le h_{x,p}(y) \le \left(|y^{(d)} - x^{(d)}| + |x^{(d)} - \phi_Q(\tilde{x})| + |\phi_Q(\tilde{x}) - \Gamma^*(\tilde{y})| \right)^p \le (|x - y| + r_0 + \Lambda |x - y|)^p \le (r_0 + 2\Lambda |x - y|)^p.
$$

Combining this and the facts that $0 \le h_p \le 1$ and $p < \alpha$, we have

$$
I_1 \le \int_{|t| \ge r_0} \frac{1 + (r_0 + 2\Lambda |t|)^p}{|t|^{1+\alpha}} dt \le 2 \left(\frac{1 + (2r_0)^p}{\alpha} r_0^{-\alpha} + \frac{(4\Lambda)^p}{\alpha - p} r_0^{p-\alpha} \right) < \infty.
$$

For $y \in A$, we have

$$
\begin{aligned}
|h_{x,p}(y)| + |h_p(y)| &\le |y^{(d)} - \Gamma^*(\tilde{y})|^p + |y^{(d)} - \phi_Q(\tilde{y})|^p \\
&\le 2|\phi_Q(\tilde{y}) - \Gamma^*(\tilde{y})|^p = 2|\phi_Q(\tilde{y}) - \phi_Q(\tilde{x}_0) - \nabla \phi_Q(\tilde{x}_0)(\tilde{y} - \tilde{x})|^p \\
&\le 2\Lambda^p |\tilde{x} - \tilde{y}|^{2p} \le 2\Lambda^p |x - y|^{2p} \qquad (3.13)
\end{aligned}
$$

(see also [11, (3.14)]). Then, $|h_{x,p}(x + te_k)| + |h_p(x + te_k)| \le 2\Lambda^p |t|^{2p}$ and since $p > \alpha/2$, we have

$$
I_2 \le \int_{|t| \le r_0} \frac{|h_{x,p}(x + te_k)| + |h_p(x + te_k)|}{|t|^{1+\alpha}} dt \le \int_0^{r_0} 2\Lambda^p |t|^{2p - \alpha - 1} dt = \frac{4\Lambda^p}{2p - \alpha} r_0^{2p-\alpha} < \infty.
$$

For $y \in E \subset D \cap B(Q, R_0)$, we have $h_p(y) = \rho_Q(y)^p$. In view of this and the following two inequalities: for $a_1, a_2 > 0$,

$$
|a_1^p - a_2^p| \le \begin{cases} |a_1 - a_2|^p, & p \in (0,1), \\ p(a_1 \vee a_2)^{p-1} |a_1 - a_2|, & p \in [1, \infty), \end{cases}
$$

we have for $y \in E$,

$$
|h_{x,p}(y) - h_p(y)| \le \begin{cases} |\overline{h}(y) - \rho_Q(y)|^p, & p \in (0,1), \\ p|\overline{h}(y) - \rho_Q(y)|, & p \in [1, \infty). \end{cases} \qquad (3.14)
$$

On the other hand, by the definitions of $\overline{h}(y)$ and $\rho_Q(y)$, we have for $y \in E$,

$$
\begin{aligned}
|\overline{h}(y) - \rho_Q(y)| &= |\phi_Q(\tilde{y}) - \Gamma^*(\tilde{y})| = |\phi_Q(\tilde{y}) - \phi_Q(\tilde{x}_0) - \nabla \phi_Q(\tilde{x}_0)(\tilde{y} - \tilde{x})| \\
&\le \Lambda |\tilde{x} - \tilde{y}|^2 \le \Lambda |x - y|^2, \qquad (3.15)
\end{aligned}
$$

(see also [11, (3.7)]). Using the last two inequalities, we have

$$
\begin{aligned}
I_3 &\le \int_{\{r_0 > |t|\} \cap \{t : x + te_k \in E\}} |h_p(x + te_k) - h_{x,p}(x + te_k)| \frac{dt}{|t|^{1+\alpha}} \\
&\le \begin{cases} 2\Lambda^p \int_0^{r_0} t^{2p-\alpha-1} dt = \frac{2\Lambda^p}{2p-\alpha} r_0^{2p-\alpha}, & \text{for } p \in (\alpha/2, 1), \\ 2p\Lambda \int_0^{r_0} t^{1-\alpha} dt = \frac{2p\Lambda}{2-\alpha} r_0^{2-\alpha}, & \text{for } p \in [1, \infty). \end{cases}
\end{aligned}
$$

Combining (3.12) and the estimates of I_1, I_2 and I_3, and using the expression $\mathcal{L} = \sum_{k=1}^d \Delta_k^{\alpha/2}$, we can prove the first part of our claim (3.9).

It remains to show the second part in (3.9) which is the case when $p=\alpha/2$. Similar to (3.12), we have by (3.4), for $1\le k\le d$ and $p=\alpha/2$,

$$\begin{aligned}
&(\mathcal{C}_{1,\alpha})^{-1}\left|\Delta_k^{\alpha/2}(h_p-h_{x,p})(x)\right| \\
&\le \int_{|t|\ge r_0}|h_p(x+te_k)-h_{x,p}(x+te_k)|\frac{dt}{|t|^{1+\alpha}} \\
&\quad+\int_{\{r_0>|t|>\rho_Q(x)/(2\sqrt{1+\Lambda^2})\}\cap\{t:x+te_k\in A\}}|h_p(x+te_k)-h_{x,p}(x+te_k)|\frac{dt}{|t|^{1+\alpha}} \\
&\quad+\int_{\{r_0>|t|>\rho_Q(x)/(2\sqrt{1+\Lambda^2})\}\cap\{t:x+te_k\in E\}}|h_p(x+te_k)-h_{x,p}(x+te_k)|\frac{dt}{|t|^{1+\alpha}} \\
&\quad+\lim_{\varepsilon\to0}\int_{\varepsilon<|t|\le\rho_Q(x)/(2\sqrt{1+\Lambda^2})}|h_p(x+te_k)-h_{x,p}(x+te_k)|\frac{dt}{|t|^{1+\alpha}} \qquad (3.16)\\
&=:J_1+J_2+J_3+J_4.
\end{aligned}$$

We can estimate J_1 similar to I_1, and have

$$J_1\le 2\left(\frac{1+(2r_0)^{\alpha/2}}{\alpha}r_0^{-\alpha}+\frac{(4\Lambda)^{\alpha/2}}{\alpha/2}r_0^{-\alpha/2}\right)<\infty.$$

Similar to I_2, by (3.13) for $p=\alpha/2\in(0,1)$, we have

$$J_2\le 2\int_{\rho_Q(x)/(2\sqrt{1+\Lambda^2})}^{r_0}2\Lambda^{\alpha/2}|t|^{\alpha-\alpha-1}dt=(4\Lambda^{\alpha/2})(\ln(2r_0\sqrt{1+\Lambda^2})-\ln\rho_Q(x)),$$

and similar to I_3, by (3.14) for $p=\alpha/2$ and (3.15),

$$J_3\le 2\int_{\rho_Q(x)/(2\sqrt{1+\Lambda^2})}^{r_0}\Lambda^{\alpha/2}|t|^{\alpha-\alpha-1}dt=(2\Lambda^{\alpha/2})(\ln(2r_0\sqrt{1+\Lambda^2})-\ln\rho_Q(x)).$$

For $t\in(0,\rho_Q(x)/(2\sqrt{1+\Lambda^2})]$, we have

$$\delta_D(x+te_k)\ge\delta_D(x)-|t|\ge\frac{\rho_Q(x)}{\sqrt{1+\Lambda^2}}-\frac{\rho_Q(x)}{2\sqrt{1+\Lambda^2}}=\frac{\rho_Q(x)}{2\sqrt{1+\Lambda^2}}>0,$$

and by the definition of $\overline{h}$, for $y=x+te_k$,

$$\begin{aligned}
\overline{h}(y)&=y^{(d)}-\phi_Q(\tilde{x}_0)-\nabla\phi_Q(\tilde{x}_0)(\tilde{y}-\tilde{x})\\
&=\overline{h}(x)+y^{(d)}-x^{(d)}-\nabla\phi_Q(\tilde{x}_0)(\tilde{y}-\tilde{x})\\
&\ge\overline{h}(x)-\Lambda|t|\ge\rho_Q(x)-\frac{\Lambda\rho_Q(x)}{2\sqrt{1+\Lambda^2}}\ge\frac{\rho_Q(x)}{2}>0. \qquad (3.17)
\end{aligned}$$

This together with (3.10) implies that $x+te_k\in E$ for all $t\in(0,\rho_Q(x)/(2\sqrt{1+\Lambda^2})]$. Furthermore, combining (3.15), (3.17) and the following inequality:

$$|a_1^{\alpha/2}-a_2^{\alpha/2}|\le a_1^{\alpha/2-1}|a_1-a_2|,\qquad a_1,a_2>0,$$

we have for $y=x+te_k$

$$|h_{x,\alpha/2}(y)-h_{\alpha/2}(y)|\le(\overline{h}(y))^{\alpha/2-1}|\overline{h}(y)-\rho_Q(y)|\le\left(\frac{\rho_Q(x)}{2}\right)^{\alpha/2-1}\Lambda|x-y|^2=\left(\frac{\rho_Q(x)}{2}\right)^{\alpha/2-1}\Lambda t^2.$$

Therefore,

$$\begin{aligned}
J_4&\le\int_0^{\rho_Q(x)/(2\sqrt{1+\Lambda^2})}2\Lambda\left(\frac{\rho_Q(x)}{2}\right)^{\alpha/2-1}t^{1-\alpha}dt\\
&=\frac{2\Lambda}{2-\alpha}(1+\Lambda^2)^{\alpha/2-1}\left(\frac{\rho_Q(x)}{2}\right)^{1-\alpha/2}\le\frac{2\Lambda r_0^{1-\alpha/2}}{2-\alpha}.
\end{aligned}$$

Combining (3.16) and the estimates of $J_1, \cdots, J_4$, and using the expression $\mathcal{L} = \sum_{k=1}^d \Delta_k^{\alpha/2}$ yields the second part of our claim (3.9). In view of (3.7), (3.8) and our claim (3.9), we get the desired results of this lemma. □

Recall that $\rho_Q(x) := x^{(d)} - \phi_Q(\tilde{x})$ for every $Q \in \partial D$ and

$$x \in B(Q,R) \cap D = \{y = (\tilde{y}, y^{(d)}) \in B(0,R) \text{ in } CS_Q : y^{(d)} > \phi_Q(\tilde{y})\}.$$

We define for $r_1, r_2 > 0$

$$D(r_1, r_2) := D_Q(r_1, r_2) := \{y \in D : r_1 > \rho_Q(y) > 0,\ |\tilde{y}| < r_2\}. \tag{3.18}$$

Recall that the constants R_0 and r_0 are defined in (3.3).

Lemma 3.3. *There are positive constants* $\delta_0 = \delta_0(d, \alpha, R, \Lambda) \in (0, r_0/(2\sqrt{1+\Lambda^2}))$ *and* $C_i = C_i(d, \alpha, R, \Lambda)$, $i = 14, 15$ *such that for every* $Q \in \partial D$ *and* $x \in D_Q(\delta_0, r_0)$ *with* $\tilde{x} = 0$,

$$\mathbb{P}_x\left(X_{\tau_{D_Q(\delta_0,r_0)}} \in D_Q(r_0/\sqrt{1+\Lambda^2}, r_0)\right) \geq C_{14}\delta_D(x)^{\alpha/2}, \tag{3.19}$$

$$\mathbb{P}_x\left(X_{\tau_{D_Q(\delta_0,r_0)}} \in D\right) \leq C_{15}\delta_D(x)^{\alpha/2}, \tag{3.20}$$

and

$$\mathbb{E}_x\left[\tau_{D_Q(\delta_0,r_0)}\right] \leq C_{15}\delta_D(x)^{\alpha/2} \tag{3.21}$$

(cf. Figure 4).

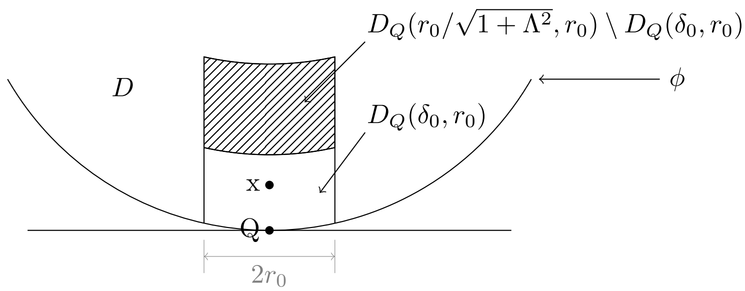

FIGURE 4. The points Q and x, and the set $D_Q(\delta_0, r_0)$, etc

Proof. Recall the notation $w = (w^{(1)}, w^{(2)}, \cdots, w^{(d)}) := (\tilde{w}, w^{(d)}) \in \mathbb{R}^d$. Since D is a $C^{1,1}$ open set with characteristics (R, Λ), let $\phi : \mathbb{R}^{d-1} \mapsto \mathbb{R}$ be the $C^{1,1}$-function satisfying (1). $\phi(0) = 0$, $\nabla\phi(0) = 0$; (2). $\|\nabla\phi\|_\infty \leq \Lambda$; (3). $|\nabla\phi(\tilde{y}) - \nabla\phi(\tilde{z})| \leq \Lambda|\tilde{y} - \tilde{z}|$. Let CS_Q be the corresponding coordinate system such that

$$B(Q,R) \cap D = \left\{(\tilde{y}, y^{(d)}) \in B(0,R) \text{ in } CS_Q : y^{(d)} > \phi(\tilde{y})\right\}.$$

Let $p \in (\alpha/2, \alpha)$ and define

$$\begin{aligned} \rho(y) &:= y^{(d)} - \phi(\tilde{y}), \\ h(y) &:= \rho(y)^{\alpha/2}\mathbb{1}_{B(Q,R_0)\cap D}(y), \\ h_p(y) &:= \rho(y)^p\mathbb{1}_{B(Q,R_0)\cap D}(y). \end{aligned}$$

Since $\rho(y) \leq \sqrt{1+\Lambda^2}\delta_D(y)$, we have

$$0 \leq h \leq (\sqrt{1+\Lambda^2}R_0)^{\alpha/2} \leq R^{\alpha/2} \leq 1.$$

Let $\psi : \mathbb{R}^d \mapsto \mathbb{R}$ be a smooth positive function with bounded first and second order derivatives such that

$$\psi(y) = \frac{2^{p+2}|\tilde{y}|^2}{r_0^2} \quad \text{for } |y - Q| < r_0/(2\sqrt{1+\Lambda^2}),$$

and

$$2^{p+2} \leq \psi(y) \leq 2^{p+3} \quad \text{for } |y - Q| \geq r_0/\sqrt{1+\Lambda^2}.$$

Then, there exists $c_1 = c_1(d, \alpha, R, \Lambda, p) > 0$ such that

$$\|\mathcal{L}\psi\|_\infty \le c_1. \tag{3.22}$$

Step 1. Constructing suitable superharmonic and subharmonic functions with respect to $\mathcal{L}$. We consider

$$u_1(y) := h(y) + h_p(y) \quad \text{and} \quad u_2(y) := h(y) + \psi(y) - h_p(y).$$

Since $p \in (\alpha/2, \alpha)$, by (3.5) and (3.6), there exists $\delta_0 := \delta_0(d, \alpha, R, \Lambda) \in (0, r_0/(2\sqrt{1+\Lambda^2}))$ such that for $y \in D_Q(\delta_0, r_0)$,

$$\mathcal{L}u_1(y) = \mathcal{L}h(y) + \mathcal{L}h_p(y) \ge -C_{11}|\ln \rho(y)| + C_{13}\rho(y)^{p-\alpha} \ge 0, \tag{3.23}$$

and by (3.22),

$$\mathcal{L}u_2(y) = \mathcal{L}h(y) + \mathcal{L}\psi(y) - \mathcal{L}h_p(y) \le C_{11}|\ln \rho(y)| + c_1 - C_{13}\rho(y)^{p-\alpha} \le -1. \tag{3.24}$$

Step 2. Translating super/subharmonic functions into super/submartingale properties for X_t.

We claim that the inequalities (3.23) and (3.24) imply that

$$t \mapsto u_2(X_{t\wedge\tau_{D(\delta_0,r_0)}}) + t \wedge \tau_{D(\delta_0,r_0)} \text{ is a non-negative bounded supermartingale,} \tag{3.25}$$

$$\mathbb{E}_x\left[\tau_{D(\delta_0,r_0)}\right] \le \rho(x)^{\alpha/2}, \tag{3.26}$$

and

$$t \mapsto u_1(X_{t\wedge\tau_{D(\delta_0,r_0)}}) \text{ is a non-negative bounded submartingale.} \tag{3.27}$$

Recall that $\tau_{D(\delta_0,r_0)}$ is the first exit time of X upon leaving the set $D(\delta_0, r_0)$.

Observe that if v is a bounded C^2-function on $\mathbb{R}^d$ with bounded second order derivatives, then, by Markov property,

$$M_t^v := v(X_t) - v(X_0) - \int_0^t \mathcal{L}v(X_s)ds \text{ is a martingale.} \tag{3.28}$$

Hence, if u_1 and u_2 are C^2-functions with bounded second order derivatives, then the above claims would just follow from (3.28), (3.23) and (3.24). However, u_1 and u_2 are not C^2-functions. We shall approximate them by smooth functions. Indeed, let g be a mollifier, and $g_n(z) := 2^{nd}g(2^n z)$, $z \in \mathbb{R}^d$ for $n \ge 1$. Define

$$u_i^{(n)}(z) := g_n * u_i(z) = \int_{\mathbb{R}^d} g_n(y)u_i(z-y)dy, \qquad i = 1, 2.$$

Since $\mathcal{L}u_i^{(n)} = g_n * \mathcal{L}u_i$ for $i = 1, 2$, we have for any $n > m \ge 1$,

$$\mathcal{L}u_1^{(n)} \ge 0, \quad \text{and} \quad \mathcal{L}u_2^{(n)} \le -1,$$

on

$$D_m(\delta_0, r_0) := \left\{y : \delta_0 - 2^{-m} > \rho(y) > 2^{-m} \text{ and } |\tilde{y}| < r_0 - 2^{-m}\right\}.$$

Since each $u_i^{(n)}$ is a bounded smooth functions with bounded second order derivatives, it follows from (3.28) that for any $n > m \ge 1$,

$$t \mapsto u_2^{(n)}(X_{t\wedge\tau_{D_m(\delta_0,r_0)}}) + t \wedge \tau_{D_m(\delta_0,r_0)} \text{ is a non-negative supermartingale,}$$

and

$$t \mapsto u_1^{(n)}(X_{t\wedge\tau_{D_m(\delta_0,r_0)}}) \text{ is a non-negative bounded submartingale.}$$

Since each u_i is bounded and continuous, $u_i^{(n)}$ converges to u_i uniformly on $D_m(\delta_0, r_0)$, and hence,

$$t \mapsto u_2(X_{t\wedge\tau_{D_m(\delta_0,r_0)}}) + t \wedge \tau_{D_m(\delta_0,r_0)} \text{ is a non-negative supermartingale,} \tag{3.29}$$

and

$$t \mapsto u_1(X_{t\wedge\tau_{D_m(\delta_0,r_0)}}) \text{ is a non-negative bounded submartingale.}$$

Since $D_m(\delta_0, r_0)$ increases to $D(\delta_0, r_0)$ as $m \to \infty$, we obtain from above (3.25) and (3.27). Moreover, it follows from (3.29) that for each $n \ge 1$ and $t > 0$,

$$\mathbb{E}_x\left[u_2(X_{t\wedge\tau_{D_m(\delta_0,r_0)}}) + t \wedge \tau_{D_m(\delta_0,r_0)}\right] \le u_2(x).$$

Since $u_2 \geq 0$ and $D_m(\delta_0, r_0)$ increases to $D(\delta_0, r_0)$, by passing the above formula to the limit as $m \to \infty$ and then $t \to \infty$, we obtain

$$\mathbb{E}_x\left[\tau_{D(\delta_0,r_0)}\right] \leq u_2(x).$$

Note that $\tilde{x} = 0$, $\psi(x) = 0$ and then, $u_2(x) \leq \rho(x)^{\alpha/2}$. This together with the above inequality implies (3.26). Consequently, (3.21) holds true.

Step 3. Deriving the estimates of the exit distributions from super/submartingale properties.

Since $\psi(y) \geq 2^{p+2}$ for $|y - Q| > r_0/\sqrt{1+\Lambda^2}$ and $\phi(x) = 0$, we have by (3.25),

$$\begin{aligned}\rho(x)^{\alpha/2} \geq u_2(x) \geq \ & \mathbb{E}_x\left[u_2(X_{\tau_{D(\delta_0,r_0)}}); X_{\tau_{D(\delta_0,r_0)}} \in D \setminus D(r_0/\sqrt{1+\Lambda^2}, r_0)\right] \\ \geq & \ (2^{p+2}-1)\mathbb{P}_x\left(X_{\tau_{D(\delta_0,r_0)}} \in D \setminus D(r_0/\sqrt{1+\Lambda^2}, r_0)\right).\end{aligned}$$

On the other hand, by (3.27), we have

$$\rho(x)^{\alpha/2} \leq \rho(x)^{\alpha/2} + \rho(x)^p = u_1(x) \leq \mathbb{E}_x\left[u_1(X_{\tau_{D(\delta_0,r_0)}})\right] \leq 2\mathbb{P}_x\left(X_{\tau_{D(\delta_0,r_0)}} \in D\right).$$

Combining the above two formulas, we obtain

$$\begin{aligned}\mathbb{P}_x\left(X_{\tau_{D(\delta_0,r_0)}} \in D(r_0/\sqrt{1+\Lambda^2}, r_0)\right) &= \mathbb{P}_x\left(X_{\tau_{D(\delta_0,r_0)}} \in D\right) - \mathbb{P}_x\left(X_{\tau_{D(\delta_0,r_0)}} \in D \setminus D(r_0/\sqrt{1+\Lambda^2}, r_0)\right) \\ &\geq \frac{2^{p+2}-3}{2(2^{p+2}-1)}\rho(x)^{\alpha/2},\end{aligned}$$

which implies (3.19).

Recall that $0 \leq h_p \leq 1$. If $|y - Q| \geq r_0/\sqrt{1+\Lambda^2}$, then, $\psi(y) \geq 2^{p+2}$, we have

$$u_2(y) = h(y) + \psi(y) - h_p(y) \geq 0 + 2^{p+2} - 1 \geq 1, \qquad y \in B(Q, r_0/\sqrt{1+\Lambda^2})^c.$$

On the other hand, we have for $y \in B(Q, r_0)$ with $\delta_0 \leq \rho(y) \leq r_0$,

$$u_2(y) \geq h(y) + \psi(y) - h_p(y) \geq \rho(y)^{\alpha/2} - \rho(y)^p \geq c_2,$$

where $c_2 = c_2(\delta_0, r_0, p) \in (0, 1)$. It follows from the above two estimates that $u_2 \geq c_2$ on $D \setminus D(\delta_0, r_0)$. Therefore, by (3.25), we have

$$\rho(x)^{\alpha/2} \geq u_2(x) \geq \mathbb{E}_x\left[u_2(X_{\tau_{D(\delta_0,r_0)}})\right] \geq c_2\mathbb{P}_x\left(X_{\tau_{D(\delta_0,r_0)}} \in D\right),$$

which implies (3.20). □

4. Dirichlet heat kernel estimates

Throughout this section, $D \subset \mathbb{R}^d$ is a $C^{1,1}$ open set with characteristics (R, Λ). Recall that we use the following convention: for $u \in \mathbb{R}^d$, $a \in \mathbb{R}$ and $1 \leq i \leq d$,

$$[u]_a^i := (u^{(1)}, \cdots, u^{(i-1)}, a, u^{(i+1)}, \cdots, u^{(d)}).$$

Define

$$j(a, b) = \frac{\mathcal{C}_{1,\alpha}}{|a-b|^{1+\alpha}} \quad \text{for } a \neq b \in \mathbb{R},$$

where $\mathcal{C}_{1,\alpha}$ is the positive constant in (1.2).

With these notation, we rewrite the Lévy system formula (1.8) as follows. For any non-negative measurable function f on $\mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d$ with $f(s, x, x) = 0$ for any $s \geq 0$ and $x \in \mathbb{R}^d$ and for any stopping time S with respect to the minimum augmented filtration generated by X, we have

$$\mathbb{E}_x\left[\sum_{s \leq S} f(s, X_{s-}, X_s)\right] = \mathbb{E}_x\left[\int_0^S \sum_{i=1}^d \int_{\mathbb{R}} f(s, X_s, [X_s]_\theta^i) j(X_s^{(i)}, \theta) d\theta ds\right]. \tag{4.1}$$

4.1. **Upper bound estimates.**

Lemma 4.1. *There is a constant $c = c(d, \alpha, R, \Lambda) > 0$ such that for any $x \in D$,*

$$\mathbb{P}_x\left(\tau_D \geq \frac{1}{4}\right) \leq c\left(1 \wedge \delta_D(x)^{\alpha/2}\right). \tag{4.2}$$

Proof. Let δ_0, r_0 be the constants from Lemma 3.3. It suffices to prove (4.2) when $\delta_D(x) < \delta_0 \wedge r_0 = \delta_0$. Indeed, let $Q \in \partial D$ be such that $\delta_D(x) = |x - Q|$, and $D(\delta_0, r_0)$ be the set defined in (3.18). In this case, $x \in D_Q(\delta_0, r_0)$ and $\tilde{x} = 0$. It follows from (3.20) and (3.21) that

$$\begin{aligned}\mathbb{P}_x\left(\tau_D \geq \frac{1}{4}\right) &\leq \mathbb{P}_x\left(\tau_D \geq \frac{1}{4}, \tau_{D(\delta_0,r_0)} \geq \frac{1}{4}\right) + \mathbb{P}_x\left(\tau_D \geq \frac{1}{4}, \tau_{D(\delta_0,r_0)} < \frac{1}{4}\right) \\ &\leq 4\mathbb{E}_x\left[\tau_{D(\delta_0,r_0)}\right] + \mathbb{P}_x\left(X_{\tau_{D(\delta_0,r_0)}} \in D\right) \leq \ 5C_{15}\delta_D(x)^{\alpha/2}.\end{aligned}$$

□

Lemma 4.2. *Let $U, U_1, U_3 \subset \mathbb{R}^d$ be three open sets with $U_1, U_3 \subset U$ and $\mathrm{dist}(U_1, U_3) > 0$. Define $U_2 := U \setminus (U_1 \cup U_3)$. We have, for any $t > 0$, $x \in U_1$ and $y \in U_3$,*

$$\begin{aligned} p_U(t,x,y) \leq\ & \frac{2}{t}\mathbb{E}_x\left[\tau_{U_1}\right] \sup_{z \in U_1} p_U(t/2, z, y) + \mathbb{P}_x\left(X_{\tau_{U_1}} \in U_2\right) \sup_{t/2 < s < t, z \in U_2} p_U(s, z, y) \\ & + \int_0^{t/2}\int_{U_1} p_{U_1}(s,x,u)\left(\sum_{i=1}^d \int_{\mathbb{R}} p_U(t-s, [u]^i_\theta, y)d\theta \cdot \sup_{u \in U_1, [u]^i_a \in U_3} j(u^{(i)}, a)\right) duds. \end{aligned} \tag{4.3}$$

Proof. Fix $t > 0$ and $x \in U_1$. Let $0 \leq f \in L^1(U) \cap L^\infty(U)$. By the strong Markov property of X and Proposition 2.1, we have

$$\begin{aligned} P^U_t f(x) &= \mathbb{E}_x\left[f(X_t); t < \tau_U\right] \\ &= \mathbb{E}_x\left[f(X_t); t < \tau_{U_1}\right] + \mathbb{E}_x\left[f(X_t); \tau_{U_1} \leq t < \tau_U\right] \\ &= P^{U_1}_t f(x) + \mathbb{E}_x\left[f(X^U_t); \tau_{U_1} \leq t\right] \\ &= P^{U_1}_t f(x) + \mathbb{E}_x\left[\mathbb{E}_{X^U_{\tau_{U_1}}}\left[f(X^U_{t-\tau_{U_1}})\right]; \tau_{U_1} < t\right] + \mathbb{E}_x\left[f(X^U_t); \tau_{U_1} = t\right] \\ &= P^{U_1}_t f(x) + \mathbb{E}_x\left[P^U_{t-\tau_{U_1}} f(X_{\tau_{U_1}}); \tau_{U_1} < t,\ X_{\tau_{U_1}} \in U_2\right] \\ &\quad + \mathbb{E}_x\left[P^U_{t-\tau_{U_1}} f(X_{\tau_{U_1}}); \tau_{U_1} < t,\ X_{\tau_{U_1}} \in U_3\right] \\ &=: P^{U_1}_t f(x) + I + II. \end{aligned} \tag{4.4}$$

Note that

$$I \leq \mathbb{P}_x\left(X_{\tau_{U_1}} \in U_2\right) \sup_{0 < s < t,\ z \in U_2} P^U_s f(z).$$

Since $\mathrm{dist}(U_1, U_3) > 0$, by (4.1), we obtain,

$$\begin{aligned} II &= \mathbb{E}_x\left[\tau_{U_1} < t,\ X_{\tau_{U_1}} \in U_3; P^U_{t-\tau_{U_1}} f(X_{\tau_{U_1}})\right] \\ &= \mathbb{E}_x\left[\int_0^t \mathbb{1}_{\{s < \tau_{U_1}\}} \cdot \left(\sum_{i=1}^d \int_{\mathbb{R}} \mathbb{1}_{\{[X_s]^i_\theta \in U_3\}} \cdot P^U_{t-s} f([X_s]^i_\theta) j(X^{(i)}_s, \theta) d\theta\right) ds\right] \\ &= \int_0^t \int_{U_1} p_{U_1}(s,x,u)\left(\sum_{i=1}^d \int_{\mathbb{R}} \mathbb{1}_{\{[u]^i_\theta \in U_3\}} \cdot P^U_{t-s} f([u]^i_\theta) j(u^{(i)}, \theta) d\theta\right) duds \\ &\leq \int_0^t \int_{U_1} p_{U_1}(s,x,u)\left(\sum_{i=1}^d \int_{\mathbb{R}} P^U_{t-s} f([u]^i_\theta) d\theta \cdot \sup_{u \in U_1, [u]^i_a \in U_3} j(u^{(i)}, a)\right) duds. \end{aligned}$$

Thus we have by (4.4)

$$P^U_t f(x) \leq P^{U_1}_t f(x) + \mathbb{P}_x\left(X_{\tau_{U_1}} \in U_2\right) \sup_{0 < s < t,\ z \in U_2} P^U_s f(z)$$

$$+\int_0^t\int_{U_1}p_{U_1}(s,x,u)\left(\sum_{i=1}^d\int_{\mathbb{R}}P^U_{t-s}f([u]^i_\theta)d\theta\cdot\sup_{u\in U_1,[u]^i_a\in U_3}j(u^{(i)},a)\right)duds.$$

For any $y\in U_3$, by setting $f=p_U(t,\cdot,y)$ in the above inequality and using semigroup property, we obtain

$$\begin{aligned}p_U(2t,x,y)\leq&\int_{U_1}p_{U_1}(t,x,z)p_U(t,z,y)dz+\mathbb{P}_x\left(X_{\tau_{U_1}}\in U_2\right)\sup_{t<s<2t,\ z\in U_2}p_U(s,z,y)\\&+\int_0^t\int_{U_1}p_{U_1}(s,x,u)\left(\sum_{i=1}^d\int_{\mathbb{R}}p_U(2t-s,[u]^i_\theta,y)d\theta\cdot\sup_{u\in U_1,[u]^i_a\in U_3}j(u^{(i)},a)\right)duds\\\leq&\ \mathbb{P}_x\left(\tau_{U_1}>t\right)\sup_{z\in U_1}p_U(t,z,y)+\mathbb{P}_x\left(X_{\tau_{U_1}}\in U_2\right)\sup_{t<s<2t,\ z\in U_2}p_U(s,z,y)\\&+\int_0^t\int_{U_1}p_{U_1}(s,x,u)\left(\sum_{i=1}^d\int_{\mathbb{R}}p_U(2t-s,[u]^i_\theta,y)d\theta\cdot\sup_{u\in U_1,[u]^i_a\in U_3}j(u^{(i)},a)\right)duds.\end{aligned}$$

By renaming $2t$ by t in the above inequality, we finish the proof. □

Lemma 4.3. *There is a constant $c=c(d,\alpha,R,\Lambda)>0$ such that for all $x,y\in D$,*

$$p_D(1/2,x,y)\leq c\left(1\wedge\delta_D(x)^{\alpha/2}\right)p(1/2,x,y).\tag{4.5}$$

Proof. It suffices to prove (4.5) when $\delta_D(x)<\delta_0$, where δ_0 is the constant from Lemma 3.3. Fix $x,y\in D$ with $\delta_D(x)<\delta_0$. Recall that $r_0=r_0(R,\Lambda)=\frac{R}{4(1+\Lambda^2)}$. Take $r=4r_0$ for simplicity.

Case 1. For all $i=1,\cdots,d$, $|x^{(i)}-y^{(i)}|<r$. By semigroup property of p_D and (4.2), we have

$$\begin{aligned}p_D(1/2,x,y)=&\int_D p_D(1/4,x,z)p_D(1/4,z,y)dz\\\leq&\sup_{z\in D}p_D(1/4,z,y)\int_D p_D(1/4,x,z)dz\\\leq&\ C_1 4^{d/\alpha}\mathbb{P}_x\left(\tau_D>1/4\right)\leq c_1\delta_D(x)^{\alpha/2}.\end{aligned}$$

On the other hand, since $|x^{(i)}-y^{(i)}|<r$, for all $i=1,\cdots,d$, we have by (1.6),

$$p(1/2,x,y)\geq C_1^{-1}\prod_{i=1}^d\left(2^{1/\alpha}\wedge\frac{1/2}{r^{1+\alpha}}\right)>0.$$

Combining the above two inequalities, we verify (4.5) in this case.

Case 2. There is some $1\leq i\leq d$ such that $|x^{(i)}-y^{(i)}|\geq r$. Let

$$\mathcal{I}:=\left\{i:|x^{(i)}-y^{(i)}|\geq r,\ 1\leq i\leq d\right\},$$

and $Q\in\partial D$ be such that $|x-Q|=\delta_D(x)$. Define

$$\begin{aligned}U_1:=&\ D_Q(\delta_0,r_0)\quad\text{(see (3.18) for the definition of the set } D_Q(\delta_0,r_0)),\\U_3:=&\left\{z\in D:\exists\ i\in\mathcal{I},\ \text{such that } |z^{(i)}-x^{(i)}|>|x^{(i)}-y^{(i)}|/2\right\},\\U_2:=&\ D\setminus U_1\setminus U_3.\end{aligned}$$

Note that $U_1\cap U_3=\emptyset$, $x\in U_1$ by $\delta_D(x)<\delta_0$ and $y\in U_3$ by the definition of $\mathcal{I}$. By Lemma 4.2 with $t=1/2$ and $U=D$, we have

$$\begin{aligned}p_D(1/2,x,y)\leq&\ 4\mathbb{E}_x\left[\tau_{U_1}\right]\sup_{z\in U_1}p_D(1/4,z,y)+\mathbb{P}_x\left(X_{\tau_{U_1}}\in U_2\right)\sup_{\frac14<s<\frac12,z\in U_2}p_D(s,z,y)\\&+\int_0^{1/4}\int_{U_1}p_{U_1}(s,x,u)\left(\sum_{i=1}^d\int_{\mathbb{R}}p_D(1/2-s,[u]^i_\theta,y)d\theta\cdot\sup_{u\in U_1,[u]^i_a\in U_3}j(u^{(i)},a)\right)duds\\=:&\ I_1+I_2+I_3.\end{aligned}\tag{4.6}$$

We estimate I_1, I_2, I_3 separately. Indeed, for any $i\in\mathcal{I}$ and $z\in U_1$, since $|z^{(i)}-x^{(i)}|<2r_0$, we have

$$|z^{(i)}-y^{(i)}|\geq|x^{(i)}-y^{(i)}|-|z^{(i)}-x^{(i)}|\geq|x^{(i)}-y^{(i)}|-2r_0\geq|x^{(i)}-y^{(i)}|/2.$$

This together with the upper bound of $p(1/4, z, y)$, the lower bound of $p(1/4, x, y)$ in (1.6) and (3.21) yields that

$$\begin{aligned}
I_1 &= 4\mathbb{E}_x\left[\tau_{U_1}\right] \sup_{z\in U_1} p_D(1/4, z, y) \\
&\le c_2 C_1 \delta_D(x)^{\alpha/2} \sup_{z\in U_1} \prod_{i\in\mathcal{I}} \left(4^{1/\alpha} \wedge \frac{1/4}{|z^{(i)}-y^{(i)}|^{1+\alpha}}\right) \prod_{i\notin\mathcal{I}} \left(4^{1/\alpha} \wedge \frac{1/4}{|z^{(i)}-y^{(i)}|^{1+\alpha}}\right) \\
&\le c_2 C_1 \delta_D(x)^{\alpha/2} \prod_{i\in\mathcal{I}} \left(4^{1/\alpha} \wedge \frac{2^{1+\alpha}(1/4)}{|x^{(i)}-y^{(i)}|^{1+\alpha}}\right) \prod_{i\notin\mathcal{I}} \left(4^{1/\alpha}\right) \\
&\le c_3 \delta_D(x)^{\alpha/2} \prod_{i\in\mathcal{I}} \left(4^{1/\alpha} \wedge \frac{1/4}{|x^{(i)}-y^{(i)}|^{1+\alpha}}\right) \prod_{i\notin\mathcal{I}} \left(4^{1/\alpha} \wedge \frac{1/4}{|x^{(i)}-y^{(i)}|^{1+\alpha}}\right) \\
&\le c_3 C_1 \delta_D(x)^{\alpha/2} p(1/4, x, y),
\end{aligned}$$

where in the second to the last inequality, we have used the fact that for all $i \notin \mathcal{I}$, $|x^{(i)} - y^{(i)}|^{1+\alpha} < r^{1+\alpha} < \infty$. By the definition of U_2, for any $z \in U_2$ and any $i \in \mathcal{I}$, we have $|z^{(i)} - x^{(i)}| \le |x^{(i)} - y^{(i)}|/2$, and then

$$|z^{(i)} - y^{(i)}| \ge |x^{(i)} - y^{(i)}| - |z^{(i)} - x^{(i)}| \ge |x^{(i)} - y^{(i)}| - |x^{(i)} - y^{(i)}|/2 = |x^{(i)} - y^{(i)}|/2.$$

This together with the upper bound of $p(s, z, y)$, the lower bound of $p(1/4, x, y)$ in (1.6) and (3.20) yields

$$\begin{aligned}
I_2 &= \mathbb{P}_x\left(X_{\tau_{U_1}} \in U_2\right) \sup_{\frac14<s<\frac12, z\in U_2} p_D(s, z, y) \\
&\le \mathbb{P}_x\left(X_{\tau_{U_1}} \in D\right) \sup_{\frac14<s<\frac12, z\in U_2} p(s, z, y) \\
&\le c_4 \delta_D(x)^{\alpha/2} \sup_{\frac14<s<\frac12, z\in U_2} \prod_{i\in\mathcal{I}} \left(s^{-1/\alpha} \wedge \frac{s}{|z^{(i)}-y^{(i)}|^{1+\alpha}}\right) \prod_{i\notin\mathcal{I}} s^{-1/\alpha} \\
&\le c_5 \delta_D(x)^{\alpha/2} \prod_{i\in\mathcal{I}} \left(4^{1/\alpha} \wedge \frac{2^{1+\alpha}(1/2)}{|x^{(i)}-y^{(i)}|^{1+\alpha}}\right) \prod_{i\notin\mathcal{I}} \left(4^{1/\alpha} \wedge \frac{1/2}{|x^{(i)}-y^{(i)}|^{1+\alpha}}\right) \\
&\le c_6 \delta_D(x)^{\alpha/2} p(1/2, x, y).
\end{aligned}$$

It remains to estimate I_3. Note that for all $u \in U_1$, we have $|u^{(k)} - x^{(k)}| < 2r_0$ for $k = 1, \cdots, d$. Hence, for all $i \notin \mathcal{I}$ and $u \in U_1$, by definition of U_3, it is not possible that there exists $a \in \mathbb{R}$ such that $[u]_a^i \in U_3$. In this case,

$$\sup_{u\in U_1, [u]_a^i \in U_3} j(u^{(i)}, a) = 0. \tag{4.7}$$

On the other hand, for any $i \in \mathcal{I}$ and $u \in U_1$, if $[u]_a^i \in U_3$ for some $a \in \mathbb{R}$, then the number a must satisfy $|a - x^{(i)}| > |x^{(i)} - y^{(i)}|/2 > 2r_0$, and so

$$|a - u^{(i)}| \ge |a - x^{(i)}| - |x^{(i)} - u^{(i)}| \ge |x^{(i)} - y^{(i)}|/2 - r_0 \ge |x^{(i)} - y^{(i)}|/4.$$

In this case,

$$\sup_{u\in U_1, [u]_a^i \in U_3} j(u^{(i)}, a) \le \sup_{u\in U_1, [u]_a^i \in U_3} \frac{\mathcal{C}_{1,\alpha}}{|a-u^{(i)}|^{1+\alpha}} \le \frac{4^{1+\alpha}\mathcal{C}_{1,\alpha}}{|x^{(i)}-y^{(i)}|^{1+\alpha}} \le c_7 \left(2^{1/\alpha} \wedge \frac{1/2}{|x^{(i)}-y^{(i)}|^{1+\alpha}}\right), \tag{4.8}$$

and, for $k \in \mathcal{I}$ with $k \ne i$,

$$|u^{(k)} - y^{(k)}| \ge |x^{(k)} - y^{(k)}| - |u^{(k)} - x^{(k)}| \ge |x^{(k)} - y^{(k)}| - 2r_0 \ge |x^{(k)} - y^{(k)}|/2. \tag{4.9}$$

Hence,

$$\begin{aligned}
I_3 &= \int_0^{1/4} \int_{U_1} p_{U_1}(s, x, u) \left(\sum_{i=1}^d \int_{\mathbb{R}} p_D(1/2 - s, [u]_\theta^i, y) d\theta \cdot \sup_{u\in U_1, [u]_a^i \in U_3} j(u^{(i)}, a)\right) duds \\
&\le \int_{1/4}^{1/2} \int_{U_1} p_{U_1}(1/2 - s, x, u) \left(\sum_{i\in\mathcal{I}} \int_{\mathbb{R}} p(s, [u]_\theta^i, y) d\theta \cdot \sup_{u\in U_1, [u]_a^i \in U_3} j(u^{(i)}, a)\right) duds \quad \text{(by (4.7))}
\end{aligned}$$

$$
\begin{aligned}
&\le c_7 C_1 \int_{1/4}^{1/2}\int_{U_1} p_{U_1}(1/2-s,x,u)\Bigg(\sum_{i\in\mathcal{I}} \sup_{\frac14<s<\frac12} \prod_{\substack{k\in\mathcal{I}\\ k\neq i}}\left(s^{-1/\alpha}\wedge \frac{s}{|u^{(k)}-y^{(k)}|^{1+\alpha}}\right)\prod_{k\notin\mathcal{I}}\left(s^{-1/\alpha}\right)\\
&\quad\cdot \int_{\mathbb{R}}\left(s^{-1/\alpha}\wedge\frac{s}{|\theta-y^{(i)}|^{1+\alpha}}\right)d\theta\cdot\left(2^{1/\alpha}\wedge\frac{1/2}{|x^{(i)}-y^{(i)}|^{1+\alpha}}\right)\Bigg)duds \quad \text{(by (4.8))}\\
&\le c_8 \int_{1/4}^{1/2}\int_{U_1} p_{U_1}(1/2-s,x,u)\Bigg(\sum_{i\in\mathcal{I}} \prod_{\substack{k\in\mathcal{I}\\ k\neq i}}\left(4^{1/\alpha}\wedge \frac{2^{1+\alpha}(1/2)}{|x^{(k)}-y^{(k)}|^{1+\alpha}}\right)\prod_{k\notin\mathcal{I}}\left(4^{1/\alpha}\right) \quad \text{(by (4.9))}\\
&\quad\cdot\left(2^{1/\alpha}\wedge\frac{1/2}{|x^{(i)}-y^{(i)}|^{1+\alpha}}\right)\Bigg)duds\\
&\le c_8 \int_{0}^{1/4}\int_{U_1} p_{U_1}(s,x,u)du\, ds\Bigg(\sum_{i\in\mathcal{I}} \prod_{\substack{k\in\mathcal{I}\\ k\neq i}}\left(4^{1/\alpha}\wedge \frac{2^{1+\alpha}(1/2)}{|x^{(k)}-y^{(k)}|^{1+\alpha}}\right)\prod_{k\notin\mathcal{I}}\left(4^{1/\alpha}\wedge\frac{1/2}{|x^{(k)}-y^{(k)}|^{1+\alpha}}\right)\\
&\quad\cdot\left(2^{1/\alpha}\wedge\frac{1/2}{|x^{(i)}-y^{(i)}|^{1+\alpha}}\right)\Bigg)\\
&\le c_9\int_0^{1/4}\mathbb{P}_x\left(\tau_{U_1}>s\right)ds\cdot p(1/2,x,y) \quad \text{(by the lower bound in (1.6))}\\
&\le c_9\left(1\wedge \mathbb{E}_x\left[\tau_{U_1}\right]\right)\cdot p(1/2,x,y)\le c_9\left(1\wedge\delta_D(x)^{\alpha/2}\right)\cdot p(1/2,x,y) \quad \text{(by (3.21))}.
\end{aligned}
$$

Combining (4.6) and the estimates of I_1, I_2, I_3, we finish the case 2, and then this lemma. □

Lemma 4.4. *There is a constant $c = c(d, \alpha, R, \Lambda) > 0$ such that for all $x, y \in D$,*

$$
p_D(1,x,y)\le c\left(1\wedge\delta_D(x)^{\alpha/2}\right)\left(1\wedge\delta_D(y)^{\alpha/2}\right)p(1,x,y). \tag{4.10}
$$

Proof. By the semigroup property of p_D and (4.5), since $p_D(t, x, y)$ is symmetric in x, y, we have

$$
\begin{aligned}
p_D(1,x,y) &= \int_D p_D(1/2,x,z)p_D(1/2,z,y)dz\\
&\le c\left(1\wedge\delta_D(x)^{\alpha/2}\right)\int_{\mathbb{R}^d}p(1/2,x,z)p(1/2,z,y)dz\left(1\wedge\delta_D(y)^{\alpha/2}\right)\\
&= c\left(1\wedge\delta_D(x)^{\alpha/2}\right)\left(1\wedge\delta_D(y)^{\alpha/2}\right)p(1,x,y).
\end{aligned}
$$

□

Lemma 4.5. *There is a constant $c = c(d, \alpha, R, \Lambda) > 0$ such that for all $t \in (0, 1]$, $x, y \in D$,*

$$
p_D(t,x,y)\le c\left(1\wedge\frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}}\right)\left(1\wedge\frac{\delta_D(y)^{\alpha/2}}{\sqrt{t}}\right)p(t,x,y).
$$

Proof. Note that for $t \in (0, 1]$, $D_t := t^{-1/\alpha}D = \{t^{-1/\alpha}z : z \in D\}$ is also a $C^{1,1}$ open set with the same characteristics of D. Hence, by scaling properties (2.6) and (2.5), and applying (4.10) for D_t, we obtain for all $x, y \in D$

$$
\begin{aligned}
p_D(t,x,y) &= t^{-d/\alpha}p_{D_t}(1,t^{-1/\alpha}x,t^{-1\alpha}y)\\
&\le ct^{-d/\alpha}\left(1\wedge\delta_{D_t}(t^{-1/\alpha}x)^{\alpha/2}\right)\left(1\wedge\delta_{D_t}(t^{-1/\alpha}y)^{\alpha/2}\right)p(1,t^{-1/\alpha}x,t^{-1/\alpha}y)\\
&= c\left(1\wedge\frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}}\right)\left(1\wedge\frac{\delta_D(y)^{\alpha/2}}{\sqrt{t}}\right)p(t,x,y).
\end{aligned}
$$

□

4.2. **Lower bound estimates.** The following near diagonal lower estimate of p_B for balls B is also called *localized lower estimate* in some literatures (cf. [17, 18]).

Lemma 4.6. *Let $B := B(x_0, r)$ be a ball with radius $r > 0$. For any $a_1 > 0$, there exists $a_2 := a_2(d, \alpha, a_1) \in (0, 1/a_1)$ such that for any $t^{1/\alpha} \le a_1 r$,*

$$p_B(t, x, y) \ge ct^{-d/\alpha}, \qquad x, y \in B(x_0, a_2 t^{1/\alpha}),$$

where $c = c(d, \alpha, a_1) > 0$.

Proof. **Step 1.** We first show that for any $x, y \in B$ and $t > 0$,

$$p(t, x, y) \le p_B(t, x, y) + 2 \sup_{s\in(t/2,t]} \sup_{\substack{z\in\{x,y\} \\ w\in B^c}} p(s, z, w). \tag{4.11}$$

By the Markov property of X, (2.1), the symmetry of $p(t, x, y)$ and $p_B(t, x, y)$ in x and y, and Theorem 2.5, we have for any $x, y \in B$,

$$\begin{aligned}
p(2t, x, y) &= \int_{\mathbb{R}^d} p(t, x, z)p(t, z, y)dz \\
&= \int_{\mathbb{R}^d} p_B(t, x, z)p(t, z, y)dz + \int_{\mathbb{R}^d} \mathbb{E}_x \left[p(t - \tau_B, X_{\tau_B}, z); \tau_B < t\right] p(t, z, y)dz \\
&= \int_B p(t, y, z)p_B(t, z, x)dz + \mathbb{E}_x \left[p(2t - \tau_B, X_{\tau_B}, y); \tau_B < t\right] \\
&= \int_B p_B(t, y, z)p_B(t, z, x)dz + \int_B \mathbb{E}_y \left[p(t - \tau_B, X_{\tau_B}, z); \tau_B < t\right] p_B(t, z, x)dz \\
&\quad + \mathbb{E}_x \left[p(2t - \tau_B, X_{\tau_B}, y); \tau_B < t\right] \\
&\le \int_B p_B(t, y, z)p_B(t, z, x)dz + \int_{\mathbb{R}^d} \mathbb{E}_y \left[p(t - \tau_B, X_{\tau_B}, z); \tau_B < t\right] p(t, z, x)dz \\
&\quad + \mathbb{E}_x \left[p(2t - \tau_B, X_{\tau_B}, y); \tau_B < t\right] \\
&= p_B(2t, y, x) + \mathbb{E}_y \left[p(2t - \tau_B, X_{\tau_B}, x); \tau_B < t\right] \\
&\quad + \mathbb{E}_x \left[p(2t - \tau_B, X_{\tau_B}, y); \tau_B < t\right] \\
&\le p_B(2t, x, y) + \sup_{s\in(t,2t]} \sup_{z\in B^c} p(s, z, x) + \sup_{s\in(t,2t]} \sup_{z\in B^c} p(s, z, y).
\end{aligned}$$

This establishes the claim (4.11) after replacing $2t$ by t.

Step 2. We next show that there exists $a := a(d, \alpha) \in (0, 1)$ and $c_1 = c_1(d, \alpha) > 0$ such that for any $t^{1/\alpha} \le ar$,

$$p_B(t, x, y) \ge c_1 t^{-d/\alpha}, \qquad x, y \in B(x_0, t^{1/\alpha}/2) \subset B. \tag{4.12}$$

Indeed, we have by (4.11) that for all $x, y \in B(x_0, t^{1/\alpha}/2)$,

$$|x^{(i)} - y^{(i)}| \le |x - y| \le t^{1/\alpha}, \quad i = 1, 2, \cdots, d,$$

and then, by (1.6),

$$p(t, x, y) \ge C_1^{-1} \prod_{i=1}^d \left(t^{-1/\alpha} \wedge \frac{t}{(t^{1/\alpha})^{1+\alpha}}\right) = C_1^{-1} t^{-d/\alpha}. \tag{4.13}$$

On the other hand, for all $t^{1/\alpha} \le ar$ (where a is to be determined later), $z \in \{x, y\}$ and $w \in B^c$, we have

$$|z - w| \ge |x_0 - w| - |x_0 - z| \ge r - \frac{t^{1/\alpha}}{2} \ge \frac{r}{2} \ge \frac{t^{1/\alpha}}{2a},$$

and then, there exists $k \in \{1, 2, \cdots, d\}$ so that

$$|z^{(k)} - w^{(k)}| \ge \frac{r}{2\sqrt{d}} \ge \frac{t^{1/\alpha}}{2\sqrt{d}a}.$$

Consequently,

$$\begin{aligned}\sup_{s\in[t/2,t)}\sup_{z\in\{x,y\},w\in B^c}p(s,z,w)&\le C_1\left(\prod_{\substack{i=1\\i\neq k}}^{d}s^{-1/\alpha}\right)\left(\frac{s}{|z^{(k)}-w^{(k)}|^{1+\alpha}}\right)\\&\le C_1 2^{(d-1)/\alpha}t^{-(d-1)/\alpha}\left(\frac{t}{(t^{1/\alpha}/(2\sqrt{d}a))^{1+\alpha}}\right)\\&=c_2a^{1+\alpha}t^{-d/\alpha},\end{aligned}\tag{4.14}$$

where $c_2:=2^{(d-1)/\alpha+1+\alpha}C_1d^{(1+\alpha)/2}>0$. Combining (4.11), (4.13) and (4.14), we obtain

$$p_B(t,x,y)\ge(C_1^{-1}-2c_2a^{1+\alpha})t^{-d/\alpha}.$$

Setting $a:=(4C_1c_2)^{-1/(1+\alpha)}$, we obtain (4.12) with $c_1=(2C_1)^{-1}$.

Step 3. When $a_1\le a$, this lemma follows directly from (4.12) with $c=c_1$ and $a_2=\frac{1}{2}$. So it suffices to consider the case that $a_1>a$. Let

$$n:=\left[\left(\frac{a_1}{a}\right)^{\alpha}\right]+1,\quad\text{ and }\quad a_2:=\frac{1}{2n^{1/\alpha}}.$$

For all $t^{1/\alpha}\le a_1r$, we have

$$\left(\frac{t}{n}\right)^{1/\alpha}\le\frac{a_1r}{n^{1/\alpha}}\le\frac{a_1r}{((a_1/a)^{\alpha})^{1/\alpha}}=ar<r,$$

and then,

$$B(x_0,a_2t^{1/\alpha})=B(x_0,(t/n)^{1/\alpha}/2)\subset B.$$

Hence, by Step 2 and semigroup property of $p_B(t,x,y)$, we have for all $t^{1\alpha}\le a_1r$ and $x,y\in B(x_0,a_2t^{1/\alpha})$,

$$\begin{aligned}p_B(t,x,y)=&\int_{B^{n-1}}p_B(t/n,x,z_1)p_B(t/n,z_1,z_2)\cdots p_B(t/n,z_{d-1},y)dz_1dz_2\cdots dz_{d-1}\\\ge&\int_{B(x_0,a_2t^{1/\alpha})^{n-1}}p_B(t/n,x,z_1)p_B(t/n,z_1,z_2)\cdots p_B(t/n,z_{d-1},y)dz_1dz_2\cdots dz_{d-1}\\\ge&\left(\prod_{i=1}^{n}c_1\,(t/n)^{-d/\alpha}\right)|B(x_0,a_2t^{1/\alpha})|^{n-1}\\=&\,c_1^nn^{dn/\alpha}t^{-dn/\alpha}\cdot|B(0,1)|^{n-1}(a_2t^{1/\alpha})^{d(n-1)}\\=:&\,ct^{-d/\alpha},\end{aligned}$$

where $c=c_1^nn^{dn/\alpha}|B(0,1)|^{n-1}a_2^{d(n-1)}>0$. This completes the proof of the lemma. □

Lemma 4.7. *Let $U\subset\mathbb{R}^d$ be a non-empty open set. For any $a_1>0$, there exists $a_2:=a_2(d,\alpha,,a_1)>0$, $c=c(d,\alpha,a_1)>0$ such that for all $t>0$ and $x,y\in U$ with $\delta_U(x)\wedge\delta_U(y)\ge a_1t^{1/\alpha}$ and $|x-y|<a_2t^{1/\alpha}$, we have*

$$p_U(t,x,y)\ge ct^{-d/\alpha}.$$

Proof. Fix $t>0$ and let $r:=a_1t^{1/\alpha}$. Then, we have for $x\in U$ with $\delta_U(x)\ge a_1t^{1/\alpha}=r$,

$$t^{1/\alpha}\le a_1^{-1}r\quad\text{ and }\quad B:=B(x,r)\subset U.$$

By Lemma 4.6 (with a_1 being replaced by a_1^{-1}), there exists $a_2:=a_2(d,\alpha,a_1)\in(0,a_1)$ and $c=c(d,\alpha,a_1)>0$ such that

$$p_B(t,z,w)\ge ct^{-d/\alpha},\quad z,w\in B(x,a_2t^{1/\alpha}).\tag{4.15}$$

On the other hand, by shrinking a_2 if necessary, we have for all $x,y\in U$ with $\delta_U(x)\wedge\delta_U(y)\ge a_1t^{1/\alpha}$ and $|x-y|<a_2t^{1/\alpha}$,

$$y\in B(x,a_2t^{1/\alpha})\quad\text{ and }\quad B(x,a_2t^{1/\alpha})\subset B(x,r)\subset U.$$

Consequently, by (4.15), we have $p_U(t,x,y)\ge p_B(t,x,y)\ge ct^{-d/\alpha}$. □

The following lemma gives the so-called *survival estimate* in some literatures (cf. [17]). Recall that $\tau_{B(x,r)}$ is the first exit time of X for the ball $B(x,r)$, and $\tau_{Q(x,r)}$ is the first exit time of X for the cube $Q(x,r)$.

Lemma 4.8. *For any $t>0$, $r>0$ and $x\in\mathbb{R}^d$, there exists $c=c(d,\alpha)>0$ such that*

$$\mathbb{P}_x\left(\tau_{B(x,r)}\le t\right)\le \frac{ct}{r^\alpha} \tag{4.16}$$

and

$$\mathbb{P}_x\left(\tau_{Q(x,r)}\le t\right)\le \frac{ct}{r^\alpha}. \tag{4.17}$$

Proof. Fix $t>0$, $r>0$ and $x\in\mathbb{R}^d$. By the strong Markov property (cf. [4, p. 43-44]), one can prove that

$$\mathbb{P}_x\left(\tau_{B(x,r)}\le t\right)\le 2\sup_{s\le t}\sup_{y\in\mathbb{R}^d}\mathbb{P}_y\left(|X_s-X_0|\ge r/2\right)$$

(see also [10, (3.1), p. 2494]). By the scaling property (2.5), we have

$$\begin{aligned}\mathbb{P}_y\left(|X_s-X_0|\ge r/2\right)&=\int_{|y-z|\ge r/2}p(s,y,z)dz=\int_{|z|\ge r/2}p(s,0,z)dz\\&=s^{-d/\alpha}\int_{|z|\ge r/2}p(1,0,s^{-1/\alpha}z)dz=\int_{|z|\ge r/(2s^{1/\alpha})}p(1,0,z)dz.\end{aligned}$$

Noting that for any $z=(z^{(1)},z^{(2)},\cdots,z^{(d)})$ with $|z|\ge r/(2s^{1/\alpha})$, there exists some $1\le k\le d$ such that $|z^{(k)}|\ge r/(2\sqrt{d}s^{1/\alpha})$, and then,

$$\{|z|\ge r/(2s^{1/\alpha})\}\subset\bigcup_{k=1}^d\{|z^{(k)}|\ge r/(2\sqrt{d}s^{1/\alpha})\}.$$

Consequently, by (1.5) and (1.4)

$$\begin{aligned}\mathbb{P}_y\left(|X_s-X_0|\ge r/2\right)&\le\sum_{k=1}^d\int_{|z^{(k)}|\ge r/(2\sqrt{d}s^{1/\alpha})}p^{(1,\alpha)}(1,0,z^{(k)})dz^{(k)}\\&\le c_1(d,\alpha)\int_{|\theta|\ge r/(2\sqrt{d}s^{1/\alpha})}\frac{d\theta}{|\theta|^{1+\alpha}}=\frac{(2\sqrt{d})^\alpha c_1}{\alpha}\frac{s}{r^\alpha}.\end{aligned}$$

Combining the above three formulas, we obtain the first inequality (4.16).

Finally, since $B(x,r)\subset Q(x,r)$, we have $\{\tau_{Q(x,r)}\le t\}\subset\{\tau_{B(x,r)}\le t\}$ for all $t>0$. Then, the second inequality (4.17) follows from this and (4.16). □

Lemma 4.9. *Assume that U is an open set satisfying condition $(\mathbf{H}_\gamma)$ for some $\gamma\in(0,1]$. For any $a_1>0$, there exists $c=c(d,\alpha,\gamma,a_1)>0$ such that for all $t>0$ and $x,y\in U$ with $\delta_U(x)\wedge\delta_U(y)\ge a_1t^{1/\alpha}$, we have*

$$p_U(t,x,y)\ge cp(t,x,y). \tag{4.18}$$

Proof. Fix $t>0$ and $x,y\in U$ with $\delta_U(x)\wedge\delta_U(y)\ge a_1t^{1/\alpha}$. By Lemma 4.7, there exists $a_2>0$ and $c_1=c_1(d,\alpha)>0$ such that if $|x-y|<a_2t^{1/\alpha}$, then $p_U(t,x,y)\ge c_1t^{-d/\alpha}$, which together with (1.6) yields (4.18). It remains to consider the case when $|x-y|\ge a_2t^{1/\alpha}$. Without loss of generality, we may assume that $a_2<a_1$.

Step 1. Let $a\in(0,a_2/2]$ and $\delta\in(0,1)$ be determined later. By semigroup property of $p_U(t,x,y)$, we have

$$\begin{aligned}p_U(t,x,y)&=\int_U p_U(\delta t,x,z)p_U((1-\delta)t,z,y)dz\\&\ge\int_{B(y,at^{1/\alpha})}p_U(\delta t,x,z)p_U((1-\delta)t,z,y)dz\\&\ge\inf_{z\in B(y,at^{1/\alpha})}p_U((1-\delta)t,z,y)\mathbb{P}_x\left(X^U_{\delta t}\in B(y,at^{1/\alpha})\right).\end{aligned}\tag{4.19}$$

Note that for all $z\in B(y,at^{1/\alpha})$,

$$\delta_U(z)\wedge\delta_U(y)\ge(a_1-a)t^{1/\alpha}\ge\frac{a_1}{2}t^{1/\alpha}.$$

Hence, by Lemma 4.7, there exists $a_3 \in (0, a_2)$ such that if

$$|z-y| < at^{1/\alpha} < a_3((1-\delta)t)^{1/\alpha}, \tag{4.20}$$

then,

$$\inf_{z\in B(y,at^{1/\alpha})} p_U((1-\delta)t, z, y) \geq c_2((1-\delta)t)^{-d/\alpha} = c_2(1-\delta)^{-d/\alpha} t^{-d/\alpha},$$

for some $c_2 = c_2(d, \alpha, a_1) > 0$. Consequently, under condition (4.20), the above inequality together with (4.19) implies

$$p_U(t,x,y) \geq c_2(1-\delta)^{-d/\alpha} t^{-d/\alpha} \mathbb{P}_x \left(X^U_{\delta t} \in B(y, at^{1/\alpha}) \right). \tag{4.21}$$

Note that, inequality (4.20) can be achieved by choosing a, δ small enough such that

$$a < a_3(1-\delta)^{1/\alpha}. \tag{4.22}$$

Step 2 We next derive a lower bound of $\mathbb{P}_x \left(X^U_{\delta t} \in B(y, at^{1/\alpha}) \right)$. Since U satisfies condition $(\mathbf{H}_\gamma)$ and $\delta_U(x) \wedge \delta_U(y) \geq a_1 t^{1/\alpha}$, there exists a permutation $\{i_1, i_2, \cdots, i_d\}$ of $\{1, 2, \cdots, d\}$, such that

$$B(\overline{xy}_k, \gamma a_1 t^{1/\alpha}) \subset U, \qquad k = 1, 2, \cdots, d, \tag{4.23}$$

where $\overline{xy}_1 := [x]^{i_1}_{y^{(i_1)}}$, $\overline{xy}_2 := [\overline{xy}_1]^{i_2}_{y^{(i_2)}}, \cdots, \overline{xy}_d := [\overline{xy}_{d-1}]^{i_d}_{y^{(i_d)}} = y$. Set $r := at^{1/\alpha}/\sqrt{d}$, where a is chosen to be small enough such that

$$a < \gamma a_1/2. \tag{4.24}$$

Then $Q_0 := Q(x, r) \subset U$ and

$$Q_k := Q(\overline{xy}_k, r) \subset B_k := B(\overline{xy}_k, \sqrt{d} r) \subset U, \qquad k = 1, 2, \cdots, d. \tag{4.25}$$

In the rest of the proof, for a number $\lambda > 0$ and a cube $Q := Q(z, r)$, we use the notation λQ to denote the set $Q(z, \lambda r)$, that is $\lambda Q(z, r) = Q(z, \lambda r)$. By semigroup property, we have

$$\begin{aligned}
&\mathbb{P}_x \left(X^U_{\delta t} \in B(y, at^{1/\alpha}) \right) = \int_{B(y,at^{1/\alpha})} p_U(\delta t, x, z_d) dz_d \\
&= \int_{B(y,at^{1/\alpha})} \left(\int_{U^{d-1}} p_U(\delta t/d, x, z_1) p_U(\delta t/d, z_1, z_2) \cdots p_U(\delta t/d, z_{d-1}, z_d) dz_1 \cdots dz_{d-1} \right) dz_d \\
&\geq \int_{Q_d} \int_{(2^{1-d}Q_1)\times(2^{2-d}Q_2)\times\cdots\times(2^{-1}Q_{d-1})} \\
&\qquad p_U(\delta t/d, x, z_1) p_U(\delta t/d, z_1, z_2) \cdots p_U(\delta t/d, z_{d-1}, z_d) dz_1 \cdots dz_{d-1} dz_d \\
&\geq \mathbb{P}_x \left(X^U_{\delta t/d} \in 2^{1-d} Q_1 \right) \inf_{z_1 \in 2^{1-d}Q_1} \mathbb{P}_{z_1} \left(X^U_{\delta t/d} \in 2^{2-d} Q_2 \right) \\
&\qquad \times \cdots \times \inf_{z_{d-2}\in 2^{-2}Q_{d-2}} \mathbb{P}_{z_{d-2}} \left(X^U_{\delta t/d} \in 2^{-1} Q_{d-1} \right) \inf_{z_{d-1}\in 2^{-1}Q_{d-1}} \mathbb{P}_{z_{d-1}} \left(X^U_{\delta t/d} \in Q_d \right) \\
&\geq \inf_{z_0\in 2^{-d}Q_0} \mathbb{P}_{z_0} \left(X^U_{\delta t/d} \in 2^{1-d} Q_1 \right) \inf_{z_1 \in 2^{1-d}Q_1} \mathbb{P}_{z_1} \left(X^U_{\delta t/d} \in 2^{2-d} Q_2 \right) \\
&\qquad \times \cdots \times \inf_{z_{d-2}\in 2^{-2}Q_{d-2}} \mathbb{P}_{z_{d-2}} \left(X^U_{\delta t/d} \in 2^{-1} Q_{d-1} \right) \inf_{z_{d-1}\in 2^{-1}Q_{d-1}} \mathbb{P}_{z_{d-1}} \left(X^U_{\delta t/d} \in Q_d \right).
\end{aligned} \tag{4.26}$$

Step 3. We estimate the lower bound of each term on the right hand side of (4.26). In fact, they can be estimated similarly. We claim that for each $1 \leq k \leq d$,

$$\inf_{z\in 2^{k-1-d}Q_{k-1}} \mathbb{P}_z \left(X^U_{\delta t/d} \in 2^{k-d} Q_k \right) \geq c_3 \left(1 \wedge \frac{t^{1+1/\alpha}}{|y^{(i_k)} - x^{(i_k)}|^{1+\alpha}} \right), \tag{4.27}$$

where the constant $c_3 > 0$ is independent of t, x, y.

Fix $z \in 2^{(k-1)-d} Q_{k-1}$. Let $\widehat{Q} := Q(z, r/2^{d-(k-1)})$ and

$$\begin{aligned}
\widetilde{Q} &:= \widehat{Q} + (y^{(i_k)} - x^{(i_k)}) e_{i_k} \\
&= \left\{ w \in \mathbb{R}^d : |w^{(i)} - z^{(i)}| < \frac{r}{2^{d-(k-1)}} \text{ for } i \neq i_k; |w^{(i_k)} - (z^{(i_k)} + y^{(i_k)} - x^{(i_k)})| < \frac{r}{2^{d-(k-1)}} \right\}.
\end{aligned}$$

Then, we have $\widehat{Q} \subset 2^{k-d}Q_{k-1}$ and $\widetilde{Q} \subset 2^{k-d}Q_k$. Indeed, since $z \in 2^{(k-1)-d}Q_{k-1}$ and

$$2^{k-d}Q_k = 2^{k-d}Q_{k-1} + (y^{(i_k)} - x^{(i_k)})e_{i_k},$$

we have, for any $w \in \widetilde{Q}$ and $i \neq i_k$,

$$\begin{aligned}|w^{(i)} - \overline{xy}_k^{(i)}| \le |w^{(i)} - z^{(i)}| + |z^{(i)} - \overline{xy}_k^{(i)}| &= |w^{(i)} - z^{(i)}| + |z^{(i)} - \overline{xy}_{k-1}^{(i)}| \\ &< \frac{r}{2^{d-(k-1)}} + \frac{r}{2^{d-(k-1)}} = \frac{r}{2^{d-k}},\end{aligned}$$

and for i_k,

$$\begin{aligned}|w^{(i_k)} - \overline{xy}_k^{(i_k)}| = |w^{(i_k)} - y^{(i_k)}| &\le |w^{(i_k)} - (z^{(i_k)} + y^{(i_k)} - x^{(i_k)})| + |z^{(i_k)} - x^{(i_k)}| \\ &= |w^{(i_k)} - (z^{(i_k)} + y^{(i_k)} - x^{(i_k)})| + |z^{(i_k)} - \overline{xy}_{k-1}^{(i_k)}| \\ &< \frac{r}{2^{d-(k-1)}} + \frac{r}{2^{d-(k-1)}} = \frac{r}{2^{d-k}}.\end{aligned}$$

This shows that $\widetilde{Q} \subset 2^{k-d}Q_k$. Since $\delta_U(z) \wedge \delta_U(w) \ge \frac{\gamma a_1}{2} t^{1/\alpha}$ for all $w \in \widetilde{Q}$ by (4.23), (4.25) and (4.24), it follows from Lemma 4.7 that there exists $a_4 \in (0, \frac{1}{2})$ such that for all $|z - w| < a_4(\delta t/d)^{1/\alpha}$,

$$p_U(\delta t/d, z, w) \ge c_4 \, (\delta t/d)^{-d/\alpha}. \tag{4.28}$$

Let c_5 be the constant in (4.17), and choose a, δ small enough such that

$$(c_5 \vee 1)\frac{\delta}{d} < \left(\frac{a}{2^{d+2}\sqrt{d}}\right)^{\alpha}. \tag{4.29}$$

Let $a_5 := a_4 \, (\delta/d)^{1/\alpha}$.

Case 1. $|y^{(i_k)} - x^{(i_k)}| \le a_5 t^{1/\alpha}$. In this case, by (4.29),

$$|y^{(i_k)} - x^{(i_k)}| \le a_5 t^{1/\alpha} = a_4 \left(\frac{\delta}{d}\right)^{1/\alpha} t^{1/\alpha} \le \frac{1}{2} \cdot \frac{a t^{1/\alpha}}{2^{d+2}\sqrt{d}} < \frac{r}{2^{d-k+2}}.$$

Hence, by the definitions of $\widehat{Q}$ and $\widetilde{Q}$, we have $B(z, a_5 t^{1/\alpha}) \subset 2^{-1}\widehat{Q} \subset \widehat{Q} \cap \widetilde{Q}$; see Figure 5.

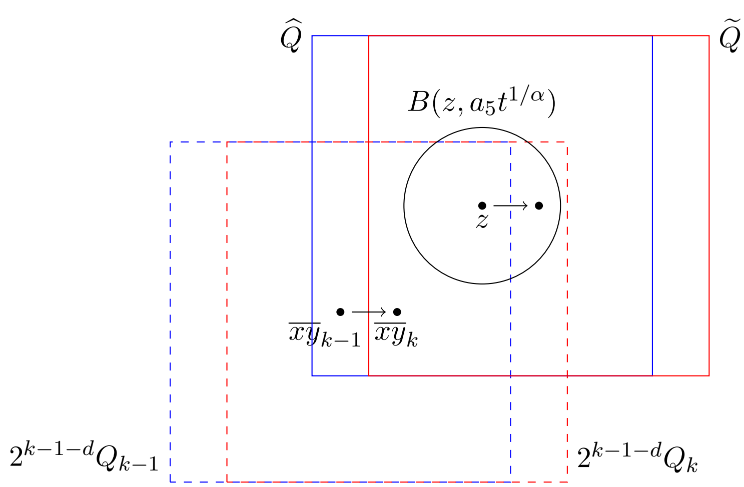

FIGURE 5. The case when $|y^{(i_k)} - x^{(i_k)}| \le a_5 t^{1/\alpha}$

Consequently, by (4.28), we have

$$\begin{aligned}\mathbb{P}_z\left(X^U_{\delta t/d} \in 2^{k-d}Q_k\right) \ge \mathbb{P}_z\left(X^U_{\delta t/d} \in \widetilde{Q}\right) &= \int_{\widetilde{Q}} p_U(\delta t/d, z, w)dw \\ &\ge \int_{B(z, a_5 t^{1/\alpha})} p_U(\delta t/d, z, w)dw \ge c_4 \, (\delta t/d)^{-d/\alpha} \, |B(z, a_5 t^{1/\alpha})| \\ &= c_4 \, (\delta/d)^{-d/\alpha} \, a_5^d |B(0,1)|\end{aligned}$$

$$\geq c_4 \left(\frac{\delta}{d}\right)^{-d/\alpha} a_5^d |B(0,1)| \left(1 \wedge \frac{t^{1+1/\alpha}}{|y^{(i_k)} - x^{(i_k)}|^{1+\alpha}}\right),$$

which is (4.27).

Case 2. $|y^{(i_k)} - x^{(i_k)}| > a_5 t^{1/\alpha}$. Without loss of generality, we may and do assume that $y^{(i_k)} > x^{(i_k)}$. In this case, let

$$A := \left\{\theta \in \mathbb{R} : \left(\frac{r}{2^{d-k+2}} - |y^{(i_k)} - x^{(i_k)}|\right) \vee \left(-\frac{r}{2^{d-k+2}}\right) < \theta - (z^{(i_k)} + y^{(i_k)} - x^{(i_k)}) < \frac{r}{2^{d-k+2}}\right\},$$

and, for any $w \in 2^{-1}\widetilde{Q} \setminus \overline{\left(2^{-1}\widehat{Q}\right)}$, we have

$$2^{-1}\widetilde{Q} \setminus \overline{\left(2^{-1}\widehat{Q}\right)} = \left\{w \in \mathbb{R}^d : |w^{(i)} - z^{(i)}| < \frac{r}{2^{d-k+2}} \text{ for } i \neq i_k; w^{(i_k)} \in A\right\}. \tag{4.30}$$

See Figure 6.

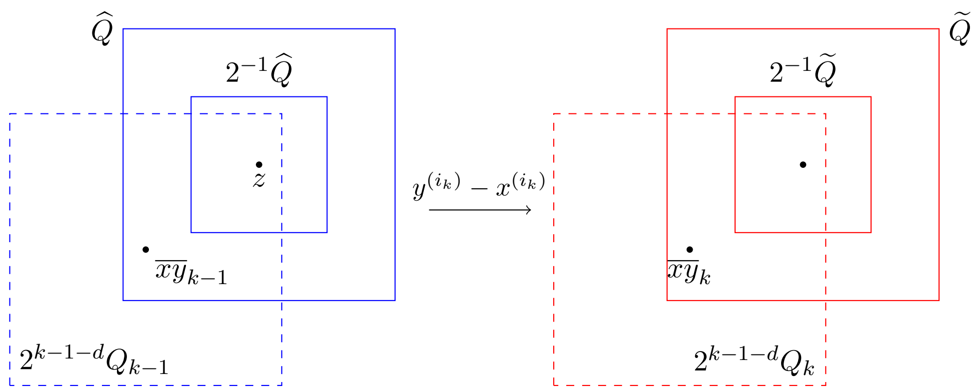

FIGURE 6. The case when $|y^{(i_k)} - x^{(i_k)}| > a_5 t^{1/\alpha}$

We are to apply (4.1) to estimate $\mathbb{P}_z(X^U_{\delta t/d} \in 2^{k-d}Q_k)$. Indeed, by (4.17) and (4.29), we have

$$\mathbb{P}_z\left(\tau_{2^{-1}\widehat{Q}} \leq \delta t/d\right) \leq \frac{c_5 \delta t}{d} \frac{1}{(r/2^{d-k+2})^\alpha} < \left(\frac{a}{2^{d+2}\sqrt{d}}\right)^\alpha \cdot t \cdot \frac{2^{(d-k+2)\alpha}\sqrt{d}^\alpha}{a^\alpha t} < \frac{1}{2^\alpha} < 1. \tag{4.31}$$

This implies that

$$\mathbb{E}_z\left[\frac{\delta t}{2d} \wedge \tau_{2^{-1}\widehat{Q}}\right] \geq \frac{\delta t}{2d}\mathbb{P}_z\left(\tau_{2^{-1}\widehat{Q}} > \frac{\delta t}{2d}\right) \geq \frac{(1-2^{-\alpha})\delta t}{2d}. \tag{4.32}$$

Denote by $\sigma^U_{2^{-1}\widetilde{Q}}$ the first hitting time of X^U for the set $\frac{1}{2}\widetilde{Q}$:

$$\sigma^U_{2^{-1}\widetilde{Q}} := \inf\{s > 0 : X^U_s \in 2^{-1}\widetilde{Q}\}.$$

Since $(2^{-1}\widehat{Q}) \cup (2^{-1}\widetilde{Q}) \subset U$, the above inequality together with (4.1) yields that

$$\mathbb{P}_z\left(\sigma^U_{2^{-1}\widetilde{Q}} < \delta t/d\right) \geq \mathbb{P}_z\left(X^U_{(\frac{\delta t}{2d})\wedge\tau^U_{2^{-1}\widehat{Q}}} \in 2^{-1}\widetilde{Q} \setminus \overline{\left(2^{-1}\widehat{Q}\right)}\right) = \mathbb{P}_z\left(X_{(\frac{\delta t}{2d})\wedge\tau_{2^{-1}\widehat{Q}}} \in 2^{-1}\widetilde{Q} \setminus \overline{\left(2^{-1}\widehat{Q}\right)}\right)$$

$$= \mathbb{E}_z\left[\int_0^{(\frac{\delta t}{2d})\wedge\tau_{2^{-1}\widehat{Q}}} \sum_{i=1}^d \int_{\mathbb{R}^d} \mathbb{1}_{2^{-1}\widetilde{Q}\setminus\overline{(2^{-1}\widehat{Q})}}(X_s^{(1)}, X_s^{(2)}, \cdots, X_s^{(i-1)}, w^{(i)}, X_s^{(i+1)}, \cdots, X_s^{(d)}) j(X_s^{(i)}, w^{(i)}) dw^{(i)} ds\right]$$

$$\geq \mathbb{E}_z\left[\int_0^{(\frac{\delta t}{2d})\wedge\tau_{2^{-1}\widehat{Q}}} \int_{\mathbb{R}} \mathbb{1}_{2^{-1}\widetilde{Q}\setminus\overline{(2^{-1}\widehat{Q})}}(X_s^{(1)}, X_s^{(2)}, \cdots, X_s^{(i_k-1)}, w^{(i_k)}, X_s^{(i_k+1)}, \cdots, X_s^{(d)}) j(X_s^{(i_k)}, w^{(i_k)}) dw^{(i_k)} ds\right].$$

For any $i = 1, 2, \cdots, d$ and $s < \left(\frac{\delta t}{2d}\right) \wedge \tau_{2^{-1}\widehat{Q}}$, we have $|X_s^{(i)} - z^{(i)}| < \frac{r}{2^{d-k+2}}$. For any $|w^{(i_k)} - (z^{(i_k)} + y^{(i_k)} - x^{(i_k)})| < \frac{r}{2^{d-k+2}}$, we have

$$|X_s^{(i_k)} - w^{(i_k)}| \leq |X_s^{(i_k)} - z^{(i_k)}| + |w^{(i_k)} - (z^{(i_k)} + y^{(i_k)} - x^{(i_k)})| + |y^{(i_k)} - x^{(i_k)}|$$

$$
\begin{aligned}
&\le \frac{r}{2^{d-k+2}} + \frac{r}{2^{d-k+2}} + |y^{(i_k)} - x^{(i_k)}| \\
&\le at^{1/\alpha} + |y^{(i_k)} - x^{(i_k)}| \le (a/a_5 + 1)|y^{(i_k)} - x^{(i_k)}|.
\end{aligned}
$$

Combining the above three formulas, (4.32) and (4.30), we obtain

$$
\begin{aligned}
&\mathbb{P}_z\left(\sigma^U_{2^{-1}\widetilde{Q}} < \delta t/d\right) \\
\ge\ & \mathbb{E}_z\left[\int_0^{\left(\frac{\delta t}{2d}\right)\wedge\tau_{2^{-1}\widehat{Q}}}\int_{\mathbb{R}} \mathbb{1}_{2^{-1}\widetilde{Q}\setminus\overline{(2^{-1}\widehat{Q})}}(X_s^{(1)},\cdots,X_s^{(i_k-1)},w^{(i_k)},X_s^{(i_k+1)},\cdots,X_s^{(d)})\cdot j(X_s^{(i_k)},w^{(i_k)})dw^{(i_k)}ds\right] \\
\ge\ & \mathbb{E}_z\left[\int_0^{\left(\frac{\delta t}{2d}\right)\wedge\tau_{2^{-1}\widehat{Q}}}\int_{\{w^{(i_k)}\in A\}} j(X_s^{(i_k)},w^{(i_k)})dw^{(i_k)}ds\right] \\
\ge\ & \mathbb{E}_z\left[\int_0^{\left(\frac{\delta t}{2d}\right)\wedge\tau_{2^{-1}\widehat{Q}}}\int_{\{w^{(i_k)}\in A\}} dw^{(i_k)}\frac{\mathcal{C}_{1,\alpha}}{(a/a_5+1)^{1+\alpha}|y^{(i_k)}-x^{(i_k)}|^{1+\alpha}}ds\right] \\
=\ & \mathbb{E}_z\left[\left(\frac{\delta t}{2d}\right)\wedge\tau_{2^{-1}\widehat{Q}}\right]\frac{\mathcal{C}_{1,\alpha}}{(a/a_5+1)^{1+\alpha}|y^{(i_k)}-x^{(i_k)}|^{1+\alpha}}\cdot|\{w^{(i_k)}\in A\}| \\
=\ & \mathbb{E}_z\left[\left(\frac{\delta t}{2d}\right)\wedge\tau_{2^{-1}\widehat{Q}}\right]\frac{\mathcal{C}_{1,\alpha}}{(a/a_5+1)^{1+\alpha}|y^{(i_k)}-x^{(i_k)}|^{1+\alpha}} \\
&\times\left(\frac{r}{2^{d-k+2}} - \left(\frac{r}{2^{d-k+2}} - |y^{(i_k)}-x^{(i_k)}|\right)\vee\left(-\frac{r}{2^{d-k+2}}\right)\right) \quad \text{(by (4.30))} \\
\ge\ & \frac{(1-2^{-\alpha})\delta t}{2d}\frac{\mathcal{C}_{1,\alpha}}{(a/a_5+1)^{1+\alpha}|y^{(i_k)}-x^{(i_k)}|^{1+\alpha}}\left(|y^{(i_k)}-x^{(i_k)}|\wedge\frac{r}{2^{d-k+1}}\right) \quad \text{(by (4.32))} \\
\ge\ & \frac{(1-2^{-\alpha})\delta t}{2d}\frac{\mathcal{C}_{1,\alpha}}{(a/a_5+1)^{1+\alpha}|y^{(i_k)}-x^{(i_k)}|^{1+\alpha}}\left(a_5 t^{1/\alpha}\wedge\frac{at^{1/\alpha}}{2^d\sqrt{d}}\right) \\
\ge\ & \frac{c_6 t^{1+1/\alpha}}{|y^{(i_k)}-x^{(i_k)}|^{1+\alpha}}.
\end{aligned}
$$

Furthermore, by the strong Markov property, the above inequality and an inequality similar to (4.31) yield that

$$
\begin{aligned}
\mathbb{P}_z\left(X^U_{\delta t/d}\in 2^{k-d}Q_k\right) &\ge \mathbb{P}_z\left(X^U_{\delta t/d}\in\widetilde{Q}\right) \\
&\ge \mathbb{P}_z\left(X^U \text{ hits } 2^{-1}\widetilde{Q} \text{ before time } \delta t/d \text{ and stays in } \widetilde{Q} \text{ for at least } \delta t/d \text{ units of time}\right) \\
&\ge \mathbb{P}_z\left(\sigma^U_{2^{-1}\widetilde{Q}} < \delta t/d;\ \tau^U_{\widetilde{Q}}\circ\theta_{\sigma^U_{2^{-1}\widetilde{Q}}} > \delta t/d\right) \\
&= \mathbb{P}_z\left(\sigma^U_{2^{-1}\widetilde{Q}} < \delta t/d;\ \mathbb{E}_{X_{\sigma^U_{2^{-1}\widetilde{Q}}}}\left[\tau^U_{\widetilde{Q}} > \delta t/d\right]\right) \\
&= \mathbb{P}_z\left(\sigma^U_{2^{-1}\widetilde{Q}} < \delta t/d\right)\inf_{w\in 2^{-1}\widetilde{Q}}\mathbb{P}_w\left(\tau_{\widetilde{Q}} > \delta t/d\right) \\
&= \mathbb{P}_z\left(\sigma^U_{2^{-1}\widetilde{Q}} < \delta t/d\right)\inf_{w\in 2^{-1}\widetilde{Q}}\mathbb{P}_w\left(\tau_{Q(w,r/2^{d-k+2})} > \delta t/d\right) \\
&\ge \frac{c_6 t^{1+1/\alpha}}{|y^{(i_k)}-x^{(i_k)}|^{1+\alpha}}\cdot\left(1-\frac{1}{2^\alpha}\right) \quad \text{(by an inequality similar to (4.31))},
\end{aligned}
$$

which gives (4.27).

Step 4. Note that $\{i_1, i_2, \cdots, i_d\}$ is a permutation of $\{1, 2, \cdots, d\}$. By choosing a, δ small enough such that all the conditions (4.22), (4.24), (4.29) are satisfied, it follows from (4.21), (4.26) and (4.27) that

$$
p_U(t,x,y) \ge c_2(1-\delta)^{-d/\alpha}t^{-d/\alpha}\prod_{k=1}^d c_3\left(1\wedge\frac{t^{1+1/\alpha}}{|y^{(i_k)}-x^{(i_k)}|^{1+\alpha}}\right)
$$

$$= c_2 c_3^d (1-\delta)^{-d/\alpha} \prod_{k=1}^d \left(t^{-1/\alpha} \wedge \frac{t}{|y^{(k)} - x^{(k)}|^{1+\alpha}} \right),$$

which together with (1.6) finishes the proof. □

Remark 4.10. Suppose that $t > 0$, $a \in \mathbb{R}$ and $U \subset \mathbb{R}^d$ is an open set. Let $Q_1 := Q(x, c_1 t^{1/\alpha})$ and $Q_2 := Q(x + a e_i, c_1 t^{1/\alpha})$ be two cubes with $Q_1 \cup Q_2 \subset U$, where $1 \le i \le d$, $c_1 > 0$ is a small constant and e_i is the unit vector in the positive $x^{(i)}$-direction. Then, by the same arguments that lead to (4.27), we can in fact prove the following more general inequality: there exists $c_2 > 0$ independent of t, a, x such that

$$\inf_{z \in 2^{-1} Q_1} \mathbb{P}_z \left(X_t^U \in Q_2 \right) \ge c_2 \left(1 \wedge \frac{t^{1+1/\alpha}}{|a|^{1+\alpha}} \right).$$

Lemma 4.11. *Let $D \subset \mathbb{R}^d$ be a $C^{1,1}$ open set with characteristics (R, Λ) satisfying condition $(\mathbf{H}_\gamma)$ for some $\gamma \in (0,1]$. There exist constants $c = c(d, \alpha, R, \Lambda, \gamma) > 0$ and $t_* = t_*(d, \alpha, R, \Lambda, \gamma) > 0$ such that for all $x, y \in D$, we have*

$$p_D(3t_*, x, y) \ge c \left(1 \wedge \frac{\delta_D(x)^{\alpha/2}}{\sqrt{t_*}} \right) \left(1 \wedge \frac{\delta_D(y)^{\alpha/2}}{\sqrt{t_*}} \right) p(t_*, x, y).$$

Proof. Let δ_0, r_0 be the constants from Lemma 3.3. For any $z \in D$, we can choose $Q_z \in \partial D$ such that $|z - Q_z| = \delta_D(z)$. Let $D_{Q_z}(\delta_0, r_0)$ be the set from Lemma 3.3. Define

$$\delta_1 := \frac{\delta_0}{2\sqrt{1+\Lambda^2}}, \quad \text{and} \quad r_1 := \frac{r_0}{\sqrt{1+\Lambda^2}}.$$

Note that

$$\delta_0 < \frac{r_0}{2\sqrt{1+\Lambda^2}} = \frac{r_1}{2} < \frac{r_0}{2}.$$

For the above z, we use the following notation:

$$\begin{aligned} U_z^0 &:= D_{Q_z}(r_1, r_0) \setminus D_{Q_z}(\delta_0, r_0), \\ U_z &:= \bigcup_{u \in U_z^0} B(u, \delta_1), \\ E_z &:= \begin{cases} B(z, \delta_1), & z \notin D_{Q_z}(\delta_0, r_0) \ \ (\text{ i.e. } \delta_D(z) \ge \delta_0), \\ U_z, & z \in D_{Q_z}(\delta_0, r_0) \ \ (\text{ i.e. } \delta_D(z) < \delta_0). \end{cases} \end{aligned}$$

Let C_{14}, C_{15} be the constants from Lemma 3.3, and define

$$t_* = \frac{2C_{15}}{C_{14}} \vee 1. \tag{4.33}$$

By semigroup property, we have for any $x, y \in D$,

$$\begin{aligned} p_D(3t_*, x, y) &= \int_{D \times D} p_D(t_*, x, u) p_D(t_*, u, v) p_D(t_*, v, y) du dv \\ &\ge \int_{E_x \times E_y} p_D(t_*, x, u) p_D(t_*, u, v) p_D(t_*, v, y) du dv \\ &\ge \inf_{u \in E_x, v \in E_y} p_D(t_*, u, v) \int_{E_x} p_D(t_*, x, u) du \int_{E_y} p_D(t_*, v, y) dv \\ &= I_1 \cdot I_2 \cdot I_3. \end{aligned} \tag{4.34}$$

We estimate I_1, I_2, I_3 separately. For $1 \le i \le d$ and $(u, v) \in E_x \times E_y$, if $|y^{(i)} - x^{(i)}| < 3r_0$, then,

$$|u^{(i)} - v^{(i)}| \le |u^{(i)} - x^{(i)}| + |x^{(i)} - y^{(i)}| + |y^{(i)} - v^{(i)}| < 3r_0 + 3r_0 + 3r_0 = 9r_0,$$

and

$$t_*^{-1/\alpha} \wedge \frac{t_*}{|u^{(i)} - v^{(i)}|^{1+\alpha}} \ge t_*^{-1/\alpha} \wedge \frac{t_*}{(9r_0)^{1+\alpha}} \ge \left(1 \wedge \frac{t_*^{1+1/\alpha}}{(9r_0)^{1+\alpha}} \right) \left(t_*^{-1/\alpha} \wedge \frac{t_*}{|x^{(i)} - y^{(i)}|^{1+\alpha}} \right).$$

If $|y^{(i)} - x^{(i)}| \ge 3r_0$, then,

$$|u^{(i)} - v^{(i)}| \le |u^{(i)} - x^{(i)}| + |x^{(i)} - y^{(i)}| + |y^{(i)} - v^{(i)}| < 3r_0 + |x^{(i)} - y^{(i)}| + 3r_0 \le 3|x^{(i)} - y^{(i)}|,$$

and

$$t_*^{-1/\alpha} \wedge \frac{t_*}{|u^{(i)}-v^{(i)}|^{1+\alpha}} \geq \left(t_*^{-1/\alpha} \wedge \frac{t_*}{(3|x^{(i)}-y^{(i)}|)^{1+\alpha}}\right) \geq \frac{1}{3^{1+\alpha}} \left(t_*^{-1/\alpha} \wedge \frac{t_*}{|x^{(i)}-y^{(i)}|^{1+\alpha}}\right).$$

Combining the above two cases, we obtain

$$t_*^{-1/\alpha} \wedge \frac{t_*}{|u^{(i)}-v^{(i)}|^{1+\alpha}} \geq c_1 \left(t_*^{-1/\alpha} \wedge \frac{t_*}{|x^{(i)}-y^{(i)}|^{1+\alpha}}\right) \tag{4.35}$$

for some $c_1 = c_1(d, \alpha, R, \Lambda) > 0$. Since for all $(u, v) \in E_x \times E_y$,

$$\delta_D(u) \wedge \delta_D(v) \geq \frac{\rho_{Q_x}(u) \wedge \rho_{Q_y}(v)}{\sqrt{1+\Lambda^2}} \geq \frac{\delta_0 - \delta_1}{\sqrt{1+\Lambda^2}} \geq \delta_1 = (\delta_1 t_*^{1/\alpha}) \cdot t_*^{-1/\alpha}, \tag{4.36}$$

the above inequality (4.35) together with Lemma 4.9 (with $a_1 = \delta_1 t_*^{1/\alpha}$) yields that, there exists $c_2 = c_2(d, \alpha, R, \Lambda, \gamma) > 0$ such that

$$\begin{aligned} I_1 &= \inf_{u \in E_x, v \in E_y} p_D(t_*, u, v) \geq c_2 \inf_{u \in E_x, v \in E_y} p(t_*, u, v) \\ &\geq c_2 C_1^{-1} \prod_{i=1}^d \left(t_*^{-1/\alpha} \wedge \frac{t_*}{|u^{(i)}-v^{(i)}|^{1+\alpha}}\right) \\ &\geq c_1^d c_2 C_1^{-1} \prod_{i=1}^d \left(t_*^{-1/\alpha} \wedge \frac{t_*}{|x^{(i)}-y^{(i)}|^{1+\alpha}}\right) \\ &\geq c_1^d c_2 C_1^{-2} p(t_*, x, y). \end{aligned}$$

We next estimate the lower bound of I_2. The estimate for I_3 is similar so it will be omitted. If $\delta_D(x) \geq \delta_0$, then $E_x = B(x, \delta_1)$ and, $\delta_D(x) \wedge \delta_D(u) \geq \delta_1 = (\delta_1 t_*^{1/\alpha}) t_*^{-1/\alpha}$ for $u \in E_x$ by (4.36). Hence, by Lemma 4.9 with $a_1 = \delta_1 t_*^{1/\alpha}$, we obtain

$$\begin{aligned} I_2 &= \int_{E_x} p_D(t_*, x, u) du = \int_{B(x,\delta_1)} p_D(t_*, x, u) du \\ &\geq c_3 \int_{B(x,\delta_1)} p(t_*, x, u) du = c_3 \int_{B(0,\delta_1)} p(t_*, 0, u) du \\ &\geq c_3 \int_{B(0,\delta_1)} p(t_*, 0, u) du \left(1 \wedge \frac{\delta_D(x)^{\alpha/2}}{\sqrt{t_*}}\right), \end{aligned}$$

where $\int_{B(0,\delta_1)} p(t_*, 0, u) du$ is a positive constant independent of x and D.

Recall that $\sigma^D_{U_x^0} := \inf\{t > 0 : X_t^D \in U_x^0\}$ is the first entry time of X^D for the set U_x^0. If $\delta_D(x) < \delta_0$, then $E_x = U_x = \cup_{u \in U_x^0} B(u, \delta_1)$, and by Markov property, we have

$$\begin{aligned} I_2 &= \int_{E_x} p_D(t_*, x, u) du = \mathbb{P}_x\left(X_{t_*}^D \in U_x\right) \\ &\geq \mathbb{P}_x\Big(X^D \text{ hits } U_x^0 \text{ before time } t_* \text{ and stays in } U_x \text{ for at least } t_* \text{ units of time}\Big) \\ &= \mathbb{E}_y\left[\sigma^D_{U_x^0} < t_*, \mathbb{P}_{X^D_{\sigma^D_{U_x^0}}}\left(\tau^D_{U_x} > t_*\right)\right] \geq \inf_{z \in U_x^0} \mathbb{P}_z\left(\tau^D_{U_x} > t_*\right) \cdot \mathbb{P}_x\left(\sigma^D_{U_x^0} < t_*\right). \end{aligned}$$

Similar to (4.36), we have for $z \in U_x^0$,

$$\delta_D(z) \geq \frac{\rho_{Q_x}(z)}{\sqrt{1+\Lambda^2}} \geq \frac{\delta_0}{\sqrt{1+\Lambda^2}} = 2\delta_1,$$

and hence $B := B(z, \delta_1) \subset U_x \subset D$. By Lemma 4.6 with $t = t_*$, $r = \delta_1$ and $a_1 = t_*^{1/\alpha}/\delta_1$, there exists $a_2 = a_2(d, \alpha, R, \Lambda) > 0$ and $c_4 = c_4(d, \alpha, R, \Lambda) > 0$ such that $B(z, a_2 t_*^{1/\alpha}) \subset B$ and $p_B(t_*, z, v) \geq c_4 t_*^{-d/\alpha}$ for $v \in B(z, a_2 t_*^{1/\alpha})$. Hence,

$$\mathbb{P}_z\left(\tau^D_{U_x} > t_*\right) = \mathbb{P}_z\left(\tau_{U_x} > t_*\right) = \int_{U_x} p_{U_x}(t_*, z, v) dv$$

$$
\begin{aligned}
&\geq \int_B p_B(t_*, z, v)dv \geq \int_{B(z, a_2 t_*^{1/\alpha})} p_B(t_*, z, v)dv \\
&\geq c_4 t_*^{-d/\alpha} |B(z, a_2 t_*^{1/\alpha})| = c_4 t_*^{-d/\alpha} \cdot a_2^d t_*^{d/\alpha} |B(0,1)| \\
&= c_4 a_2^d |B(0,1)| > 0.
\end{aligned}
$$

On the other hand, by the definition (4.33) of t_*, (3.19) and (3.20), we obtain

$$
\begin{aligned}
\mathbb{P}_x\left(\sigma^D_{U^0_x} < t_*\right) &\geq \mathbb{P}_x\left(\tau^D_{D_{Q_x(\delta_0, r_0)}} < t_*, X^D_{\tau^D_{D_{Q_x(\delta_0, r_0)}}} \in U^0_x\right) \\
&= \mathbb{P}_x\left(X_{\tau_{D_{Q_x(\delta_0, r_0)}}} \in U^0_x\right) - \mathbb{P}_x\left(X_{\tau_{D_{Q_x(\delta_0, r_0)}}} \in U^0_x, \tau_{D_{Q_x(\delta_0, r_0)}} \geq t_*\right) \\
&\geq \mathbb{P}_x\left(X_{\tau_{D_{Q_x(\delta_0, r_0)}}} \in U^0_x\right) - \mathbb{P}_x\left(\tau_{D_{Q_x(\delta_0, r_0)}} \geq t_*\right) \\
&\geq \mathbb{P}_x\left(X_{\tau_{D_{Q_x(\delta_0, r_0)}}} \in D_{Q_x(r_1, r_0)}\right) - \frac{\mathbb{E}_x[\tau_{D_{Q_x(\delta_0, r_0)}}]}{t_*} \\
&\geq C_{14}\delta_D(x)^{\alpha/2} - \frac{C_{15}\delta_D(x)^{\alpha/2}}{t_*} \geq \sqrt{\frac{C_{14}C_{15}}{2}}\left(1 \wedge \frac{\delta_D(x)^{\alpha/2}}{\sqrt{t_*}}\right).
\end{aligned}
$$

(See [7, p. 36] for details.) Combining the above four inequalities, we obtain

$$
I_2 \geq c_5 \left(1 \wedge \frac{\delta_D(x)^{\alpha/2}}{\sqrt{t_*}}\right).
$$

Finally, combining the estimates of I_1, I_2, I_3 and (4.34), we finish the proof. □

Note that the constant t_* from Lemma 4.11 is greater than or equal to 1.

Lemma 4.12. *Let $D \subset \mathbb{R}^d$ be a $C^{1,1}$ open set with characteristics (R, Λ) satisfying condition $(\mathbf{H}_\gamma)$ for some $\gamma \in (0, 1]$. There is a constant $c = c(d, \alpha, R, \Lambda, \gamma) > 0$ such that for all $t \in (0, 3t_*]$, $x, y \in D$,*

$$
p_D(t, x, y) \geq c\left(1 \wedge \frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}}\right)\left(1 \wedge \frac{\delta_D(y)^{\alpha/2}}{\sqrt{t}}\right)p(t, x, y).
$$

Proof. For $t \in (0, 3t_*]$, let

$$
\lambda := \left(\frac{3t_*}{t}\right)^{1/\alpha} \geq 1.
$$

Note that $\lambda D = \{\lambda z : z \in D\}$ is also a $C^{1,1}$ open set with the same characteristics of D and satisfies condition $(\mathbf{H}_\gamma)$. Then, by the scaling property (2.6), (2.5), (1.6) and Lemma 4.11 for λD, we obtain for all $x, y \in D$

$$
\begin{aligned}
p_D(t, x, y) &= \lambda^d p_{\lambda D}(\lambda^\alpha t, \lambda x, \lambda y) = \lambda^d p_{\lambda D}(3t_*, \lambda x, \lambda y) \\
&\geq c\lambda^d \left(1 \wedge \frac{\delta_{\lambda D}(\lambda x)^{\alpha/2}}{\sqrt{t_*}}\right)\left(1 \wedge \frac{\delta_{\lambda D}(\lambda y)^{\alpha/2}}{\sqrt{t_*}}\right)p(t_*, \lambda x, \lambda y) \\
&= c\left(1 \wedge \frac{\lambda^{\alpha/2}\delta_D(x)^{\alpha/2}}{\sqrt{t_*}}\right)\left(1 \wedge \frac{\lambda^{\alpha/2}\delta_D(y)^{\alpha/2}}{\sqrt{t_*}}\right)\lambda^d p(3^{-1}\lambda^\alpha t, \lambda x, \lambda y) \\
&= c\left(1 \wedge \frac{\sqrt{3}\delta_D(x)^{\alpha/2}}{\sqrt{t}}\right)\left(1 \wedge \frac{\sqrt{3}\delta_D(y)^{\alpha/2}}{\sqrt{t}}\right)p(3^{-1}t, x, y) \\
&\geq \frac{c}{3^d C_1^2}\left(1 \wedge \frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}}\right)\left(1 \wedge \frac{\delta_D(y)^{\alpha/2}}{\sqrt{t}}\right)p(t, x, y).
\end{aligned}
$$

□

We need the following lemma to estimate Dirichlet heat kernel $p_D(t, x, y)$ for large time.

Lemma 4.13. *Suppose that $U \subset \mathbb{R}^d$ is a bounded open set and let $\kappa \geq \mathrm{diam}(U)$. Then, there are two positive constants $c_i = c_i(d, \alpha, \kappa), i = 1, 2$ such that*

$$
p_U(t, x, y) \leq c_1 e^{-c_2 t}, \quad t > 1, \ x, y \in U.
$$

Proof. Since $\{|z| \geq \kappa\} \supset \{z = (z^{(1)}, \cdots, z^{(d)}) : |z^{(1)}| \geq \kappa \vee 1\}$, by (1.5), for every $x \in U$, we have

$$\begin{aligned}\mathbb{P}_x(\tau_U \leq 1) &\geq \mathbb{P}_x(X_1 \in \mathbb{R}^d \setminus U) = \int_{\mathbb{R}^d \setminus U} p(1, x, z)dz \geq \int_{|z| \geq \kappa} p(1, 0, z)dz \\ &\geq \int_{|z^{(1)}| \geq \kappa \vee 1} p(1, 0, z)dz = \int_{|z^{(1)}| \geq \kappa \vee 1} p^{(1,\alpha)}(1, 0, z^{(1)})dz^{(1)} := c_3(\alpha, \kappa).\end{aligned}$$

Then,

$$\sup_{x \in U} \int_U p_U(1, x, y)dy = \sup_{x \in U} \mathbb{P}_x(\tau_U > 1) \leq (1 - c_3) =: c_4 < 1.$$

For $t \in (0, 1]$ and $x \in U$, we have $\int_U p_U(t, x, z)dz \leq \int_U p(t, x, z)dz \leq 1 \leq ee^{-t}$. When $t \in (n, n+1]$ for some integer $n \geq 1$, we set $x_0 = x$ and then obtain by semigroup property,

$$\begin{aligned}\int_U p_U(t, x, z)dz &= \int_{U^{n+1}} \prod_{k=1}^{n} p_U(1, x_{k-1}, x_k)p_U(t - n, x_n, z)dx_1 \cdots dx_n dz \\ &= \int_{U^{n+1}} p_U(t - n, x_n, z)dz \; p_U(1, x_{n-1}, x_n)dx_n \cdots p_U(1, x_0, x_1)dx_1 \leq c_4^n \leq c_4^{-1} c_4^t.\end{aligned}$$

Combining the above two inequalities, we have for all $(t, x) \in (0, \infty) \times U$,

$$\int_U p_U(t, x, y)dy \leq (e \vee c_4^{-1})e^{-c_2 t},$$

where $c_2 := \ln(c_4^{-1} \wedge e)$. Note that by (1.6), $p_U(1, z, y) \leq p(1, z, y) \leq C_1$ for all $z, y \in U$. Thus, for all $t > 1$ and $z, y \in U$ we have

$$p_U(t, x, y) = \int_U p_U(t - 1, x, z)p_U(1, z, y)dz \leq C_1 \int_U p_U(t - 1, x, z)dz \leq C_1(e \vee c_4^{-1})e^{-c_2(t-1)}.$$

This proves the lemma with $c_1 := C_1(e \vee c_4^{-1})e^{c_2}$. □

4.3. **Proof of Theorem 1.5.**

Proof of Theorem 1.5. (i) By Lemmas 4.5 and 4.12, we obtain (1.12) and (1.13) for all $t \in (0, 1]$. By the semigroup property and (1.12) for $t \in (1/2, 1]$, we have for $x, y \in D$,

$$\begin{aligned}p_D(2t, x, y) &= \int_D p_D(t, x, z)p_D(t, z, y)dz \\ &\leq c_1 \int_D \left(1 \wedge \frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}}\right) p(t, x, z) \left(1 \wedge \frac{\delta_D(z)^{\alpha/2}}{\sqrt{t}}\right) \left(1 \wedge \frac{\delta_D(z)^{\alpha/2}}{\sqrt{t}}\right) p(t, z, y) \left(1 \wedge \frac{\delta_D(y)^{\alpha/2}}{\sqrt{t}}\right) dz \\ &\leq c_1 \left(1 \wedge \frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}}\right) \left(1 \wedge \frac{\delta_D(y)^{\alpha/2}}{\sqrt{t}}\right) \int_{\mathbb{R}^d} p(t, x, z)p(t, z, y)dz \\ &\leq 2c_1 \left(1 \wedge \frac{\delta_D(x)^{\alpha/2}}{\sqrt{2t}}\right) \left(1 \wedge \frac{\delta_D(y)^{\alpha/2}}{\sqrt{2t}}\right) p(2t, x, y).\end{aligned}$$

This shows that (1.12) holds for $t \in (1, 2]$. For general $T > 0$, one can repeat the above arguments for $\lceil \log_2 T \rceil$ times to prove (1.12) for $t \in (0, T]$. Here for $a \in \mathbb{R}$, the notation $\lceil a \rceil$ stands for the smallest integer greater than or equal to a.

(ii) We next estimate the lower bound of $p_D(2t, x, y)$. Note that D is a $C^{1,1}$ open set with characteristics (R, Λ). Fix $x \in D$ and $t \in (1/2, 1]$, and let $Q \in \partial D$ be such that $\delta_D(x) = |x - Q|$. Define

$$x_0 = \begin{cases} Q + \frac{R}{4|x-Q|}(x - Q), & \text{if } \delta_D(x) < \frac{R}{4}, \\ x, & \text{if } \delta_D(x) \geq \frac{R}{4}, \end{cases}$$

and, let $r := R/8$. We have $B := B(x_0, r) \subset B(x_0, R/4) \subset D$ and $\delta_D(z) \geq r$ for all $z \in B$. By semigroup property of $p_D(2t, x, y)$ and (1.13) for $t \in (1/2, 1]$, we have

$$p_D(2t, x, y) = \int_D p_D(t, x, z)p_D(t, z, y)dz$$

$$\geq c_2 \int_D \left(1 \wedge \frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}}\right) p(t,x,z) \left(1 \wedge \frac{\delta_D(z)^{\alpha/2}}{\sqrt{t}}\right) \left(1 \wedge \frac{\delta_D(z)^{\alpha/2}}{\sqrt{t}}\right) p(t,z,y) \left(1 \wedge \frac{\delta_D(y)^{\alpha/2}}{\sqrt{t}}\right) dz$$

$$\geq c_2 r^\alpha \left(1 \wedge \frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}}\right) \left(1 \wedge \frac{\delta_D(y)^{\alpha/2}}{\sqrt{t}}\right) \int_B p(t,x,z)p(t,z,y)dz. \tag{4.37}$$

Note that $x \in B(x_0, R/4)$ and $|x-z| < R/4 + R/8 = 3R/8$ for all $z \in B = B(x_0, r)$. We have, by (1.6), for $t \in (1/2, 1]$,

$$p(t,x,z) \geq C_1^{-1} \prod_{i=1}^d \left(t^{-\frac{1}{\alpha}} \wedge \frac{t}{|x^{(i)} - z^{(i)}|^{1+\alpha}}\right) \geq C_1^{-1} \prod_{i=1}^d \left(1 \wedge \frac{1/2}{(3R/8)^{1+\alpha}}\right) = C_1^{-1} > 0. \tag{4.38}$$

On the other hand, for $1 \leq i \leq d$, if $|x^{(i)} - y^{(i)}| < R/2$, then

$$|z^{(i)} - y^{(i)}| \leq |z^{(i)} - x^{(i)}| + |x^{(i)} - y^{(i)}| \leq 3R/8 + R/2 < R,$$

and

$$t^{-\frac{1}{\alpha}} \wedge \frac{t}{|z^{(i)} - y^{(i)}|^{1+\alpha}} \geq (2t)^{-\frac{1}{\alpha}} \wedge \frac{2t}{2R^{1+\alpha}} = \left(1 \wedge \frac{(2t)^{1+1/\alpha}}{2R^{1+\alpha}}\right) (2t)^{-\frac{1}{\alpha}}$$
$$\geq \frac{1}{2} \left((2t)^{-\frac{1}{\alpha}} \wedge \frac{2t}{|x^{(i)} - y^{(i)}|^{1+\alpha}}\right).$$

If $|x^{(i)} - y^{(i)}| \geq R/2$, then

$$|z^{(i)} - y^{(i)}| \leq |z^{(i)} - x^{(i)}| + |x^{(i)} - y^{(i)}| \leq 3R/8 + |x^{(i)} - y^{(i)}| \leq 2|x^{(i)} - y^{(i)}|,$$

and

$$t^{-\frac{1}{\alpha}} \wedge \frac{t}{|z^{(i)} - y^{(i)}|^{1+\alpha}} \geq (2t)^{-\frac{1}{\alpha}} \wedge \frac{2t}{2|z^{(i)} - y^{(i)}|^{1+\alpha}}$$
$$\geq (2t)^{-\frac{1}{\alpha}} \wedge \frac{2t}{2(2|x^{(i)} - y^{(i)}|)^{1+\alpha}}$$
$$\geq \frac{1}{2^{2+\alpha}} \left((2t)^{-\frac{1}{\alpha}} \wedge \frac{2t}{|x^{(i)} - y^{(i)}|^{1+\alpha}}\right).$$

Combining the above two cases, we always have

$$t^{-\frac{1}{\alpha}} \wedge \frac{t}{|z^{(i)} - y^{(i)}|^{1+\alpha}} \geq c_3 \left((2t)^{-\frac{1}{\alpha}} \wedge \frac{2t}{|x^{(i)} - y^{(i)}|^{1+\alpha}}\right),$$

for some constant $c_3 > 0$ independent of t, x, y. Hence, by (1.6), we have

$$p(t,z,y) \geq C_1^{-1} \prod_{i=1}^d \left(t^{-\frac{1}{\alpha}} \wedge \frac{t}{|z^{(i)} - y^{(i)}|^{1+\alpha}}\right) \geq c_3^d C_1^{-1} \prod_{i=1}^d \left((2t)^{-\frac{1}{\alpha}} \wedge \frac{2t}{|x^{(i)} - y^{(i)}|^{1+\alpha}}\right) \geq c_3^d C_1^{-2} p(2t,x,y).$$

Combining this, (4.37) and (4.38), we have

$$p_D(2t,x,y) \geq c_2 r^\alpha \left(1 \wedge \frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}}\right) \left(1 \wedge \frac{\delta_D(y)^{\alpha/2}}{\sqrt{t}}\right) \int_B C_1^{-1} \cdot c_3^d C_1^{-2} p(2t,x,y) dz$$
$$\geq c_2 c_3^d r^\alpha C_1^{-3} \left(1 \wedge \frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}}\right) \left(1 \wedge \frac{\delta_D(y)^{\alpha/2}}{\sqrt{t}}\right) |B| \; p(2t,x,y)$$
$$\geq c_2 c_3^d r^{d+\alpha} C_1^{-3} \left(1 \wedge \frac{\delta_D(x)^{\alpha/2}}{\sqrt{2t}}\right) \left(1 \wedge \frac{\delta_D(y)^{\alpha/2}}{\sqrt{2t}}\right) |B(0,1)| \; p(2t,x,y).$$

Therefore, we have proved (1.13) for $t \in (1, 2]$. Similarly, one can repeat the above arguments for $\lceil \log_2 T \rceil$ times to prove (1.13) for $t \in (0, T]$.

(iii) Now, assume in addition that D is a *bounded* $C^{1,1}$ open set and satisfies $(\mathbf{H}_\gamma)$ for some $\gamma \in (0, 1]$. Recall that $\mathcal{L}^D$ is the infinitesimal generator of the semigroup $\{P_t^D, t \geq 0\}$ on $L^2(D, dx)$. Since for each $t > 0$, the heat kernel $p_D(t,x,y)$ is bounded on $D \times D$, it follows from Jentzsch's Theorem ([25, Theorem V.6.6, p. 337]) that the value $-\lambda_1(D) = \sup(\sigma(\mathcal{L}^D))$ is an eigenvalue of multiplicity 1 for $\mathcal{L}^D$ and that the eigenfunction ϕ_D associated with $\lambda_1(D)$ can be chosen to be strictly positive with $\|\phi_D\|_{L^2(D)} = 1$. In the rest of the proof, we write $\lambda_1(D)$ as λ_1 for simplicity.

Step 1. We first prove the second inequality in (1.14). Since ϕ_D is the eigenfunction of $\mathcal{L}^D$ associated with λ_1, we have for all $t > 0$ and $x \in D$,

$$\phi_D(x) = e^{\lambda_1 t} P_t^D \phi_D(x) = e^{\lambda_1 t} \int_D p_D(t,x,y)\phi_D(y)dy. \tag{4.39}$$

Setting $t = 1/4$ in (4.39), by (1.12) with $T = 1$, (1.6) and Hölder inequality, we have for all $x \in D$,

$$\begin{aligned}\phi_D(x) &\le c_4 e^{\frac{1}{4}\lambda_1}(1 \wedge 2\delta_D(x)^{\alpha/2}) \int_D p(1/4,x,y)\phi_D(y)dy \\ &\le 2c_4 e^{\frac{1}{4}\lambda_1}(1 \wedge \delta_D(x)^{\alpha/2}) \sqrt{\int_D p(1/4,x,y)^2 dy} \cdot \|\phi_D\|_{L^2(D)} \\ &\le 2c_4 e^{\frac{1}{4}\lambda_1}(1 \wedge \delta_D(x)^{\alpha/2}) \sqrt{p(1/2,x,x)} \\ &\le 2c_4 \sqrt{2^{d/\alpha}C_1}\, e^{\frac{1}{4}\lambda_1}(1 \wedge \delta_D(x)^{\alpha/2}) =: c_5 e^{\frac{1}{4}\lambda_1}(1 \wedge \delta_D(x)^{\alpha/2}),\end{aligned} \tag{4.40}$$

where $c_i = c_i(d,\alpha,R,\Lambda,\gamma) > 0$, $i = 4,5$.

On the other hand, set $\kappa := \mathrm{diam}(D)$ to be the diameter of D for simplicity. Setting $t = 1$ in (4.39), by (1.13) with $T = 1$ and (1.6), we have for all $x \in D$,

$$\begin{aligned}\phi_D(x) &\ge c_4^{-1}(1 \wedge \delta_D(x)^{\alpha/2})e^{\lambda_1} \int_D (1 \wedge \delta_D(y)^{\alpha/2}) p(1,x,y)\phi_D(y)dy \\ &\ge c_4^{-1} C_1^{-1} \left(1 \wedge \kappa^{-d(1+\alpha)}\right)(1 \wedge \delta_D(x)^{\alpha/2})e^{\lambda_1} \int_D (1 \wedge \delta_D(y)^{\alpha/2})\phi_D(y)dy \\ &=: c_6 (1 \wedge \delta_D(x)^{\alpha/2})e^{\lambda_1} \int_D (1 \wedge \delta_D(y)^{\alpha/2})\phi_D(y)dy,\end{aligned}$$

where $c_6 = c_6(d,\alpha,R,\Lambda,\gamma,\kappa) > 0$. Combining this and (4.40), we have for $x \in D$,

$$\begin{aligned}\phi_D(x) &\ge c_6 (1 \wedge \delta_D(x)^{\alpha/2})e^{\lambda_1} \int_D c_5^{-1} e^{-\frac{1}{4}\lambda_1}(\phi_D(y))^2 dy \\ &\ge c_6 c_5^{-1} e^{\frac{3}{4}\lambda_1}(1 \wedge \delta_D(x)^{\alpha/2}) \int_D (\phi_D(y))^2 dy \ge c_6 c_5^{-1} e^{\frac{3}{4}\lambda_1}(1 \wedge \delta_D(x)^{\alpha/2}).\end{aligned} \tag{4.41}$$

Combining (4.40) and (4.41), we have for $x \in D$,

$$c_6 c_5^{-1} e^{\frac{3}{4}\lambda_1}(1 \wedge \delta_D(x)^{\alpha/2}) \le \phi_D(x) \le c_5 e^{\frac{1}{4}\lambda_1}(1 \wedge \delta_D(x)^{\alpha/2}),$$

which implies that

$$\lambda_1 \le 2\ln(c_5^2/c_6) < \infty. \tag{4.42}$$

This is exactly the second inequality in (1.14).

Step 2. We next show (1.15). In view of (4.40), (4.41) and (4.42), we have for $x \in D$,

$$c_7^{-1}(1 \wedge \delta_D(x)^{\alpha/2}) \le \phi_D(x) \le c_7 (1 \wedge \delta_D(x)^{\alpha/2}), \tag{4.43}$$

where $c_7 = c_7(d,\alpha,R,\Lambda,\gamma,\kappa) \ge 1$.

Now, multiplying (4.39) by $\phi_D(x)$ and integrating it over D with respect to dx, we have for $t > 0$,

$$1 = \int_D (\phi_D(x))^2 dx = e^{\lambda_1 t} \int_{D\times D} \phi_D(x) p_D(t,x,y)\phi_D(y)dxdy,$$

which implies that

$$\int_{D\times D} \phi_D(x) p_D(t,x,y)\phi_D(y)dxdy = e^{-\lambda_1 t}.$$

Combining this and (4.43), we have for $t > 0$,

$$c_7^{-2} e^{-\lambda_1 t} \le \int_{D\times D} (1 \wedge \delta_D(x)^{\alpha/2}) p_D(t,x,y)(1 \wedge \delta_D(y)^{\alpha/2})dxdy \le c_7^2 e^{-\lambda_1 t}. \tag{4.44}$$

For any $T > 0$, set $t_0 = (T \wedge 1)/4$. Note that, by (1.12) and (1.13)(with $T = 1$), there is a constant $c_8 = c_8(d, \alpha, R, \Lambda, \gamma) \geq 1$ such that for $(u, v) \in D \times D$,

$$p_D(t_0, u, v) \leq c_8 t_0^{-d/\alpha - 1}(1 \wedge \delta_D(u)^{\alpha/2})(1 \wedge \delta_D(v)^{\alpha/2}), \tag{4.45}$$

$$p_D(t_0, u, v) \geq c_8^{-1} \left(t_0^{-1/\alpha} \wedge (t_0 \kappa^{-(1+\alpha)})\right)^d (1 \wedge \delta_D(u)^{\alpha/2})(1 \wedge \delta_D(v)^{\alpha/2}). \tag{4.46}$$

Combining (4.45), the semigroup property of p_D and (4.44), we have for all $(t, x, y) \in (T, \infty) \times D \times D$,

$$\begin{aligned} p_D(t, x, y) &= \int_{D \times D} p_D(t_0, x, u) p_D(t - 2t_0, u, v) p_D(t_0, v, y) du dv \\ &\leq c_8^2 t_0^{-2d/\alpha - 2} \delta_D(x)^{\alpha/2} \delta_D(y)^{\alpha/2} \int_{D \times D} (1 \wedge \delta_D(u)^{\alpha/2}) p_D(t - 2t_0, u, v)(1 \wedge \delta_D(v)^{\alpha/2}) du dv \\ &\leq c_7^2 c_8^2 t_0^{-2d/\alpha - 2} e^{-(t - 2t_0)\lambda_1} \delta_D(x)^{\alpha/2} \delta_D(y)^{\alpha/2}. \end{aligned} \tag{4.47}$$

Similarly, by (4.46), the semigroup property of p_D and (4.44), we have for all $(t, x, y) \in (T, \infty) \times D \times D$,

$$\begin{aligned} p_D(t, x, y) &= \int_{D \times D} p_D(t_0, x, u) p_D(t - 2t_0, u, v) p_D(t_0, v, y) du dv \\ &\geq c_8^{-2} \left(t_0^{-1/\alpha} \wedge (t_0 \kappa^{-(1+\alpha)})\right)^{2d} (\kappa \vee 1)^{-\alpha} \delta_D(x)^{\alpha/2} \delta_D(y)^{\alpha/2} \\ &\qquad \cdot \int_{D \times D} (1 \wedge \delta_D(u)^{\alpha/2}) p_D(t - 2t_0, u, v)(1 \wedge \delta_D(v)^{\alpha/2}) du dv \\ &\geq c_8^{-2} \left(t_0^{-1/\alpha} \wedge (t_0 \kappa^{-(1+\alpha)})\right)^{2d} (\kappa \vee 1)^{-\alpha} e^{-(t - 2t_0)\lambda_1} \delta_D(x)^{\alpha/2} \delta_D(y)^{\alpha/2}. \end{aligned} \tag{4.48}$$

Combining (4.42), (4.47) and (4.48), we obtain (1.15).

Step 3. For the first inequality in (1.14), by Lemma 4.13, there exist $c_i = c_i(d, \alpha, \kappa) > 0$, $i = 9, 10$ such that

$$p_D(t, x, y) \leq c_9 e^{-c_{10} t}, \qquad t > 0, \ x, y \in D.$$

This together with the first inequality in (4.44) yields that for all $t > 1$,

$$\begin{aligned} c_7^{-2} e^{-\lambda_1 t} &\leq \int_{D \times D} (1 \wedge \delta_D(x)^{\alpha/2}) p_D(t, x, y)(1 \wedge \delta_D(y)^{\alpha/2}) dx dy \\ &\leq c_9 e^{-c_{10} t} \int_{D \times D} (1 \wedge \delta_D(x)^{\alpha/2})(1 \wedge \delta_D(y)^{\alpha/2}) dx dy \leq c_9 e^{-c_{10} t} |D|^2. \end{aligned}$$

Rewriting the above inequality, we have for all $t > 1$, $e^{(c_{10} - \lambda_1)t} \leq c_7^2 c_9 |D|^2 < \infty$. Since this inequality holds for all $t > 1$, we obtain that $\lambda_1 \geq c_{10}$, which is exactly the first inequality in (1.14). □

5. Irreducibility

Let $D \subset \mathbb{R}^d$ be a non-empty open set. In this section, we study the irreducibility of the subprocess X^D and prove Theorem 1.3 and Corollary 1.4. Before that, we need some auxiliary results on the positivity of $p_D(t, x, y)$. For $z \in D$, define

$$\begin{aligned} U_z := \{w \in D : &\text{ there exist finitely many points } \{x_i\}_{i=0}^N \subset D \text{ with } x_0 = z \text{ and } x_N = w \text{ so that} \\ &\text{each pair } (x_{i-1}, x_i),\ 1 \leq i \leq N, \text{ has only one different coordinate}\}. \end{aligned} \tag{5.1}$$

Note that $U_z \supset B(z, r)$ for any $r > 0$ such that $B(z, r) \subset D$.

Lemma 5.1. *For each $z \in D$, U_z is both open and closed in D.*

Proof. For any $w \in U_z$, there is $r > 0$ so that $B(w, r) \subset D$ and so $B(w, r) \subset U_z$ by the definition of U_z. This shows that U_z is an open subset of D. If $U_z = D$, then U_z is clearly closed in D. Suppose now $U_z \neq D$. Then for any $w \in D \setminus U_z$, there is some $r > 0$ so that $B(w, r) \subset D$. Note that $B(w, r) \cap U_z = \emptyset$, as otherwise there would be some $w_1 \in B(w, r) \cap U_z$ which would imply that $w \in U_z$. Hence U_z is a closed subset of D. □

Theorem 5.2. *Suppose that x_0 and y_0 are two distinct points in D so that $y_0 \in U_{x_0}$. Then $p_D(t, x_0, y_0) > 0$ for every $t > 0$.*

Proof. Fix an arbitrary $t > 0$. Set $c_1 = t^{-1/\alpha}(\delta_D(x_0) \wedge \delta_D(y_0)) > 0$. According to Lemma 4.7, there is some constant $c_2 > 0$ so that $p_D(t, x_0, y_0) > 0$ whenever $|x_0 - y_0| \le c_2 t^{1/\alpha}$. Suppose that $|x_0 - y_0| > c_2 t^{1/\alpha}$. We have by (4.19),

$$p_D(t, x_0, y_0) \ge \inf_{z \in B(y_0, c_3 t^{1/\alpha})} p_D(t/2, z, y_0) \mathbb{P}_{x_0}\left(X^D_{t/2} \in B(y_0, c_3 t^{1/\alpha})\right), \tag{5.2}$$

where $c_3 \in (0, (c_1 \wedge c_2)/2)$ is to be chosen sufficiently small later. Note that for $z \in B(y_0, c_3 t^{1/\alpha})$,

$$\delta_D(z) \wedge \delta_D(y_0) \ge (c_1 - c_3) t^{1/\alpha} \ge \frac{c_1}{2} t^{1/\alpha}.$$

By Lemma 4.7, there exists $c_4 > 0$ and $c_5 > 0$ depending on d, α and c_1 such that

$$p_D(t/2, z, y_0) \ge c_5 t^{-d/\alpha}$$

for every z with $|z - y_0| < c_4 t^{1/\alpha}$. Taking $c_3 \in (0, (c_1 \wedge c_2)/2)$ small enough so that $c_3 < c_4$, we get by (5.2) that

$$p_D(t, x_0, y_0) \ge c_5 t^{-d/\alpha} \mathbb{P}_{x_0}\left(X^D_{t/2} \in B(y_0, c_3 t^{1/\alpha})\right). \tag{5.3}$$

We next show that $\mathbb{P}_{x_0}\left(X^D_{t/2} \in B(y_0, c_3 t^{1/\alpha})\right) > 0$. Let $\{x_i\}_{i=1}^N$ be a finite sequence of points in D in the definition for $y_0 \in U_{x_0}$. Define

$$r := \frac{1}{4\sqrt{d}} \min\left\{ \min_{1 \le i \le N} \{|x_{i-1} - x_i|\}, \min_{0 \le i \le N} \{\delta_D(x_i)\}, c_3 t^{1/\alpha} \right\},$$

and

$$Q_i := Q(x_i, r) \quad \text{for } 0 \le i \le N.$$

For each $0 \le i \le N$, $Q_i \subset D$ and $\delta_D(z) \ge c_3 t^{1/\alpha}$ for all $z \in Q_i$. Note also that $Q_N \subset B(y_0, c_3 t^{1/\alpha})$.

For $\lambda > 0$ and a cube $Q := Q(z, \bar{r})$, we use the notation λQ to denote the cube $Q(z, \lambda \bar{r})$, that is $\lambda Q(z, \bar{r}) = Q(z, \lambda \bar{r})$. It follows from the similar arguments that lead to (4.26) that

$$\begin{aligned} &\mathbb{P}_{x_0}\left(X^D_{t/2} \in B(y_0, c_3 t^{1/\alpha})\right) \\ &\ge \inf_{z_0 \in 2^{-N} Q_0} \mathbb{P}_{z_0}\left(X^D_{t/(2N)} \in 2^{1-N} Q_1\right) \inf_{z_1 \in 2^{1-N} Q_1} \mathbb{P}_{z_1}\left(X^D_{t/(2N)} \in 2^{2-N} Q_2\right) \\ &\quad \times \cdots \times \inf_{z_{N-2} \in 2^{-2} Q_{N-2}} \mathbb{P}_{z_{N-2}}\left(X^D_{t/(2N)} \in 2^{-1} Q_{N-1}\right) \inf_{z_{N-1} \in 2^{-1} Q_{N-1}} \mathbb{P}_{z_{N-1}}\left(X^D_{t/(2N)} \in Q_N\right). \end{aligned} \tag{5.4}$$

The reader can find the details of the derivation in the arXiv version of this paper [7, Eq. (5.4) on p. 42].

We next estimate the lower bound of the right hand side of the above inequality. Let $1 \le k \le N$. Note that the centers x_{k-1} and x_k of Q_{k-1} and Q_k differ by only one coordinate. Thus there exists some $1 \le i_k \le d$ and $a_k \neq 0$ so that $Q_k = Q_{k-1} + a_k e_{i_k}$. Hence, by Remark 4.10 (with $c_1 = r t^{-1/\alpha}$), we have

$$\inf_{z \in 2^{k-1-N} Q_{k-1}} \mathbb{P}_z\left(X^D_{t/(2N)} \in 2^{k-N} Q_k\right) \ge c_6 \left(1 \wedge \frac{t^{1+1/\alpha}}{|a_k|^{1+\alpha}}\right) > 0,$$

where the constant $c_6 > 0$ that may depend on t. This together with (5.3)-(5.4) yields that $p_D(t, x_0, y_0) > 0$ when $|x_0 - y_0| \ge c_2 t^{1/\alpha}$. Combining the above two cases, we see that $p_D(t, x_0, y_0) > 0$ for any $t > 0$. □

We next establish a converse of Theorem 5.2.

Theorem 5.3. *Suppose that $x_0 \in D$ and $y_0 \in D \setminus U_{x_0}$. Then $p_D(t, x, y) = 0$ on $(0, \infty) \times U_{x_0} \times U_{y_0}$. In particular, $p_D(t, x_0, y_0) = 0$ for all $t > 0$.*

Proof. Suppose $x_0 \in D$ and $y_0 \in D \setminus U_{x_0}$. Then $U_{x_0} \cap U_{y_0} = \emptyset$ and

$$z + a e_i \notin D \setminus U_{x_0} \quad \text{for every } a \in \mathbb{R},\ z \in U_{x_0} \text{ and } 1 \le i \le d. \tag{5.5}$$

Since U_{x_0} is both open and closed by Lemma 5.1, we have for every $x \in U_{x_0}$, with $\tau := \tau_{U_{x_0}}$,

$$X_{\tau-} \in U_{x_0} \quad \text{and} \quad X_\tau \in D \setminus U_{x_0} \qquad \mathbb{P}_x\text{-a.s. on } \{\tau < \tau_D\}.$$

By the strong Markov property of X^D, (4.1) and (5.5), we have for every $t>0$ and $x\in U_{x_0}$ and $y\in U_{y_0}$,

$$p_D(t,x,y)=\mathbb{E}_x\left[p_D(t-\tau,X^D_\tau,y);\tau<t\right]=0. \tag{5.6}$$

□

Proof of Theorem 1.3. (i) (Sufficient condition) Suppose that the property (1.11) holds. We have by Theorem 5.2 that $p_D(t,x,y)>0$ for every $t>0$ and $x,y\in D$. This in particular implies that X^D is irreducible as for any non-empty open subset $U\subset D$, $\mathbb{P}_x(X^D_t\in U)=\int_U p_D(t,x,y)dy>0$ for every $t>0$ and $x\in D$.

(ii) (Necessary condition) Suppose that X^D is irreducible in D. Recall that for $z\in D$, U_z is the set defined by (5.1). Were there two distinct $x,y\in D$ so that $U_x\cap U_y=\emptyset$, we would have by Theorem 5.3

$$p_D(t,x,z)=0 \quad \text{for all } t>0 \text{ and } z\in U_y.$$

Consequently,

$$\mathbb{E}_x\int_0^{\tau_D}\mathbb{1}_{U_y}(X_s)ds=\int_0^\infty\int_{U_y}p_D(t,x,z)dzdt=0.$$

Since X^D is irreducible, we have, by Lemma 5.1 that $\mathbb{P}_x(\sigma_{U_y}<\tau_D)>0$. Since U_y is a non-empty open subset of D, $\mathbb{P}_x$-a.s. on $\{\sigma_{U_y}<\tau_D\}$, X^D spends positive Lebesgue amount of the time in U_y in view of the right continuity of the sample paths of X. Thus $\mathbb{E}_x\int_0^{\tau_D}\mathbb{1}_{U_y}(X_s)ds>0$. This contradiction proves that $U_x\cap U_y\neq\emptyset$ for every $x\neq y$ in D. In other words, $U_x=D$ for every $x\in D$. Thus the property (1.11) holds. This completes the proof of the theorem. □

Proof of Corollary 1.4. (i) This follows directly from Theorem 1.3 and the connectedness of D_1 as $p_D(t,x,y)\geq p_{D_1}(t,x,y)>0$ for $(t,x,y)\in(0,\infty)\times D_1\times D_1$.

(ii) If there are some $x_0\in D_1$ and $y_0\in D_2$ so that $y_0\in U_{x_0}$ (which is equivalent to $x_0\in U_{y_0}$), then, since D_1 and D_2 are connected, $U_z\supset D_1\cup D_2$ for every $z\in D_1\cup D_2$. In this case, we have by Theorem 5.2 that $p(t,x,y)>0$ for any $(t,x,y)\times D_1\times D_2$. Otherwise, $U_x\cap U_y=\emptyset$ for any $x\in D_1$ and $y\in D_2$. We then have by Theorem 5.3 that $p_D(t,x,y)=0$ for every $(t,x,y)\in(0,\infty)\times D_1\times D_2$. □

6. Examples

In this section, we present three more examples and present a proof for Theorem 1.6. The first two show that the lower bound estimate (1.13) in Theorem 1.5(ii) may fail for some smooth bounded connected open sets that do not satisfy condition $(\mathbf{H}_\gamma)$ for any $\gamma\in(0,1]$. The third presents a bounded $C^{1,1}$ open set that does not satisfy the irreducibility condition (1.11) but for which we can derive two-sided sharp estimates for its Dirichlet heat kernel. Recall that

$$j(a,b)=\frac{\mathcal{C}_{1,\alpha}}{|a-b|^{1+\alpha}} \quad \text{for } a\neq b\in\mathbb{R},$$

where $\mathcal{C}_{1,\alpha}$ is the positive constant in (1.2).

Example 6.1. Let $U_i\subset\mathbb{R}^d$, $i=1,2,\cdots,5$ be sets and $x,y\in\mathbb{R}^d$ be two points as shown in Figure 7. Set

$$U=\bigcup_{1\leq i\leq 5}U_i\subset\mathbb{R}^d,$$

which is a bounded connected smooth open set in $\mathbb{R}^2$ that does not satisfy condition $(\mathbf{H}_\gamma)$ for any $\gamma\in(0,1]$ as swapping any coordinate of $x=(0,0)$ by that of $y=(4,4)$ results a point falling outside D.

Claim: The lower bound estimate (1.13) in Theorem 1.5(ii) fails for this open set U.

Proof of the Claim. Fix $x=(0,0)$ and $y=(4,4)$. Since $|x^{(i)}-y^{(i)}|=4>0$ for $i=1,2$, if the inequality (1.13) in Theorem 1.5 (ii) does hold for the set U shown in Figure 7, then there exists $c_1>0$ such that

$$p_U(t,x,y)\geq c_1t^2 \qquad \text{for } t\in(0,1]. \tag{6.1}$$

We will show that there exists $c_2>0$ such that

$$p_U(t,x,y)\leq c_2t^3 \quad \text{for } t\in(0,1], \tag{6.2}$$

which will contradict (6.1) and finish our claim.

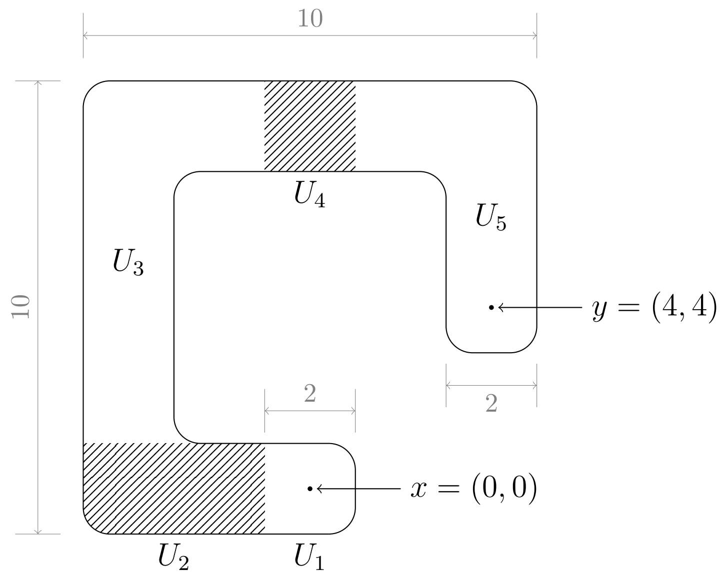

FIGURE 7. The set $U \subset \mathbb{R}^2$

Indeed, since $(U_3 \cup U_5) \cap \overline{U_1} = \emptyset$, we have by (4.1),

$$\mathbb{P}_x\left(X_{\tau_{U_1}} \in U_3 \cup U_5\right) = \mathbb{E}_x\left[\int_0^{\tau_{U_1}} \sum_{i=1}^2 \int_{\mathbb{R}^d} \mathbb{1}_{U_3 \cup U_5}([X_s]^i_\theta) j(X_s^{(i)}, \theta) d\theta ds\right].$$

Note that by the definitions of U_3, U_5 and U_1, we have that for any $z \in U_3$, $y \in U_5$ and $w \in U_1$,

$$z^{(1)} < w^{(1)} < y^{(1)} \quad \text{and} \quad z^{(2)} > w^{(2)},\ y^{(2)} > w^{(2)}.$$

Hence, since $X_s \in U_1$ for $s < \tau_{U_1}$, it is not possible that $[X_s]^i_\theta \in U_3 \cup U_5$ for any $\theta \in \mathbb{R}$. This together with the above identity implies that

$$\mathbb{P}_x\left(X_{\tau_{U_1}} \in U_3 \cup U_5\right) = 0. \tag{6.3}$$

By the above identity and the strong Markov property, we obtain, for almost every $w \in U$,

$$\begin{aligned} p_U(t, x, w) &= \mathbb{E}_x\left[p_U(t - \tau_{U_1}, X^U_{\tau_{U_1}}, w); \tau_{U_1} < t\right] \\ &= \mathbb{E}_x\left[p_U(t - \tau_{U_1}, X_{\tau_{U_1}}, w); \tau_{U_1} < t, X_{\tau_{U_1}} \in U_2 \cup U_4\right]. \end{aligned} \tag{6.4}$$

By the positions of U_2, U_4 and y, for any $z \in U_2 \cup U_4$, we have $|z^{(i)} - y^{(i)}| \geq 3$, $i = 1, 2$, and, then by (1.6),

$$\begin{aligned} p_U(t - \tau_{U_1}, X_{\tau_{U_1}}, y) \cdot \mathbb{1}_{\{\tau_{U_1} < t, X_{\tau_{U_1}} \in U_2 \cup U_4\}} &\leq C_1 \prod_{i=1}^2 \frac{t - \tau_{U_1}}{(X_{\tau_{U_1}})^{(i)} - y^{(i)}|^{1+\alpha}} \mathbb{1}_{\{\tau_{U_1} < t, X_{\tau_{U_1}} \in U_2 \cup U_4\}} \\ &\leq \frac{C_1 t^2}{3^{2(1+\alpha)}} \mathbb{1}_{\{\tau_{U_1} < t\}}. \end{aligned}$$

Combining (6.4), the continuity of p_U and dominated convergence theorem, we obtain for any $t \in (0, 1]$,

$$p_U(t, x, y) \leq C_1 \mathbb{E}_x\left[\prod_{i=1}^2 \frac{t - \tau_{U_1}}{(X_{\tau_{U_1}})^{(i)} - y^{(i)}|^{1+\alpha}}; \tau_{U_1} < t, X_{\tau_{U_1}} \in U_2 \cup U_4\right] \leq \frac{C_1 t^2}{3^{2(1+\alpha)}} \mathbb{P}_x\left(\tau_{U_1} < t\right).$$

Furthermore, by (4.16) with $r = 1$, we have

$$\mathbb{P}_x\left(\tau_{U_1} < t\right) \leq \mathbb{P}_x\left(\tau_{B(x,1)} < t\right) \leq c_3 t.$$

Combining the above two inequalities, we get (6.2) with $c_2 := \frac{c_3 C_1}{3^{2(1+\alpha)}}$. □

In the above example, the domain U is connected but not convex. The ideas used in the above example can be refined to show that the lower bound estimate (1.13) for the Dirichlet heat kernel may still fail for some smooth bounded convex domains.

Example 6.2. Let D be the tilted rectangle with rounded corners shown in Figure 8. The points x, y have the coordinates $(-4,-4)$ and $(4,4)$ respectively. The connected open set $D \subset \mathbb{R}^2$ that does not satisfy $(\mathbf{H}_\gamma)$ for any $\gamma \in (0,1)$ as swapping any coordinate of x by that of y results a point falling outside D.

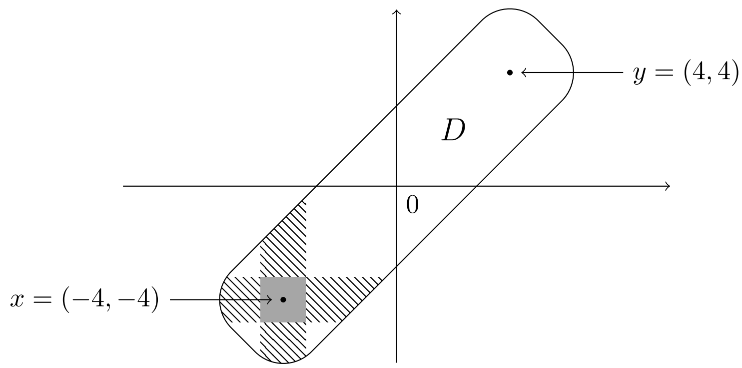

FIGURE 8. The convex set $D \subset \mathbb{R}^2$ that does not satisfies condition $(\mathbf{H}_\gamma)$ for any $\gamma \in (0,1)$

The similar arguments in Example 6.1 show that the lower bound estimate (1.13) in Theorem 1.5(ii) fails for this convex open set D. We skip the details here. The reader can find details in the arXiv version of this paper at [7, Example 6.2 on p. 45–47].

The following is an example of a bounded smooth open set $D \subset \mathbb{R}^2$ that does not satisfy the irreducibility condition (1.11) but for which we can derive two-sided sharp estimates for the Dirichlet heat kernel in D.

Example 6.3. Let $r > 0$ and D be the union of two disjoint balls sitting in diagonal quadrants:

$$D := B(O_1, r) \cup B(O_2, r), \tag{6.5}$$

(see Figure 9) where the two points $O_1, O_2 \in \mathbb{R}^d$ satisfy

$$O_1^{(i)} < -r \quad \text{and} \quad O_2^{(i)} > r \quad \text{for } 1 \le i \le d.$$

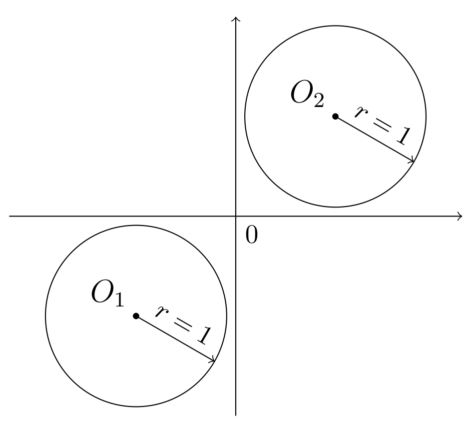

FIGURE 9. The set $D := B(O_1, r) \cup B(O_2, r)$ with $r = 1$ on $\mathbb{R}^2$

The open set D clearly does not satisfy the condition (1.11) as for any $x \in O_1$ and $y \in O_2$, swapping any coordinate of x by that of y results a point falling outside D. So X^D is not irreducible. It follows from Corollary 1.4 that the following holds with $B_1 := B(O_1, r)$ and $B_2 := B(O_2, r)$.

(i) $p_D(t, x, y) = 0$ for all $t > 0$ and x, y that are not in the same connected component of D.

(ii) For $i = 1, 2$, $p_D(t, x, y) = p_{B_i}(t, x, y)$ for all $t > 0$ and $x, y \in B_i$, and $p_{B_i}(t, x, y)$ has the two-sided estimates given by Theorem 1.5 with B_i in place of D there.

The above example clearly can be extended to more general open sets D that is the union of two disjoint $C^{1.1}$-smooth connected open subsets that at most one of them has non-empty intersection with any line that is parallel to the coordinate axes.

We conclude this paper by presenting the proof of Theorem 1.6, using the techniques from Section 4.

Proof of Theorem 1.6. Fix $T>0$. Note that $p_D(t,x,y)$ is symmetric. For any $x,y\in D$, if y does not belong to $A_1\cup A_4$, then x,y belong to the case in (i).

(i). For any $x\in A_i$, $y\in A_j$ with $|i-j|\le 2$, we have that x,y belong to some $C^{1,1}$ open subset U of D and U satisfies the condition $(\mathbf{H_1})$. In this case, we also have

$$\delta_D(x)=\delta_U(x),\quad \delta_D(y)=\delta_U(y). \tag{6.6}$$

For example, for $x\in A_1$ and $y\in A_3$, we can set $U=A_1\cup A_2\cup A_3$. For $x\in A_4$ and $y\in A_2$, we can set $U=A_2\cup A_3\cup A_4$.

The upper bound of $p_D(t,x,y)$ in (1.16) follows directly from Theorem 1.5 (i) since D is a $C^{1,1}$ open set. For the lower bound of $p_D(t,x,y)$ in (1.16), note that $U\subset D$ is a $C^{1,1}$ open set satisfying the condition $(\mathbf{H_1})$. By Theorem 1.5 (ii) and (6.6), we have

$$\begin{aligned}p_D(t,x,y)\ge p_U(t,x,y)&\ge\ c_1\left(1\wedge\frac{\delta_U(x)^{\alpha/2}}{\sqrt{t}}\right)\left(1\wedge\frac{\delta_U(y)^{\alpha/2}}{\sqrt{t}}\right)p(t,x,y)\\&=c_1\left(1\wedge\frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}}\right)\left(1\wedge\frac{\delta_D(y)^{\alpha/2}}{\sqrt{t}}\right)p(t,x,y).\end{aligned}$$

(ii). Fix $x=(x^{(1)},x^{(2)})\in A_1$ and $y=(y^{(1)},y^{(2)})\in A_4$. Similar to (6.3), one can apply (4.1) to prove that

$$\mathbb{P}_x\left(X_{\tau_{A_1}}\in A_3\cup A_4\right)=0.$$

Hence, by (1.6), the function $p_D(t-\tau_{A_1},X_{\tau_{A_1}},y)\mathbb{1}_{\{\tau_{A_1}<t\}}$ is uniformly bounded in $y\in A_4$, and then, by the strong Markov property of X, the continuity of p_D and dominated convergence theorem, we have

$$p_D(t,x,y)=\mathbb{E}_x\left[p_D(t-\tau_{A_1},X_{\tau_{A_1}},y);\tau_{A_1}<t\right]=\mathbb{E}_x\left[p_D(t-\tau_{A_1},X_{\tau_{A_1}},y);\tau_{A_1}<t,X_{\tau_{A_1}}\in A_2\right].$$

Furthermore, by (4.1) again, we have for $t>0$,

$$\begin{aligned}p_D(t,x,y)&=\mathbb{E}_x\left[p_D(t-\tau_{A_1},X_{\tau_{A_1}},y);\tau_{A_1}<t,X_{\tau_{A_1}}\in A_2\right]\\&=\mathbb{E}_x\left[\int_0^t\mathbb{1}_{\{s<\tau_{A_1}\}}\cdot\left(\sum_{i=1}^2\int_{\mathbb{R}}\mathbb{1}_{\{[X_s]^i_\theta\in A_2\}}\cdot p_D(t-s,[X_s]^i_\theta,y)j(X_s^{(i)},\theta)d\theta\right)ds\right]\\&=\int_0^t\int_{A_1}p_{A_1}(s,x,u)\left(\int_{\mathbb{R}}\mathbb{1}_{\{[u]^1_\theta\in A_2\}}\cdot p_D(t-s,[u]^1_\theta,y)j(u^{(1)},\theta)d\theta\right)duds.\end{aligned}\tag{6.7}$$

Note that

- for any $u=(u^{(1)},u^{(2)})\in A_1$ and $\theta\in\mathbb{R}$ with $[u]^1_\theta\in A_2$, we have
$$|u^{(1)}-\theta|\ge 1,\quad |y^{(1)}-\theta|\ge 1\quad\text{and}\quad |u^{(2)}-y^{(2)}|\ge 1;$$
- by (1.16), for any $[u]^1_\theta\in A_2$ and $0<s<t$, we have

$$\begin{aligned}p_D(t-s,[u]^1_\theta,y)&\le C_1C_6\left(1\wedge\frac{\delta_D(y)^{\alpha/2}}{\sqrt{t-s}}\right)\left((t-s)^{-\frac{1}{\alpha}}\wedge\frac{t-s}{|\theta-y^{(1)}|^{1+\alpha}}\right)\left((t-s)^{-\frac{1}{\alpha}}\wedge\frac{t-s}{|u^{(2)}-y^{(2)}|^{1+\alpha}}\right)\\&\le C_1C_6\left(1\wedge\frac{\delta_D(y)^{\alpha/2}}{\sqrt{t-s}}\right)t^2.\end{aligned}$$

Combining the above two inequalities with (6.7) and Theorem 1.5 (i), we obtain, for all $t\in(0,T]$,

$$\begin{aligned}p_D(t,x,y)&\le c_2\int_0^t\int_{A_1}p_{A_1}(s,x,u)\left(\int_{O_2^{(1)}-1}^{O_2^{(1)}+1}d\theta\right)\left(1\wedge\frac{\delta_D(y)^{\alpha/2}}{\sqrt{t-s}}\right)t^2duds\\&\le 2c_2t^2\int_0^t\left(\int_{A_1}p(s,x,u)du\right)\left(1\wedge\frac{\delta_{A_1}(x)^{\alpha/2}}{\sqrt{s}}\right)\left(1\wedge\frac{\delta_D(y)^{\alpha/2}}{\sqrt{t-s}}\right)ds\end{aligned}$$

$$\le 2c_2t^2\left(\int_0^{t/2}+\int_{t/2}^t\right)\left(1\wedge\frac{\delta_D(x)^{\alpha/2}}{\sqrt{s}}\right)\left(1\wedge\frac{\delta_D(y)^{\alpha/2}}{\sqrt{t-s}}\right)ds$$
$$= c_3t^3\left(1\wedge\frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}}\right)\left(1\wedge\frac{\delta_D(y)^{\alpha/2}}{\sqrt{t}}\right).$$

which is exactly the upper bound in (1.17).

We apply (6.7) to establish the lower bound of $p_D(t,x,y)$. Fix $t\in(0,T]$. Let $\delta<(16T^{1/\alpha})^{-1}$ and $Q_x\in\partial D$ be such that $\delta_D(x)=|x-Q_x|$. Set

$$x_0=\begin{cases}Q_x+\frac{2\delta t^{1/\alpha}}{|x-Q_x|}(x-Q_x), & \text{if } \delta_D(x)<2\delta t^{1/\alpha},\\ x, & \text{if } \delta_D(x)\ge 2\delta t^{1/\alpha}.\end{cases}$$

Define $E_x=B(x_0,\delta t^{1/\alpha})$. Note that $E_x\subset A_1\subset D$. Observe that

- for any $s\in(0,t)$ and $u\in E_x$, we have $\delta_D(u)\ge\delta t^{1/\alpha}$ and so by Theorem 1.5 (ii),

$$p_{A_1}(s,x,u)$$
$$\ge c_4\left(1\wedge\frac{\delta_{A_1}(x)^{\alpha/2}}{\sqrt{s}}\right)\left(1\wedge\frac{\delta_{A_1}(u)^{\alpha/2}}{\sqrt{s}}\right)\left(s^{-\frac{1}{\alpha}}\wedge\frac{s}{|x^{(1)}-u^{(1)}|^{1+\alpha}}\right)\left(s^{-\frac{1}{\alpha}}\wedge\frac{s}{|x^{(2)}-u^{(2)}|^{1+\alpha}}\right)$$
$$\ge c_4\left(1\wedge\frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}}\right)(1\wedge\delta^{\alpha/2})\left(s^{-\frac{1}{\alpha}}\wedge\frac{s}{(3\delta t^{1/\alpha})^{1+\alpha}}\right)^2\ge c_5\left(1\wedge\frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}}\right)s^2t^{-2-2/\alpha};$$

- for any $u\in E_x$ and $\theta\in[O_2^{(1)}-\frac12,O_2^{(1)}+\frac12]$, we have $[u]^1_\theta\in A_2$, $|u^{(1)}-\theta|\le 3$ and $\delta_D([u]^1_\theta)\ge c_6t^{1/\alpha}$. Furthermore, by (1.16), we have for any $s\in(0,\frac{t}{2}]$,

$$p_D(t-s,[u]^1_\theta,y)$$
$$\ge c_7\left(1\wedge\frac{\delta_D([u]^1_\theta)^{\alpha/2}}{\sqrt{t-s}}\right)\left(1\wedge\frac{\delta_D(y)^{\alpha/2}}{\sqrt{t-s}}\right)\left((t-s)^{-\frac{1}{\alpha}}\wedge\frac{t-s}{|\theta-y^{(1)}|^{1+\alpha}}\right)\left((t-s)^{-\frac{1}{\alpha}}\wedge\frac{t-s}{|u^{(2)}-y^{(2)}|^{1+\alpha}}\right)$$
$$\ge c_8\left(1\wedge\frac{\delta_D(y)^{\alpha/2}}{\sqrt{t}}\right)t^2.$$

Combining the above two inequalities with (6.7), we have for any $t\in(0,T]$,

$$p_D(t,x,y)\ge\int_0^{t/2}\int_{E_x}p_{A_1}(s,x,u)\left(\int_{\mathbb{R}}\mathbb{1}_{\{\theta\in[O_2^{(1)}-\frac12,O_2^{(1)}+\frac12]\}}\cdot p_D(t-s,[u]^1_\theta,y)j(u^{(1)},\theta)d\theta\right)duds$$
$$\ge c_9\int_0^{t/2}\int_{E_x}\left(1\wedge\frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}}\right)s^2t^{-2-2/\alpha}\cdot\left(\int_{O_2^{(1)}-\frac12}^{O_2^{(1)}+\frac12}\left(1\wedge\frac{\delta_D(y)^{\alpha/2}}{\sqrt{t}}\right)t^2\frac{1}{|u^{(1)}-\theta|^{1+\alpha}}d\theta\right)duds$$
$$\ge c_{10}t^{-2/\alpha}\left(1\wedge\frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}}\right)\left(1\wedge\frac{\delta_D(y)^{\alpha/2}}{\sqrt{t}}\right)|E_x|\int_0^{t/2}s^2ds$$
$$= c_{11}t^3\left(1\wedge\frac{\delta_D(x)^{\alpha/2}}{\sqrt{t}}\right)\left(1\wedge\frac{\delta_D(y)^{\alpha/2}}{\sqrt{t}}\right),$$

which is the lower bound in (1.17).

On the other hand, for $x=(x^{(1)},x^{(2)})\in A_1$ and $y=(y^{(1)},y^{(2)})\in A_4$, we have

$$1\le|x^{(k)}-y^{(k)}|\le 3+3+2=8\quad\text{for } k=1,2.$$

Hence by (1.6),

$$p(t,x,y)\overset{c_{13}}{\asymp}\prod_{k=1}^2\left(t^{-1/\alpha}\wedge\frac{t}{|x^{(k)}-y^{(k)}|^{1+\alpha}}\right)\overset{c_{14}}{\asymp}t^2\quad\text{for any } t\in(0,T],$$

where c_{13} and c_{14} are positive constants that depend only on α and T. So we get (1.18) from (1.17). This completes the proof of the theorem. □

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