Constantly curved holomorphic two-spheres in the complex Grassmannian $G(2,6)$ with constant square norm of the second fundamental form

Jie Fei [*] Ling He [†] Jun Wang [‡]

ABSTRACT. We completely classify all noncongruent linearly full totally unramified constantly curved holomorphic two-spheres in $G(2,6)$ with constant square norm of the second fundamental form. They turn out to be homogeneous.

Keywords and Phrases. Complex Grassmannian, Holomorphic two-sphere, Second fundamental form.

Mathematics Subject Classification (2010). 53C40; 53C42; 53C55.

1 Introduction

Motivated by Calabi's rigidity principle about holomorphic isometric embedding from one complex manifold into the complex projective space in [3], one can study the holomorphic isometric embedding from one complex manifold into the complex Grassmannian. A holomorphic isometry starting from Riemann sphere S^2 with metric of constant curvature is also called a constantly curved holomorphic two-sphere. In differential geometry, the question of classification of noncongruent constantly curved holomorphic two-spheres in the complex Grassmannians is an important and difficult problem. More interestingly, it has close contact with the Grassmannian sigma models in theoretical physics (for example, see [9]).

We denote by g_{FS} the Fubini-Study metric of constant holomorphic sectional curvature 4. From Calabi's rigidity principle (cf. [3]; see also [27], Sec.5 in [1]), we know that if $\varphi : S^2 \to (\mathbb{C}P^n, g_{FS})$ is a linearly full (the image does not lie in some subspace $\mathbb{C}^k$ for $k < n+1$) constantly curved holomorphic immersion of degree d, then d $= n$ and φ is the well-known Veronese embedding (up to $U(n+1)$) defined by

$$V_0^{(n)}(z) = \left[1, \sqrt{\binom{n}{1}}z, \cdots, \sqrt{\binom{n}{n}}z^n\right].$$

It is interesting that $V_0^{(n)}(z)$ can generate the harmonic sequence $V_0^{(n)}, V_1^{(n)}, \cdots, V_n^{(n)}$ in

[*] J. Fei

School of Mathematics and Physics, Xi'an Jiaotong-Liverpool University, Suzhou 215123, P. R. China
E-mail: jie.fei@xjtlu.edu.cn

[†] L. He (Corresponding author)

Center for Applied Mathematics and KL-AAGDM, Tianjin University, Tianjin, 300072, P. R. China
E-mail: heling@tju.edu.cn

[‡] J. Wang

School of Mathematics Sciences and Institute of Mathematics, Nanjing Normal University, Nanjing 210023, P. R. China
E-mail: wangjun706@njnu.edu.cn

$\mathbb{C}P^n$, where $V_i^{(n)} = [f_{i,0}, \cdots, f_{i,p}, \cdots, f_{i,n}]$ with $f_{i,p}$ being explicitly given by

$$f_{i,p}(z) = \frac{i!}{(1+z\bar{z})^i}\sqrt{\binom{n}{p}} z^{p-i} \sum_k (-1)^k \binom{p}{i-k}\binom{n-p}{k}(z\bar{z})^k.$$

Such a map $V_i^{(n)} : S^2 \to \mathbb{C}P^n$ is a conformal minimal immersion with constant curvature and constant Kähler angle, which are given by

$$K_i^{(n)} = \frac{4}{n+2i(n-i)}, \quad \cos\alpha_i^{(n)} = \frac{n-2i}{n+2i(n-i)}.$$

This harmonic sequence is well known as *Veronese sequence* (cf. [1], Sec.5).

We denote by $ds^2_{G(m,m+n)}$ the standard Kähler metric of complex Grassmannian $G(m, m+n)$ ($m \leq n$), which can also be induced from g_{FS} by the Plücker embedding. Chi-Zheng [7] classified the constantly curved holomorphic two-spheres of degree 4 in $\left(G(2,4), ds^2_{G(2,4)}\right)$ into two families (up to $U(4)$) by using the method of moving frames and Cartan's theory of higher order invariants (cf.[19]). This result means that the constantly curved holomorphic two-spheres in $\left(G(m, m+n), ds^2_{G(m,m+n)}\right)$ are more complicated. Li-Yu [23] (see [22] for a detailed proof) showed that if $\varphi : S^2 \to G(2,4)$ is a linearly full constantly curved holomorphic immersion of degree d, then d $= 2, 3, 4$ and φ is explicitly characterized (up to $U(4)$). For the case of $G(2,5)$, Jiao-Peng ([20], [21]) proved that if $\varphi : S^2 \to G(2,5)$ is a non-singular constantly curved holomorphic immersion, then d $= 1, 2, 3, 4, 5$ and φ is explicitly characterized (up to $U(5)$) by using Plücker embedding.

Delisle-Hussin-Zakrzewski [9] recovered the known classification results in $G(2,4)$ and $G(2,5)$ mentioned above from the viewpoint of Grassmannian sigma models, and proposed two conjectures, which we refer to as DHZ conjectures in the following.

Conjecture 1. *If $\varphi : S^2 \to G(m, m+n)$ is a constantly curved holomorphic immersion of degree d, then d $\leq mn$.*

Conjecture 2. *For m, n fixed, the holomorphic immersion φ can be constructed for $1 \leq$ d $\leq mn$.*

Until now, there are few results about these two problems. Under the assumption of homogeneity (the image is an orbit of an isometry subgroup of the target space), Peng-Xu [26] and Fei [10] independently used the representations of $SU(2)$ to give a complete classification of linearly full constantly curved holomorphic two-spheres of degree d in $G(2, n+2)$ and obtained that d takes n or $2n$. Delisle-Hussin-Zakrzewski ([9], Proposition 1) declared that d ≤ 6 for $G(2,5)$. Recently the second named author gave some detailed discussions in [17]. Interestingly, Chi-Xie-Xu [8] constructed many non-homogeneous and singular (in Jiao-Peng's sense) constantly curved holomorphic two-spheres of degree 6 in $G(2,5)$ and described the moduli space of such two-spheres.

S.S. Chern et al.([4], [5]) showed the interest of the study of compact minimal submanifolds on the unit sphere S^{n+p} with constant square norm of the second fundamental form, which led to the famous Chern conjecture for discreteness of such constant values. N. Mok et al.([24], [25]) indicated that the second fundamental form can be used as a tool for studying the non-totally geodesic holomorphic isometric embeddings of the Poincaré disk into bounded symmetric domains. Inspired by these works, the second fundamental

forms are expected to play an important role for studying constantly curved holomorphic two-spheres in $G(m, m+n)$. Recently, the first and third named authors and Xu ([29],[13]) applied the method of moving frames and harmonic sequences to completely classify constantly curved holomorphic two-spheres in $G(2, N)$ and $G(3, N)$ with the square norm of the second fundamental form satisfying a certain pinching condition.

The current article is an attempt to study linearly full constantly curved holomorphic two-spheres of degree d in $G(2, n+2)$ with constant square norm of the second fundamental form. For the reducible case, the first named author, Jiao and Xu in [12] and the second named author in her Ph.D. Thesis [16] showed that d $= n$ with $\varphi = V_0^{(n+1)} \oplus V_1^{(n+1)}$ (up to $U(n+2)$) or d $= 2n$ with $\varphi = V_0^{(n)} \oplus v$ (up to $U(n+2)$), where v is a non-zero constant vector. Notice that no assumption with respect to the second fundamental form is needed, and these two holomorphic curves are both homogeneous.

For the irreducible case, we have the Gauss equation (cf.[10],[14])

$$K = 4 - 8|\det A_1|^2 - \frac{S}{2}, \qquad (1.1)$$

where $K = 4/$d, S is the square norm of the second fundamental form, and $|\det A_1|^2\phi^2\bar{\phi}^2$ is a global defined $(2,2)$-form (see section 2). The key idea is to transform the condition of S being constant into the one that $|\det A_1|^2$ is a constant. Notice that S involving the second derivative of the holomorphic mapping is hard to compute, while $|\det A_1|^2$ is only related to the first derivative of the holomorphic mapping. This makes it possible to calculate explicitly. Thus by solving algebraic equations in the congruence class, we get the following main theorem.

Theorem 1.1 *Let $\varphi : S^2 \to G(2, n+2)$ be a linearly full irreducible constantly curved holomorphic two-sphere of degree d $= 4$ with constant square norm of the second fundamental form S. Then $n = 4$ and the corresponding holomorphic two-spheres are given as follows, up to $U(6)$,*

$$\varphi = \begin{bmatrix} 1 & 0 & |t-2|z^3 & \sqrt{t}z & \sqrt{\frac{3-t}{4-t}}|t-2|z^2 & \sqrt{\frac{t}{4-t}}z^2 \\ 0 & 1 & \sqrt{3-t}z^2 & 0 & \sqrt{4-t}z & 0 \end{bmatrix} \subset G(2,6), \ 0 < t \le 3 \qquad (1.2)$$

with $S = t^2 - 4t + 6$. Here when $t = 3$, $\varphi = V_0^{(1)} \oplus V_0^{(3)}$ with $S = 3$; when $t = 2$, $\varphi = V_0^{(2)} \oplus V_0^{(2)}$ with $S = 2$.

Applying Theorem 1.1 and appropriate harmonic sequences, we obtain the classification theorem of constantly curved holomorphic two-spheres in $G(2, 6)$ with constant square norm of the second fundamental form, as follows.

Theorem 1.2 *Let $\varphi : S^2 \to G(2, 6)$ be a linearly full constantly curved holomorphic two-sphere of degree d with constant square norm of the second fundamental form.*
(1) If φ is reducible, then d $= 4$ with $\varphi = V_0^{(5)} \oplus V_1^{(5)}$ (up to $U(6)$), or d $= 8$ with $\varphi = V_0^{(4)} \oplus v$ (up to $U(6)$), where v is a non-zero constant vector.
(2) If φ is irreducible and totally unramified, then d $= 4$ and φ is unitary congruent to (1.2).

Remark 1.3 *It follows from Theorem 1.2 that DHZ's conjecture 1 is true for $G(2,6)$ in the case of totally unramified and constant square norm of the second fundamental form.*

Recently, we [11] give a local rigidity characterization of all homogeneous holomorphic two-spheres in $G(2,N)$ in terms of a new global invariant κ defined by the square norm of $(1,0)$ part of the second order covariant differential of the first ∂-transform for a holomorphic curve in $G(2,N)$. Specifically, we showed that a linearly full irreducible constantly curved holomorphic two-sphere in $G(2,N)$ with constant square norm of the second fundamental form and κ vanishing identically is unitary congruent to

$$V_0^{(n)} \oplus \left(\cos\theta V_1^{(n)} + \sin\theta V_0^{(n-2)} \right) \ \text{or} \ V_0^{(n_1)} \oplus V_0^{(n_2)},$$

where $\theta \in (0, \frac{\pi}{2}]$, $N = 2n$ or $N = n_1 + n_2 + 2$. By virtue of the explicit expression given by (1.2) and straightforward calculations, we checked that the embedding (1.2) has vanishing κ for each t, which concludes that the holomorphic curve given by (1.2) is unitary congruent to the homogeneous one

$$V_0^{(3)} \oplus \left(\cos\theta V_1^{(3)} + \sin\theta V_0^{(1)} \right)$$

and the correspondence between the two parameters is that $t = 3\sin^2\theta$. Therefore, the classification results of Theorem 1.2 tell us that

Theorem 1.4 *A linearly full totally unramified constantly curved holomorphic two-sphere in $G(2,6)$ with constant square norm of the second fundamental form must be homogeneous.*

The requirement that a constantly curved holomorphic two-sphere is totally unramified is a somehow strong assumption. Thus one is optimistic about generalizing Theorem 1.4 to any complex Grassmannian, which deserves further investigation. In addition, whether is the assumption that the constantly curved holomorphic two-sphere is totally unramified in Theorem 1.2 and Theorem 1.4 necessary? Since the condition of constant square norm of the second fundamental form leads to that the constantly curved holomorphic two-sphere is unramified, then can one construct an example of non-homogeneous constantly curved holomorphic two-sphere with constant square norm of the second fundamental form satisfying that the second element of the harmonic sequence has ramified points?

This paper is organized as follows. In section 2, we give a characterization of constantly curved holomorphic two-sphere in $G(2, n+2)$ with constant square norm of the second fundamental form in terms of a system of algebraic equations. In section 3, we firstly solve the algebraic system to prove Theorem 1.1. Applying this result, together with the theory of harmonic sequence, we verify Theorem 1.2.

2 Constantly curved holomorphic two-spheres in $G(2, n+2)$

Let $\varphi : S^2 \to G(2, n+2)$ be a linearly full constantly curved holomorphic two-sphere of degree d. Let $\varphi(0) = \mathcal{Z}_0 \in G(2, n+2)$. We choose a rectangular coordinate system in $\mathbb{C}^{n+2}$ such that the extended matrix $\mathcal{Z}_0 = (I_2, \mathbf{0})$. Then on the neighborhood of $\mathcal{Z}_0$ denoted by $\mathcal{V}_{\mathcal{Z}_0}$, we can write

$$\varphi(z) = [I_2, F(z)],$$

where $F(z) = \begin{pmatrix} F_1(z) \\ F_2(z) \end{pmatrix}$ is a $2 \times n$ matrix-valued holomorphic function satisfying $F(0) = 0$.

Let

$$Pl : \left(G(2, n+2), ds^2_{G(2,n+2)} \right) \to \left(\mathbb{C}P^N, g_{FS} \right) \ \ (N = \frac{(n+2)(n+1)}{2} - 1)$$

be the standard Plücker embedding, which is a holomorphic isometric embedding (cf. [15]). Then $Pl \circ \varphi : S^2 \to \mathbb{C}P^N$ given by

$$Pl \circ \varphi = [v_1 \wedge v_2] = \begin{bmatrix} 1 & F_2 & -F_1 & F_1 \wedge F_2 \end{bmatrix}$$

is a constantly curved holomorphic two-sphere, which may be not linearly full. It follows from Calabi's rigidity theorem (cf. [3]; see also [27], Sec.5 in [1]) that there exists a constant matrix $U \in U(N+1)$ such that $Pl \circ \varphi = V_0^{(d)} \cdot U$, where $V_0^{(d)} : S^2 \to \mathbb{C}P^N$ is the holomorphic Veronese embedding given by

$$V_0^{(d)}(z) = \begin{bmatrix} 1 & \sqrt{\binom{d}{1}}z & \cdots & \sqrt{\binom{d}{k}}z^k & \cdots & \sqrt{\binom{d}{d}}z^d & 0 & \cdots & 0 \end{bmatrix}.$$

We immediately conclude that $d \le N$ and

$$1 + |F_1|^2 + |F_2|^2 + |F_1 \wedge F_2|^2 = (1 + z\bar{z})^d. \tag{2.1}$$

Moreover, we have the Gauss equation (cf. [10],[14])

$$K = 4 - 8|\det A_1|^2 - \frac{S}{2}, \tag{2.2}$$

where $K = 4/d$, S is the square norm of the second fundamental form, and $|\det A_1|^2 \phi^2 \bar{\phi}^2$ is a globally defined $(2, 2)$-form specified as follows. Here, ϕ is a local unitary coframe of $(1, 0)$ type with respect to the induced metric $ds^2 = \varphi^*(ds^2_{G(2,n+2)})$.

Let $\varphi = \begin{bmatrix} I_2 & F \end{bmatrix}$, $F(z) = \begin{bmatrix} F_1(z) \\ F_2(z) \end{bmatrix}$. Set $v_1 = \begin{bmatrix} 1 & 0 & F_1 \end{bmatrix}$, $v_2 = \begin{bmatrix} 0 & 1 & F_2 \end{bmatrix}$, $e_1 = \frac{v_1}{|v_1|}$, $e_2 = \frac{v_2 - \frac{\langle v_2, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1}{\left| v_2 - \frac{\langle v_2, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1 \right|}$. Let e_3, e_4 be a unitary basis of the orthogonal projection plane $\varphi^{\perp}(\partial_z \varphi)$. Then we have

$$\partial_z e_1 = a_{11}e_1 + a_{12}e_2 + a_{13}e_3 + a_{14}e_4,$$
$$\partial_z e_2 = a_{21}e_1 + a_{22}e_2 + a_{23}e_3 + a_{24}e_4.$$

It follows that

$$|\det A_1|^2 \phi^2 \bar{\phi}^2 = |a_{13}a_{24} - a_{14}a_{23}|^2 dz^2 d\bar{z}^2.$$

If $|\det A_1|^2\phi^2\overline{\phi}^2$ is identically equal to zero on S^2, then φ is called *reducible*. If $|\det A_1|^2\phi^2\overline{\phi}^2$ is not identically equal to zero on S^2, then φ is called *irreducible* (see sec.3 in [2] for general harmonic map). In the latter case, $|\det A_1|^2\phi^2\overline{\phi}^2$ has isolated zeros, which is called *ramified points*; the order of zeros is called *multiplicity of the corresponding ramified points*. In particular, if $|\det A_1|^2\phi^2\overline{\phi}^2$ has no zeros, then φ is called *unramified*. Similarly, we can give the definition of unramified minimal two-sphere. If every element of the harmonic sequence generated by φ is unramified, then φ is called *totally unramified* (cf. [18]).

Now we give a characterization of constantly curved holomorphic two-spheres with constant square norm of the second fundamental form as follows:

Theorem 2.1 *Let $\varphi : S^2 \to G(2, n+2)$ be a linearly full constantly curved holomorphic two-sphere of degree* d *with constant square norm of the second fundamental form. Locally, set*

$$\varphi = \begin{bmatrix} I_2 & F \end{bmatrix}, \quad F(z) = \begin{bmatrix} F_1(z) \\ F_2(z) \end{bmatrix}$$

where $F(z)$ is a $2 \times n$ matrix-valued holomorphic function satisfying $F(0) = 0$. Then

$$1 + |F_1|^2 + |F_2|^2 + |F_1 \wedge F_2|^2 = (1 + z\bar{z})^{\mathrm{d}}, \tag{2.3}$$

and

$$|\partial_z F_1 \wedge \partial_z F_2|^2 + |\partial_z F_1 \wedge \partial_z F_2 \wedge F_1|^2 + |\partial_z F_1 \wedge \partial_z F_2 \wedge F_2|^2 + |\partial_z F_1 \wedge \partial_z F_2 \wedge F_1 \wedge F_2|^2 = c(1 + z\bar{z})^{2\mathrm{d}-4}, \tag{2.4}$$

where c is a non-negative constant.

Proof: If S is constant, then $|\det A_1|^2$ is constant by (2.2). Here $|v_1 \wedge v_2|^2 = (1+z\bar{z})^{\mathrm{d}}$ and $e_1 \wedge e_2 = \frac{v_1 \wedge v_2}{|v_1 \wedge v_2|}$. On the one hand,

$$\begin{aligned} \partial_z(e_1 \wedge e_2) &= \partial_z e_1 \wedge e_2 + e_1 \wedge \partial_z e_2 \\ &= (a_{11} + a_{22})e_1 \wedge e_2 - a_{13}e_2 \wedge e_3 - a_{14}e_2 \wedge e_4 + a_{23}e_1 \wedge e_3 + a_{24}e_1 \wedge e_4, \end{aligned}$$

which means

$$\partial_z(e_1 \wedge e_2) \wedge \partial_z(e_1 \wedge e_2) = 2(-a_{13}a_{24} + a_{14}a_{23})e_1 \wedge e_2 \wedge e_3 \wedge e_4.$$

On the other hand,

$$\partial_z(e_1 \wedge e_2) \wedge \partial_z(e_1 \wedge e_2) = \frac{-2v_1 \wedge v_2 \wedge \partial_z v_1 \wedge \partial_z v_2}{|v_1 \wedge v_2|^2}.$$

Since the induced metric is $ds^2 = \phi\overline{\phi} = \frac{\mathrm{d}}{(1+z\bar{z})^2}dzd\bar{z}$, then we have

$$|\det A_1|^2 = \frac{|v_1 \wedge v_2 \wedge \partial_z v_1 \wedge \partial_z v_2|^2}{\mathrm{d}^2(1 + z\bar{z})^{2\mathrm{d}-4}}.$$

Setting $|\det A_1|^2 = c/\mathrm{d}^2$, where c is a non-negative constant, we get

$$|v_1 \wedge v_2 \wedge \partial_z v_1 \wedge \partial_z v_2|^2 = c(1 + z\bar{z})^{2\mathrm{d}-4}. \tag{2.5}$$

6

A straightforward calculation verifies (2.4). Furthermore we have

$$S = 8 - \frac{16c}{\mathrm{d}^2} - \frac{8}{\mathrm{d}}.$$

$\square$

Let $F = \sum_{\alpha=1}^{N} A_\alpha z^\alpha$, $A_\alpha = \begin{bmatrix} a_1^{(\alpha)} \\ a_2^{(\alpha)} \end{bmatrix}$, $W_\alpha = \begin{bmatrix} a_1^{(\alpha)} & a_2^{(\alpha)} \end{bmatrix}$. Then we have

$$F_1 = \sum_{\alpha=1}^{N} a_1^{(\alpha)} z^\alpha, \ F_2 = \sum_{\alpha=1}^{N} a_2^{(\alpha)} z^\alpha.$$

Set

$$F_1 \wedge F_2 = \sum_{j=2}^{2N} V_j z^j, \ \partial_z F_1 \wedge \partial_z F_2 = \sum_{j=2}^{2N} R_j z^{j-2}.$$

A straightforward calculation shows

$$\partial_z F_1 \wedge \partial_z F_2 \wedge F_1 = \sum_{j=2}^{2N} \sum_{\alpha=1}^{N} R_j \wedge a_1^{(\alpha)} z^{j+\alpha-2} = \sum_{p=4}^{3N} S_p z^{p-2},$$

$$\partial_z F_1 \wedge \partial_z F_2 \wedge F_2 = \sum_{j=2}^{2N} \sum_{\alpha=1}^{N} R_j \wedge a_2^{(\alpha)} z^{j+\alpha-2} = \sum_{p=4}^{3N} T_p z^{p-2},$$

$$\partial_z F_1 \wedge \partial_z F_2 \wedge F_1 \wedge F_2 = \sum_{j,k=2}^{2N} R_j \wedge V_k z^{j+k-2} = \sum_{p=6}^{4N} X_p z^{p-2}.$$

Set

$$U = \begin{bmatrix} 1 & \mathbf{0} & \mathbf{0} \\ 0 & W_1 & \mathbf{0} \\ \vdots & \vdots & \vdots \\ 0 & W_{\mathrm{d}} & V_{\mathrm{d}} \end{bmatrix}, \ Q = \begin{bmatrix} R_2 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ R_3 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ R_4 & S_4 & T_4 & \mathbf{0} \\ R_5 & S_5 & T_5 & \mathbf{0} \\ R_6 & S_6 & T_6 & X_6 \\ \vdots & \vdots & \vdots & \vdots \\ R_{2\mathrm{d}-2} & S_{2\mathrm{d}-2} & T_{2\mathrm{d}-2} & X_{2\mathrm{d}-2} \end{bmatrix}.$$

Then we see that $U \in \mathbb{C}^{(\mathrm{d}+1)\times(N+1)}$ and $Q \in \mathbb{C}^{(2\mathrm{d}-3)\times\left(\binom{n}{2}+2\binom{n}{3}+\binom{n}{4}\right)}$.

Applying Theorem 2.1, we know that (2.3) is equivalent to

$$UU^* = \Lambda_1, \tag{2.6}$$

where $*$ denotes conjugate transpose and

$$\Lambda_1 = \mathrm{diag}\left\{1, \binom{\mathrm{d}}{1}, \cdots, \binom{\mathrm{d}}{\mathrm{d}}\right\},$$

and (2.4) is equivalent to

$$QQ^* = \Lambda_2, \tag{2.7}$$

7

where

$$\Lambda_2 = \mathrm{diag}\left\{c, \binom{2\mathrm{d}-4}{1}c, \cdots, \binom{2\mathrm{d}-4}{2\mathrm{d}-4}c\right\}.$$

In order to determine the linearly full constantly curved holomorphic two-spheres of degree d in $G(2, n+2)$ with constant square norm of the second fundamental form, we need to solve (2.6) and (2.7), modulo extrinsically the ambient unitary $U(n+2)$-congruence. Considering the case of d = 4, we can get our main result.

3 Proof of Theorems 1.1 and 1.2

Proof of Theorem 1.1 Substituting d = 4 into (2.6) and (2.7) respectively yields

$$U = \begin{bmatrix} 1 & \mathbf{0} & \mathbf{0} \\ 0 & W_1 & \mathbf{0} \\ 0 & W_2 & V_2 \\ 0 & W_3 & V_3 \\ 0 & W_4 & V_4 \end{bmatrix}, \ UU^* = \mathrm{diag}\left\{1, 4, 6, 4, 1\right\} \tag{3.1}$$

and

$$Q = \begin{bmatrix} R_2 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ R_3 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ R_4 & S_4 & T_4 & \mathbf{0} \\ R_5 & S_5 & T_5 & \mathbf{0} \\ R_6 & S_6 & T_6 & X_6 \end{bmatrix}, \ QQ^* = \mathrm{diag}\left\{c, 4c, 6c, 4c, c\right\}. \tag{3.2}$$

In the following, we will discuss (3.1) and (3.2) by case I: $W_4 = 0 = W_3$, case II: $W_4 = 0, W_3 \neq 0$ and case III: $W_4 \neq 0$ respectively.

Case I: $W_4 = 0 = W_3$. In this case, from (3.1) and (3.2), we get $|V_3|^2 = 4$ and $|R_3|^2 = 4c$ respectively. Since $R_3 = 2V_3$, then we obtain $c = 4$. Using (3.1) and (3.2) again, we have $|W_1|^2 = |R_2|^2 = 4$, which implies

$$|a_1^{(1)}|^2 + |a_2^{(1)}|^2 = 4, \ |a_1^{(1)} \wedge a_2^{(1)}|^2 = 4. \tag{3.3}$$

In the congruent class, using the singular-value decomposition of complex matrix A_1, we can take

$$a_1^{(1)} = \begin{pmatrix} a_{11}^{(1)} & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, a_2^{(1)} = \begin{pmatrix} 0 & a_{22}^{(1)} & 0 & 0 & \cdots & 0 \end{pmatrix},$$

$$a_1^{(2)} = \begin{pmatrix} a_{11}^{(2)} & a_{12}^{(2)} & a_{13}^{(2)} & a_{14}^{(2)} & \cdots & a_{1n}^{(2)} \end{pmatrix}, a_2^{(2)} = \begin{pmatrix} a_{21}^{(2)} & a_{22}^{(2)} & a_{23}^{(2)} & a_{24}^{(2)} & \cdots & a_{2n}^{(2)} \end{pmatrix},$$

where $a_{11}^{(1)} \geq a_{22}^{(1)} > 0$. It follows from (3.3) that

$$a_{11}^{(1)} = a_{22}^{(1)} = \sqrt{2}. \tag{3.4}$$

From (3.4), together with $\langle W_1, W_2 \rangle = 0$ by (3.1), we obtain

$$a_{11}^{(2)} + a_{22}^{(2)} = 0. \tag{3.5}$$

Since $V_3 = a_1^{(1)} \wedge a_2^{(2)} + a_1^{(2)} \wedge a_2^{(1)}$, then $|V_3|^2 = 4$ gives us $|a_1^{(1)} \wedge a_2^{(2)} + a_1^{(2)} \wedge a_2^{(1)}|^2 = 4$. Combining this with (3.4) and (3.5), we have

$$|a_1^{(2)}|^2 + |a_2^{(2)}|^2 - 2|a_{11}^{(2)}|^2 - |a_{21}^{(2)}|^2 - |a_{12}^{(2)}|^2 = 2. \tag{3.6}$$

It follows from $V_2 = R_2$ that $|V_2|^2 = 4$. Since $|W_2|^2 + |V_2|^2 = 6$ by (3.1), then we get $|W_2|^2 = 2$, i.e.,

$$|a_1^{(2)}|^2 + |a_2^{(2)}|^2 = 2. \tag{3.7}$$

Substituting (3.7) into (3.6) yields

$$2|a_{11}^{(2)}|^2 + |a_{21}^{(2)}|^2 + |a_{12}^{(2)}|^2 = 0,$$

which implies

$$a_{11}^{(2)} = 0 = a_{21}^{(2)} = a_{12}^{(2)} = a_{22}^{(2)}. \tag{3.8}$$

From (3.1), we know $|V_4|^2 = 1$, i.e., $|a_1^{(2)} \wedge a_2^{(2)}|^2 = 1$. Combining this with (3.7) yields

$$|a_1^{(2)}|^2 = |a_2^{(2)}|^2 = 1, \ \left\langle a_1^{(2)}, \ a_2^{(2)} \right\rangle = 0. \tag{3.9}$$

Hence using (3.8) and (3.9), we can take, in the congruent class,

$$a_1^{(1)} = \begin{pmatrix} \sqrt{2} & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, a_2^{(1)} = \begin{pmatrix} 0 & \sqrt{2} & 0 & 0 & \cdots & 0 \end{pmatrix},$$
$$a_1^{(2)} = \begin{pmatrix} 0 & 0 & 1 & 0 & \cdots & 0 \end{pmatrix}, a_2^{(2)} = \begin{pmatrix} 0 & 0 & 0 & 1 & \cdots & 0 \end{pmatrix}.$$

Then in this case, φ is congruent to

$$\varphi = \begin{bmatrix} 1 & 0 & \sqrt{2}z & 0 & z^2 & 0 \\ 0 & 1 & 0 & \sqrt{2}z & 0 & z^2 \end{bmatrix} = V_0^{(2)} \oplus V_0^{(2)} \subset G(2,6).$$

Case II: $W_4 = 0, W_3 \neq 0$. In this case, we have $V_6 = a_1^{(3)} \wedge a_2^{(3)} = 0$ and $V_5 = a_1^{(2)} \wedge a_2^{(3)} + a_1^{(3)} \wedge a_2^{(2)} = 0$. From (3.1), it follows that $\langle W_1, W_3 \rangle = 0$. Then in the congruent class, we can take

$$a_1^{(3)} = \begin{pmatrix} a_{11}^{(3)} & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, a_2^{(3)} = 0,$$
$$a_1^{(2)} = \begin{pmatrix} a_{11}^{(2)} & a_{12}^{(2)} & a_{13}^{(2)} & a_{14}^{(2)} & \cdots & a_{1n}^{(2)} \end{pmatrix}, a_2^{(2)} = \begin{pmatrix} a_{21}^{(2)} & 0 & 0 & 0 & \cdots & 0 \end{pmatrix},$$
$$a_1^{(1)} = \begin{pmatrix} 0 & a_{12}^{(1)} & a_{13}^{(1)} & a_{14}^{(1)} & \cdots & a_{1n}^{(1)} \end{pmatrix}, a_2^{(1)} = \begin{pmatrix} a_{21}^{(1)} & a_{22}^{(1)} & a_{23}^{(1)} & a_{24}^{(1)} & \cdots & a_{2n}^{(1)} \end{pmatrix},$$

where $a_{11}^{(3)} > 0$. From (3.1) and (3.2), we get $\langle V_2, V_4 \rangle = 0$ and $\langle R_2, R_4 \rangle = 0$ respectively. Since $R_2 = V_2 = a_1^{(1)} \wedge a_2^{(1)}$, $V_4 = a_1^{(2)} \wedge a_2^{(2)} + a_1^{(3)} \wedge a_2^{(1)}$ and $R_4 = 4a_1^{(2)} \wedge a_2^{(2)} + 3a_1^{(3)} \wedge a_2^{(1)}$, then we get

$$\left\langle a_1^{(1)} \wedge a_2^{(1)}, a_1^{(3)} \wedge a_2^{(1)} \right\rangle = 0,$$

and

$$\left\langle a_1^{(1)} \wedge a_2^{(1)}, a_1^{(2)} \wedge a_2^{(2)} \right\rangle = 0,$$

9

which implies by using $\left\langle a_1^{(1)},\ a_1^{(3)} \right\rangle = 0$ and $\left\langle a_1^{(1)},\ a_2^{(2)} \right\rangle = 0$ respectively,

$$\left\langle a_1^{(1)},\ a_2^{(1)} \right\rangle a_{21}^{(1)} \overline{a_{11}^{(3)}} = 0, \tag{3.10}$$

and

$$\left\langle a_1^{(1)},\ a_1^{(2)} \right\rangle a_{21}^{(1)} \overline{a_{21}^{(2)}} = 0. \tag{3.11}$$

In (3.1), we have $\langle W_1,\ W_2 \rangle = 0$, i.e.,

$$\left\langle a_1^{(1)},\ a_1^{(2)} \right\rangle + a_{21}^{(1)} \overline{a_{21}^{(2)}} = 0. \tag{3.12}$$

It follows from (3.11) and (3.12) that

$$\left\langle a_1^{(1)},\ a_1^{(2)} \right\rangle = 0 = a_{21}^{(1)} \overline{a_{21}^{(2)}}. \tag{3.13}$$

Claim: $a_{21}^{(1)} = 0$.

Otherwise if $a_{21}^{(1)} \neq 0$, then $\left\langle a_1^{(1)},\ a_2^{(1)} \right\rangle = 0$ by (3.10) and $a_{21}^{(2)} = 0$ by (3.13). The latter tells us $a_2^{(2)} = 0$. In (3.1), we have $|V_4|^2 = 1$. Since $V_4 = a_1^{(3)} \wedge a_2^{(1)}$, then

$$|a_1^{(3)} \wedge a_2^{(1)}|^2 = 1. \tag{3.14}$$

In (3.2), we have $|R_5|^2 + |S_5|^2 + |T_5|^2 = 4c$. Since $R_5 = T_5 = 0,\ S_5 = -2a_1^{(1)} \wedge a_2^{(1)} \wedge a_1^{(3)}$, then

$$|a_1^{(1)}|^2 |a_1^{(3)} \wedge a_2^{(1)}|^2 = c. \tag{3.15}$$

It follows from (3.14) and (3.15) that

$$|a_1^{(1)}|^2 = c. \tag{3.16}$$

In (3.1) and (3.2), we have $|W_1|^2 = 4$ and $|R_2|^2 = c$ respectively, i.e.,

$$|a_1^{(1)}|^2 + |a_2^{(1)}|^2 = 4,\ |a_1^{(1)}|^2 |a_2^{(1)}|^2 = c. \tag{3.17}$$

Combining (3.16) and (3.17), we obtain $|a_2^{(1)}|^2 = 1$ and $|a_1^{(1)}|^2 = c = 3$. From (3.1) and (3.2), we obtain $|W_3|^2 + |V_3|^2 = 4$ and $|R_3|^2 = 4c = 12$ respectively. Using these, together with $R_3 = 2V_3$, we get $|W_3|^2 = 1$, i.e., $|a_1^{(3)}|^2 = 1$. Substituting this and $|a_2^{(1)}|^2 = 1$ into (3.14) yields $\left\langle a_1^{(3)},\ a_2^{(1)} \right\rangle = 0$, which shows $a_{11}^{(3)} \overline{a_{21}^{(1)}} = 0$. It follows that $a_{21}^{(1)} = 0$, which contradicts our assumption. Hence we verify $a_{21}^{(1)} = 0$.

Now we know $\left\langle a_1^{(3)},\ a_2^{(1)} \right\rangle = \left\langle a_2^{(2)},\ a_2^{(1)} \right\rangle = 0$. The latter gives us $\left\langle a_1^{(2)},\ a_1^{(1)} \right\rangle = 0$ by $\langle W_2,\ W_1 \rangle = 0$ in (3.1).

From (3.1) and (3.2), we get $\langle V_3,\ V_4 \rangle = 0$ and $\langle R_3,\ R_4 \rangle = 0$ respectively. Since $R_3 = 2V_3 = 2a_1^{(1)} \wedge a_2^{(2)} + 2a_1^{(2)} \wedge a_2^{(1)},\ V_4 = a_1^{(2)} \wedge a_2^{(2)} + a_1^{(3)} \wedge a_2^{(1)}$ and $R_4 = 4a_1^{(2)} \wedge a_2^{(2)} + 3a_1^{(3)} \wedge a_2^{(1)}$, then we get

$$\left\langle a_1^{(1)} \wedge a_2^{(2)} + a_1^{(2)} \wedge a_2^{(1)}, a_1^{(2)} \wedge a_2^{(2)} \right\rangle = 0,$$

10

and
$$\left\langle a_1^{(1)} \wedge a_2^{(2)} + a_1^{(2)} \wedge a_2^{(1)}, a_1^{(3)} \wedge a_2^{(1)} \right\rangle = 0,$$

which implies, respectively,

$$\left\langle a_2^{(1)}, \ a_1^{(2)} \right\rangle a_{11}^{(2)} \overline{a_{21}^{(2)}} = 0, \tag{3.18}$$

and

$$|a_2^{(1)}|^2 a_{11}^{(2)} \overline{a_{11}^{(3)}} - \left\langle a_1^{(1)}, \ a_2^{(1)} \right\rangle a_{21}^{(2)} \overline{a_{11}^{(3)}} = 0. \tag{3.19}$$

In the following, we will discuss this case from two subcases of $a_{21}^{(2)} = 0$ and $a_{21}^{(2)} \neq 0$ respectively.

Subcase II1: If $a_{21}^{(2)} = 0$, i.e., $a_2^{(2)} = 0$, then since $a_2^{(1)}$ is non-zero due to that φ is unramified, we obtain $a_{11}^{(2)} = 0$ by using (3.19). It follows from the proof of the above Claim, we know that (3.14), (3.15) and (3.17) become, respectively,

$$|a_1^{(3)}|^2 |a_2^{(1)}|^2 = 1. \tag{3.20}$$

$$|a_1^{(1)} \wedge a_2^{(1)}|^2 |a_1^{(3)}|^2 = c. \tag{3.21}$$

$$|a_1^{(1)}|^2 + |a_2^{(1)}|^2 = 4, \ |a_1^{(1)} \wedge a_2^{(1)}|^2 = c. \tag{3.22}$$

In light of (3.20)-(3.22), we assert immediately that $|a_1^{(3)}|^2 = |a_2^{(1)}|^2 = 1$ and $|a_1^{(1)}|^2 = 3$. On the one hand, since $|W_3|^2 + |V_3|^2 = 4$, then we get by $|W_3|^2 = |a_1^{(3)}|^2 = 1$ that $|V_3|^2 = 3$, which implies $|R_3|^2 = 4|V_3|^2 = 12$. In terms of this, together with $|R_3|^2 = 4c$, we get $c = 3$. Substituting it into (3.22) yields $|a_1^{(1)} \wedge a_2^{(1)}|^2 = 3$. Combining this with $|a_1^{(1)}|^2 = 3$ and $|a_2^{(1)}|^2 = 1$, we obtain $\left\langle a_1^{(1)}, \ a_2^{(1)} \right\rangle = 0$.

On the other hand, since $|W_2|^2 + |V_2|^2 = 6$ and $|V_2|^2 = 3$, then we have $|W_2|^2 = 3$, i.e., $|a_1^{(2)}|^2 = 3$. In light of this, together with $|V_3|^2 = |a_1^{(2)} \wedge a_2^{(1)}|^2 = 3$ and $|a_2^{(1)}|^2 = 1$, we obtain that $\left\langle a_1^{(2)}, \ a_2^{(1)} \right\rangle = 0$. Hence in the congruent class, we can take

$$a_1^{(3)} = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, a_2^{(3)} = 0,$$
$$a_1^{(2)} = \begin{pmatrix} 0 & \sqrt{3} & 0 & 0 & \cdots & 0 \end{pmatrix}, a_2^{(2)} = 0,$$
$$a_1^{(1)} = \begin{pmatrix} 0 & 0 & \sqrt{3} & 0 & \cdots & 0 \end{pmatrix}, a_2^{(1)} = \begin{pmatrix} 0 & 0 & 0 & 1 & \cdots & 0 \end{pmatrix}.$$

Then in this subcase, φ is congruent to

$$\varphi = \begin{bmatrix} 1 & 0 & z^3 & \sqrt{3}z^2 & \sqrt{3}z & 0 \\ 0 & 1 & 0 & 0 & 0 & z \end{bmatrix} = V_0^{(1)} \oplus V_0^{(3)} \subset G(2,6).$$

Subcase II2: If $a_{21}^{(2)} \neq 0$, then we first claim $a_{11}^{(2)} = 0$. Otherwise if $a_{11}^{(2)} \neq 0$, then (3.18) leads to $\left\langle a_2^{(1)}, a_1^{(2)} \right\rangle = 0$. Combining this with $\left\langle a_1^{(1)}, a_1^{(2)} \right\rangle = \left\langle a_1^{(1)}, a_2^{(2)} \right\rangle = \left\langle a_2^{(1)}, a_2^{(2)} \right\rangle = 0$, we find $\langle V_2, V_3 \rangle = 0$, which implies $\langle W_2, W_3 \rangle = 0$ by applying $\langle W_2, W_3 \rangle + \langle V_2, V_3 \rangle = 0$ in (3.1). It follows that $a_{11}^{(2)} = 0$, which is a contradiction. Thus we verify $a_{11}^{(2)} = 0$.

Now (3.19) yields $\left\langle a_1^{(1)}, a_2^{(1)} \right\rangle = 0$. In (3.1), we have $|W_1|^2 = 4$ and $|W_3|^2 + |V_3|^2 = 4$, i.e.,

$$|a_1^{(1)}|^2 + |a_2^{(1)}|^2 = 4 \tag{3.23}$$

and

$$|a_{11}^{(3)}|^2 + |a_1^{(1)}|^2|a_{21}^{(2)}|^2 + |a_1^{(2)} \wedge a_2^{(1)}|^2 = 4. \tag{3.24}$$

In (3.2), we know $|R_2|^2 = c$ and $|R_6|^2 + |S_6|^2 + |T_6|^2 + |X_6|^2 = c$. Since $R_6 = T_6 = 0$, $S_6 = -a_1^{(2)} \wedge a_2^{(1)} \wedge a_1^{(3)}$ and $X_6 = a_1^{(1)} \wedge a_2^{(1)} \wedge a_1^{(2)} \wedge a_2^{(2)}$, then

$$|a_1^{(1)}|^2|a_2^{(1)}|^2 = c \tag{3.25}$$

and

$$|a_1^{(2)} \wedge a_2^{(1)}|^2 \left(|a_{11}^{(3)}|^2 + |a_1^{(1)}|^2|a_{21}^{(2)}|^2 \right) = c. \tag{3.26}$$

Observing (3.23)-(3.26), we find that $|a_1^{(1)}|^2, |a_2^{(1)}|^2$ and $|a_{11}^{(3)}|^2 + |a_1^{(1)}|^2|a_{21}^{(2)}|^2, |a_2^{(1)} \wedge a_1^{(2)}|^2$ are respectively the roots of equation $x^2 - 4x + c = 0$. Then the following two cases will happen: (i) $|a_1^{(1)}|^2 = |a_{11}^{(3)}|^2 + |a_1^{(1)}|^2|a_{21}^{(2)}|^2, |a_2^{(1)}|^2 = |a_2^{(1)} \wedge a_1^{(2)}|^2$, or (ii) $|a_1^{(1)}|^2 = |a_2^{(1)} \wedge a_1^{(2)}|^2, |a_2^{(1)}|^2 = |a_{11}^{(3)}|^2 + |a_1^{(1)}|^2|a_{21}^{(2)}|^2$.

Set $|a_1^{(1)}|^2 = t (0 < t < 4)$. Then $|a_2^{(1)}|^2 = 4 - t = \frac{c}{t}$, i.e., $c = 4t - t^2$. It follows from $|W_3|^2 + |V_3|^2 = 4$ and $|R_3|^2 = 4|V_3|^2 = 4c$ that $|W_3|^2 = 4 - c$, i.e., $|a_{11}^{(3)}|^2 = 4 - c = (t-2)^2$. If (i) happens, then $|a_{21}^{(2)}|^2 = 1 - \frac{4-c}{t}$. Using $|R_5|^2 + |S_5|^2 + |T_5|^2 = 4c$ in (3.2) and $R_5 = 0$, $S_5 = -2a_1^{(1)} \wedge a_2^{(1)} \wedge a_1^{(3)} + 2a_1^{(1)} \wedge a_1^{(2)} \wedge a_2^{(2)}, T_5 = 2a_2^{(1)} \wedge a_1^{(2)} \wedge a_2^{(2)}$, we get $|a_1^{(1)}|^2|V_4|^2 + |a_2^{(1)} \wedge a_1^{(2)}|^2|a_{21}^{(2)}|^2 = c$. Combining this with $|V_4|^2 = 1$ by (3.1), we obtain $|a_1^{(1)}|^2 + |a_2^{(1)}|^2|a_{21}^{(2)}|^2 = c$. Then

$$t + (4 - t)\left(1 - \frac{4-c}{t} \right) = c,$$

which implies $t = 2$. It follows that $a_{11}^{(3)} = 0$, which contradicts $W_3 \neq 0$. So this case of (i) will not happen.

If (ii) happens, then we get

$$|a_{21}^{(2)}|^2 = \frac{4-t}{t} - \frac{4-c}{t} = \frac{c}{t} - 1 = 3 - t, 0 < t < 3, t \neq 2. \tag{3.27}$$

Using $|W_2|^2 + |V_2|^2 = 6$ in (3.1), we have

$$|a_1^{(2)}|^2 + |a_{21}^{(2)}|^2 + |a_1^{(1)}|^2|a_2^{(1)}|^2 = 6,$$

which gives us by (3.25) and (3.27)

$$|a_1^{(2)}|^2 = 6 - c - (3 - t) = t^2 - 3t + 3. \tag{3.28}$$

Since $\left\langle a_1^{(1)}, a_1^{(2)} \right\rangle = 0 = \left\langle a_1^{(1)}, a_2^{(1)} \right\rangle$, then in the congruent class, we can take

$$a_1^{(3)} = \begin{pmatrix} |t-2| & 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, a_2^{(3)} = 0,$$

$$a_1^{(2)} = \begin{pmatrix} 0 & 0 & a_{13}^{(2)} & a_{14}^{(2)} & 0 & \cdots & 0 \end{pmatrix}, a_2^{(2)} = \begin{pmatrix} \sqrt{3-t} & 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix},$$

$$a_1^{(1)} = \begin{pmatrix} 0 & \sqrt{t} & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, a_2^{(1)} = \begin{pmatrix} 0 & 0 & \sqrt{4-t} & 0 & 0 & \cdots & 0 \end{pmatrix},$$

12

where $a_{13}^{(2)} \in \mathbb{C}, a_{14}^{(2)} > 0$. It follows from $|a_1^{(1)}|^2 = |a_2^{(1)} \wedge a_1^{(2)}|^2$ that $t = |a_{14}^{(2)}|^2|a_{23}^{(1)}|^2$, which implies

$$a_{14}^{(2)} = \sqrt{\frac{t}{4-t}}. \tag{3.29}$$

Then (3.28) and (3.29) tell us

$$|a_{13}^{(2)}|^2 = |a_1^{(2)}|^2 - |a_{14}^{(2)}|^2 = t^2 - 3t + 3 - \frac{t}{4-t} = \frac{(3-t)(2-t)^2}{4-t}. \tag{3.30}$$

On the other hand, substituting $V_4 = a_1^{(2)} \wedge a_2^{(2)} + a_1^{(3)} \wedge a_2^{(1)}$ into $|V_4|^2 = 1$ leads to

$$|a_1^{(2)}|^2|a_{21}^{(2)}|^2 + |a_{11}^{(3)}|^2|a_2^{(1)}|^2 - a_{13}^{(2)}\overline{a_{23}^{(1)}}a_{11}^{(3)}\overline{a_{21}^{(2)}} - \overline{a_{13}^{(2)}}a_{23}^{(1)}a_{11}^{(3)}\overline{a_{21}^{(2)}} = 1,$$

which yields by a straightforward calculation

$$a_{13}^{(2)} + \overline{a_{13}^{(2)}} = 2\sqrt{\frac{3-t}{4-t}}|t-2|. \tag{3.31}$$

Comparing (3.30) and (3.31), we find

$$\operatorname{Re} a_{13}^{(2)} = \sqrt{\frac{3-t}{4-t}}|t-2| = |a_{13}^{(2)}|,$$

which implies $a_{13}^{(2)} \in \mathbb{R}$ and

$$a_{13}^{(2)} = \sqrt{\frac{3-t}{4-t}}|t-2|.$$

Thus in this subcase, φ is congruent to

$$\varphi = \begin{bmatrix} 1 & 0 & |t-2|z^3 & \sqrt{t}tz & \sqrt{\frac{3-t}{4-t}}|t-2|z^2 & \sqrt{\frac{t}{4-t}}z^2 \\ 0 & 1 & \sqrt{3-t}z^2 & 0 & \sqrt{4-t}z & 0 \end{bmatrix} \subset G(2,6),\ 0 < t < 3, t \neq 2.$$

Case III: $W_4 \neq 0$. We will prove that there does not exist this case by contradiction. Suppose that this case happens, then we have $V_8 = a_1^{(4)} \wedge a_2^{(4)} = 0$ and $V_7 = a_1^{(3)} \wedge a_2^{(4)} + a_1^{(4)} \wedge a_2^{(3)} = 0$. From (3.1), it follows that $\langle W_1,\ W_4 \rangle = 0$. Then in the congruent class, we can take

$$a_1^{(4)} = \begin{pmatrix} a_{11}^{(4)} & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, a_2^{(4)} = 0,$$

$$a_1^{(3)} = \begin{pmatrix} a_{11}^{(3)} & a_{12}^{(3)} & a_{13}^{(3)} & a_{14}^{(3)} & \cdots & a_{1n}^{(3)} \end{pmatrix}, a_2^{(3)} = \begin{pmatrix} a_{21}^{(3)} & 0 & 0 & 0 & \cdots & 0 \end{pmatrix},$$

$$a_1^{(2)} = \begin{pmatrix} a_{11}^{(2)} & a_{12}^{(2)} & a_{13}^{(2)} & a_{14}^{(2)} & \cdots & a_{1n}^{(2)} \end{pmatrix}, a_2^{(2)} = \begin{pmatrix} a_{21}^{(2)} & a_{22}^{(2)} & a_{23}^{(2)} & a_{24}^{(2)} & \cdots & a_{2n}^{(2)} \end{pmatrix},$$

$$a_1^{(1)} = \begin{pmatrix} 0 & a_{12}^{(1)} & a_{13}^{(1)} & a_{14}^{(1)} & \cdots & a_{1n}^{(1)} \end{pmatrix}, a_2^{(1)} = \begin{pmatrix} a_{21}^{(1)} & a_{22}^{(1)} & a_{23}^{(1)} & a_{24}^{(1)} & \cdots & a_{2n}^{(1)} \end{pmatrix},$$

where $a_{11}^{(4)} > 0$. Since $V_5 = a_1^{(2)} \wedge a_2^{(3)} + a_1^{(3)} \wedge a_2^{(2)} + a_1^{(4)} \wedge a_2^{(1)} = 0$, then we get $R_5 = 6a_1^{(2)} \wedge a_2^{(3)} + 6a_1^{(3)} \wedge a_2^{(2)} + 4a_1^{(4)} \wedge a_2^{(1)} = -2a_1^{(4)} \wedge a_2^{(1)}$. Substituting this and $R_2 = a_1^{(1)} \wedge a_2^{(1)}$ into $\langle R_2,\ R_5 \rangle = 0$ in (3.2), we obtain

$$\left\langle a_1^{(1)} \wedge a_2^{(1)}, a_2^{(1)} \wedge a_1^{(4)} \right\rangle = 0,$$

which implies by using $\left\langle a_1^{(1)}, a_1^{(4)} \right\rangle = 0$ that

$$\left\langle a_1^{(1)}, \ a_2^{(1)} \right\rangle a_{21}^{(1)} \overline{a_{11}^{(4)}} = 0. \tag{3.32}$$

Similarly from $V_6 = a_1^{(3)} \wedge a_2^{(3)} + a_1^{(4)} \wedge a_2^{(2)} = 0$, we know $R_6 = 9a_1^{(3)} \wedge a_2^{(3)} + 8a_1^{(4)} \wedge a_2^{(2)} = a_1^{(3)} \wedge a_2^{(3)}$. Substituting this and $R_2 = a_1^{(1)} \wedge a_2^{(1)}$ into $\langle R_2, \ R_6 \rangle = 0$ in (3.2), we get

$$\left\langle a_1^{(1)} \wedge a_2^{(1)}, a_1^{(3)} \wedge a_2^{(3)} \right\rangle = 0,$$

which implies by $\left\langle a_1^{(1)}, a_2^{(3)} \right\rangle = 0$ that

$$\left\langle a_1^{(1)}, \ a_1^{(3)} \right\rangle a_{21}^{(1)} \overline{a_{21}^{(3)}} = 0. \tag{3.33}$$

In (3.2), we have $\langle W_1, \ W_3 \rangle = 0$, i.e.,

$$\left\langle a_1^{(1)}, \ a_1^{(3)} \right\rangle + a_{21}^{(1)} \overline{a_{21}^{(3)}} = 0. \tag{3.34}$$

It follows from (3.33) and (3.34) that

$$\left\langle a_1^{(1)}, \ a_1^{(3)} \right\rangle = 0 = a_{21}^{(1)} \overline{a_{21}^{(3)}}. \tag{3.35}$$

Claim 1: $a_{21}^{(1)} = 0$.

Otherwise if $a_{21}^{(1)} \neq 0$, then $\left\langle a_1^{(1)}, \ a_2^{(1)} \right\rangle = 0$ by (3.32) and $a_{21}^{(3)} = 0$ by (3.35). The latter tells us $a_2^{(3)} = 0$. Then by using $V_6 = a_1^{(4)} \wedge a_2^{(2)} = 0$ we have $a_2^{(2)} = a_{21}^{(2)} / a_{11}^{(4)} \cdot a_1^{(4)}$. This gives us $\left\langle a_1^{(1)}, \ a_2^{(2)} \right\rangle = 0$. At this time, since $R_2 = a_1^{(1)} \wedge a_2^{(1)}$ and $R_4 = 4a_1^{(2)} \wedge a_2^{(2)} + 3a_1^{(3)} \wedge a_2^{(1)}$, then $\langle R_2, \ R_4 \rangle = 0$ in (3.2) gives

$$4 \left\langle a_1^{(1)} \wedge a_2^{(1)}, a_1^{(2)} \wedge a_2^{(2)} \right\rangle + 3 \left\langle a_1^{(1)} \wedge a_2^{(1)}, a_1^{(3)} \wedge a_2^{(1)} \right\rangle = 0. \tag{3.36}$$

It follows from $\left\langle a_1^{(1)}, \ a_1^{(3)} \right\rangle = \left\langle a_1^{(1)}, \ a_2^{(1)} \right\rangle = 0$ that

$$\left\langle a_1^{(1)} \wedge a_2^{(1)}, a_1^{(3)} \wedge a_2^{(1)} \right\rangle = 0. \tag{3.37}$$

Substituting (3.37) into (3.36) yields

$$\left\langle a_1^{(1)} \wedge a_2^{(1)}, a_1^{(2)} \wedge a_2^{(2)} \right\rangle = 0,$$

which implies by $\left\langle a_1^{(1)}, \ a_2^{(2)} \right\rangle = 0$, that

$$\left\langle a_1^{(1)}, \ a_1^{(2)} \right\rangle a_{21}^{(1)} \overline{a_{21}^{(2)}} = 0. \tag{3.38}$$

On the other hand, in (3.1), we have $\langle W_1,\ W_2 \rangle = 0$, i.e.,

$$\left\langle a_1^{(1)},\ a_1^{(2)} \right\rangle + a_{21}^{(1)} \overline{a_{21}^{(2)}} = 0. \tag{3.39}$$

It follows from (3.38) and (3.39) that

$$\left\langle a_1^{(1)},\ a_1^{(2)} \right\rangle = 0 = a_{21}^{(1)} \overline{a_{21}^{(2)}}, \tag{3.40}$$

which means $a_{21}^{(2)} = 0$ by $a_{21}^{(1)} \neq 0$, i.e., $a_2^{(2)} = 0$. Now by using $V_5 = a_1^{(4)} \wedge a_2^{(1)} = 0$, we have $a_2^{(1)} = a_{21}^{(1)}/a_{11}^{(4)} \cdot a_1^{(4)}$. And V_2, V_3, V_4 reduce to, respectively,

$$V_2 = a_1^{(1)} \wedge a_2^{(1)},\ V_3 = a_1^{(2)} \wedge a_2^{(1)},\ V_4 = a_1^{(3)} \wedge a_2^{(1)}.$$

Observing that $\langle V_2,\ V_3 \rangle = \langle V_2,\ V_4 \rangle = 0$. Substituting this into $\langle W_2,\ W_3 \rangle + \langle V_2,\ V_3 \rangle = 0$ and $\langle W_2,\ W_4 \rangle + \langle V_2,\ V_4 \rangle = 0$ in (3.1) respectively yields $\langle W_2,\ W_3 \rangle = \langle W_2,\ W_4 \rangle = 0$, i.e., $a_{11}^{(2)} = 0$, $\left\langle a_1^{(2)},\ a_1^{(3)} \right\rangle = 0$. This just gives us $\langle V_3,\ V_4 \rangle = 0$. Substituting it into $\langle W_3,\ W_4 \rangle + \langle V_3,\ V_4 \rangle = 0$ in (3.1) yields $\langle W_3,\ W_4 \rangle = 0$, i.e., $a_{11}^{(3)} = 0$.

Since $R_5 = T_5 = 0$ and $S_5 = -2a_1^{(1)} \wedge a_2^{(1)} \wedge a_1^{(3)}$, then $|R_5|^2 + |S_5|^2 + |T_5|^2 = 4c$ in (3.2) leads to $|R_2|^2 |a_1^{(3)}|^2 = c$, which implies $|a_1^{(3)}|^2 = 1$ by $|R_2|^2 = c$ in (3.2). This tells us $|W_3|^2 = 1$. Substituting it into $|W_3|^2 + |V_3|^2 = 4$ in (3.1) yields $|V_3|^2 = 3$. On the other hand, it follows from $|R_3|^2 = 4|V_3|^2 = 4c$ in (3.2) that $|V_3|^2 = c$. Hence we get $c = 3$, which implies $|V_2|^2 = 3$. Substituting it into $|W_2|^2 + |V_2|^2 = 6$ in (3.1) yields $|W_2|^2 = 3$, i.e., $|a_1^{(2)}|^2 = 3$. Using this and $|V_3|^2 = |a_1^{(2)}|^2 |a_2^{(1)}|^2 = 3$ leads to $|a_2^{(1)}|^2 = 1$. In terms of this, together with $|a_1^{(3)}|^2 = 1$, we obtain $|V_4|^2 = |a_1^{(3)} \wedge a_2^{(1)}|^2 = 1$. Substituting it into $|W_4|^2 + |V_4|^2 = 1$ in (3.1) yields $|W_4|^2 = 0$, which contradicts $W_4 \neq 0$. Hence we verify $a_{21}^{(1)} = 0$.

Now we know $\left\langle a_1^{(4)},\ a_2^{(1)} \right\rangle = \left\langle a_2^{(3)},\ a_2^{(1)} \right\rangle = 0$. The latter gives us $\left\langle a_1^{(3)},\ a_1^{(1)} \right\rangle = 0$ by $\langle W_3,\ W_1 \rangle = 0$ in (3.1).

Since $V_6 = a_1^{(3)} \wedge a_2^{(3)} + a_1^{(4)} \wedge a_2^{(2)} = \left(a_{21}^{(3)}/a_{11}^{(4)} \cdot a_1^{(3)} - a_2^{(2)} \right) \wedge a_1^{(4)} = 0$, then there exists $\lambda \in \mathbb{C}$ such that

$$a_2^{(2)} = a_{21}^{(3)}/a_{11}^{(4)} \cdot a_1^{(3)} + \lambda a_1^{(4)}, \tag{3.41}$$

which implies $\left\langle a_1^{(1)},\ a_2^{(2)} \right\rangle = 0$ by using $\left\langle a_1^{(1)},\ a_1^{(3)} \right\rangle = 0 = \left\langle a_1^{(1)},\ a_1^{(4)} \right\rangle$. From (3.41), we have $V_5 = a_1^{(2)} \wedge a_2^{(3)} + a_1^{(3)} \wedge a_2^{(2)} + a_1^{(4)} \wedge a_2^{(1)} = \left(a_{21}^{(3)}/a_{11}^{(4)} \cdot a_1^{(2)} + \lambda a_1^{(3)} - a_2^{(1)} \right) \wedge a_1^{(4)} = 0$, then there exists $\mu \in \mathbb{C}$ such that

$$a_2^{(1)} = a_{21}^{(3)}/a_{11}^{(4)} \cdot a_1^{(2)} + \lambda a_1^{(3)} + \mu a_1^{(4)}. \tag{3.42}$$

Applying (3.41) and (3.42), a straightforward calculation shows

$$\begin{aligned}
V_4 &= a_1^{(1)} \wedge a_2^{(3)} + a_1^{(2)} \wedge a_2^{(2)} + a_1^{(3)} \wedge a_2^{(1)} \\
&= \left(a_{21}^{(3)}/a_{11}^{(4)} \cdot a_1^{(1)} + \lambda a_1^{(2)} + \mu a_1^{(3)} \right) \wedge a_1^{(4)}.
\end{aligned}$$

From this, together with $V_2 = a_1^{(1)} \wedge a_2^{(1)}$ and $\left\langle a_1^{(4)}, a_1^{(1)} \right\rangle = \left\langle a_1^{(4)}, a_2^{(1)} \right\rangle = 0$, we find $\langle V_2, V_4 \rangle = 0$. Substituting it into $\langle W_2, W_4 \rangle + \langle V_2, V_4 \rangle = 0$ in (3.1) yields $\langle W_2, W_4 \rangle = 0$, which leads to $\left\langle a_1^{(2)}, a_1^{(4)} \right\rangle = 0$, i.e., $a_{11}^{(2)} = 0$.

On the other hand, combining $\langle V_2, V_4 \rangle = 0$ and $\langle R_2, R_4 \rangle = 0$ in (3.2), we obtain

$$\left\langle a_1^{(1)} \wedge a_2^{(1)}, a_1^{(2)} \wedge a_2^{(2)} \right\rangle = 0. \tag{3.43}$$

In terms of this, together with $\left\langle a_1^{(1)}, a_2^{(2)} \right\rangle = 0$, we get

$$\left\langle a_1^{(1)}, a_1^{(2)} \right\rangle \left\langle a_2^{(1)}, a_2^{(2)} \right\rangle = 0. \tag{3.44}$$

Since $\langle W_1, W_2 \rangle = 0$ in (3.1), then

$$\left\langle a_1^{(1)}, a_1^{(2)} \right\rangle + \left\langle a_2^{(1)}, a_2^{(2)} \right\rangle = 0. \tag{3.45}$$

It follows from (3.44) and (3.45) that

$$\left\langle a_1^{(1)}, a_1^{(2)} \right\rangle = \left\langle a_2^{(1)}, a_2^{(2)} \right\rangle = 0, \tag{3.46}$$

which implies respectively by (3.41) and (3.42)

$$\left\langle a_1^{(1)}, a_2^{(1)} \right\rangle = 0 \tag{3.47}$$

and

$$\overline{a_{21}^{(3)} / a_{11}^{(4)}} \left\langle a_2^{(1)}, a_1^{(3)} \right\rangle = 0. \tag{3.48}$$

It is worth noting that $\left\langle a_1^{(1)}, a_i^{(\alpha)} \right\rangle = 0$ for $(\alpha, i) \neq (1, 1)$.

Claim 2: $a_{21}^{(3)} = 0$.

Otherwise if $a_{21}^{(3)} \neq 0$, then $\left\langle a_2^{(1)}, a_1^{(3)} \right\rangle = 0$ by (3.48). Since $S_4 = -a_1^{(1)} \wedge a_2^{(1)} \wedge a_1^{(2)}, T_4 = -a_1^{(1)} \wedge a_2^{(1)} \wedge a_2^{(2)}$ and $S_6 = a_1^{(1)} \wedge a_2^{(1)} \wedge a_1^{(4)} + a_1^{(1)} \wedge a_2^{(3)} \wedge a_1^{(2)} - a_1^{(2)} \wedge a_2^{(1)} \wedge a_1^{(3)}, T_6 = a_1^{(3)} \wedge a_2^{(1)} \wedge a_2^{(2)} - a_1^{(1)} \wedge a_2^{(2)} \wedge a_2^{(3)}$, then a straightforward calculation shows

$$\langle S_4, S_6 \rangle = 0 = \langle T_4, T_6 \rangle.$$

From $\langle R_4, R_6 \rangle + \langle S_4, S_6 \rangle + \langle T_4, T_6 \rangle = 0$ in (3.2), we have $\langle R_4, R_6 \rangle = 0$, which implies $\left\langle a_1^{(2)} \wedge a_2^{(2)}, a_1^{(3)} \wedge a_2^{(3)} \right\rangle = 0$, i.e.,

$$\left\langle a_1^{(2)}, a_1^{(3)} \right\rangle a_{21}^{(2)} \overline{a_{21}^{(3)}} = 0.$$

Since $\langle R_2, R_3 \rangle = 0$, then $\langle V_2, V_3 \rangle = 0$, which means $\langle W_2, W_3 \rangle = 0$, i.e.,

$$\left\langle a_1^{(2)}, a_1^{(3)} \right\rangle + a_{21}^{(2)} \overline{a_{21}^{(3)}} = 0.$$

16

So that we obtain
$$\left\langle a_1^{(2)},\ a_1^{(3)}\right\rangle = 0 = a_{21}^{(2)}.$$

Since $\left\langle a_1^{(1)} \wedge a_2^{(2)} + a_1^{(2)} \wedge a_2^{(1)},\ a_1^{(2)} \wedge a_2^{(2)}\right\rangle = 0$, then by $\langle R_3,\ R_4\rangle = 0$ we have $\langle V_3,\ V_4\rangle = 0$, which implies $\langle W_3,\ W_4\rangle = 0$, i.e., $\left\langle a_1^{(3)},\ a_1^{(4)}\right\rangle = 0$. Hence we get $a_{11}^{(3)} = 0$. Using $a_{21}^{(2)} = 0 = a_{11}^{(3)}$ in (3.41), we know $\lambda = 0$, i.e.,
$$a_2^{(2)} = a_{21}^{(3)}/a_{11}^{(4)} \cdot a_1^{(3)}.$$

Then by $a_{21}^{(1)} = 0 = a_{11}^{(2)}$ in (3.42), we have $\mu = 0$, i.e.,
$$a_2^{(1)} = a_{21}^{(3)}/a_{11}^{(4)} \cdot a_1^{(2)}.$$

Since $V_4 = a_1^{(1)} \wedge a_2^{(3)}$ and
$$R_5 = -2a_1^{(4)} \wedge a_2^{(1)}, S_5 = 0, T_5 = -2a_1^{(1)} \wedge a_2^{(1)} \wedge a_2^{(3)},$$

then using $|W_4|^2 + |V_4|^2 = 1$ in (3.1) and $|R_5|^2 + |S_5|^2 + |T_5|^2 = 4c$ in (3.2), we obtain
$$|a_{11}^{(4)}|^2 + |a_1^{(1)}|^2|a_{21}^{(3)}|^2 = 1,\ |a_{11}^{(4)}|^2|a_2^{(1)}|^2 + |a_1^{(1)}|^2|a_2^{(1)}|^2|a_{21}^{(3)}|^2 = c,$$

which implies $|a_2^{(1)}|^2 = c$. Substituting this into $|R_2|^2 = |a_1^{(1)}|^2|a_2^{(1)}|^2 = c$ in (3.2) yields $|a_1^{(1)}|^2 = 1$. From $|W_1|^2 = |a_1^{(1)}|^2 + |a_2^{(1)}|^2 = 4$ in (3.1), we have $|a_2^{(1)}|^2 = 3$, which implies $c = 3$. It follows that $|V_2|^2 = |R_2|^2 = 3$, which leads to $|W_2|^2 = |a_1^{(2)}|^2 + |a_2^{(2)}|^2 = 3$ by $|W_2|^2 + |V_2|^2 = 6$ in (3.1). On the other hand, that $|R_3|^2 = 4|a_1^{(1)}|^2|a_2^{(2)}|^2 = 12$ in (3.2) and $|a_1^{(1)}|^2 = 1$ tells us $|a_2^{(2)}|^2 = 3$, which shows $|a_1^{(2)}|^2 = 0$, i.e., $a_2^{(1)} = 0$. It contradicts $|a_2^{(1)}|^2 = 3$. Thus we verify $a_{21}^{(3)} = 0$.

At this time, equations (3.41) and (3.42) reduce to
$$a_2^{(2)} = a_{21}^{(2)}/a_{11}^{(4)} \cdot a_1^{(4)},\ a_2^{(1)} = a_{21}^{(2)}/a_{11}^{(4)} \cdot a_1^{(3)} + \mu a_1^{(4)}.$$

Since $V_3 = a_1^{(1)} \wedge a_2^{(2)} + a_1^{(2)} \wedge a_2^{(1)}$, $V_4 = a_1^{(2)} \wedge a_2^{(2)} + a_1^{(3)} \wedge a_2^{(1)}$, then we observe that $\langle V_3,\ V_4\rangle = 0$, which implies $\langle W_3,\ W_4\rangle = 0$, i.e., $a_{11}^{(3)} = 0$. So that $a_2^{(1)} = a_{21}^{(2)}/a_{11}^{(4)} \cdot a_1^{(3)}$.

Since $V_2 = R_2 = a_1^{(1)} \wedge a_2^{(1)}$, then using $|V_2|^2 = |R_2|^2 = c$ in (3.2) yields
$$|a_1^{(1)}|^2|a_2^{(1)}|^2 = c. \tag{3.49}$$

From $|W_2|^2 + |V_2|^2 = 6$ in (3.1), we know that $|W_2|^2 = 6 - c$, i.e.,
$$|a_1^{(2)}|^2 + |a_{21}^{(2)}|^2 = 6 - c. \tag{3.50}$$

Because $R_4 = 4a_1^{(2)} \wedge a_2^{(2)}, S_4 = -a_1^{(1)} \wedge a_2^{(1)} \wedge a_1^{(2)}, T_4 = -a_1^{(1)} \wedge a_2^{(1)} \wedge a_2^{(2)}$, then applying $|R_4|^2 + |S_4|^2 + |T_4|^2 = 6c$ in (3.2) gives us
$$16|a_1^{(2)}|^2|a_{21}^{(2)}|^2 + |a_1^{(1)}|^2|a_2^{(1)}|^2|a_1^{(2)}|^2 + |a_1^{(1)}|^2|a_2^{(1)}|^2|a_{21}^{(2)}|^2 = 6c. \tag{3.51}$$

17

Substituting (3.49) and (3.50) into (3.51) leads to

$$|a_1^{(2)}|^2 |a_{21}^{(2)}|^2 = \frac{c^2}{16}, \tag{3.52}$$

which implies $|V_4|^2 = \frac{c^2}{16}$. From this, together with $|W_4|^2 + |V_4|^2 = 1$ in (3.1), we get

$$|a_{11}^{(4)}|^2 = 1 - \frac{c^2}{16}, \tag{3.53}$$

which means that $0 < c < 4$ by $a_{11}^{(4)} > 0$. On the other hand, since $R_5 = -2a_1^{(4)} \wedge a_2^{(1)}, S_5 = 2a_1^{(1)} \wedge a_1^{(2)} \wedge a_2^{(2)}, T_5 = 2a_2^{(1)} \wedge a_1^{(2)} \wedge a_2^{(2)}$, then it follows from $|R_5|^2 + |S_5|^2 + |T_5|^2 = 4c$ in (3.2) that

$$|a_{11}^{(4)}|^2 |a_2^{(1)}|^2 + |a_1^{(1)}|^2 |a_1^{(2)}|^2 |a_{21}^{(2)}|^2 + |a_2^{(1)}|^2 |a_1^{(2)}|^2 |a_{21}^{(2)}|^2 = c. \tag{3.54}$$

Now substituting (3.52), (3.53) and $|W_1|^2 = |a_1^{(1)}|^2 + |a_2^{(1)}|^2 = 4$ into (3.54) yields

$$|a_2^{(1)}|^2 = \frac{4c}{c+4}, \tag{3.55}$$

which implies $|a_1^{(1)}|^2 = \frac{16}{c+4}$. Then it follows from (3.49) that $\frac{64c}{(c+4)^2} = c$, which contradicts that $0 < c < 4$. Thus there does not exist the case of $W_4 \neq 0$. □

Proof of Theorem 1.2 It is enough to consider the irreducible case. Assume that φ generates the following harmonic sequence (cf. [2],[6])

$$0 \xleftarrow{A''_{\varphi_0}} \underline{\varphi}_0 = \underline{\varphi} \xrightarrow{A'_{\varphi_0}} \underline{\varphi}_1 \xrightarrow{A'_{\varphi_1}} \underline{\varphi}_2 \xrightarrow{A'_{\varphi_2}} \underline{\varphi}_3 \xrightarrow{A'_{\varphi_3}} 0, \tag{3.56}$$

where $\operatorname{rank}(\underline{\varphi}_0) = \operatorname{rank}(\underline{\varphi}_1) = 2$ and $\left(\operatorname{rank}(\underline{\varphi}_2), \operatorname{rank}(\underline{\varphi}_3)\right) = (1,1)$ or $(2,0)$. We will prove d = 4 in the above two cases.

If $\left(\operatorname{rank}(\underline{\varphi}_2), \operatorname{rank}(\underline{\varphi}_3)\right) = (1,1)$, then $\underline{\varphi}_2 = \underline{f}_{m-1}^{(m)}, \underline{\varphi}_3 = \underline{f}_m^{(m)}$ belong the following harmonic sequence in $\mathbb{C}P^m$

$$0 \xleftarrow{A''_{f_0^{(m)}}} \underline{f}_0^{(m)} \xrightarrow{A'_{f_0^{(m)}}} \underline{f}_1^{(m)} \xrightarrow{A'_{f_1^{(m)}}} \cdots \xrightarrow{A'_{f_{m-2}^{(m)}}} \underline{f}_{m-1}^{(m)} \xrightarrow{A'_{f_{m-1}^{(m)}}} \underline{f}_m^{(m)} \xrightarrow{A'_{f_m^{(m)}}} 0 \tag{3.57}$$

for $m = 5, 4, 3$.

When $m = 5$, we have a diagram below about the harmonic sequence in $\mathbb{C}P^5$ (cf. [2]),

$$\underline{e}_2 \longrightarrow \underline{e}_4 = \underline{f}_3^{(5)} \longrightarrow \underline{\varphi}_2 = \underline{f}_4^{(5)} \longrightarrow \underline{\varphi}_3 = \underline{f}_5^{(5)} \longrightarrow 0.$$

$$\underline{e}_1 \longrightarrow \underline{e}_3$$

It follows that

$$v_1 \wedge v_2 \wedge \partial_z v_1 \wedge \partial_z v_2 = h(z) f_0^{(5)} \wedge \partial_z f_0^{(5)} \wedge \partial_z^2 f_0^{(5)} \wedge \partial_z^3 f_0^{(5)},$$

18

where $h(z)$ is a polynomial. Notice that $f_0^{(5)} \wedge \partial_z f_0^{(5)} \wedge \partial_z^2 f_0^{(5)} \wedge \partial_z^3 f_0^{(5)}$ is the 3-osculating curve σ_3 of $f_0^{(5)}$. It follows from (2.5) that h is a non-zero constant and σ_3 has constant curvature. Using Y.B. Shen's result ([28], Theorem 6.2), we know the harmonic sequence (3.57) in $\mathbb{C}P^5$ is the Veronese sequence, up to $U(6)$. Then we obtain by (2.5) and (3.24) in [1]

$$2\mathrm{d} - 4 = \delta_3^{(5)} = 8,$$

where $\delta_3^{(5)}$ is the degree of σ_3. It follows that $\mathrm{d} = 6$. In this case, there exist polynomials P_1, P_2, P_3 such that

$$v_1 \wedge v_2 = P_1 f_0^{(5)} \wedge \partial_z f_0^{(5)} + P_2 f_0^{(5)} \wedge \partial_z^2 f_0^{(5)} + P_3 \partial_z f_0^{(5)} \wedge \partial_z^2 f_0^{(5)}.$$

We choose a rectangular coordinate system in $\mathbb{C}^6$ such that

$$f_0^{(5)} = \epsilon_1 + \sqrt{5}z\epsilon_2 + \sqrt{10}z^2\epsilon_3 + \sqrt{10}z^3\epsilon_4 + \sqrt{5}z^4\epsilon_5 + z^5\epsilon_6.$$

A straightforward calculation shows that the degree of $f_0^{(5)} \wedge \partial_z f_0^{(5)}$, $f_0^{(5)} \wedge \partial_z^2 f_0^{(5)}$ and $\partial_z f_0^{(5)} \wedge \partial_z^2 f_0^{(5)}$ is 8, 7 and 6 respectively, and

$$
\begin{aligned}
f_0^{(5)} \wedge \partial_z f_0^{(5)} = {} & \sqrt{5}\epsilon_1 \wedge \epsilon_2 + 2\sqrt{10}z\epsilon_1 \wedge \epsilon_3 + 3\sqrt{10}z^2\epsilon_1 \wedge \epsilon_4 + 4\sqrt{5}z^3\epsilon_1 \wedge \epsilon_5 + 5z^4\epsilon_1 \wedge \epsilon_6 \\
& + \sqrt{50}z^2\epsilon_2 \wedge \epsilon_3 + 2\sqrt{50}z^3\epsilon_2 \wedge \epsilon_4 + 15z^4\epsilon_2 \wedge \epsilon_5 + 4\sqrt{5}z^5\epsilon_2 \wedge \epsilon_6 \\
& + 10z^4\epsilon_3 \wedge \epsilon_4 + 2\sqrt{50}z^5\epsilon_3 \wedge \epsilon_5 + 3\sqrt{10}z^6\epsilon_3 \wedge \epsilon_6 \\
& + \sqrt{50}z^6\epsilon_4 \wedge \epsilon_5 + 2\sqrt{10}z^7\epsilon_4 \wedge \epsilon_6 + \sqrt{5}z^8\epsilon_5 \wedge \epsilon_6,
\end{aligned}
$$

$$
\begin{aligned}
f_0^{(5)} \wedge \partial_z^2 f_0^{(5)} = {} & 2\sqrt{10}\epsilon_1 \wedge \epsilon_3 + 6\sqrt{10}z\epsilon_1 \wedge \epsilon_4 + 12\sqrt{5}z^2\epsilon_1 \wedge \epsilon_5 + 20z^3\epsilon_1 \wedge \epsilon_6 \\
& + 2\sqrt{50}z\epsilon_2 \wedge \epsilon_3 + 6\sqrt{50}z^2\epsilon_2 \wedge \epsilon_4 + 60z^3\epsilon_2 \wedge \epsilon_5 + 20\sqrt{5}z^4\epsilon_2 \wedge \epsilon_6 \\
& + 40z^3\epsilon_3 \wedge \epsilon_4 + 10\sqrt{50}z^4\epsilon_3 \wedge \epsilon_5 + 18\sqrt{10}z^5\epsilon_3 \wedge \epsilon_6 \\
& + 6\sqrt{50}z^5\epsilon_4 \wedge \epsilon_5 + 14\sqrt{10}z^6\epsilon_4 \wedge \epsilon_6 + 8\sqrt{5}z^7\epsilon_5 \wedge \epsilon_6,
\end{aligned}
$$

$$
\begin{aligned}
\partial_z f_0^{(5)} \wedge \partial_z^2 f_0^{(5)} = {} & 2\sqrt{50}\epsilon_2 \wedge \epsilon_3 + 6\sqrt{50}z\epsilon_2 \wedge \epsilon_4 + 60z^2\epsilon_2 \wedge \epsilon_5 + 20\sqrt{5}z^3\epsilon_2 \wedge \epsilon_6 \\
& + 60z^2\epsilon_3 \wedge \epsilon_4 + 16\sqrt{50}z^3\epsilon_3 \wedge \epsilon_5 + 30\sqrt{10}z^4\epsilon_3 \wedge \epsilon_6 \\
& + 12\sqrt{50}z^4\epsilon_4 \wedge \epsilon_5 + 30\sqrt{10}z^5\epsilon_4 \wedge \epsilon_6 + 20\sqrt{5}z^6\epsilon_5 \wedge \epsilon_6.
\end{aligned}
$$

Since $[v_1 \wedge v_2]$ is a constantly curved holomorphic curve of degree 6, then under the standard basis $\epsilon_i \wedge \epsilon_j$, $1 \le i < j \le 6$ in the lexicographic order, we can write $v_1 \wedge v_2 = PHV_0^{(6)}$, where P is a polynomial, $V_0^{(6)} = (1, \sqrt{6}z, \sqrt{15}z^2, \sqrt{20}z^3, \sqrt{15}z^4, \sqrt{6}z^5, z^6)^T$ and $H = (H_{\alpha j})_{\alpha=0,\cdots,14, j=0,\cdots,6}$ is a 15×7 matrix whose column vectors are mutually orthonormal. A straightforward calculation shows

$$\sqrt{5}P_1 = P(H_{00} + \cdots + H_{06}z^6), \quad 2\sqrt{10}zP_1 + 2\sqrt{10}P_2 = P(H_{10} + \cdots + H_{16}z^6),$$

$$\sqrt{50}z^2 P_1 + 2\sqrt{50}zP_2 + 2\sqrt{50}P_3 = P(H_{50} + \cdots + H_{56}z^6),$$

which implies $P|P_j (j = 1, 2, 3)$. Set $P_j = P\tilde{P}_j$, then we have

$$\tilde{P}_1 f_0^{(5)} \wedge \partial_z f_0^{(5)} + \tilde{P}_2 f_0^{(5)} \wedge \partial_z^2 f_0^{(5)} + \tilde{P}_3 \partial_z f_0^{(5)} \wedge \partial_z^2 f_0^{(5)} = HV_0^{(6)}.$$

It follows from the above formulas that the degree of $\tilde{P}_1, \tilde{P}_2, \tilde{P}_3$ is $0, 1, 2$ respectively. Set $\tilde{P}_1 = \tilde{p}_{10} \neq 0$, $\tilde{P}_2 = \tilde{p}_{20} + \tilde{p}_{21}z$ and $\tilde{P}_3 = \tilde{p}_{30} + \tilde{p}_{31}z + \tilde{p}_{32}z^2$, then that the terms of z^7, z^8 in $v_1 \wedge v_2$ are vanishing tells us

$$\tilde{p}_{10} + 7\tilde{p}_{21} + 15\tilde{p}_{32} = 0, \tag{3.58}$$

$$\tilde{p}_{10} + 8\tilde{p}_{21} + 20\tilde{p}_{32} = 0, \tag{3.59}$$

and

$$2\tilde{p}_{20} + 5\tilde{p}_{31} = 0. \tag{3.60}$$

From (3.58) and (3.59), we get

$$\tilde{p}_{21} = -\frac{1}{4}\tilde{p}_{10}, \ \tilde{p}_{32} = \frac{1}{20}\tilde{p}_{10}. \tag{3.61}$$

At this time H is given by

$$H = \begin{pmatrix} H_{00} & 0 & 0 & 0 & 0 & 0 & 0 \\ H_{10} & H_{11} & 0 & 0 & 0 & 0 & 0 \\ 0 & H_{21} & H_{22} & 0 & 0 & 0 & 0 \\ 0 & 0 & H_{32} & H_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & H_{43} & H_{44} & 0 & 0 \\ H_{50} & H_{51} & H_{52} & 0 & 0 & 0 & 0 \\ 0 & H_{61} & H_{62} & H_{63} & 0 & 0 & 0 \\ 0 & 0 & H_{72} & H_{73} & H_{74} & 0 & 0 \\ 0 & 0 & 0 & H_{83} & H_{84} & H_{85} & 0 \\ 0 & 0 & H_{92} & H_{93} & H_{94} & 0 & 0 \\ 0 & 0 & 0 & H_{10,3} & H_{10,4} & H_{10,5} & 0 \\ 0 & 0 & 0 & 0 & H_{11,4} & H_{11,5} & H_{11,6} \\ 0 & 0 & 0 & 0 & H_{12,4} & H_{12,5} & H_{12,6} \\ 0 & 0 & 0 & 0 & 0 & H_{13,5} & H_{13,6} \\ 0 & 0 & 0 & 0 & 0 & 0 & H_{14,6} \end{pmatrix}.$$

Using (3.61), we know that $H_{11}H_{52} \neq 0$. Since the column vectors of H are mutually orthonormal, then $\overline{H_{50}}H_{52} = 0$, which implies $H_{50} = 0$, i.e., $\tilde{p}_{30} = 0$. From $\overline{H_{10}}H_{11} = 0$, we have $H_{10} = 0$, i.e., $\tilde{p}_{20} = 0$. Substituting this into (3.60) yields $\tilde{p}_{31} = 0$. Now H

becomes

$$
H = \begin{pmatrix}
H_{00} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & H_{11} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & H_{22} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & H_{33} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & H_{44} & 0 & 0 \\
0 & 0 & H_{52} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & H_{63} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & H_{74} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & H_{85} & 0 \\
0 & 0 & 0 & 0 & H_{94} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & H_{10,5} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & H_{11,6} \\
0 & 0 & 0 & 0 & 0 & 0 & H_{12,6} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.
$$

Since $H_{00} = \sqrt{5}\tilde{p}_{10}$ and $H_{11} = \frac{3\sqrt{10}}{2}\tilde{p}_{10}$, then it follows from $|H_{00}|^2 = 1$ that $|\tilde{p}_{10}|^2 = \frac{1}{5}$, which implies that $|H_{11}|^2 = \frac{3}{2}$. It contradicts that $|H_{11}|^2 = 1$. Thus there does not exist the case of $m = 5$.

When $m = 4$, the trivial bundle $S^2 \times \mathbb{C}^6$ over S^2 has a decomposition $S^2 \times \mathbb{C}^6 = S^2 \times \mathbb{C}^5 \oplus S^2 \times \mathbb{C}$. In this case, the diagram reduces to

where $\mathrm{span}_{\mathbb{C}}\{e_1, e_2, e_3\} = \mathrm{span}_{\mathbb{C}}\left\{v_0, f_0^{(4)}, f_1^{(4)}\right\}$ with $v_0 = (0,0,0,0,0,1)$. Similarly we get

$$2\mathrm{d} - 4 = \delta_2^{(4)} = 6,$$

which implies $\mathrm{d} = 5$. In this case, there exist polynomials P_1, P_2, P_3 such that

$$v_1 \wedge v_2 = P_1 f_0^{(4)} \wedge v_0 + P_2 \partial_z f_0^{(4)} \wedge v_0 + P_3 f_0^{(4)} \wedge \partial_z f_0^{(4)}.$$

We choose a rectangular coordinate system in $\mathbb{C}^5$ such that

$$f_0^{(4)} = \epsilon_1 + 2z\epsilon_2 + \sqrt{6}z^2\epsilon_3 + 2z^3\epsilon_4 + z^4\epsilon_5.$$

A straightforward calculation shows that the degree of $f_0^{(4)} \wedge v_0$, $\partial_z f_0^{(4)} \wedge v_0$ and $f_0^{(4)} \wedge \partial_z f_0^{(4)}$ is 4, 3 and 6 respectively, and

$$f_0^{(4)} \wedge v_0 = \epsilon_1 \wedge \epsilon_6 + 2z\epsilon_2 \wedge \epsilon_6 + \sqrt{6}z^2\epsilon_3 \wedge \epsilon_6 + 2z^3\epsilon_4 \wedge \epsilon_6 + z^4\epsilon_5 \wedge \epsilon_6,$$

$$\partial_z f_0^{(4)} \wedge v_0 = 2\epsilon_2 \wedge \epsilon_6 + 2\sqrt{6}z\epsilon_3 \wedge \epsilon_6 + 6z^2\epsilon_4 \wedge \epsilon_6 + 4z^3\epsilon_5 \wedge \epsilon_6,$$

$$
\begin{aligned}
f_0^{(4)} \wedge \partial_z f_0^{(4)} = \; & 2\epsilon_1 \wedge \epsilon_2 + 2\sqrt{6}z\epsilon_1 \wedge \epsilon_3 + 6z^2\epsilon_1 \wedge \epsilon_4 + 4z^3\epsilon_1 \wedge \epsilon_5 \\
& + 2\sqrt{6}z^2\epsilon_2 \wedge \epsilon_3 + 8z^3\epsilon_2 \wedge \epsilon_4 + 6z^4\epsilon_2 \wedge \epsilon_5 \\
& + 2\sqrt{6}z^4\epsilon_3 \wedge \epsilon_4 + 2\sqrt{6}z^5\epsilon_3 \wedge \epsilon_5 + 2z^6\epsilon_4 \wedge \epsilon_5.
\end{aligned}
$$

Since $[v_1 \wedge v_2]$ is a constantly curved holomorphic curve of degree 5, then under the standard basis $\epsilon_i \wedge \epsilon_j$, $1 \le i < j \le 6$ in the lexicographic order, we can write $v_1 \wedge v_2 = PHV_0^{(5)}$, where P is a polynomial, $V_0^{(5)} = (1, \sqrt{5}z, \sqrt{10}z^2, \sqrt{10}z^3, \sqrt{5}z^4, z^5)^T$ and $H = (H_{\alpha j})_{\alpha = 0, \cdots, 14, j = 0, \cdots, 5}$ is a 15×6 matrix whose column vectors are mutually orthonormal. A similar discussion shows that $P | P_j (j = 1, 2, 3)$. Set $P_j = P\tilde{P}_j$, then we have

$$
\tilde{P}_1 f_0^{(4)} \wedge v_0 + \tilde{P}_2 \partial_z f_0^{(4)} \wedge v_0 + \tilde{P}_3 f_0^{(4)} \wedge \partial_z f_0^{(4)} = HV_0^{(5)}.
$$

It follows from the above formulas that $\tilde{P}_3 = 0$. So $\underline{\varphi} = \mathrm{span}\left\{v_0, \tilde{P}_1 f_0^{(4)} + \tilde{P}_2 \partial_z f_0^{(4)}\right\}$, which implies that φ is reducible. Hence there does not exist the case of $m = 4$.

When $m = 3$, then the trivial bundle $S^2 \times \mathbb{C}^6$ over S^2 has a decomposition $S^2 \times \mathbb{C}^6 = S^2 \times \mathbb{C}^4 \oplus S^2 \times \mathbb{C}^2$. In this case, we have

$$
2\mathrm{d} - 4 = \delta_1^{(3)} = 4,
$$

which implies $\mathrm{d} = 4$.

If $\left(\mathrm{rank}(\underline{\varphi}_2), \mathrm{rank}(\underline{\varphi}_3)\right) = (2, 0)$, then we choose a local unitary frame $e = \{e_1, \cdots, e_6\}$ along φ so that

$$
\underline{\varphi}_0 = \mathrm{span}\{e_1, e_2\}, \quad \underline{\varphi}_1 = \mathrm{span}\{e_3, e_4\}, \quad \underline{\varphi}_2 = \mathrm{span}\{e_5, e_6\}.
$$

Under such frame, the pull back of (right invariant) Maurer-Cartan forms which are denoted by $\omega = (\omega_{AB})$ are

$$
\begin{pmatrix}
\Omega_1 & A_1\phi & \\
-A_1^*\bar{\phi} & \Omega_2 & A_2\phi \\
& -A_2^*\bar{\phi} & \Omega_3
\end{pmatrix}.
$$

Notice that the unitary frame we choose is determined up to a transformation of the group $U(2) \times U(2) \times U(2)$, so $|\det A_i|$ $(i = 1, 2)$ are global invariants of analytic type on S^2 vanishing only at isolated points, and away from their zeros, they satisfy (cf. [7], [12])

$$
\Delta \log |\det A_1| = 2K - 4L_1 + 2L_2, \tag{3.62}
$$

$$
\Delta \log |\det A_2| = 2K + 2L_1 - 4L_2, \tag{3.63}
$$

where $K = 4/\mathrm{d}$, $L_i = tr A_i A_i^*$ $(i = 1, 2)$ are also globally defined invariants on S^2 with $L_1 = 1$, and Δ is Laplace-Beltrami operator with respect to $\mathrm{d} \cdot ds_{S^2}^2$. It follows from (3.62) and (3.63) that

$$
\Delta \log |\det A_1|^2 |\det A_2| = \frac{6(4 - \mathrm{d})}{\mathrm{d}}. \tag{3.64}
$$

Since φ is totally unramified, then $|\det A_1|^2 |\det A_2|$ has no zeros on S^2. Applying the maximum principle to (3.64), we get $\mathrm{d} = 4$. The conclusion follows from Theorem 1.1. $\square$

Remark 3.1 *In the irreducible case of Theorem 1.2, the condition of constant square norm of the second fundamental form is necessary. Jiao [18] gave an example of irreducible constantly curved holomorphic two-sphere of* d = 4 *in* $G(2,6)$*, as follows,*

$$\varphi = \begin{bmatrix} 1 & 0 & \frac{1}{\sqrt{2}}z & \frac{\sqrt{31}}{2\sqrt{7}}z^2 & \frac{9}{2\sqrt{7}}z^2 & 0 \\ 0 & 1 & 0 & 0 & \frac{\sqrt{7}}{\sqrt{2}}z & \frac{1}{2}z^2 \end{bmatrix} \tag{3.65}$$

with

$$|\det A_1|^2 = \frac{112 + 1024z\bar{z} + 1176z^2\bar{z}^2 + 376z^3\bar{z}^3 + 31z^4\bar{z}^4}{1024(1 + z\bar{z})^4},$$

which is not a constant.

References

[1] J. Bolton, G.R. Jensen, M. Rigoli and L.M. Woodward, *On conformal minimal immersions of S^2 into $\mathbb{C}P^n$*, Math. Ann., 279(1988), 599-620.

[2] F.E. Burstall and J.C. Wood, *The construction of harmonic maps into complex Grassmannians*, J. Diff. Geom., 23(1986), 255-297.

[3] E. Calabi, *Isometric embeddings of complex manifolds*, Ann. Math., 58(1953), 1-23.

[4] S.S. Chern, *Minimal submanifolds in a Riemannian manifold*, Mineographed Lecture Note, Univ. of Kansas, 1968.

[5] S.S. Chern, M. do Carmo, S. Kobayashi, *Minimal submanifolds of a sphere with second fundamental form of constant length*, Functional Analysis and Related Fields(Proc. conf. in honor of M. Stone at Univ. of Chicago, 1968), Springer, Berlin, 59-75, 1970.

[6] S.S. Chern and J.G. Wolfson, *Harmonic maps of the 2-spheres in a complex Grassmann manifold II*, Ann. Math., 125(1987), 301-335.

[7] Q.S. Chi and Y.B. Zheng, *Rigidity of pseudo-holomorphic curves of constant curvature in Grassmann manifolds*, Trans. Amer. Math. Soc., 313(1989), 393-406.

[8] Q.S. Chi, Z.X. Xie and Y. Xu, *Fano 3-folds and classification of constantly curved holomorphic 2-spheres in the complex Grassmannian $G(2,5)$* , arXiv:2208.08525v2.

[9] L. Delisle, V. Hussin and W.J. Zakrzewski, *Constant curvature solutions of Grassmannian sigma models: (1) Holomorphic solutions*, J. Geom. Phys., 66(2013), 24-36.

[10] J. Fei, *Classification of homogeneous holomorphic two-spheres in complex Grassmann manifolds*, Differ. Geom. Appl., 62(2019), 1-38.

[11] J. Fei, L. He, J. Wang, *Rigidity of homogeneous holomorphic two-spheres in a complex Grassmann manifold $G(2,N)$*, J. Geom. Anal., 33(2023):324.

[12] J. Fei, X.X. Jiao and X.W. Xu, *On conformal minimal 2-spheres in complex Grassmann manifold $G(2,n)$*, Proc. Indian Acad. Sci.(Math. Sci), 121(2)(2011), 181-199.

[13] J. Fei and J. Wang, *Pinched Constantly Curved Holomorphic Two-Spheres in the complex Grassmann manifolds*, Results Math., 79(2024), 209.

[14] J. Fei and X.W. Xu, *Local rigidity of holomorphic curves in the complex Grassmann manifold $G(2,6)$*, J. Geom. Phys., 121(2017), 438-451.

[15] P. Griffiths and J. Harris, *Principles of algebraic geometry*, Wiley, New York, 1978.

[16] L. He, *The conformal minimal immersions of S^2 in $G(k,n)$*, Ph.D Thesis, University of Chinese Academy Sciences, 2014.

[17] L. He, *Degree of constantly curved holomorphic 2-spheres in the complex Grassmannians $G(2, n+2; \mathbb{C})$*, arXiv: 2204.08116.

[18] X.X. Jiao, *Pseudo-holomorphic curves of constant curvature in complex Grassmannians*, Isr. J. Math., 163(2008), 45-60.

[19] G. Jensen, *Higher Order Contact of Submanifolds of Homogeneous Spaces*, Lecture Notes in Mathematics, vol.610, Springer-Verlag, Berlin, Heidelberg, New York, 1977.

[20] X.X. Jiao and J.G. Peng, *Classification of holomorphic spheres of constant curvature in complex Grassmann manifold $G_{2,5}$*, Diff. Geom. Appl., 20(2004), 267-277.

[21] X.X. Jiao and J.G. Peng, *On holomorphic curves of constant curvature in the complex Grassmann manifold $G(2,5)$*, Acta Math. Sin., 31B(1)(2011), 237-248.

[22] Z.Q. Li and M.M. Jin, *Constant curved holomorphic 2-spheres in $G(2,4)$*, J. Math. Sci. Adv. Appl., 1(2008), 547-561.

[23] Z.Q. Li and Z.H. Yu, *Constant curved minimal 2-spheres in $G(2,4)$*, Manuscr. Math., 100(1999), 305-316.

[24] N. Mok, *On the asymptotic behavior of holomorphic isometries of the Poincaré disk into bounded symmetric domains*, Acta Math Sci, Engl Ser, 29(2009), 881-902.

[25] N. Mok and S.C. Ng, *Second fundamental forms of holomorphic isometries of the Poincaré disk into bounded symmetric domains and their boundary behavior along the unit circle*, Sci China Ser A, 52(2009), 2628-2646.

[26] C.K. Peng and X.W. Xu, *Classification of minimal homogeneous two spheres in the complex Grassmann manifold $G(2,n)$*, J. Math. Pures Appl., 103(2) (2014), 374-399.

[27] M. Rigoli, *A rigidity result for holomorphic immersions of surfaces in $\mathbb{C}P^n$*, Proc. Amer. Math. Soc., 93(1985), 317-320.

[28] Y.B. Shen, *The Riemannian geometry of superminimal surfaces in complex spaces forms*, Acta Math. Sin., 12(1996), 298-313.

[29] J. Wang, J. Fei, X.W. Xu, *Pinching for holomorphic curves in a complex Grassmann manifold $G(2, n; \mathbb{C})$*, Differ. Geom. Appl., 80(2022), 101840.