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Weak Type Estimates for Square Functions of Dunkl Heat Flows

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Abstract We establish the weak $(1,1)$ boundedness of the Littlewood–Paley–Stein square function associated with the Dunkl heat flow, which is generated by the non-local Dunkl Laplacian. Our proof relies on precise integral estimates for derivatives of the Dunkl heat kernel, combined with the technique of Caldrón–Zygmund decomposition. This is the continuity of the recent work [Math. Nachr. 296: 1225–1243 (2023)], where dimension-free L^p boundedness for the same square functions was proved.

Keywords Dunkl operator; heat kernel; square function; jump process
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1 Introduction to main results

In this section, we aim to recall some necessary basics on the Dunkl operator and then present the main results of this work. The Dunkl operator, initially introduced by C.F. Dunkl in the seminal paper[16] (see also [15]), has been studied intensively. For a general overview on its development and more details, refer to the survey papers[4, 33] and the monographs[14, 18].

Consider the d-dimensional Euclidean space $\mathbb{R}^d$ endowed with the standard inner product $\langle\cdot,\cdot\rangle$ and the induced norm $|\cdot|$. For every $\alpha\in\mathbb{R}^d\setminus\{0\}$, define

$$r_\alpha x = x - 2\frac{\langle\alpha,x\rangle}{|\alpha|^2}\alpha,\qquad x\in\mathbb{R}^d,$$

where r_α is the so-called reflection operator with respect to (abbrev w.r.t.) the hyperplane orthogonal to α.

Let $\mathfrak{R}$ denote the root system, which is a finite subset of $\mathbb{R}^d\setminus\{0\}$ and satisfies that, for every $\alpha\in\mathfrak{R}$, $r_\alpha(\mathfrak{R})=\mathfrak{R}$ and $\alpha\mathbb{R}\cap\mathfrak{R}=\{\alpha,-\alpha\}$, where $\alpha\mathbb{R}:=\{\alpha b:\ b\in\mathbb{R}\}$. Without loss of generality, we may assume that $|\alpha|=\sqrt{2}$ for all $\alpha\in\mathfrak{R}$. Let G be the reflection (or Weyl) group generated by the family of finitely many reflection operators $\{r_\alpha:\ \alpha\in\mathfrak{R}\}$. Note that G is a finite subgroup of the orthogonal group $O(d)$, i.e., the group of $d\times d$ orthogonal matrices, and $\{r_\alpha:\ \alpha\in\mathfrak{R}\}\subset G$ (see e.g. [18, Theorem 6.2.7] for a proof). Let $\mathfrak{R}_+$ be an arbitrary chosen positive subsystem such that $\mathfrak{R}$ can be written as the disjoint union of $\mathfrak{R}_+$ and $-\mathfrak{R}_+$, where $-\mathfrak{R}_+:=\{-\alpha:\ \alpha\in\mathfrak{R}_+\}$.

Let $\kappa_\cdot:\mathfrak{R}\to\mathbb{R}_+$ be the multiplicity function such that it is G-invariant, i.e., $\kappa_{g\alpha}=\kappa_\alpha$ for every $g\in G$ and every $\alpha\in\mathfrak{R}$.

Let $\xi\in\mathbb{R}^d$. Define the (non-local) Dunkl operator D_ξ along ξ associated with the root system $\mathfrak{R}$ and the multiplicity function κ as follows:

$$\mathrm{D}_\xi f(x)=\partial_\xi f(x)+\sum_{\alpha\in\mathfrak{R}_+}\kappa_\alpha\langle\alpha,\xi\rangle\frac{f(x)-f(r_\alpha x)}{\langle\alpha,x\rangle},\qquad f\in C^1(\mathbb{R}^d),\quad x\in\mathbb{R}^d,$$

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where ∂_ξ denotes the directional derivative along ξ. It is important to mention that, for every $\xi,\eta\in\mathbb{R}^d$, $\mathrm{D}_\eta\circ\mathrm{D}_\xi=\mathrm{D}_\xi\circ\mathrm{D}_\eta$. However, due to the existence of difference parts, the Leibniz rule and the chain rule may not hold for D_ξ.

Let $\{e_j:\ j=1,\cdots,d\}$ be the standard orthonormal basis of $\mathbb{R}^d$, and write D_j instead of D_{e_j} for short, $j=1,\cdots,d$. We denote $\nabla_\kappa=(\mathrm{D}_1,\cdots,\mathrm{D}_d)$ and $\Delta_\kappa=\sum_{j=1}^d\mathrm{D}_j^2$ the Dunkl gradient operator and the Dunkl Laplacian, respectively. By a straightforward calculation, for every $f\in C^2(\mathbb{R}^d)$,

$$\Delta_\kappa f(x)=\Delta f(x)+2\sum_{\alpha\in\mathfrak{R}_+}\kappa_\alpha\Big(\frac{\langle\alpha,\nabla f(x)\rangle}{\langle\alpha,x\rangle}-\frac{f(x)-f(r_\alpha x)}{\langle\alpha,x\rangle^2}\Big),\qquad x\in\mathbb{R}^d.$$

Obviously, when $\kappa=0$, then $\nabla_0=\nabla$ and $\Delta_0=\Delta$, the standard gradient operator and the Laplacian on $\mathbb{R}^d$, respectively.

Similar as the Laplacian case, define the *carré du champ* (i.e.,squared (modulus) of the (vector) field in English) Γ (see e.g. [6]) by

$$\Gamma(f,g):=\frac{1}{2}\big[\Delta_\kappa(fg)-f\Delta_k g-g\Delta_\kappa f\big],\qquad f,g\in C^2(\mathbb{R}^d).$$

Set $\Gamma(f)=\Gamma(f,f)$ for convenience. It is easy to see that, for every $f,g\in C^2(\mathbb{R}^d)$ and $x\in\mathbb{R}^d$,

$$\Gamma(f,g)(x)=\langle\nabla f(x),\nabla g(x)\rangle+\sum_{\alpha\in\mathfrak{R}_+}\kappa_\alpha\frac{\big(f(x)-f(r_\alpha x)\big)\big(g(x)-g(r_\alpha x)\big)}{\langle\alpha,x\rangle^2},\tag{1.1}$$

and hence $\Gamma(f)\geq 0$. Let

$$\chi=\sum_{\alpha\in\mathfrak{R}_+}\kappa_\alpha.$$

From[?]Remark 1.4(i)]LZ2020), we have the following pointwise inequality, i.e.,

$$|\nabla_\kappa f|^2\leq(1+2\chi)\Gamma(f),\qquad f\in C^2(\mathbb{R}^d),\tag{1.2}$$

and in general, the converse inequality is not true (see [30, Theorem 3.5]).

The natural measure associated with the Dunkl operator is $w_\kappa\mathscr{L}_d$, where for every $x\in\mathbb{R}^d$,

$$w_\kappa(x):=\prod_{\alpha\in\mathfrak{R}_+}|\langle\alpha,x\rangle|^{2\kappa_\alpha},$$

and $\mathscr{L}_d$ stands for the Lebesgue measure on $\mathbb{R}^d$. Let $\mu_\kappa=w_\kappa\mathscr{L}_d$. For each $p\in[1,\infty]$, we denote the L^p-space on $\mathbb{R}^d$ w.r.t. μ_κ by $L^p(\mu_\kappa):=L^p(\mathbb{R}^d,\mu_\kappa)$ and the corresponding norm by $\|\cdot\|_{L^p(\mu_\kappa)}$. It is well known that Δ_κ is an essentially self-adjoint operator defined on a suitable domain in $L^2(\mu_\kappa)$.

Let $H_\kappa(t):=e^{t\Delta_\kappa}$, $t\geq 0$, be the Dunkl heat flow, which is self-adjoint in $L^2(\mu_\kappa)$. For $1\leq p<\infty$, $(H_\kappa(t))_{t\geq 0}$ can be extended uniquely to a strongly continuous contraction semigroup in $L^p(\mu_\kappa)$, for which, with some abuse of notation, we keep the same notation. See [31, 33, 34] for further properties for the Dunkl heat flow.

We are concerned with square functions corresponding to the Dunkl heat flow. For $f\in C_c^\infty(\mathbb{R}^d)$ and $x\in\mathbb{R}^d$, we define the vertical Littlewood–Paley–Stein (abbrev LPS) square functions by

$$\mathcal{V}_\Gamma(f)(x)=\Big(\int_0^\infty\Gamma\big(H_\kappa(t)f\big)(x)\ dt\Big)^{1/2},$$

$$\mathcal{V}_{\nabla_\kappa}(f)(x) = \Big(\int_0^\infty |\nabla_\kappa H_\kappa(t)f|^2(x)\, dt\Big)^{1/2},$$
$$\mathcal{V}_{\nabla}(f)(x) = \Big(\int_0^\infty |\nabla H_\kappa(t)f|^2(x)\, dt\Big)^{1/2},$$

and the horizontal LPS square function by

$$\mathcal{H}(f)(x) \quad = \quad \Big(\int_0^\infty t\big|\partial_t H_\kappa(t)f\big|^2(x)\, dt\Big)^{1/2}.$$

It is easy to see that, being initially defined on $C_c^\infty(\mathbb{R}^d)$, operators $\mathcal{V}_\Gamma, \mathcal{V}_{\nabla_\kappa}, \mathcal{V}_\nabla$ and $\mathcal{H}$ are all sublinear.

In this work, we concentrate on the study of weak $(1,1)$ boundedness of the square functions defined above. From (1.2) and the definition of Γ, we see that, for every $f \in C_c^\infty(\mathbb{R}^d)$, both $\mathcal{V}_{\nabla_\kappa}(f)$ and $\mathcal{V}_\nabla(f)$ are controlled by $\mathcal{V}_\Gamma(f)$ in the pointwise sense. Since the Dunkl heat flow $\{H_\kappa(t)\}_{t\ge 0}$ is a symmetric diffusion semigroup in the sense of [37, page 65], $\mathcal{H}$ is always bounded in $L^p(\mu_\kappa)$ for all $p \in (1,\infty)$ as a particular example of Corollary 1 on page 120 of [37]. So, it is more interesting to us to study the weak $(1,1)$ boundedness of $\mathcal{V}_\Gamma$ and $\mathcal{H}$.

With these preparations in hand, we can present our main results in the following theorems. The first one is on the vertical LPS square function.

Theorem 1.1. *The operator $\mathcal{V}_\Gamma$ is weak $(1,1)$ bounded.*

The second one is on the horizontal LPS square function.

Theorem 1.2. *The operator $\mathcal{H}$ is weak $(1,1)$ bounded.*

We remark that bounds in Theorems 1.1 and 1.2 are dimension-dependent, due to the approach we employed.

It is well known that the square function, which corresponds to the quadratic variation of martingales, is one of the most fundamental objects in harmonic analysis and plays important roles in probability theory; see e.g. the survey paper[38] and the books[37, 39]. Despite extensive studies on LPS square functions in various settings in the literature, we recall known results in the Dunkl setting here. For L^p boundedness $(1 < p < \infty)$, see [35] and [29] in the one-dimensional case, and see [1, 36] and the recent[19, 27] in the multidimensional case. We should point out that the results in [27] are dimension-free, although restricted to the $\mathbb{Z}_2^d$ case when $p > 2$. However, the weak $(1,1)$ boundedness seems not widely studied. We mention that, the approach via the vector-valued Calderón–Zygmund theory, which crucially depends on pointwise Dunkl heat kernel estimates and is different from the approach employed below, should imply the weak $(1,1)$ boundedness; see the proof of [19, Proposition 3.1] for more details.

We should point out that the LPS associated with Markovian jump processes is getting increasing attention in recent years. As for the study on L^p boundedness $(1 < p < \infty)$ of LPS square functions associated with pure jump Lévy processes, we should mention the paper[7], and see also the recent works[25, 26] for further extensions to non-local pure jump Dirichlet forms (with killing) on metric measure spaces. The recent paper[24] showed the L^p boundedness $(1 < p < \infty)$ of LPS square functions associated with the fractional discrete Laplacian on the one-dimensional lattice, where the underlying graph is not locally finite. However, we notice that the Markovian jump process corresponding to the Dunkl Laplacian, the so-called Dunkl process, is in general not a Lévy process; see e.g. [20, 27, 34] for further studies on the Dunkl process.

For the proof of our Theorems 1.1 and 1.2, the overall idea is motivated by the study on LPS square functions associated with the (local) Laplace–Beltrami operator on Riemannian manifolds in [10], which mainly depends on the Caldrón–Zygmund decomposition and integral estimates on derivatives of the heat kernel. However, in the present Dunkl setting, we need some new ideas to overcome difficulties caused by the non-local nature. Recently, a different type of integral bounds on Dunkl gradient of the corresponding kernel has been employed in [3] to prove the weak $(1,1)$ boundedness of the Riesz transform associated with the Dunkl-Schrödinger operator $-\Delta_\kappa + V$ with $0 \le V \in L^2_{\rm loc}(\mathbb{R}^d)$ and the Dunkl gradient operator ∇_κ; see also [2] for more details on the Dunkl–Schrödinger operator. To mention in passing, an interesting question for future research is how to extend our approach used below to study the weak $(1,1)$ boundedness of LPS square functions associated with Dunkl–Schrödinger operators.

The present article is organized as follows. In Section 2, we recall necessary known facts and establish several lemmata that are important to prove our main results. In Section 3, we present the proofs of our main results.

We should point out that the constants $c, C, C', C'', \cdots$, used in what follows, may vary from one location to another.

2 Preparations

In this section, we recall necessary known facts, present some preliminary results and establish the crucial integral bounds of derivatives of the Dunkl heat kernel, which will be used to prove the main results. Let $B(x,r)$ denote the open ball in $\mathbb{R}^d$ with center $x \in \mathbb{R}^d$ and radius $r > 0$ w.r.t. the Euclidean distance $|\cdot - \cdot|$, and for every $g \in G$ and every $A \subset \mathbb{R}^d$, let $gA = \{gx \in \mathbb{R}^d : \ x \in A\}$.

Let $d_\kappa = d + 2\chi$. It is known that μ_κ is G-invariant, i.e., for every $g \in G$ and every ball $B \subset \mathbb{R}^d$, $\mu_\kappa(gB) = \mu_\kappa(B)$, and the volume comparison property (see e.g. [5, (3.2)]) holds: there is a constant $\theta \ge 1$ such that, for every $x \in \mathbb{R}^d$ and every $0 < r \le R < \infty$,

$$\frac{1}{\theta}\Big(\frac{R}{r}\Big)^d \le \frac{\mu\big(B(x,R)\big)}{\mu\big(B(x,r)\big)} \le \theta\Big(\frac{R}{r}\Big)^{d_\kappa}. \tag{2.1}$$

In particular, μ_κ satisfies the volume doubling property. However, generally speaking, μ_κ is not Ahlfors regular. We remark that we do not use the left inequality of (2.1) in the proof below.

Let $\xi \in \mathbb{R}^d$ and let ${\rm Lip}_{\rm loc}(\mathbb{R}^d)$ be the space of locally Lipschitz continuous function on $\mathbb{R}^d$ w.r.t. the Euclidean distance. With respect to μ_κ, the following integration-by-parts formula holds: for every $u \in C^1(\mathbb{R}^d)$ and every $v \in C^1_c(\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d} v \mathrm{D}_\xi u \, \mathrm{d}\mu_\kappa = -\int_{\mathbb{R}^d} u \mathrm{D}_\xi v \, \mathrm{d}\mu_\kappa. \tag{2.2}$$

See [17, Lemma 2.9] and [33, Proposition 2.1]. It is easy to see that (2.2) holds true when $C^1(\mathbb{R}^d)$ is replaced by ${\rm Lip}_{\rm loc}(\mathbb{R}^d)$. Although we may not expect that the Dunkl operator satisfies the Leibniz rule in general, the following particular case is useful (see e.g. [33, (2.1)] and see [18, Proposition 6.4.12] for the general situation): for every $u, v \in C^1(\mathbb{R}^d)$ with at lest one of them being G-invariant,

$$\mathrm{D}_\xi(uv) = v\mathrm{D}_\xi u + u \mathrm{D}_\xi v. \tag{2.3}$$

For every $x \in \mathbb{R}^d$, let $G(x) = \{gx : \ g \in G\}$, which denotes the G-orbit of x. Let

$$\rho(x,y) = \min_{g\in G} |x - gy|, \qquad x, y \in \mathbb{R}^d,$$

which is the distance between G-orbits $G(x)$ and $G(y)$. Note that ρ is also G-invariant in each variable by definition. However, ρ may fail to be a pseudo-distance on $\mathbb{R}^d$ in the sense of [8, page 66] or a quasi-distance in the sense of [39, Section 2.4]. Consequently, the triple $(\mathbb{R}^d, \rho, \mu_\kappa)$

generally does not constitute a space of homogeneous type in the classical sense studied in harmonic analysis. Nevertheless, the following simple observation will prove useful. Given any point $x_0 \in \mathbb{R}^d$, we let $\rho_{x_0}(\cdot) = \rho(x_0, \cdot)$.

Lemma 2.1. *For an arbitrarily fixed point $x_0 \in \mathbb{R}^d$,*

$$|\nabla \rho_{x_0}(x)| \le 1, \qquad \mu_\kappa\text{-a.e. } x \in \mathbb{R}^d.$$

Proof. By the definition of ρ, we have

$$|\rho_{x_0}(y) - \rho_{x_0}(z)| \le |y - z|, \qquad y, z \in \mathbb{R}^d,$$

which implies that $\rho_{x_0}(\cdot)$ is Lipschitz continuous with respect to $|\cdot - \cdot|$ with Lipschitz constant 1. Then, by the well known Rademacher theorem, $\rho_{x_0}(\cdot)$ is differentiable almost everywhere with respect to $\mathscr{L}_d$; furthermore,

$$|\nabla \rho_{x_0}(x)| \le 1, \qquad \mathscr{L}_d\text{-a.e.} \quad x \in \mathbb{R}^d.$$

Since μ_κ is clearly absolutely continuous with respect to $\mathscr{L}_d$, we complete the proof. □

Let $h_t(x, y)$ be the Dunkl heat kernel of $H_\kappa(t)$, which is a C^∞ function of all variables $x, y \in \mathbb{R}^d$ and $t > 0$, and satisfies that

$$\partial_t h_t(x, y) = \Delta_\kappa h_t(\cdot, y)(x), \qquad x, y \in \mathbb{R}^d, \quad t > 0,$$
$$h_t(x, y) = h_t(y, x) > 0, \qquad x, y \in \mathbb{R}^d,\ t > 0,$$

and, moreover, there exist positive constants c, C such that

$$h_t(x, y) \le \frac{C}{V(x, y, t)} \exp\Big(- c\frac{\rho(x, y)^2}{t}\Big), \qquad x, y \in \mathbb{R}^d, \quad t > 0. \tag{2.4}$$

Here and in what follows, we use the notation

$$V(x, y, r) = \max\big\{\mu_\kappa\big(B(x, r)\big), \mu_\kappa\big(B(y, r)\big)\big\}.$$

Recently, the estimate on time derivative of the Dunkl heat kernel is established, i.e., for every nonnegative integer m, there exist positive constants c, C such that

$$|\partial_t^m h_t(x, y)| \le \frac{c}{t^m V(x, y, t)} \exp\Big(- C\frac{\rho(x, y)^2}{t}\Big), \qquad x, y \in \mathbb{R}^d, \quad t > 0. \tag{2.5}$$

See e.g. [5, 33] for the above estimates and more details on the Dunkl heat kernel. The proof of (2.5) employs the integral representation of the Dunkl translation operator first obtained in the paper[32] (see also [13, Lemma 3.4]). However, we shall give a remark here.

Remark 2.2. Although ρ may not be a true metric,the analyticity of the mapping $t \mapsto h_t(x, y)$, combined with the kernel estimate (2.4) and the right-hand inequality of (2.1), allows us to derive (2.5) via an alternative approach. Specifically, we may apply the general result from [11, Theorem 4] alongside (2.4) to establish the desired bound.

Let $|G|$ denote the order of the reflection group G. For $x \in \mathbb{R}^d$ and $r \ge 0$, define

$$B^\rho(x, r) = \{y \in \mathbb{R}^d : \ \rho(x, y) < r\},$$

where $B^\rho(x, 0) := \{y \in \mathbb{R}^d : \ \rho(x, y) = 0\}$ and it is at most a finite subset of $\mathbb{R}^d$. From the volume comparison property (2.1) and the Dunkl heat kernel estimate (2.4), we can immediately obtain the following lemma. The proof is standard and short, and we present it here for the sake of completeness (see e.g. the proof of [9, Lemma 2.1]).

Lemma 2.3. *For every $\delta>0$, there exists a positive constant C, depending on $|G|$ and δ, such that*

$$\int_{\mathbb{R}^d\setminus B^\rho(y,r)} \exp\Big(-2\delta\frac{\rho(x,y)^2}{s}\Big)\,\mathrm{d}\mu_\kappa(x)\le C\mu_\kappa\big(B(y,\sqrt{s})\big)e^{-\delta r^2/s},$$

for every $s>0$, $r\ge 0$ and $y\in\mathbb{R}^d$.

Proof. Let

$$\mathrm{I}=\int_{\mathbb{R}^d} e^{-\delta\rho(x,y)^2/s}\,\mathrm{d}\mu_\kappa(x),\qquad s>0,\quad y\in\mathbb{R}^d.$$

Then

$$\begin{aligned}\mathrm{I}=&\sum_{n=0}^{\infty}\int_{B^\rho(y,(n+1)\sqrt{s})\setminus B^\rho(y,n\sqrt{s})} e^{-\delta\rho(x,y)^2/s}\,\mathrm{d}\mu_\kappa(x)\\ \le&\sum_{n=0}^{\infty}e^{-\delta n^2}\mu_\kappa\big(B^\rho(y,(n+1)\sqrt{s})\big).\end{aligned}$$

Since for any $x\in\mathbb{R}^d$ and any $r>0$,

$$B^\rho(x,r)=\bigcup_{g\in G}\{y\in\mathbb{R}^d:\ |x-gy|<r\}=\bigcup_{g\in G}gB(x,r),$$

we have, by the G-invariance of μ_κ and the right inequality of (2.1),

$$\begin{aligned}\mathrm{I}\le&\sum_{n=0}^{\infty}e^{-\delta n^2}\mu_\kappa\big(\cup_{g\in G}\,gB(y,(n+1)\sqrt{s})\big)\\ \le&\sum_{n=0}^{\infty}e^{-\delta n^2}|G|\mu_\kappa\big(B(y,(n+1)\sqrt{s})\big)\\ \le&|G|\sum_{n=0}^{\infty}e^{-\delta n^2}(n+1)^{d_\kappa}\mu_\kappa\big(B(y,\sqrt{s})\big)\\ \le&C\mu_\kappa\big(B(y,\sqrt{s})\big),\end{aligned}$$

for some constant $C>0$. Thus

$$\int_{\mathbb{R}^d\setminus B^\rho(y,r)} \exp\Big(-2\delta\frac{\rho(x,y)^2}{s}\Big)\,\mathrm{d}\mu_\kappa(x)\le e^{-\delta r^2/s}\,\mathrm{I}\le C\mu_\kappa\big(B(y,\sqrt{s})\big)e^{-\delta r^2/s},$$

which completes the proof of Lemma 2.3. □

The next result is on integral estimates of the gradient of the Dunkl heat kernel, which is motivated by [10, Lemma 3.3]. However, here the gradient is induced by the carré-du-champ operator Γ. Due to the lack of the Leibniz rule and the chain rule for the general Dunkl Laplacian, the method used in the aforementioned reference is no longer directly applicable.

Lemma 2.4. *For every nonnegative integer m and every small enough $\epsilon>0$, there exists a positive constant $c_{\epsilon,m}$ such that*

$$\int_{\mathbb{R}^d}\Gamma\big(\Delta_\kappa^m h_s(\cdot,y)\big)(x)\exp\Big(2\epsilon\frac{\rho(x,y)^2}{s}\Big)\,\mathrm{d}\mu_\kappa(x)\le\frac{c_{\epsilon,m}}{s^{2m+1}\mu_\kappa\big(B(y,\sqrt{s})\big)},\tag{2.6}$$

and

$$\int_{\mathbb{R}^d\setminus B^\rho(y,\sqrt{t})} \Gamma\big(\Delta_\kappa^m h_s(\cdot,y)\big)(x)\exp\Big(\epsilon\frac{\rho(x,y)^2}{s}\Big)\,\mathrm{d}\mu_\kappa(x)\le \frac{c_{\epsilon,m}\,e^{-\epsilon t/s}}{s^{2m+1}\mu_\kappa\big(B(y,\sqrt{s})\big)}, \quad (2.7)$$

for all $y\in\mathbb{R}^d,\ s>0,\ t\ge 0.$

Proof. Let $x,y\in\mathbb{R}^d$, $\epsilon,s,R>0$ and let m be a nonnegative integer. For every $\alpha\in\mathfrak{R}$, we denote $\alpha=(\alpha_1,\cdots,\alpha_d)$. For convenience, we let

$$f(x)=\partial_s^m h_s(x,y),\qquad \eta(x)=e^{2\epsilon\rho(x,y)^2/s}.$$

Then, it is clear that $f\in C^\infty(\mathbb{R}^d)$ and $\eta\in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}^d)$. Take

$$\phi_R(x)=\min\Big\{1,\Big(3-\frac{|x|}{R}\Big)^+\Big\},\qquad x\in\mathbb{R}^d,$$

where for any $a\in\mathbb{R}$, $a^+:=\max\{a,0\}$. Then, $0\le\phi_R\le 1$ on $\mathbb{R}^d$, $\phi_R=1$ on $B(0,2R)$, $\phi_R=0$ outside $B(0,3R)$; moreover, ϕ_R is Lipschitz continuous with respect to $|\cdot-\cdot|$, G-invariant, increasing as R grows up and $|\nabla\phi_R|\le 1/R$. Note that η is G-invariant. Hence, $\eta\phi_R$ is G-invariant and it is clear that $\eta\phi_R\in\mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}^d)$. Set

$$J=\int_{\mathbb{R}^d}\Gamma(f)\eta\,\mathrm{d}\mu_\kappa,\qquad J_R=\int_{\mathbb{R}^d}\Gamma(f)\eta\phi_R^2\,\mathrm{d}\mu_\kappa,$$

and

$$J_{R,1}=\frac12\int_{\mathbb{R}^d}\Delta_\kappa(f^2)\eta\phi_R^2\,\mathrm{d}\mu_\kappa,\qquad J_{R,2}=-\int_{\mathbb{R}^d} f\langle\nabla_\kappa f,\nabla_\kappa(\eta\phi_R^2)\rangle\,\mathrm{d}\mu_\kappa.$$

By (2.2) and (2.3), we have

$$\begin{aligned}
J_{R,1}=&-\frac12\sum_{j=1}^d\int_{\mathbb{R}^d}\mathrm{D}_j(f^2)\partial_j(\eta\phi_R^2)\,\mathrm{d}\mu_\kappa\\
=&-\frac12\sum_{j=1}^d\int_{\mathbb{R}^d}\Big[2f(x)\partial_j f(x)+\sum_{\alpha\in\mathfrak{R}_+}\kappa_\alpha\alpha_j\frac{f^2(x)-f^2(r_\alpha x)}{\langle\alpha,x\rangle}\Big]\\
&\times\big[\phi_R^2(x)\partial_j\eta(x)+\eta(x)\partial_j\phi_R^2(x)\big]\,\mathrm{d}\mu_\kappa(x)\\
=&-\int_{\mathbb{R}^d}\Big[f(x)\langle\nabla f(x),\nabla\eta(x)\rangle+\frac12\sum_{\alpha\in\mathfrak{R}_+}\kappa_\alpha\langle\alpha,\nabla\eta(x)\rangle\frac{f^2(x)-f^2(r_\alpha x)}{\langle\alpha,x\rangle}\Big]\phi_R^2(x)\,\mathrm{d}\mu_\kappa(x)\\
&-\int_{\mathbb{R}^d}\Big[f(x)\langle\nabla f(x),\nabla\phi_R^2(x)\rangle+\frac12\sum_{\alpha\in\mathfrak{R}_+}\kappa_\alpha\langle\alpha,\nabla\phi_R^2(x)\rangle\frac{f^2(x)-f^2(r_\alpha x)}{\langle\alpha,x\rangle}\Big]\eta(x)\,\mathrm{d}\mu_\kappa(x),
\end{aligned}$$

and

$$\begin{aligned}
J_{R,2}=&-\int_{\mathbb{R}^d}\Big(f(x)\langle\nabla f(x),\nabla\eta(x)\rangle+\sum_{\alpha\in\mathfrak{R}_+}\kappa_\alpha\langle\alpha,\nabla\eta(x)\rangle\frac{f(x)[f(x)-f(r_\alpha x)]}{\langle\alpha,x\rangle}\Big)\phi_R^2(x)\,\mathrm{d}\mu_\kappa(x)\\
&-\int_{\mathbb{R}^d}\Big(f(x)\langle\nabla f(x),\nabla\phi_R^2(x)\rangle+\sum_{\alpha\in\mathfrak{R}_+}\kappa_\alpha\langle\alpha,\nabla\phi_R^2(x)\rangle\frac{f(x)[f(x)-f(r_\alpha x)]}{\langle\alpha,x\rangle}\Big)\eta(x)\,\mathrm{d}\mu_\kappa(x).
\end{aligned}$$

Then

$$\begin{aligned} J_{R,1}-J_{R,2}=&\frac{1}{2}\int_{\mathbb{R}^d}\sum_{\alpha\in\mathfrak{R}_+}\kappa_\alpha\langle\alpha,\nabla\eta(x)\rangle\frac{[f(x)-f(r_\alpha x)]^2}{\langle\alpha,x\rangle}\phi_R^2(x)\,\mathrm{d}\mu_\kappa(x)\\ &+\frac{1}{2}\int_{\mathbb{R}^d}\sum_{\alpha\in\mathfrak{R}_+}\kappa_\alpha\langle\alpha,\nabla\phi_R^2(x)\rangle\frac{[f(x)-f(r_\alpha x)]^2}{\langle\alpha,x\rangle}\eta(x)\,\mathrm{d}\mu_\kappa(x)\\ =:&\mathrm{A}_R+\mathrm{B}_R. \end{aligned} \tag{2.8}$$

Applying (2.5), by the same method used to prove [5, (4.12)], we obtain the following estimate, i.e., there exist constants $c, C>0$ such that

$$\frac{\big[f(x)-f(r_\alpha x)\big]^2}{|\langle\alpha,x\rangle|}\leq\frac{c}{s^{2m+1/2}\mu_\kappa\big(B(y,\sqrt{s})\big)^2}\exp\Big(-C\frac{\rho(x,y)^2}{s}\Big),\qquad x,y\in\mathbb{R}^d,\quad s>0.$$

Since $0\leq\phi_R\leq 1$ and $|\nabla\phi_R|\leq 1/R$, by Lemma 2.1 and Lemma 2.3, we derive that, for any small enough $\epsilon>0$,

$$\begin{aligned} |\mathrm{A}_R|\leq&\frac{c}{s^{2m+1/2}\mu_\kappa\big(B(y,\sqrt{s})\big)^2}\int_{\mathbb{R}^d}\frac{\rho(x,y)}{s}\exp\Big(2\epsilon\frac{\rho(x,y)^2}{s}\Big)\exp\Big(-C\frac{\rho(x,y)^2}{s}\Big)\,\mathrm{d}\mu_\kappa(x)\\ \leq&\frac{c}{s^{2m+1}\mu_\kappa\big(B(y,\sqrt{s})\big)^2}\int_{\mathbb{R}^d}\exp\Big(-(C'-2\epsilon)\frac{\rho(x,y)^2}{s}\Big)\,\mathrm{d}\mu_\kappa(x)\\ \leq&\frac{c}{s^{2m+1}\mu_\kappa\big(B(y,\sqrt{s})\big)}, \end{aligned} \tag{2.9}$$

and

$$\begin{aligned} |\mathrm{B}_R|\leq&\frac{c}{Rs^{2m+1/2}\mu_\kappa\big(B(y,\sqrt{s})\big)^2}\int_{\mathbb{R}^d}\exp\Big(-(C-2\epsilon)\frac{\rho(x,y)^2}{s}\Big)\,\mathrm{d}\mu_\kappa(x)\\ \leq&\frac{c}{Rs^{2m+1/2}\mu_\kappa\big(B(y,\sqrt{s})\big)}, \end{aligned} \tag{2.10}$$

which tends to 0 as $R\to\infty$. Thus, from (2.8) and (2.9), we have

$$|J_{R,1}|\leq|J_{R,2}|+|\mathrm{A}_R|+|\mathrm{B}_R|\leq|J_{R,2}|+\frac{c}{s^{2m+1}\mu_\kappa\big(B(y,\sqrt{s})\big)}+|\mathrm{B}_R|. \tag{2.11}$$

To estimate $J_{R,2}$, we deduce that

$$\begin{aligned} |J_{R,2}|=&\Big|\int_{\mathbb{R}^d}f\langle\nabla_\kappa f,\nabla\eta\rangle\phi_R^2\,\mathrm{d}\mu_\kappa+\int_{\mathbb{R}^d}f\langle\nabla_\kappa f,\nabla\phi_R^2\rangle\eta\,\mathrm{d}\mu_\kappa\Big|\\ \leq&\int_{\mathbb{R}^d}|f||\nabla_\kappa f||\nabla\eta|\phi_R^2\,\mathrm{d}\mu_\kappa+\frac{2}{R}\int_{\mathbb{R}^d}|f||\nabla_\kappa f|\eta\phi_R\,\mathrm{d}\mu_\kappa\\ =:&J_{R,2,1}+J_{R,2,2}. \end{aligned}$$

For the estimation of $J_{R,2,1}$, we have

$$\begin{aligned} J_{R,2,1}\leq&\int_{\mathbb{R}^d}|f(x)||\nabla_\kappa f(x)|\frac{4\epsilon\rho(x,y)}{s}\exp\Big(2\epsilon\frac{\rho(x,y)^2}{s}\Big)\phi_R^2(x)\,\mathrm{d}\mu_\kappa(x)\\ \leq&\frac{c}{\sqrt{s}}\int_{\mathbb{R}^d}|f(x)||\nabla_\kappa f(x)|\exp\Big(\epsilon'\frac{\rho(x,y)^2}{s}\Big)\phi_R(x)\,\mathrm{d}\mu_\kappa(x) \end{aligned}$$

$$
\begin{aligned}
\leq & \frac{c}{\sqrt{s}}\Big(\int_{\mathbb{R}^d}|f(x)|^2 e^{\epsilon''\rho(x,y)^2/s}\,\mathrm{d}\mu_\kappa(x)\Big)^{1/2}\\
&\times\Big(\int_{\mathbb{R}^d}|\nabla_\kappa f(x)|^2 e^{2\epsilon\rho(x,y)^2/s}\phi_R^2(x)\,\mathrm{d}\mu_\kappa(x)\Big)^{1/2},
\end{aligned}
$$

where we employed Lemma 2.1 and $0\leq\phi_R\leq 1$ again in the second inequality, and the Cauchy–Schwarz inequality in the last inequality. By Lemma 2.3 and (2.5), it is easy to see that, for any small enough $\epsilon''>0$,

$$
\int_{\mathbb{R}^d}|f(x)|^2 e^{\epsilon''\rho(x,y)^2/s}\,\mathrm{d}\mu_\kappa(x)\leq\frac{c}{s^{2m}\mu_\kappa\big(B(y,\sqrt{s})\big)}.
$$

By the pointwise inequality (1.2), we have

$$
\int_{\mathbb{R}^d}|\nabla_\kappa f(x)|^2 e^{2\epsilon\rho(x,y)^2/s}\phi_R^2(x)\,\mathrm{d}\mu_\kappa(x)\leq(1+2\chi)J_R.
$$

Hence

$$
J_{R,2,1}\leq\frac{c\sqrt{J_R}}{\sqrt{s^{2m+1}\mu_\kappa\big(B(y,\sqrt{s})\big)}}.
$$

For the estimation of $J_{R,2,2}$, we have

$$
\begin{aligned}
J_{R,2,2}\leq&\frac{2}{R}\Big(\int_{\mathbb{R}^d}|f|^2\eta\,\mathrm{d}\mu_\kappa\Big)^{1/2}\Big(\int_{\mathbb{R}^d}|\nabla_\kappa f|^2\eta\phi_R^2\,\mathrm{d}\mu_\kappa\Big)^{1/2}\\
\leq&\frac{2}{R}\Big(\int_{\mathbb{R}^d}|f|^2\eta\,\mathrm{d}\mu_\kappa\Big)^{1/2}\Big((1+2\chi)\int_{\mathbb{R}^d}\Gamma(f)\eta\phi_R^2\,\mathrm{d}\mu_\kappa\Big)^{1/2}\\
\leq&\frac{2(1+2\chi)}{R^2}\int_{\mathbb{R}^d}|f|^2\eta\,\mathrm{d}\mu_\kappa+\frac{1}{2}J_R\leq\frac{c}{R^2 s^{2m}\mu_\kappa\big(B(y,\sqrt{s})\big)}+\frac{1}{2}J_R,
\end{aligned}
$$

where we used (1.2), Lemma 2.3, (2.5) and Young's inequality. Combing the estimates of $J_{R,2,1}$ and $J_{R,2,2}$, we obtain

$$
|J_{R,2}|\leq\frac{c\sqrt{J_R}}{\sqrt{s^{2m+1}\mu_\kappa\big(B(y,\sqrt{s})\big)}}+\frac{c}{R^2 s^{2m}\mu_\kappa\big(B(y,\sqrt{s})\big)}+\frac{1}{2}J_R. \tag{2.12}
$$

By applying (2.5) and Lemma 2.3 again, we get that, for small enough ϵ,

$$
\begin{aligned}
\Big|\int_{\mathbb{R}^d}(f\Delta_\kappa f)\eta\phi_R^2\,\mathrm{d}\mu_\kappa\Big|\leq&\int_{\mathbb{R}^d}|\partial_s^m h_s(x,y)||\partial_s^{m+1}h_s(x,y)|e^{2\epsilon\rho(x,y)^2/s}\,\mathrm{d}\mu_\kappa(x)\\
\leq&\frac{c}{s^{2m+1}\mu_\kappa\big(B(y,\sqrt{s})\big)}.
\end{aligned}\tag{2.13}
$$

Thus, combing (2.11), (2.12) and (2.13), we have

$$
\begin{aligned}
J_R=&\frac{1}{2}\int_{\mathbb{R}^d}\Delta_\kappa(f^2)\eta\phi_R^2\,\mathrm{d}\mu_\kappa-\int_{\mathbb{R}^d}f(\Delta_\kappa f)\eta\phi_R^2\,\mathrm{d}\mu_\kappa\\
\leq&|J_{R,1}|+\frac{c}{s^{2m+1}\mu_\kappa\big(B(y,\sqrt{s})\big)}\\
\leq&\frac{C}{s^{2m+1}\mu_\kappa\big(B(y,\sqrt{s})\big)}+\frac{1}{2}J_R+\frac{c\sqrt{J_R}}{\sqrt{s^{2m+1}\mu_\kappa\big(B(y,\sqrt{s})\big)}}
\end{aligned}
$$

$$+\frac{c}{R^2 s^{2m}\mu_\kappa\big(B(y,\sqrt{s})\big)}+|\mathrm{B}_R|.$$

By (2.10) and the monotone convergence theorem, letting $R\to\infty$, we obtain

$$J\le \frac{C}{s^{2m+1}\mu_\kappa\big(B(y,\sqrt{s})\big)}+\frac{c\sqrt{J}}{\sqrt{s^{2m+1}\mu_\kappa\big(B(y,\sqrt{s})\big)}},$$

which immediately implies that

$$J\le \frac{C}{s^{2m+1}\mu_\kappa\big(B(y,\sqrt{s})\big)}.$$

We complete the proof of (2.6).

Finally, for every $t\ge 0$,

$$\begin{aligned}
&\int_{\mathbb{R}^d\setminus B^\rho(y,\sqrt{t})}\Gamma\big(\Delta_\kappa^m h_s(\cdot,y)\big)(x)\exp\Big(\epsilon\frac{\rho(x,y)^2}{s}\Big)\,\mathrm{d}\mu_\kappa(x)\\
=&\int_{\mathbb{R}^d\setminus B^\rho(y,\sqrt{t})}\Gamma\big(\Delta_\kappa^m h_s(\cdot,y)\big)(x)\exp\Big(2\epsilon\frac{\rho(x,y)^2}{s}\Big)\exp\Big(-\epsilon\frac{\rho(x,y)^2}{s}\Big)\,\mathrm{d}\mu_\kappa(x)\\
\le& e^{-\epsilon t/s}J\le \frac{Ce^{-\epsilon t/s}}{s^{2m+1}\mu_\kappa\big(B(y,\sqrt{s})\big)},
\end{aligned}$$

which completes the proof of (2.7). □

Now we should give a remark on the proof of Lemma 2.4.

Remark 2.5. Recently, the following pointwise estimate on space-time derivative of the Dunkl heat kernel is established in [5, Theorem 4.1(c)]: for every $j=1,\cdots,d$ and every nonnegative integer m, there exist positive constants c,C such that

$$|\mathrm{D}_j\partial_t^m h_t(\cdot,y)|(x)\le \frac{c}{t^{m+1/2}V(x,y,\sqrt{t})}\exp\Big(-C\frac{\rho(x,y)^2}{t}\Big),\qquad x,y\in\mathbb{R}^d,\quad t>0. \tag{2.14}$$

Applying the same method used to obtain (2.14) (see the proof of [5, Theorem 4.1(c)]), we can obtain the following pointwise derivative bound on the Dunkl heat kernel, i.e., for every nonnegative integer m, there exist positive constants c_1,c_2 such that

$$\sqrt{\Gamma\big(\Delta_\kappa^m h_t(\cdot,y)\big)}(x)\le \frac{c_1}{t^{m+\frac{1}{2}}V(x,y,\sqrt{t})}\exp\Big(-c_2\frac{\rho(x,y)^2}{t}\Big),\qquad x,y\in\mathbb{R}^d,\quad t>0. \tag{2.15}$$

By the pointwise bound (1.2), we note that the converse inequality of (1.2) generally fails, which highlights that (2.15) is indeed stronger than (2.14). Applying Lemma 2.3, we then deduce Lemma 2.4. This approach proves particularly straightforward in our setting. However, in other settings of curved spaces, pointwise gradient kernel bounds seem not easy to get, which demand geometric conditions usually, for instance, Ricci curvature lower bounds on Riemannian manifolds; see e.g. [12, 21, 28, 40] and see also [22, 23] for the more general setting of metric measure spaces with Riemannian curvature-dimension conditions, namely RCD spaces. Our approach to prove Lemma 2.4 has the advantage that we may establish the integral bound on the gradient of the kernel, say (2.6) and (2.7), even without the pointwise gradient kernel bound.

In order to obtain the weak $(1,1)$ boundedness of the horizontal square function $\mathcal{H}$, we need the following lemma, which can be easily verified by applying Lemma 2.3 with the estimate (2.5) in hand. So details of the proof are left to interested readers..

Lemma 2.6. *For every nonnegative integer m and every small enough $\epsilon > 0$, there exists a positive constant $C_{\epsilon,m}$ such that*

$$\int_{\mathbb{R}^d\setminus B^\rho(y,\sqrt{t})} |\Delta_\kappa^m h_s(\cdot,y)(x)|^2 \exp\left(\epsilon\frac{\rho(x,y)^2}{s}\right) \mathrm{d}\mu_\kappa(x) \le \frac{C_{\epsilon,m}\, e^{-\epsilon t/s}}{s^{2m}\mu_\kappa\big(B(y,\sqrt{s})\big)},$$

for all $y \in \mathbb{R}^d$, $s > 0$, $t \ge 0$.

We also need the next "maximum principle" between the Dunkl heat flow and the Hardy–Littlewood maximum operator, presented in the following lemma. Let $L^1_{\text{loc}}(\mu_\kappa)$ denote the class of locally integrable functions on $\mathbb{R}^d$ w.r.t. μ_κ. For every $f \in L^1_{\text{loc}}(\mu_\kappa)$, recall that the Hardy–Littlewood maximum operator associated with μ_κ is defined by

$$\mathcal{M}(f)(x) = \sup_{r>0} \frac{1}{\mu_\kappa\big(B(x,r)\big)} \int_{B(x,r)} |f(z)|\, \mathrm{d}\mu_\kappa(z), \quad x \in \mathbb{R}^d.$$

Lemma 2.7. *For every $t > 0$, $x \in \mathbb{R}^d$ and every nonnegative function v from $L^1_{\text{loc}}(\mu_\kappa)$,*

$$\sup_{y\in B(x,\sqrt{t})} \big(H_\kappa(t)v\big)(y) \le C \sum_{g\in G} \inf_{y\in B(x,\sqrt{t})} \mathcal{M}(v)(gy),$$

for some constant $C > 0$.

Proof. Let $y \in B(x,\sqrt{t})$. By (2.4), we have

$$\big(H_\kappa(t)v\big)(y) \le C \int_{\mathbb{R}^d} \frac{e^{-c\rho(y,z)^2/t}}{\mu_\kappa\big(B(y,\sqrt{t})\big)} v(z)\, \mathrm{d}\mu_\kappa(z) \le C \sum_{g\in G} \int_{\mathbb{R}^d} \frac{e^{-c|gy-z|^2/t}}{\mu_\kappa\big(B(gy,\sqrt{t})\big)} v(z)\, \mathrm{d}\mu_\kappa(z).$$

For any fixed $g \in G$, let $E_1 = B(gx, 4\sqrt{t})$ and $E_j = B(gx, 2^{j+1}\sqrt{t})\setminus B(gx, 2^j\sqrt{t})$, for $j = 2, 3, \cdots$. Since $y \in B(x,\sqrt{t})$, we see that $|y-x| < \sqrt{t}$, and $2^j\sqrt{t} \le |gx - z| < 2^{j+1}\sqrt{t}$ for any $z \in E_j$ and any $j = 1, 2, \cdots$. Then the triangle inequality implies that

$$|gy - z| \ge |z - gx| - |g(y-x)| = |z - gx| - |y - x| \ge 2^{j-1}\sqrt{t}, \qquad j = 1, 2, \cdots.$$

Thus, for every $y \in B(x,\sqrt{t})$, since

$$B(gx, 2^{j+1}\sqrt{t}) \subset B(gy, 2^{j+1}\sqrt{t} + |g(x-y)|) \subset B(gy, 2^{j+2}\sqrt{t}),$$

we have

$$\begin{aligned}
\big(H_\kappa(t)v\big)(y) \le& C \sum_{g\in G}\sum_{j=1}^\infty \int_{E_j} \frac{e^{-c4^{j-1}}}{\mu_\kappa\big(B(gy,\sqrt{t})\big)} v(z)\, \mathrm{d}\mu_\kappa(z) \\
\le& C \sum_{g\in G}\sum_{j=1}^\infty e^{-c4^{j-1}} \frac{\mu_\kappa\big(B(gy, 2^{j+2}\sqrt{t})\big)}{\mu_\kappa\big(B(gy,\sqrt{t})\big)} \\
&\times \frac{1}{\mu_\kappa\big(B(gx, 2^{j+1}\sqrt{t})\big)} \int_{B(gx, 2^{j+1}\sqrt{t})} v(z)\, \mathrm{d}\mu_\kappa(z) \\
\le& C \sum_{g\in G}\sum_{j=1}^\infty e^{-c4^{j-1}} 2^{(j+2)d_\kappa} \inf_{z\in B(x,\sqrt{t})} \mathcal{M}(v)(gz) \\
\le& C \sum_{g\in G} \inf_{z\in B(x,\sqrt{t})} \mathcal{M}(v)(gz),
\end{aligned}$$

where the right inequality of (2.1) is applied. The proof is completed. □

3 Proofs of the Main Results

In this section, we devote to prove our main theorems. Theorem 1.2 can be proved by applying the same method employed in the proof of Theorem 1.1. The main difference lies in **Part III** in the *Proof of Theorem 1.1* below, where Lemma 2.6 shall be employed instead of Lemma 2.4. Due to this, we decide to omit the details of the proof for Theorem 1.2 to save some space.

Now we are in a position to prove Theorem 1.1.

Proof of Theorem 1.1. Let $f \in L^1(\mu_\kappa)$ and $\lambda > 0$. By the Caderón–Zygmund decomposition, we have

$$f = g + \sum_i b_i =: g + b,$$

and the following assertions hold: there exists a positive constant c such that

(a) $|g(x)| \le c\lambda$ for μ_κ-a.e. $x \in \mathbb{R}^d$,

(b) there exists a sequence of balls $\{B_i\}_i$ in $\mathbb{R}^d$ with $B_i = B(x_i, r_i)$ such that $r_i \in (0,1]$, $x_i \in \mathbb{R}^d$, and b_i is supported in B_i with $\|b_i\|_{L^1(\mu_\kappa)} \le c\lambda\mu_\kappa(B_i)$ for each i,

(c) $\sum_i \mu_\kappa(B_i) \le c\lambda^{-1}\|f\|_{L^1(\mu_\kappa)}$,

(d) every point of $\mathbb{R}^d$ is contained in at most finitely many balls B_i.

We shall prove that

$$\mu_\kappa\big(\{x \in \mathbb{R}^d :\ \mathcal{V}_\Gamma(f)(x) \ge \lambda\}\big) \le \frac{c}{\lambda}\|f\|_{L^1(\mu_\kappa)}. \tag{3.1}$$

By (b) and (c), we immediately get $\|b\|_{L^1(\mu_\kappa)} \le \sum_i \|b_i\|_{L^1(\mu_\kappa)} \le c\|f\|_{L^1(\mu_\kappa)}$, and hence, $\|g\|_{L^1(\mu_\kappa)} \le c\|f\|_{L^1(\mu_\kappa)}$.

We divide the proof into four parts, i.e., **Part I–VI** below.

Part I. By the sublinearity of $f \mapsto \mathcal{V}_\Gamma(f)$ and the decomposition of f, we have

$$\begin{aligned}&\mu_\kappa\big(\{x \in \mathbb{R}^d :\ \mathcal{V}_\Gamma(f)(x) \ge \lambda\}\big)\\ \le&\mu_\kappa\big(\{x \in \mathbb{R}^d :\ \mathcal{V}_\Gamma(g)(x) \ge \lambda/2\}\big) + \mu_\kappa\big(\{x \in \mathbb{R}^d :\ \mathcal{V}_\Gamma(b)(x) \ge \lambda/2\}\big).\end{aligned} \tag{3.2}$$

Since $\mathcal{V}_\Gamma$ is bounded in $L^2(\mu_\kappa)$ (see [27, Theorem 2.4]), by (a) and Chebyshev's inequality, we get

$$\mu_\kappa\big(\{x \in \mathbb{R}^d :\ \mathcal{V}_\Gamma(g)(x) \ge \lambda/2\}\big) \le \frac{c}{\lambda^2}\|\mathcal{V}_\Gamma(g)\|^2_{L^2(\mu_\kappa)} \le \frac{c}{\lambda^2}\|g\|^2_{L^2(\mu_\kappa)} \le \frac{c}{\lambda}\|f\|_{L^1(\mu_\kappa)}. \tag{3.3}$$

Part II. Let $t_i = r_i^2$ for each i and let I be the identity map. Since

$$\mathcal{V}_\Gamma(b_i) = \mathcal{V}_\Gamma\big(H_\kappa(t_i)b_i + [I - H_\kappa(t_i)]b_i\big) \le \mathcal{V}_\Gamma\big(H_\kappa(t_i)b_i\big) + \mathcal{V}_\Gamma\big([I - H_\kappa(t_i)]b_i\big),$$

we have

$$\mathcal{V}_\Gamma(b) = \mathcal{V}_\Gamma\Big(\sum_i b_i\Big) \le \mathcal{V}_\Gamma\Big(\sum_i H_\kappa(t_i)b_i\Big) + \sum_i \mathcal{V}_\Gamma\big([I - H_\kappa(t_i)]b_i\big).$$

Then

$$\mu_\kappa\big(\{x \in \mathbb{R}^d :\ \mathcal{V}_\Gamma(b)(x) \ge \lambda/2\}\big)$$

$$\leq \mu_\kappa\Big(\Big\{x\in\mathbb{R}^d:\ \mathcal{V}_\Gamma\Big(\sum_i H_\kappa(t_i)b_i\Big)(x)\geq \lambda/4\Big\}\Big)$$
$$+\,\mu_\kappa\Big(\Big\{x\in\mathbb{R}^d:\ \sum_i \mathcal{V}_\Gamma\big([I-H_\kappa(t_i)]b_i\big)(x)\geq \lambda/4\Big\}\Big). \tag{3.4}$$

By the L^2 boundedness of $\mathcal{V}_\Gamma$ (see e.g. [27]) and Chebyshev's inequality again,

$$\mu_\kappa\Big(\Big\{x\in\mathbb{R}^d:\ \mathcal{V}_\Gamma\Big(\sum_i H_\kappa(t_i)b_i\Big)(x)\geq \lambda/4\Big\}\Big)$$
$$\leq \frac{c}{\lambda^2}\Big\|\mathcal{V}_\Gamma\Big(\sum_i H_\kappa(t_i)b_i\Big)\Big\|^2_{L^2(\mu_\kappa)}\leq \frac{c}{\lambda^2}\Big\|\sum_i H_\kappa(t_i)|b_i|\Big\|^2_{L^2(\mu_\kappa)},$$

where

$$\begin{aligned}\Big\|\sum_i H_\kappa(t_i)|b_i|\Big\|_{L^2(\mu_\kappa)} &= \sup_{\|u\|_{L^2(\mu_\kappa)}=1}\Big|\int_{\mathbb{R}^d} u\sum_i H_\kappa(t_i)|b_i|\,\mathrm{d}\mu_\kappa\Big|\\ &= \sup_{\|u\|_{L^2(\mu_\kappa)}=1}\Big|\sum_i\int_{\mathbb{R}^d}|b_i|H_\kappa(t_i)u\,\mathrm{d}\mu_\kappa\Big|\\ &\leq \sup_{\|u\|_{L^2(\mu_\kappa)}=1}\sum_i\|b_i\|_{L^1(\mu_\kappa)}\Big(\sup_{B_i}H_\kappa(t_i)|u|\Big).\end{aligned}$$

By (b), (c), Lemma 2.7 and the G-invariance of μ_κ, we have, for every $u\in L^2(\mu_\kappa)$ with $\|u\|_{L^2(\mu_\kappa)}=1$,

$$\begin{aligned}\sum_i\|b_i\|_{L^1(\mu_\kappa)}\Big(\sup_{B_i}H_\kappa(t_i)|u|\Big) &\leq c\lambda\sum_i\mu_\kappa(B_i)\sum_{g\in G}\inf_{x\in B_i}\mathcal{M}(u)(gx)\\ &\leq c\lambda\sum_i\sum_{g\in G}\int_{B_i}\mathcal{M}(u)(gx)\,\mathrm{d}\mu_\kappa(x)\\ &\leq c\lambda\sum_{g\in G}\sqrt{\mu_\kappa(\cup_iB_i)}\ \|\mathcal{M}(u)\|_{L^2(\mu_\kappa)}\\ &\leq c\sqrt{\lambda\|f\|_{L^1(\mu_\kappa)}},\end{aligned}$$

where we used the fact that $\mathcal{M}$ is bounded in $L^2(\mu_\kappa)$ (see e.g. Theorem 1(c) on page 13 of [39]) since $(\mathbb{R}^d,|\cdot-\cdot|,\mu_\kappa)$ is clearly a homogeneous space according to (2.1). Hence

$$\mu_\kappa\Big(\Big\{x\in\mathbb{R}^d:\ \mathcal{V}_\Gamma\Big(\sum_i H_\kappa(t_i)b_i\Big)(x)\geq \lambda/4\Big\}\Big)\leq \frac{c}{\lambda}\|f\|_{L^1(\mu_\kappa)}. \tag{3.5}$$

Part III. It remains to estimate the last term of (3.4). For notational simplicity, for each l, we let $2B_l^\rho=B^\rho(x_l,2\sqrt{t_l})$ and $(2B_l^\rho)^c=\mathbb{R}^d\setminus 2B_l^\rho$ in the following proof. Then

$$\begin{aligned}&\mu_\kappa\Big(\Big\{x\in\mathbb{R}^d:\ \sum_i\mathcal{V}_\Gamma\big([I-H_\kappa(t_i)]b_i\big)(x)\geq\lambda/4\Big\}\Big)\\ \leq&\sum_l\mu_\kappa(2B_l^\rho)+\mu_\kappa\Big(\Big\{x\in\cap_l(2B_l^\rho)^c:\ \sum_i\mathcal{V}_\Gamma\big([I-H_\kappa(t_i)]b_i\big)(x)\geq\lambda/4\Big\}\Big)\\ =:&\sum_l\mu_\kappa(2B_l^\rho)+J.\end{aligned}\tag{3.6}$$

Note that $2B_l^\rho = \bigcup_{g\in G} gB(x_l, 2\sqrt{t_l})$. Since μ_κ is G-invariant, by (c) and the right inequality in (2.1), we derive that

$$\begin{aligned}\sum_l \mu_\kappa(2B_l^\rho) &\le \sum_l |G|\mu_\kappa\big(B(x_l, 2\sqrt{t_l})\big)\\ &\le \sum_l |G|2^{d_\kappa}\mu\big(B(x_l, r_l)\big) \le \frac{c}{\lambda}\|f\|_{L^1(\mu_\kappa)}.\end{aligned} \tag{3.7}$$

Since b_i is supported in B_i for each i by (b), it is easy to see that

$$\begin{aligned}J \le& \frac{4}{\lambda}\sum_i \int_{\cap_l (2B_l^\rho)^c} \mathcal{V}_\Gamma\big([I - H_\kappa(t_i)]b_i\big)\,\mathrm{d}\mu_\kappa\\ =& \frac{4}{\lambda}\sum_i \int_{\cap_l (2B_l^\rho)^c} \Big(\int_0^\infty \Gamma\Big(\int_{B_i} [h_s(\cdot, y) - h_{s+t_i}(\cdot, y)]b_i(y)\,\mathrm{d}\mu_\kappa(y)\Big)(x)\ ds\Big)^{1/2}\mathrm{d}\mu_\kappa(x)\\ \le& \frac{4\sqrt{2}}{\lambda}\sum_i \int_{B_i}\int_{(2B_i^\rho)^c} \Big(\int_0^\infty \Gamma\big(h_s(\cdot, y) - h_{s+t_i}(\cdot, y)\big)(x)\ ds\Big)^{1/2}\mathrm{d}\mu_\kappa(x)|b_i(y)|\,\mathrm{d}\mu_\kappa(y),\end{aligned}$$

where the last inequality can be check directly by the explicit express of Γ (see (1.1)). For each i and every $y \in \mathbb{R}^d$, let

$$J_i(y) = \int_{(2B_i^\rho)^c} \Big(\int_0^\infty \Gamma\big(h_s(\cdot, y) - h_{s+t_i}(\cdot, y)\big)(x)\ ds\Big)^{1/2}\mathrm{d}\mu_\kappa(x). \tag{3.8}$$

Then

$$J \le \frac{c}{\lambda}\sum_i \int_{B_i} J_i(y)|b_i(y)|\,\mathrm{d}\mu_\kappa(y). \tag{3.9}$$

So, by (b) and (c), it suffices to prove that, there exists a positive constant c such that, for each i,

$$\sup_{y\in B_i} J_i(y) \le c.$$

For $m = 0, 1, 2, \cdots$ and $y \in \mathbb{R}^d$, let

$$J_i^m(y) = \int_{(2B_i^\rho)^c} \Big(\int_{mt_i}^{(m+1)t_i} \Gamma\big(h_s(\cdot, y) - h_{s+t_i}(\cdot, y)\big)(x)\ ds\Big)^{1/2}\mathrm{d}\mu_\kappa(x).$$

It suffices to estimate J_i^m for each $m = 0, 1, 2, \cdots$. The conclusions are stated in the following two claims, whose proofs are presented at the end of this section.

Claim (1). There exists a constant $c > 0$ such that

$$J_i^m(y) \le \frac{c}{m^{3/2}}, \qquad y \in B_i, \quad m = 1, 2, \cdots. \tag{3.10}$$

Claim (2). There exists a constant $C > 0$ such that

$$J_i^0(y) \le C2^{d_\kappa/2}, \qquad y \in B_i. \tag{3.11}$$

Thus, combining (3.10) and (3.11) together, we obtain that for each i,

$$\sup_{y\in B_i} J_i(y) \le \sum_{m=0}^\infty \sup_{y\in B_i} J_i^m(y) \le c\Big(1 + \sum_{m=1}^\infty \frac{1}{m^{3/2}}\Big) \le c',$$

which together with (3.6) and (3.7) implies that

$$\mu_\kappa\Big(\Big\{x \in \mathbb{R}^d : \sum_i \mathcal{V}_\Gamma\big([I - H_\kappa(t_i)]b_i)(x) \geq \lambda/4\Big\}\Big) \leq \frac{c}{\lambda}\|f\|_{L^1(\mu_\kappa)}. \tag{3.12}$$

Part IV. Therefore, putting (3.2), (3.3), (3.4), (3.5) and (3.12) together, we obtain (3.1). The proof of Theorem 1.1 is completed. □

Finally, we need to prove the two claims in **Part III** above.

Proof of Claim (1). By the Cauchy-Schwarz inequality, we get

$$\begin{aligned} J_i^m(y) =& \int_{(2B_i^\rho)^c}\Big(\int_{mt_i}^{(m+1)t_i} \Gamma\big(h_s(\cdot,y) - h_{s+t_i}(\cdot,y)\big)(x)\exp\Big(2\delta\frac{\rho(x,x_i)^2}{mt_i}\Big)\,ds\Big)^{1/2} \\ &\times \exp\Big(-\delta\frac{\rho(x,x_i)^2}{mt_i}\Big)\,\mathrm{d}\mu_\kappa(x) \\ \leq& \sqrt{\tilde{J}_i^m(y)}\Big(\int_{(2B_i^\rho)^c}\exp\Big(-2\delta\frac{\rho(x,x_i)^2}{mt_i}\Big)\,\mathrm{d}\mu_\kappa(x)\Big)^{1/2}, \end{aligned} \tag{3.13}$$

where we have set

$$\tilde{J}_i^m(y) = \int_{\mathbb{R}^d}\int_{mt_i}^{(m+1)t_i} \Gamma\big(h_s(\cdot,y) - h_{s+t_i}(\cdot,y)\big)(x)\exp\Big(2\delta\frac{\rho(x,x_i)^2}{mt_i}\Big)\,ds\,\mathrm{d}\mu_\kappa(x).$$

Now we estimate $\tilde{J}_i^m$. Lemma 2.3 implies that

$$\int_{(2B_i^\rho)^c}\exp\Big(-2\delta\frac{\rho(x,x_i)^2}{mt_i}\Big)\,\mathrm{d}\mu_\kappa(x) \leq c\mu\big(B(x_i,\sqrt{mt_i})\big)e^{-\delta/(m\sqrt{t_i})}. \tag{3.14}$$

Since

$$\partial_u h_u(x,y) = \Delta_\kappa h_u(\cdot,y)(x),$$

we have

$$\begin{aligned} \tilde{J}_i^m(y) =& \int_{\mathbb{R}^d}\int_{mt_i}^{(m+1)t_i}\Gamma\Big(\int_s^{s+t_i}\Delta_\kappa h_u(\cdot,y)\,du\Big)(x)\exp\Big(2\delta\frac{\rho(x,x_i)^2}{mt_i}\Big)\,ds\,\mathrm{d}\mu_\kappa(x) \\ \leq& \int_{\mathbb{R}^d}\int_{mt_i}^{(m+1)t_i}\Big(t_i\int_s^{s+t_i}\Gamma\big(\Delta_\kappa h_u(\cdot,y)\big)(x)\,du\Big)\exp\Big(2\delta\frac{\rho(x,x_i)^2}{mt_i}\Big)\,ds\,\mathrm{d}\mu_\kappa(x) \\ =& t_i\int_{mt_i}^{(m+1)t_i}\int_s^{s+t_i}\Big(\int_{\mathbb{R}^d}\Gamma\big(\Delta_\kappa h_u(\cdot,y)\big)(x)\exp\Big(2\delta\frac{\rho(x,x_i)^2}{mt_i}\Big)\,\mathrm{d}\mu_\kappa(x)\Big)\,du\,ds, \end{aligned}$$

where we applied the Cauchy–Schwarz inequality in the second inequality and Fubini's theorem in the last equality. Since $s \leq u \leq s + t_i$, $mt_i \leq s \leq (m+1)t_i$, we get $t_i^{-1} \leq (m+2)u^{-1}$. Since $y \in B_i$, and for every $g \in G$, $|gx - x_i| \leq |gx - y| + |y - x_i|$, we have

$$\rho(x,x_i) \leq \rho(x,y) + |y - x_i| < \rho(x,y) + \sqrt{t_i}.$$

Hence

$$\tilde{J}_i^m(y) \leq t_i\int_{mt_i}^{(m+1)t_i}\int_s^{s+t_i}\Big[\int_{\mathbb{R}^d}\Gamma\big(\Delta_\kappa h_u(\cdot,y)\big)(x)\exp\Big(2\delta\frac{2\rho(x,y)^2 + 2t_i}{mt_i}\Big)\,\mathrm{d}\mu_\kappa(x)\Big]\,du\,ds$$

$$\leq c t_i \int_{mt_i}^{(m+1)t_i} \int_s^{s+t_i} \Big[\int_{\mathbb{R}^d} \Gamma\big(\Delta_\kappa h_u(\cdot,y)\big)(x) \exp\Big(12\delta \frac{\rho(x,y)^2}{u}\Big) \,\mathrm{d}\mu_\kappa(x) \Big] \,du\,ds.$$

Applying (2.6) in Lemma 2.4, we deduce that, for small enough $\delta > 0$,

$$\begin{aligned}
\tilde{J}_i^m(y) \leq & c t_i \int_{mt_i}^{(m+1)t_i} \int_s^{s+t_i} \frac{1}{u^3 \mu\big(B(y,\sqrt{u})\big)} \,du\,ds \\
\leq & c t_i^2 \int_{mt_i}^{(m+1)t_i} \frac{1}{s^3 \mu\big(B(y,\sqrt{s})\big)} \,ds \\
\leq & \frac{c}{m^3 \mu\big(B(y,\sqrt{mt_i})\big)}. \qquad (3.15)
\end{aligned}$$

Thus, combining (3.13), (3.14) and (3.15) with the fact that

$$B(x_i,\sqrt{mt_i}) \subset B(y,\sqrt{mt_i} + |x_i - y|) \subset B(y,(\sqrt{m}+1)t_i), \qquad y \in B_i,$$

we have that, by (2.1), there exists a constant $c > 0$ such that

$$\begin{aligned}
J_i^m(y) \leq & c \Big(\frac{\mu\big(B(x_i,\sqrt{mt_i})\big) e^{-\delta/(m\sqrt{t_i})}}{m^3 \mu\big(B(y,\sqrt{mt_i})\big)} \Big)^{1/2} \\
\leq & \frac{c}{m^{3/2}} \Big(\frac{1+\sqrt{m}}{\sqrt{m}} \Big)^{d_\kappa/2} \leq \frac{c}{m^{3/2}},
\end{aligned}$$

for every $y \in B_i$ and $m = 1, 2, \cdots$. □

Proof of Claim (2). Similar as the argument carried out for $J_i^m(y)$ in the proof of Claim (1) above, we have

$$J_i^0(y) \leq \sqrt{\tilde{J}_i^0(y)} \Big(\int_{(2B_i^\rho)^c} \exp\Big(-2\delta \frac{\rho(x,x_i)^2}{t_i} \Big) \,\mathrm{d}\mu_\kappa(x) \Big)^{1/2},$$

where we have let

$$\tilde{J}_i^0(y) = \int_{(2B_i^\rho)^c} \int_0^{t_i} \Gamma\big(h_s(\cdot,y)(x) - h_{s+t_i}(\cdot,y)\big)(x) \exp\Big(2\delta \frac{\rho(x,x_i)^2}{t_i}\Big) \,ds\,\mathrm{d}\mu_\kappa(x).$$

Now we estimate $\tilde{J}_i^0$. Again, by Lemma 2.3, we have

$$\int_{(2B_i^\rho)^c} \exp\Big(-2\delta \frac{\rho(x,x_i)^2}{t_i} \Big) \,\mathrm{d}\mu_\kappa(x) \leq c\mu(B_i).$$

Note that $y \in B_i$. Since for every $g \in G$, $|gx - x_i| \leq |gx - y| + |y - x_i|$, we have

$$\rho(x,x_i) \leq \rho(x,y) + |y - x_i| < \rho(x,y) + \sqrt{t_i}.$$

It is clear that for every $x \in (2B_i^\rho)^c$, $2\sqrt{t_i} \leq \rho(x,x_i) \leq \rho(x,y) + \sqrt{t_i}$; hence, $\rho(x,y) \geq \sqrt{t_i}$, which clearly implies that $(2B_i^\rho)^c \subset \mathbb{R}^d \setminus B^\rho(y,\sqrt{t_i})$. Hence

$$\begin{aligned}
\tilde{J}_i^0(y) \leq & t_i \int_0^{t_i} \int_s^{s+t_i} \int_{(2B_i^\rho)^c} \Gamma\big(\Delta_\kappa h_u(\cdot,y)\big)(x) \exp\Big(2\delta \frac{\rho(x,x_i)^2}{t_i}\Big) \,\mathrm{d}\mu_\kappa(x)\,du\,ds \\
\leq & t_i \int_0^{t_i} \int_s^{s+t_i} \int_{(2B_i^\rho)^c} \Gamma\big(\Delta_\kappa h_u(\cdot,y)\big)(x) \exp\Big(2\delta \frac{2\rho(x,y)^2 + 2t_i}{t_i}\Big) \,\mathrm{d}\mu_\kappa(x)\,du\,ds
\end{aligned}$$

$$\leq ct_i \int_0^{t_i}\int_s^{s+t_i}\int_{\mathbb{R}^d\setminus B^\rho(y,\sqrt{t_i})} \Gamma\big(\Delta_\kappa h_u(\cdot,y)\big)(x)\exp\Big(8\delta\frac{\rho(x,y)^2}{u}\Big)\,\mathrm{d}\mu_\kappa(x)\ du\ ds.$$

By (2.7), for small enough $\delta > 0$, we have

$$\int_{\mathbb{R}^d\setminus B^\rho(y,\sqrt{t_i})} \Gamma\big(\Delta_\kappa h_u(\cdot,y)\big)(x)\exp\Big(8\delta\frac{\rho(x,y)^2}{u}\Big)\,\mathrm{d}\mu_\kappa(x) \leq \frac{ce^{-ct_i/u}}{u^3\mu_\kappa\big(B(y,\sqrt{u})\big)}.$$

Hence

$$\begin{aligned}
\tilde{J}_i^0(y) \leq& ct_i \int_0^{t_i}\int_s^{s+t_i} \frac{e^{-ct_i/u}}{u^3\mu_\kappa(B(y,\sqrt{u}))}\ du\ ds \\
=& \frac{c}{t_i^2\mu_\kappa\big(B(y,\sqrt{t_i})\big)}\int_0^{t_i}\int_s^{s+t_i}\Big(\frac{t_i}{u}\Big)^3\frac{\mu_\kappa\big(B(y,\sqrt{t_i})\big)}{\mu_\kappa\big(B(y,\sqrt{u})\big)}e^{-ct_i/u}\ du\ ds \\
\leq& \frac{c}{t_i^2\mu_\kappa\big(B(y,\sqrt{t_i})\big)}\int_0^{t_i}\int_s^{s+t_i}\Big(\frac{t_i}{u}\Big)^{3+d_\kappa/2}e^{-ct_i/u}\ du\ ds \\
\leq& \frac{c}{\mu_\kappa\big(B(y,\sqrt{t_i})\big)},
\end{aligned}$$

where the last inequality is due to the fact that $\mathbb{R}_+ \ni t \mapsto t^{3+d_\kappa/2}e^{-ct}$ is bounded.

Thus, by the right inequality in (2.1), since $B_i \subset B(y,\sqrt{t_i}+|y-x_i|) \subset B(y,2\sqrt{t_i})$ for every $y \in B_i$, we arrive at

$$J_i^0(y) \leq C\Big(\frac{\mu_\kappa(B_i)}{\mu_\kappa\big(B(y,\sqrt{t_i})\big)}\Big)^{1/2} \leq C2^{d_\kappa/2}, \qquad y \in B_i,$$

for some constant $C > 0$. □

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Conflict of Interest

The authors declare no conflict of interest.

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