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Semi-log-convexity of M/M/∞ queues on $\mathbb{Z}_+$ ☆

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ABSTRACT

We solve the problem left in the recent paper by N. Gozlan et al [Potential Analysis 58, 2023, 123–158], establishing the semi-log-convexity of semigroups associated with M/M/∞ queuing processes on the set of non-negative integers. Our approach is global in nature and yields the sharp constant.

1. Introduction and main results

Let $\mathbb{Z}_+$ denote the set of non-negative integers. In this note, we are concerned with M/M/∞ queuing processes taking values in $\mathbb{Z}_+$; for further background, see e.g. Gozlan et al. (2023), Chafaï (2006), Asmussen (2003).

Let $\mathbb{N} = \mathbb{Z}_+ \setminus \{0\}$ and let $\lambda, \mu > 0$. We consider the M/M/∞ queuing processes $(X_t)_{t\geq 0}$ on $\mathbb{Z}_+$ with input (or arrival) rate λ and service rate μ. In other words, $(X_t)_{t\geq 0}$ is a continuous-time Markov chain with the infinitesimal generator $\mathcal{M}$ given by

$$\mathcal{M}f(n) := \lambda[f(n+1) - f(n)] + n\mu[f(n-1) - f(n)], \quad n \in \mathbb{Z}_+,$$

for any function $f : \mathbb{Z}_+ \to \mathbb{R}$, where $f(-1)$ is identified as $f(0)$. Let $(A_t)_{t\geq 0}$ be the semigroup generated by $\mathcal{M}$. For convenience, we define the traffic intensity $\rho = \frac{\lambda}{\mu}$ and the function $p = p_t = e^{-\mu t}$ for every $t \geq 0$. It is well known that the stationary distribution of $(X_t)_{t\geq 0}$ is the Poisson law with parameter ρ, denoted by π_ρ, where

$$\pi_\rho(k) = \frac{\rho^k}{k!}e^{-\rho}, \quad k \in \mathbb{Z}_+.$$

Recall that the discrete Laplacian $\Delta_{\rm d}$, acting on function $f : \mathbb{Z}_+ \to \mathbb{R}$, is defined as

$$\Delta_{\rm d} f(n) = f(n+1) + f(n-1) - 2f(n), \quad n \in \mathbb{N}.$$

In a recent work (Gozlan et al., 2023, Proposition 3.1), N. Gozlan et al. proved a semi-log-convexity property for the semigroup A_t. Specifically, they proved that for every $t > 0$ and every non-zero function $f : \mathbb{Z}_+ \to [0, \infty)$, the following inequality holds:

$$\Delta_{\rm d}[\log A_t f](n) \geq \log\left[\frac{1}{12}\left(1 - \frac{p^2}{\left[p + \rho(1-p)^2\right]^2}\right)\right], \quad n \in \mathbb{N}. \tag{1.1}$$

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It is easy to see that (1.1) is not sharp. Indeed, as $t \to \infty$, the right-hand side of (1.1) tends to $-\log(12)$, while the left-hand side may vanish. This follows from the fact that for any $n \in \mathbb{Z}_+$ and any bounded $f : \mathbb{Z}_+ \to [0,\infty)$, $A_t f(n) \overset{t\to\infty}{\to} \int_{\mathbb{Z}_+} f \,\mathrm{d}\pi_\rho$. Due to this discrepancy, the problem of improving (1.1) remains open, as noted in Gozlan et al. (2023, Remark 3.2). Recall that a function $h : \mathbb{Z}_+ \to [0,\infty)$ is called log-convex if for any $n \in \mathbb{N}$, $\Delta_{\mathrm{d}} h(n) \geq 0$, or equivalently, $h(n)^2 \leq h(n+1)h(n-1)$. Inequality (1.1) provides a lower bound for $\Delta_{\mathrm{d}}(\log A_t f)$, which is weaker than full log-convexity but retains a similar flavor—hence the term semi-log-convexity for A_t.

The inequality (1.1), including inequalities of this type in a broad sense, are of significant interest for several reasons. On the one hand, they play a crucial role in the approach developed in Eldan and Lee (2018) to study Talagrand's convolution conjecture (also known as the L^1-regularization effect) for the Ornstein–Uhlenbeck semigroup on $\mathbb{R}^d$; see Lehec (2016) for the complete resolution of this case. The original conjecture, formulated by Talagrand Talagrand (1989) for the Hamming (or Boolean) cube, remains open, while recent work in Gozlan et al. (2023) has extended these investigations to other models including the M/M/∞ queue. On the other hand, inequality (1.1) exhibits deep connections to non-local Li–Yau type inequalities investigated recently. For instance, in Weber and Zacher (2023), such inequalities are established for the fractional Laplacian on $\mathbb{R}^d$, with a related discrete model discussed in Section 4 of the same paper, and in Li and Qian (2023), analogous results are derived for a class of non-local Schrödinger operators on $\mathbb{R}^d$, namely, Dunkl harmonic oscillators.

Our main contribution, contained in the following theorem, improves (1.1) by eliminating the extra constant $\frac{1}{12}$.

Theorem 1.1. *For every non-zero function $f : \mathbb{Z}_+ \to [0,\infty)$ and every $t > 0$,*

$$\Delta_{\mathrm{d}}[\log A_t f](n) \geq \log\left(1 - \frac{p^2}{\left[p + \rho(1-p)^2\right]^2}\right), \quad n \in \mathbb{N}. \tag{1.2}$$

Remark 1.2. Theorem 1.1 is sharp in the following sense: for every bounded function $f : \mathbb{Z}_+ \to [0,\infty)$, both sides of (1.2) go to 0 as $t \to \infty$.

In the remaining part of this note, we aim to prove our main results.

2. Proofs of Theorem 1.1

We begin with some preliminary definitions. Let $\mathcal{B}(k,a)$ be the binomial law of parameters $k \in \mathbb{Z}_+$ and $a \in [0,1]$, adopting the conventions that $\mathcal{B}(k,0) = \delta_0$ and $\mathcal{B}(k,1) = \delta_k$, where δ_n stands for the Dirac measure at n. Recall that

$$\mathcal{B}(k,a) = \sum_{j=0}^{k} \frac{k!}{j!(k-j)!} a^j (1-a)^{k-j} \delta_j.$$

It is well known that, by the discrete analog of the Mehler formula (see equation (3) on page 321 of Chafaï (2006)), the law of the M/M/∞ queue X_t introduced above can be represented as the convolution of the binomial law and the Poisson law. More precisely, for every $t \geq 0$ and every $k \in \mathbb{Z}_+$,

$$\mathrm{Law}(X_t | X_0 = k) = \mathcal{B}(k, p_t) * \pi_{\rho q_t}, \tag{2.1}$$

where $*$ denotes the discrete convolution, $q_t = 1 - p_t$ and recall that $\rho = \frac{\lambda}{\mu}$ and $p = p_t = e^{-\mu t}$ for every $t \geq 0$. This decomposition corresponds to the sum $X_t = Y_t + Z_t$, where $(Y_t)_{t\geq 0}$ and $(Z_t)_{t\geq 0}$ are independent processes such that $\mathrm{Law}(Y_t) = \mathcal{B}(k, p_t)$ and $\mathrm{Law}(Z_t) = \pi_{\rho q_t}$. Consequently, the semigroup action admits the representation:

$$A_t f(k) := \mathbb{E}[f(X_t) | X_0 = k] = \mathbb{E}[f(Y_t + Z_t)], \quad k \in \mathbb{Z}_+, \tag{2.2}$$

for every function $f : \mathbb{Z}_+ \to [0,\infty)$. Moreover, the process $(X_t)_{t\geq 0}$ is reversible with respect to the Poisson law π_ρ:

$$\mathbb{P}(X_t = i \mid X_0 = j)\pi_\rho(j) = \mathbb{P}(X_t = j \mid X_0 = i)\pi_\rho(i), \quad i,j \in \mathbb{Z}_+. \tag{2.3}$$

To prove our main theorem, the next lemma is crucial.

Lemma 2.1. *Let $t > 0$. Denote*

$$G_k(n) = \mathbb{P}\left(X_t = n \mid X_0 = k\right), \quad k, n \in \mathbb{Z}_+, \tag{2.4}$$

and set

$$K^{-1} = \frac{n}{n+1}\left(1 - \frac{p^2}{[\rho(1-p)^2 + p]^2}\right), \quad n \in \mathbb{N}.$$

Then, for every $k \in \mathbb{Z}_+$ and every $n \in \mathbb{N}$,

$$G_k(n)^2 \leq K G_k(n+1) G_k(n-1). \tag{2.5}$$

Remark 2.2. (1) Recall that a function $h : \mathbb{Z}_+ \to [0,\infty)$ is called ultra-log-convex if the mapping $\mathbb{Z}_+ \ni n \mapsto h(n)n!$ is log-convex, or equivalently, $h(n)^2 \le \frac{n+1}{n} h(n+1)h(n-1)$ for all $n \in \mathbb{N}$. According to this, since $K \in (\frac{n+1}{n}, \infty)$ clearly, we may interpret inequality (2.5) as the semi-ultra-log-convexity of the function $\mathbb{N} \ni n \mapsto G_k(n)$ for each fixed $k \in \mathbb{Z}_+$. For further background on the log-convexity and the ultra-log-convexity, including their various properties, applications and relationships, we recommend the interested reader to the comprehensive review (Saumard and Wellner, 2014).

(2) Lemma 2.1 can be generalized as follows. Let $\widetilde{Y}$ and $\widetilde{Z}$ be independent random variables such that $\mathrm{Law}(\widetilde{Y}) = \mathcal{B}(k,a)$ and $\mathrm{Law}(\widetilde{Z}) = b$, where $a, b \in [0,1]$ and $k \in \mathbb{Z}_+$. Define

$$H_k(n) = [\mathcal{B}(k,a) * \pi_b](n), \quad k, n \in \mathbb{Z}_+,$$

and set

$$M^{-1} = \frac{n}{n+1}\left(1 - \frac{a^2}{[(1-a)b+a]^2}\right), \quad n \in \mathbb{N}.$$

Then

$$H_k(n)^2 \le M H_k(n+1) H_k(n-1), \quad k \in \mathbb{Z}_+,\ n \in \mathbb{N}. \tag{2.6}$$

The proof of (2.6) follows the same method as that used for (2.5). Furthermore, for a fixed $t > 0$, if we take $\mathrm{Law}(Y_t) = \mathrm{Law}(\widetilde{Y})$, $\mathrm{Law}(Z_t) = \mathrm{Law}(\widetilde{Z})$, $a = p$ and $b = \rho(1-p)$, then (2.6) reduces to (2.5).

Proof of Lemma 2.1. Fix $t > 0$. Let Y_t and Z_t be independent such that $\mathrm{Law}(Y_t) = \mathcal{B}(k,p)$ and $\mathrm{Law}(Z_t) = \pi_{\rho(1-p)}$. Then, by (2.2), we clearly have

$$G_k(n) = \mathbb{P}\left(Y_t + Z_t = n\right), \quad k, n \in \mathbb{Z}_+.$$

Now we prove (2.5) by induction.

(1) Let $k = 0$. Then, since $\mathbb{P}(Y_t = 0) = 1$ and $\mathrm{Law}(Z_t) = \pi_{\rho(1-p)}$, by the independence, we have

$$\begin{aligned} G_0(n) &= \mathbb{P}(Y_t + Z_t = n) = \mathbb{P}(Y_t = 0)\mathbb{P}(Z_t = n) \\ &= \mathbb{P}(Z_t = n) = \frac{[\rho(1-p)]^n}{n!} e^{-\rho(1-p)}, \quad n \in \mathbb{N}. \end{aligned}$$

Hence

$$\begin{aligned} &\frac{G_0(n+1)G_0(n-1)}{G_0(n)^2} = \frac{n}{n+1} \\ &\ge \frac{n}{n+1}\left(1 - \frac{p^2}{[\rho(1-p)^2 + p]^2}\right) = K^{-1}, \quad n \in \mathbb{N}. \end{aligned}$$

(2) Let $k = 1$. Then, since $\mathrm{Law}(Y_t) = \mathcal{B}(1,p)$ and $\mathrm{Law}(Z_t) = \pi_{\rho(1-p)}$, by the independence, we obtain

$$\begin{aligned} G_1(n) &= \mathbb{P}(Y_t = 1)\mathbb{P}(Z_t = n-1) + \mathbb{P}(Y_t = 0)\mathbb{P}(Z_t = n) \\ &= p\pi_{\rho(1-p)}(n-1) + (1-p)\pi_{\rho(1-p)}(n) \\ &= \left((1-p) + p\frac{n}{\rho(1-p)}\right)\frac{[\rho(1-p)]^n}{n!} e^{-\rho(1-p)}, \quad n \in \mathbb{N}. \end{aligned}$$

Hence

$$\begin{aligned} \frac{G_1(n+1)G_1(n-1)}{G_1(n)^2} &= \frac{n}{n+1} \frac{\left((1-p) + p\frac{n+1}{\rho(1-p)}\right)\left((1-p) + p\frac{n-1}{\rho(1-p)}\right)}{\left((1-p) + p\frac{n}{\rho(1-p)}\right)^2} \\ &= \frac{n}{n+1} \frac{\left((1-p) + p\frac{n}{\rho(1-p)}\right)^2 - \left(\frac{p}{\rho(1-p)}\right)^2}{\left((1-p) + p\frac{n}{\rho(1-p)}\right)^2} \\ &\ge \frac{n}{n+1}\left(1 - \frac{p^2}{[\rho(1-p)^2 + p]^2}\right) \\ &= K^{-1}, \quad n \in \mathbb{N}. \end{aligned}$$

(3) Let $k = m \in \mathbb{N} \setminus \{1\}$. Suppose that for any $m > 2$,

$$G_{m-1}(n)^2 \le K G_{m-1}(n+1) G_{m-1}(n-1), \quad n \in \mathbb{N}. \tag{2.7}$$

We need to prove that

$$G_m(n)^2 \le K G_m(n+1) G_m(n-1), \quad n \in \mathbb{N}. \tag{2.8}$$

Below, we divide the proof of (2.8) into two parts according to that $n = 1$ and $n \in \mathbb{N} \setminus \{1\}$.

(i) Let $n = 1$. It suffices to prove that

$$G_m(1)^2 \le K G_m(2) G_m(0). \tag{2.9}$$

For convenience, let $b = \rho(1-p)$. By the independence, we have

$$\begin{aligned}
G_m(0) &= \mathbb{P}\left(Y_t + Z_t = 0\right) = \mathbb{P}\left(Y_t = 0\right)\mathbb{P}\left(Z_t = 0\right) \\
&= (1-p)^m e^{-b}, \\
G_m(1) &= \mathbb{P}\left(Y_t + Z_t = 1\right) \\
&= \mathbb{P}\left(Y_t = 1\right)\mathbb{P}\left(Z_t = 0\right) + \mathbb{P}\left(Y_t = 0\right)\mathbb{P}\left(Z_t = 1\right) \\
&= mp(1-p)^{m-1}e^{-b} + (1-p)^m b e^{-b}, \\
G_m(2) &= \mathbb{P}\left(Y_t + Z_t = 2\right) \\
&= \mathbb{P}\left(Y_0 = 0\right)\mathbb{P}\left(Z_t = 2\right) + \mathbb{P}\left(Y_t = 1\right)\mathbb{P}\left(Z_t = 1\right) + \mathbb{P}\left(Y_t = 2\right)\mathbb{P}\left(Z_t = 0\right) \\
&= (1-p)^m \frac{b^2}{2} e^{-b} + mp(1-p)^{m-1} b e^{-b} + \frac{m(m-1)}{2} p^2 (1-p)^{m-2} e^{-b}.
\end{aligned}$$

Hence, to show that (2.9) holds, it is equivalent to prove that

$$[mp + (1-p)b]^2 \le K\left[\frac{(1-p)^2 b^2}{2} + mp(1-p)b + \frac{m(m-1)}{2}p^2\right], \tag{2.10}$$

where

$$K^{-1} = \frac{1}{2}\left(1 - \frac{p^2}{[b(1-p)+p]^2}\right).$$

Let $u = b(1-p)$. Then, (2.10) can be rewritten as

$$(mp+u)^2\left(u^2 + 2pu\right) \le (u+p)^2\left[u^2 + 2mpu + m(m-1)p^2\right],$$

which is clearly equivalent to

$$(1-m)(p^2u^2 - mp^4) \ge 0. \tag{2.11}$$

By (2.7), $G_{m-1}(1)^2 \le K G_{m-1}(2) G_{m-1}(0)$ for any $m \in \mathbb{N}$ such that $m > 2$, which means that

$$(2-m)[p^2u^2 - (m-1)p^4] \ge 0.$$

Hence

$$p^2u^2 \le (m-1)p^4 \le mp^4.$$

Thus, (2.11) holds for any $m \in \mathbb{N} \setminus \{1\}$, from which we conclude that (2.9) holds.

(ii) Let $n \in \mathbb{N} \setminus \{1\}$. Let $Y_t = Y_t' + \varepsilon_t$, where $\mathrm{Law}(Y_t') = \mathcal{B}(m-1, p)$ and $\mathrm{Law}(\varepsilon_t) = \mathcal{B}(1, p)$ such that Y_t' and ε_t are independent and also independent of Z_t with $\mathrm{Law}(Z_t) = \pi_{\rho(1-p)}$. Then

$$\begin{aligned}
G_m(n) &= \mathbb{P}(Y_t + Z_t = n) = \mathbb{P}(Y_t' + \varepsilon_t + Z_t = n) \\
&= \sum_{j=0}^{n} \mathbb{P}(Y_t' + Z_t = j)\mathbb{P}(\varepsilon_t = n-j) \\
&= \sum_{j=1}^{n} \mathbb{P}(Y_t' + Z_t = j)\mathbb{P}(\varepsilon_t = n-j),
\end{aligned}$$

where the last equality is due to the fact that $\mathbb{P}(\varepsilon_t = n) = 0$ since $n > 1$. Thus, by the Cauchy–Schwarz inequality, we derive that

$$
\begin{aligned}
G_m(n) &= \sum_{j=1}^{n} \mathbb{P}(Y_t' + Z_t = j)\mathbb{P}(\varepsilon_t = n-j) \\
&\le \sum_{j=1}^{n} \sqrt{K}\, \mathbb{P}(Y_t' + Z_t = j+1)^{\frac{1}{2}} \mathbb{P}(Y_t' + Z_t = j-1)^{\frac{1}{2}}\, \mathbb{P}(\varepsilon_t = n-j) \\
&\le \sqrt{K} \left(\sum_{j=1}^{n} \mathbb{P}(Y_t' + Z_t = j+1)\mathbb{P}(\varepsilon_t = n-j) \right)^{\frac{1}{2}} \left(\sum_{j=1}^{n} \mathbb{P}(Y_t' + Z_t = j-1)\mathbb{P}(\varepsilon_t = n-j) \right)^{\frac{1}{2}} \\
&= \sqrt{K} \left(\sum_{j=2}^{n+1} \mathbb{P}(Y_t' + Z_t = j)\mathbb{P}(\varepsilon_t = n-j+1) \right)^{\frac{1}{2}} \left(\sum_{j=0}^{n-1} \mathbb{P}(Y_t' + Z_t = j)\mathbb{P}(\varepsilon_t = n-j-1) \right)^{\frac{1}{2}} \\
&\le \sqrt{K} \left(\sum_{j=0}^{n+1} \mathbb{P}(Y_t' + Z_t = j)\mathbb{P}(\varepsilon_t = n-j+1) \right)^{\frac{1}{2}} \left(\sum_{j=0}^{n-1} \mathbb{P}(Y_t' + Z_t = j)\mathbb{P}(\varepsilon_t = n-j-1) \right)^{\frac{1}{2}} \\
&= \sqrt{K}\, \mathbb{P}(Y_t' + Z_t + \varepsilon_t = n+1)^{\frac{1}{2}} \mathbb{P}(Y_t' + Z_t + \varepsilon_t = n-1)^{\frac{1}{2}} \\
&= \sqrt{K}\, G_m(n+1)^{\frac{1}{2}} G_m(n-1)^{\frac{1}{2}}.
\end{aligned}
$$

Putting (i) and (ii) together, we prove (2.8).

Therefore, combining (1), (2) and (3) together, we complete the proof of (2.5). □

Now we are ready to prove our main result.

Proof of Theorem 1.1. Let $t > 0$ and let $f : \mathbb{N} \to [0, \infty)$ be not identical to 0. By the proof of Gozlan et al. (2023, Proposition 3.5), we have

$$
\Delta_{\mathrm{d}}[\log A_t f](n) = \log \frac{A_t f(n+1) A_t f(n-1)}{A_t f(n)^2}, \quad n \in \mathbb{N}.
$$

Then, combining this with (2.2) and (2.3), we derive that

$$
\begin{aligned}
&\Delta_{\mathrm{d}}[\log A_t f](n) \\
&= \log \frac{\left(\sum_{k=0}^{\infty} f(k)\mathbb{P}(X_t = k \mid X_0 = n+1)\right)\left(\sum_{k=0}^{\infty} f(k)\mathbb{P}(X_t = k \mid X_0 = n-1)\right)}{\left(\sum_{k=0}^{\infty} f(k)\mathbb{P}(X_t = k \mid X_0 = n)\right)^2} \\
&= \log \left[\frac{n+1}{n} \cdot \frac{\left(\sum_{k=0}^{\infty} f(k)\pi_\rho(k) G_k(n+1)\right)\left(\sum_{k=0}^{\infty} f(k)\pi_\rho(k) G_k(n-1)\right)}{\left(\sum_{k=0}^{\infty} f(k)\pi_\rho(k) G_k(n)\right)^2} \right] \\
&\ge \log \left[\frac{n+1}{n} \cdot \frac{\left(\sum_{k=0}^{\infty} f(k)\pi_\rho(k) \sqrt{G_k(n+1)}\sqrt{G_k(n-1)}\right)^2}{\left(\sum_{k=0}^{\infty} f(k)\pi_\rho(k) G_k(n)\right)^2} \right], \quad n \in \mathbb{N},
\end{aligned} \tag{2.12}
$$

where $G_k(n)$ is defined in (2.4) and the Cauchy–Schwarz inequality is applied in the last inequality. Thus, by (2.5) and (2.12), we immediately have

$$
\Delta_{\mathrm{d}}[\log A_t f](n) \ge \log \left(\frac{n+1}{n} K^{-1} \right) = \log \left(1 - \frac{p^2}{[\rho(1-p)^2 + p]^2} \right), \quad n \in \mathbb{N}.
$$

The proof is completed. □

Data availability

No data was used for the research described in the article.

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