

HOLOMORPHIC KOSZUL–BRYLINSKI HOMOLOGIES OF POISSON BLOW-UPS

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ABSTRACT. We derive a blow-up formula for holomorphic Koszul–Brylinski homologies of compact holomorphic Poisson manifolds. As applications, we investigate the invariance of the E_1 -degeneracy of the Dolbeault–Koszul–Brylinski spectral sequence under Poisson blow-ups, and compute the holomorphic Koszul–Brylinski homology for del Pezzo surfaces and two complex nilmanifolds with holomorphic Poisson structures.

CONTENTS

| | |
|---|----|
| 1. Introduction | 1 |
| 2. Preliminaries | 4 |
| 3. Blow-ups and modifications in the Poisson category | 7 |
| 4. Comparison under Poisson projective bundles | 11 |
| 5. Comparison under Poisson blow-ups | 14 |
| 6. Examples | 20 |
| Appendix A. Hodge diamond of \mathbb{I}_6 | 26 |
| References | 28 |

1. INTRODUCTION

Historically, Poisson structures arise from classical mechanics. In mathematics, the Poisson structures emerge from many fields, such as generalized complex geometry, geometric representation theory, integrable systems, and algebraic geometry. In many situations, the Poisson structures are actually holomorphic; see [7, 19, 25, 16, 20, 8, 11] etc.. The study of Poisson structures from the viewpoint of algebraic geometry can be traced back at least to Bondal [4] and Polishchuk [29]. More generally, we refer the readers to [32] for an introduction to the algebraic geometry of holomorphic Poisson structures. The purpose of this paper is to study holomorphic Poisson structures from an algebro-geometric point of view, and we focus on the homological aspects of compact holomorphic Poisson manifolds.

Let (X, \mathcal{O}_X) be a complex manifold or a scheme of finite type over the complex number field \mathbb{C} . By a Poisson structure on X , we mean a \mathbb{C} -bilinear sheaf morphism:

$$\{-, -\} : \mathcal{O}_X \times \mathcal{O}_X \longrightarrow \mathcal{O}_X$$

which satisfies the usual axioms for a Poisson bracket, and we call $(X, \{-, -\})$ a holomorphic Poisson manifold or a Poisson scheme. In particular, holomorphic Poisson structures are closely

related to generalized complex geometry. On the one hand, a holomorphic Poisson structure naturally defines a generalized complex structure of special type; see [17]. On the other hand, by Bailey's local classification theorem of generalized complex structures, each generalized complex manifold is locally equivalent to the product of a symplectic manifold and a holomorphic Poisson manifold; see [2]. We refer the readers to [18, 19, 15, 25, 16, 9, 17, 5, 3] and references therein for more results on the applications of holomorphic Poisson structures in generalized complex geometry and the relationships with other geometries.

Assume that $(X, \{-, -\})$ is a compact holomorphic Poisson manifold of complex dimension n , and let $\pi \in H^0(X, \wedge^2 \mathcal{T}_X)$ be the holomorphic Poisson bi-vector field determined by the given Poisson bracket. For a cohomological study of the Poisson structure of X , we have the *holomorphic Lichnerowicz–Poisson cohomology* $H^\bullet(X, \pi)$ defined to be the hypercohomology of the sheaf complex

$$0 \longrightarrow \mathcal{O}_X \xrightarrow{b_\pi} \mathcal{T}_X \xrightarrow{b_\pi} \wedge^2 \mathcal{T}_X \xrightarrow{b_\pi} \wedge^3 \mathcal{T}_X \xrightarrow{b_\pi} \cdots \xrightarrow{b_\pi} \wedge^n \mathcal{T}_X \longrightarrow 0 ,$$

where the differential operator $b_\pi(-) = [\pi, -]_S$ is the adjoint action of π with respect to the Schouten bracket; see [27, 25]. This cohomology has been widely studied; see, for example, [22, 14, 11, 8, 30, 31, 21] and references therein. Dually, from a homological point of view, we have the so called *holomorphic Koszul–Brylinski complex*:

$$0 \longrightarrow \Omega_X^n \xrightarrow{\partial_\pi} \Omega_X^{n-1} \xrightarrow{\partial_\pi} \Omega_X^{n-2} \xrightarrow{\partial_\pi} \Omega_X^{n-3} \xrightarrow{\partial_\pi} \cdots \xrightarrow{\partial_\pi} \mathcal{O}_X \longrightarrow 0 ,$$

where $\partial_\pi = [\iota_\pi, \partial]$. The hypercohomology of the sheaf complex above, denoted by $H_\bullet(X, \pi)$, is called the *holomorphic Koszul–Brylinski homology* of X . Most notably, there exists a holomorphic version of Evens–Lu–Weinstein duality for $H_\bullet(X, \pi)$, which is a generalization of Serre duality for Dolbeault cohomology; see [37, Theorem 4.4]. Furthermore, there is a canonical Fröhlicher-type spectral sequence, called the Dolbeault–Koszul–Brylinski spectral sequence (see Definition 5.6), which converges to $H_\bullet(X, \pi)$. However, it is not so easy to compute the holomorphic Koszul–Brylinski homology for a specific holomorphic Poisson manifold.

In algebraic and complex geometry, the blow-up transformation plays a central role in the study of algebraic varieties and complex manifolds. In the Poisson category, it was Polishchuk [29] who first gave the construction of blow-ups for Poisson schemes. Polishchuk's construction of blow-up transformations for Poisson schemes adapts to holomorphic Poisson manifolds without any essential changes. Our starting point is to understand the homological aspect of holomorphic Poisson manifolds under a Poisson blow-up transformation. Particularly, if the holomorphic Poisson structure π is trivial, then the holomorphic Koszul–Brylinski homology is isomorphic to the Hochschild homology of the complex manifold X :

$$H_k(X, 0) \cong \bigoplus_{p-q=n-k} H^q(X, \Omega_X^p) \cong \mathrm{HH}_{n-k}(X).$$

The blow-up formula for the Hochschild homology has been established in [33]. To be more specific, suppose $Z \subset X$ is a closed complex manifold of codimension $c \geq 2$ and \tilde{X} is the blow-up of X along Z , then there exists an isomorphism of Hochschild homologies

$$\mathrm{HH}_{n-k}(\tilde{X}) \cong \mathrm{HH}_{n-k}(X) \oplus \mathrm{HH}_{n-k}(Z)^{\oplus c-1}.$$

So a natural question that arises now is:

Question. *For a non-trivial holomorphic Poisson structure, can we describe explicitly the variance of the holomorphic Koszul–Brylinski homology under a Poisson blow-up?*

Using a sheaf-theoretic approach, we establish a blow-up formula for holomorphic Koszul–Brylinski homology as follows.

Theorem 1.1. *Suppose (X, π) is a compact holomorphic Poisson manifold of complex dimension $n \geq 2$, and $(Z, \pi|_Z) \subset (X, \pi)$ is a closed holomorphic Poisson submanifold of codimension $c \geq 2$ with trivial transverse Poisson structure. Let $\varphi : \tilde{X} \rightarrow X$ be the blow-up of X along Z with exceptional divisor $E := \varphi^{-1}(Z)$, and let $\tilde{\pi}$ be the unique holomorphic Poisson structure on \tilde{X} such that φ is a Poisson morphism, i.e., $\varphi_* \tilde{\pi} = \pi$. Then there exists an isomorphism of holomorphic Koszul–Brylinski homologies:*

$$H_k(\tilde{X}, \tilde{\pi}) \cong H_k(X, \pi) \oplus \left(H_{k-1}(E, \tilde{\pi}|_E) / \rho^* H_{k-c}(Z, \pi|_Z) \right)$$

for any $0 \leq k \leq 2n$. Furthermore, if Z satisfies the $\partial\bar{\partial}$ -lemma, then we get

$$H_k(\tilde{X}, \tilde{\pi}) \cong H_k(X, \pi) \oplus H_{k-c}(Z, \pi|_Z)^{\oplus c-1}.$$

In particular, there exists an isomorphism

$$H_k(\tilde{X}, \tilde{\pi}) \cong H_k(X, \pi)$$

for $0 \leq k \leq c-1$ or $2n-c+1 \leq k \leq 2n$.

Observe that the first page of the Dolbeault–Koszul–Brylinski spectral sequence of the holomorphic Poisson manifold (X, π) is the Dolbeault cohomology:

$$E_1^{s,t} = H^t(X, \Omega_X^{n-s}) \cong H_{\bar{\partial}}^{n-s,t}(X) \implies H_{n-s+t}(X, \pi).$$

The study of the degeneracy of the Dolbeault–Koszul–Brylinski spectral sequence at E_1 -page may be of independent interest. As an application of Theorem 1.1, we investigate the invariance of such degeneracy under Poisson blow-ups.

Theorem 1.2. *With the assumption of Theorem 1.1, if Z satisfies the $\partial\bar{\partial}$ -lemma then the Dolbeault–Koszul–Brylinski spectral sequence for $(\tilde{X}, \tilde{\pi})$ degenerates at E_1 -page if and only if it does so for (X, π) and $(Z, \pi|_Z)$.*

It is worth noting that if X is a projective manifold or Kähler manifold then the closed complex submanifold Z automatically satisfies the $\partial\bar{\partial}$ -lemma, and therefore both Theorem 1.1 and Theorem 1.2 are applicable to these situations.

This paper is organized as follows. In § 2, we review some basics on holomorphic Poisson manifolds and the holomorphic Koszul–Brylinski homology. We devote § 3 to Poisson blow-ups and modifications. In § 4 we derive the Poisson projective bundles formula for holomorphic Koszul–Brylinski homology, a key part of the proof of the main theorems. In § 5, the proofs of the main theorems are given. In § 6, the holomorphic Koszul–Brylinski homologies of some compact holomorphic Poisson manifolds are computed. Finally, the Appendix A gives the Hodge diamond of a six-dimensional complex nilmanifold in § 6.3.

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2. PRELIMINARIES

In this section, we review some basic facts on holomorphic Poisson manifolds and the Koszul–Brylinski homology of holomorphic Poisson manifolds.

2.1. Holomorphic Poisson manifolds. Let X be a complex manifold and let \mathcal{O}_X be its structure sheaf (i.e., the sheaf of holomorphic functions), Ω_X^p be the sheaf of holomorphic p -forms, \mathcal{T}_X be the sheaf of holomorphic vector fields.

Definition 2.1. A complex manifold X is called a holomorphic Poisson manifold if X admits a holomorphic bi-vector field $\pi \in H^0(X, \wedge^2 \mathcal{T}_X)$ such that $[\pi, \pi]_S = 0$, where $[-, -]_S$ is the Schouten bracket.

Such a holomorphic bi-vector field π is called the *holomorphic Poisson bi-vector field* of the holomorphic Poisson manifold X , and the holomorphic Poisson manifold X is also denoted by (X, π) . In particular, for any open subset $U \subset X$, the ring $\mathcal{O}_X(U)$ is equipped with a Poisson bracket $\{ -, - \}$ via π such that for any open subset $V \subset U$ of X , the restriction map $\mathcal{O}_X(U) \rightarrow \mathcal{O}_X(V)$ is a morphism of Poisson algebras; the holomorphic Poisson bi-vector field π induces a sheaf morphism $\pi^\sharp : \Omega_X^1 \rightarrow \mathcal{T}_X$ by contraction with π . For any fixed point $p \in X$, $\text{Rank}(\pi)|_p$ is defined to be the rank of the linear map $\pi^\sharp|_p$. Naturally, $\text{Rank}(\pi)|_p$ is even and the following theorem describes the local structure of a holomorphic Poisson structure (c.f. [26, Theorem 1.25]).

Theorem 2.2 (Weinstein’s splitting theorem). *Let (X, π) be a holomorphic Poisson manifold and p is an arbitrary point of X . Suppose $\text{Rank}(\pi)|_p = 2r$. Then there exists a neighborhood U of p with holomorphic coordinates $\{z_1, \dots, z_s, z_{s+1}, \dots, z_{s+2r}\}$ centered at p , such that on U ,*

$$\pi = \sum_{1 \leq i, j \leq s} \phi_{ij}(z_1, \dots, z_s) \frac{\partial}{\partial z_i} \wedge \frac{\partial}{\partial z_j} + \sum_{i=1}^r \frac{\partial}{\partial z_{s+i}} \wedge \frac{\partial}{\partial z_{s+r+i}}$$

where the functions ϕ_{ij} are holomorphic functions of (z_1, \dots, z_s) satisfying $\phi_{ij}(p) = 0$. Such a local coordinate $\{z_1, \dots, z_s, z_{s+1}, \dots, z_{s+2r}\}$ is called a splitting coordinate centered at p .

A holomorphic map $f : Y \rightarrow X$ of holomorphic Poisson manifolds (Y, π_Y) and (X, π_X) is a Poisson morphism if and only if $f_*(\pi_Y|_p) = \pi_X|_{f(p)}$ for every $p \in Y$; in this case, we write $f_*\pi_Y = \pi_X$. In particular, let $\jmath : Z \hookrightarrow X$ be a closed complex submanifold of holomorphic Poisson manifold X . Suppose that Z is also a holomorphic Poisson, then we say that Z is a *closed holomorphic Poisson submanifold* of X if the inclusion \jmath is a Poisson morphism.

Analogous to the real case, there are some intrinsic restrictions on the existence of holomorphic Poisson submanifolds in a holomorphic Poisson manifold. For example, due to Weinstein's splitting theorem, one can prove that each holomorphic symplectic leaf (which is hyper-Kähler) is a Poisson submanifold, and every holomorphic Poisson submanifold is the union of some symplectic leaves. Therefore, if the holomorphic Poisson bi-vector field of X is induced by a holomorphic symplectic form, then only open subsets of X are Poisson submanifolds. More precisely, consider a closed holomorphic Poisson submanifold $\jmath : (Z, \pi|_Z) \hookrightarrow (X, \pi)$, for any $p \in Z$, we can choose a neighborhood U of p in X with splitting coordinates $\{z_1, \dots, z_s, z_{s+1}, \dots, z_{s+2r}\}$ centered at p satisfying

$$\pi|_U = \sum_{1 \leq i, j \leq s} \phi_{ij}(z_1, \dots, z_s) \frac{\partial}{\partial z_i} \wedge \frac{\partial}{\partial z_j} + \sum_{i=1}^r \frac{\partial}{\partial z_{s+i}} \wedge \frac{\partial}{\partial z_{s+r+i}},$$

such that there exists a neighborhood $V = U \cap Z = \{z_1 = 0, \dots, z_c = 0\} \subset U$ of p in Z satisfying

$$(\pi|_Z)|_V = \sum_{c+1 \leq i, j \leq s} j^* \phi_{ij} \frac{\partial}{\partial z_i} \wedge \frac{\partial}{\partial z_j} + \sum_{i=1}^r \frac{\partial}{\partial z_{s+i}} \wedge \frac{\partial}{\partial z_{s+r+i}}.$$

2.2. Koszul–Brylinski homology. Koszul–Brylinski homology is introduced independently by Koszul [24] and Brylinski [6]. Let (X, π) be a holomorphic Poisson manifold. The Koszul–Brylinski operator of (X, π) on the sheaves of holomorphic forms is given as follows:

$$\partial_\pi := [\iota_\pi, \partial] : \Omega_X^p \longrightarrow \Omega_X^{p-1},$$

where Ω_X^p is the sheaf of holomorphic p -forms, ∂ is the Dolbeault operator and ι_π is the contraction operator with respect to holomorphic Poisson bi-vector field π .

Lemma 2.3. *Let (X, π) be a holomorphic Poisson manifold. Then we have $\iota_\pi \bar{\partial} = \bar{\partial} \iota_\pi$ and $\bar{\partial} \partial_\pi + \partial_\pi \bar{\partial} = 0$*

Proof. To prove the first statement, it suffices to verify the assertion on an arbitrary coordinate neighborhood of X . Let $(U; z_1, \dots, z_n)$ be a coordinate neighborhood of X . Locally, the holomorphic Poisson bi-vector field can be expressed as $\pi = \sum_{i,j} c_{ij} \frac{\partial}{\partial z_i} \wedge \frac{\partial}{\partial z_j}$, where c_{ij} are holomorphic functions on U . By definition, for any smooth (p, q) -form $\alpha = f dz_{k_1} \wedge \dots \wedge dz_{k_p} \wedge d\bar{z}_{l_1} \wedge \dots \wedge d\bar{z}_{l_q}$ on U , we have

$$\begin{aligned} (\iota_\pi \bar{\partial} - \bar{\partial} \iota_\pi) \alpha &= \sum_{i,j} c_{ij} \cdot \iota_{\frac{\partial}{\partial z_i} \wedge \frac{\partial}{\partial z_j}} \left(\sum_s \frac{\partial f}{\partial \bar{z}_s} d\bar{z}_s \wedge dz_{k_1} \wedge \dots \wedge dz_{k_p} \wedge d\bar{z}_{l_1} \wedge \dots \wedge d\bar{z}_{l_q} \right) \\ &\quad - \bar{\partial} \left(\sum_{i,j} f c_{ij} \cdot \iota_{\frac{\partial}{\partial z_i} \wedge \frac{\partial}{\partial z_j}} (dz_{k_1} \wedge \dots \wedge dz_{k_p}) \wedge d\bar{z}_{l_1} \wedge \dots \wedge d\bar{z}_{l_q} \right) \\ &= (-1)^p \sum_s \sum_{i,j} \frac{\partial f}{\partial \bar{z}_s} c_{ij} \cdot \iota_{\frac{\partial}{\partial z_i} \wedge \frac{\partial}{\partial z_j}} (dz_{k_1} \wedge \dots \wedge dz_{k_p}) \wedge d\bar{z}_s \wedge d\bar{z}_{l_1} \wedge \dots \wedge d\bar{z}_{l_q} \\ &\quad - \sum_s \sum_{i,j} \frac{\partial f}{\partial \bar{z}_s} d\bar{z}_s \wedge c_{ij} \cdot \iota_{\frac{\partial}{\partial z_i} \wedge \frac{\partial}{\partial z_j}} (dz_{k_1} \wedge \dots \wedge dz_{k_p}) \wedge d\bar{z}_{l_1} \wedge \dots \wedge d\bar{z}_{l_q} \\ &= 0. \end{aligned}$$

Equivalently, we get $\bar{\partial}_\pi := [\iota_\pi, \bar{\partial}] = \iota_\pi \bar{\partial} - \bar{\partial} \iota_\pi = 0$.

For the second statement, since $[\bar{\partial}, \partial] = \bar{\partial}\partial + \partial\bar{\partial} = 0$, by the first statement, we have that $\bar{\partial}\partial_\pi + \partial_\pi\bar{\partial} = [\bar{\partial}, \partial_\pi] = [\bar{\partial}, [\iota_\pi, \partial]] = [[\bar{\partial}, \iota_\pi], \partial] + [\iota_\pi, [\bar{\partial}, \partial]] = 0$, and the lemma is proved. \square

According to the Cartan formulae, we have $\partial_\pi^2 = 0$, and

$$\partial_\pi(\alpha \wedge \beta) = (\partial_\pi\alpha) \wedge \beta + (-1)^k \alpha \wedge (\partial_\pi\beta) + (-1)^k [\alpha, \beta]_{\partial_\pi}$$

for any $\alpha \in \Omega_X^k$ and $\beta \in \Omega_X^l$. Here $[-, -]_{\partial_\pi}$ is a graded Lie bracket on Ω_X^\bullet obtained by Leibniz rule via

$$[\alpha, \beta]_{\partial_\pi} := L_{\pi^\sharp(\alpha)}\beta - L_{\pi^\sharp(\beta)}\alpha - \partial(\pi(\alpha, \beta)), \quad \forall \alpha, \beta \in \Omega_X^1. \quad (2.1)$$

The *holomorphic Koszul–Brylinski complex* of X is the sheaf complex:

$$0 \longrightarrow \Omega_X^n \xrightarrow{\partial_\pi} \cdots \xrightarrow{\partial_\pi} \Omega_X^{s+1} \xrightarrow{\partial_\pi} \Omega_X^s \xrightarrow{\partial_\pi} \Omega_X^{s-1} \xrightarrow{\partial_\pi} \cdots \xrightarrow{\partial_\pi} \mathcal{O}_X \longrightarrow 0. \quad (2.2)$$

Definition 2.4. Let (X, π) be a holomorphic Poisson manifold. The k -th *holomorphic Koszul–Brylinski homology* of (X, π) is defined to be

$$H_k(X, \pi) := \mathbb{H}^k(X, (\Omega_X^\bullet, \partial_\pi)), \quad (2.3)$$

the k -th hyperhomology of the holomorphic Koszul–Brylinski complex.

Proposition 2.5. Suppose (X, π) is a holomorphic Poisson manifold. Then its holomorphic Koszul–Brylinski complex admits a fine resolution which is the total complex of the Koszul–Brylinski double complex $(\mathcal{A}_X^{\bullet, \bullet}, \partial_\pi, \bar{\partial})$, where $\mathcal{A}_X^{p, q}$ is the sheaf of (p, q) -forms on X . In particular, the Koszul–Brylinski homology is isomorphic to the hypercohomology of the associated total complex.

Proof. Since the sheaf complex $\mathcal{A}_X^{p, \bullet}$ gives rise to a fine resolution of Ω_X^p , the assertion follows from the fact that the Koszul–Brylinski operator ∂_π commutes with $\bar{\partial}$; see also [37, Theorem 5.1]. \square

This proposition immediately yields the natural morphism of Koszul–Brylinski homology under Poisson morphisms.

Corollary 2.6. Suppose that $f : (Y, \pi_Y) \rightarrow (X, \pi_X)$ is a Poisson morphism of holomorphic Poisson manifolds. Then the pullback of differential forms naturally induces a morphism of the holomorphic Koszul–Brylinski homologies

$$f^* : H_k(X, \pi_X) \longrightarrow H_k(Y, \pi_Y).$$

Proof. Note that on the space of (p, q) -forms, we have

$$f^* \circ \partial_{\pi_X} = f^* \circ \partial_{f^*\pi_Y} = \partial_{\pi_Y} \circ f^* \text{ and } f^* \circ \bar{\partial} = \bar{\partial} \circ f^*.$$

Hence, the corollary follows immediately from Proposition 2.5. \square

By a result of Stiénon [37, Theorem 6.4], the holomorphic Evans–Lu–Weinstein pairing on the holomorphic Koszul–Brylinski homology is non-degenerate. More precisely, if (X, π) is a compact holomorphic Poisson manifold of complex dimension n , then there is an isomorphism

$$H_{2n-k}(X, \pi) \cong H_k(X, \pi) \quad (2.4)$$

for $0 \leq k \leq 2n$. In the dual aspect, there exists a *holomorphic Lichnerowicz–Poisson complex* $(\wedge^\bullet \mathcal{T}_X, b_\pi)$:

$$0 \longrightarrow \mathcal{O}_X \xrightarrow{b_\pi} \cdots \xrightarrow{b_\pi} \wedge^{s-1} \mathcal{T}_X \xrightarrow{b_\pi} \wedge^s \mathcal{T}_X \xrightarrow{b_\pi} \wedge^{s+1} \mathcal{T}_X \xrightarrow{b_\pi} \cdots \xrightarrow{b_\pi} \wedge^n \mathcal{T}_X \longrightarrow 0$$

where $b_\pi(-) = [\pi, -]_S$. The k -th hyperhomology of $(\wedge^\bullet \mathcal{T}_X, b_\pi)$ is called the k -th *holomorphic Lichnerowicz–Poisson cohomology*, i.e.,

$$H^k(X, \pi) := \mathbb{H}^k(X, (\wedge^\bullet \mathcal{T}_X, b_\pi)).$$

Assume that X admits a holomorphic volume form $\omega \in \Gamma(X, \Omega_X^n)$. Then there is the natural morphism of sheaves

$$\iota_{(-)}\omega : \wedge^s \mathcal{T}_X \longrightarrow \Omega_X^{n-s}$$

for each $s \in \{0, 1, \dots, n\}$. However, it does not induce a morphism of sheaf complexes between $(\wedge^\bullet \mathcal{T}_X, b_\pi)$ and $(\Omega_X^\bullet, \partial_\pi)$. The reason lies in the fact that the diagram

$$\begin{array}{ccc} \wedge^s \mathcal{T}_X & \xrightarrow{\iota_{(-)}\omega} & \Omega_X^{n-s} \\ b_\pi \downarrow & & \downarrow \partial_\pi \\ \wedge^{s+1} \mathcal{T}_X & \xrightarrow{\iota_{(-)}\omega} & \Omega_X^{n-s-1} \end{array}$$

is not commutative in general. This motivates the following definition.

Definition 2.7 (c.f. [38, 7]). A holomorphic Poisson manifold (X, π) is called *unimodular* if there is a holomorphic volume form ω such that the morphism $\iota_{(-)}\omega$ induces a morphism of sheaf complexes from $(\wedge^\bullet \mathcal{T}_X, b_\pi)$ to $(\Omega_X^\bullet, \partial_\pi)$.

Equivalently, a holomorphic Poisson manifold (X, π) is unimodular if and only if $\partial_\pi \omega = 0$, or the modular vector field, introduced by Weinstein [38] and Brylinski–Zuckerman [7], vanishes. In particular, we have

Proposition 2.8 ([37, Proposition 4.7]). *If the holomorphic Poisson manifold (X, π) is unimodular, then there is an isomorphism*

$$H_k(X, \pi) \cong H^{2n-k}(X, \pi),$$

for any $k \in \mathbb{Z}$, where $n = \dim_{\mathbb{C}} X$.

3. BLOW-UPS AND MODIFICATIONS IN THE POISSON CATEGORY

In this section, we give a rapid review on the blow-ups and modifications in the holomorphic Poisson category.

3.1. Poisson blow-ups. Given a complex manifold X and a closed complex submanifold $\jmath : Z \hookrightarrow X$ with complex codimension $c \geq 2$. Let $\varphi : \tilde{X} \rightarrow X$ be the blow-up of X along Z . Then the holomorphic map

$$\varphi : \tilde{X} - E \longrightarrow X - Z$$

is biholomorphic, where $E := \varphi^{-1}(Z)$ is the exceptional divisor, which is the projective bundle of the normal bundle of Z in X . Moreover, we have a commutative diagram

$$\begin{array}{ccc}
E & \xhookrightarrow{\tilde{\jmath}} & \tilde{X} \\
\downarrow \rho := \varphi|_E & & \downarrow \varphi \\
Z & \xhookrightarrow{\jmath} & X.
\end{array} \tag{3.1}$$

In the Poisson category, if X is a holomorphic Poisson manifold and Z is a closed holomorphic Poisson submanifold of X , then the existence of the holomorphic Poisson structure on the complex blow-up \tilde{X} is not an unconditional result. In fact, there exist some restrictions on the existence of the holomorphic Poisson structure on \tilde{X} . Let us recall the result which was originally studied by Polishchuk [29]. Assume $(Z, \pi|_Z)$ is a closed holomorphic Poisson submanifold of (X, π) . Then for any point $z \in Z$, the conormal space N_z^*Z is a Lie algebra induced by the bracket (2.1), or equivalently, the normal space N_zZ admits a linear Poisson structure which defines the transverse Poisson structure $\pi_N \in \Gamma(Z, N^*Z \otimes \wedge^2 NZ)$.

Definition 3.1. The transverse Poisson structure π_N of a closed holomorphic Poisson submanifold $(Z, \pi|_Z)$ in (X, π_X) is said to be *degenerate* if, for any point $z \in Z$, the map

$$\begin{aligned}
\wedge^3 N_z^*Z &\longrightarrow S^2 N_z^*Z \\
\alpha \wedge \beta \wedge \gamma &\longmapsto [\alpha, \beta]\gamma + [\beta, \gamma]\alpha + [\gamma, \alpha]\beta
\end{aligned}$$

is identically to zero.

It follows from [29, Proposition 8.1] that a degenerate Lie algebra is either abelian or isomorphic to the Lie algebra $\text{Span}\{e_1, \dots, e_{c-1}, f\}$ with Lie bracket $[e_i, e_j] = 0, [f, e_i] = e_i$.

Example 3.2 (c.f. [32, § 2.5.2]). Let π be a holomorphic Poisson bi-vector field on \mathbb{C}^2 , and $\text{Bl}_o\mathbb{C}^2 \xrightarrow{\varphi} \mathbb{C}^2$ the blow-up of \mathbb{C}^2 at the origin $o = (0, 0) \in \mathbb{C}^2$. Choose coordinates z_1, z_2 , and suppose $\{z_1, z_2\} = f(z_1, z_2)$ for some holomorphic function f . Set

$$u = \varphi^*(z_1), \text{ and } v = \frac{\varphi^*(z_2)}{\varphi^*(z_1)} = \varphi^*(z_1^{-1}z_2).$$

Suppose we can define a holomorphic Poisson bracket on $\text{Bl}_o\mathbb{C}^2$ which is compatible with the one determined by π on \mathbb{C}^2 ; then

$$\begin{aligned}
\{u, v\} &= \{\varphi^*(z_1), \varphi^*(z_1^{-1}z_2)\} = \varphi^*\{z_1, z_1^{-1}z_2\} = \varphi^*(z_1^{-1}f(z_1, z_2)) \\
&= u^{-1}f(u, uv) = u^{-1}(f(0, 0) + ug(u, v)),
\end{aligned}$$

where g is holomorphic near the locus $u = 0$. Therefore the holomorphic Poisson bracket given by $\{z_1, z_2\} = f(z_1, z_2)$ on \mathbb{C}^2 can be lifted to $\text{Bl}_o\mathbb{C}^2$ if and only if $f(0, 0) = 0$.

Now, let us return to the construction of Poisson blow-ups. The blow-up of a Poisson scheme was originally clarified in the work of Polishchuk [29]. Here, we review the blow-up of holomorphic Poisson manifolds along closed holomorphic Poisson submanifolds; see also [15, Section 2].

Proposition 3.3 ([29, Propositions 8.2 & 8.3] or [3, Proposition 3.15]). *Let (X, π) be a holomorphic Poisson manifold. Suppose $\jmath : (Z, \pi|_Z) \hookrightarrow (X, \pi)$ is a closed holomorphic Poisson submanifold. If the associated transverse Poisson structure π_N vanishes, then the following statements hold:*

- (i) there exists a unique holomorphic Poisson structure $\tilde{\pi}$ on \tilde{X} such that φ is Poisson morphism (i.e., $\varphi_*\tilde{\pi} = \pi$);
- (ii) E is a holomorphic Poisson manifold such that $\varphi|_E : E \rightarrow Z$ is a Poisson morphism;
- (iii) the diagram (3.1) of holomorphic Poisson manifolds is commutative.

3.2. Poisson modifications. This subsection is devoted to the study of the behavior of the holomorphic Koszul–Brylinski homology under Poisson modifications of compact holomorphic Poisson manifolds. Recall that a *modification* of compact complex manifolds is a holomorphic map $\psi : Y \rightarrow X$ of compact complex manifolds satisfying:

- (i) $\dim Y = \dim X$; and
- (ii) there is an analytic subset $S \subset X$ of codimension ≥ 2 such that the restriction

$$\psi : Y - \psi^{-1}(S) \longrightarrow X - S$$

is biholomorphic.

Definition 3.4. A *Poisson modification* is a Poisson morphism $\psi : (Y, \pi_Y) \rightarrow (X, \pi_X)$ of compact holomorphic Poisson manifolds (X, π_X) and (Y, π_Y) such that ψ is also a modification of compact complex manifolds.

Note that the holomorphic Poisson blow-ups are important examples of Poisson modifications. To study the behavior of the holomorphic Koszul–Brylinski homology under Poisson modifications of compact holomorphic Poisson manifolds, we need to reinterpret the Koszul–Brylinski homology in terms of currents. Let (X, π) be a holomorphic Poisson manifold, and $\mathcal{C}_X^{s,t}$ be the sheaf of (s, t) -currents on X . Then the operators ∂_π and $\bar{\partial}$ naturally induce the dual operators ∂_π^* and $\bar{\partial}^*$ acting on $\mathcal{C}_X^{s,t}$, respectively. Since ∂_π^* commutes with $\bar{\partial}^*$, we obtain a double complex $(\mathcal{C}_X^{\bullet,\bullet}, \partial_\pi^*, \bar{\partial}^*)$. In particular, there exists a natural morphism of double complexes

$$\tau_X : (\mathcal{A}_X^{\bullet,\bullet}, \partial_\pi, \bar{\partial}) \hookrightarrow (\mathcal{C}_X^{\bullet,\bullet}, \partial_\pi^*, \bar{\partial}^*). \quad (3.2)$$

Denote by $H_k^C(X, \pi)$ the k -hypercohomology of the total complex of the double complex $(\mathcal{C}_X^{\bullet,\bullet}, \partial_\pi^*, \bar{\partial}^*)$.

Lemma 3.5. *The natural morphism τ_X induces an isomorphism*

$$\tau_X : H_k(X, \pi) \longrightarrow H_k^C(X, \pi),$$

for any $k \in \mathbb{Z}$.

Proof. To prove the assertion, it suffices to verify that $(\mathcal{A}_X^{\bullet,\bullet}, \partial_\pi, \bar{\partial})$ is quasi-isomorphic to $(\mathcal{C}_X^{\bullet,\bullet}, \partial_\pi^*, \bar{\partial}^*)$ under the morphism τ_X . By the spectral sequence theory for double complexes, there exists a sequence $\{E_r, d_r\}$ for $(\mathcal{A}_X^{\bullet,\bullet}, \partial_\pi, \bar{\partial})$ such that

$$E_1 = H^\bullet(\mathcal{A}_X^{\bullet,\bullet}, \bar{\partial}) = H_{\bar{\partial}}^{\bullet,\bullet}(X) \implies E_\infty = H_\bullet(X, \pi).$$

Similarly, the double complex $\{\mathcal{C}_X^{\bullet,\bullet}, \partial_\pi^*, \bar{\partial}^*\}$ admits a spectral sequence $(\tilde{E}_r, \tilde{d}_r)$ satisfying

$$\tilde{E}_1 = H^\bullet(\mathcal{C}_X^{\bullet,\bullet}, \bar{\partial}^*) \implies \tilde{E}_\infty = H_\bullet^C(X, \pi).$$

Observe that (3.2) induces a morphism of spectral sequences

$$\tau_{X,r} : \{E_r, d_r\} \longrightarrow \{\tilde{E}_r, \tilde{d}_r\}.$$

Since the natural inclusion $\tau_X : (\mathcal{A}_X^{p,\bullet}, \bar{\partial}) \hookrightarrow (\mathcal{C}_X^{p,\bullet}, \bar{\partial}^*)$ is a quasi-isomorphism, we get that the induced map $\tau_{X,1} : E_1 \rightarrow \tilde{E}_1$ is an isomorphism and therefore $E_\infty \cong \tilde{E}_\infty$ under τ_X . This implies that (3.2) is a quasi-isomorphism and the proof is completed. \square

We are ready to present the following comparison theorem for holomorphic Koszul–Brylinski homology under Poisson modifications.

Theorem 3.6. *Let $f : (Y, \pi_Y) \rightarrow (X, \pi_X)$ be a Poisson modification of compact holomorphic Poisson manifolds. Then the natural morphism*

$$f^* : H_k(X, \pi_X) \longrightarrow H_k(Y, \pi_Y)$$

is injective, for any $k \in \mathbb{Z}$.

Proof. Since f is a Poisson morphism, by definition, we have $f_* \pi_Y = \pi_X$. This implies

$$f^* \circ \partial_{\pi_X} = \partial_{\pi_Y} \circ f^* \text{ and } f_* \circ \partial_{\pi_Y}^* = \partial_{\pi_X}^* \circ f_*.$$
 (3.3)

In particular, we obtain a diagram

$$\begin{array}{ccc} (\Gamma(X, \mathcal{A}_X^{\bullet, \bullet}), \partial_{\pi_X}, \bar{\partial}) & \xhookrightarrow{\tau_X} & (\Gamma(X, \mathcal{C}_X^{\bullet, \bullet}), \partial_{\pi_X}^*, \bar{\partial}^*) \\ f^* \downarrow & & \uparrow f_* \\ (\Gamma(Y, \mathcal{A}_Y^{\bullet, \bullet}), \partial_{\pi_Y}, \bar{\partial}) & \xhookrightarrow{\tau_Y} & (\Gamma(Y, \mathcal{C}_Y^{\bullet, \bullet}), \partial_{\pi_Y}^*, \bar{\partial}^*). \end{array} \quad (3.4)$$

However, it is not *a priori* clear that the diagram (3.4) is commutative. We now show the commutativity of (3.4). As f is a modification of compact complex manifolds, its degree is 1; moreover, f is a biholomorphism outside of two sets with Lebesgue measure zero. As a result, let α be a differential k -form on X , then we have

$$\begin{aligned} \langle f_* \circ \tau_Y \circ f^*(\alpha), \beta \rangle &= \int_X (f_* \circ \tau_Y \circ f^*(\alpha)) \wedge \beta \\ &= \int_Y f^*(\alpha \wedge \beta) = \int_X \alpha \wedge \beta \\ &= \langle \tau_X(\alpha), \beta \rangle, \end{aligned}$$

where β is an arbitrary differential $(2n - k)$ -form on X . It follows that $\tau_X(\alpha) = f_* \circ \tau_Y \circ f^*(\alpha)$; see the proof of [13, Theorem 12.9]. Combining it with (3.3) yields that (3.4) is a commutative diagram. Applying Lemma 3.5 to X and Y , we obtain two natural isomorphisms

$$\tau_Y : H_k(Y, \pi_Y) \longrightarrow H_k^C(Y, \pi_Y) \text{ and } \tau_X : H_k(X, \pi_X) \longrightarrow H_k^C(X, \pi_X).$$

Consequently, we obtain a commutative diagram

$$\begin{array}{ccc} H_k(X, \pi_X) & \xrightarrow{\simeq \tau_X} & H_k^C(X, \pi_X) \\ f^* \downarrow & & \uparrow f_* \\ H_k(Y, \pi_Y) & \xrightarrow{\simeq \tau_Y} & H_k^C(Y, \pi_Y). \end{array}$$

and hence the morphism

$$f^* : H_k(X, \pi_X) \longrightarrow H_k(Y, \pi_Y)$$

is injective. \square

4. COMPARISON UNDER POISSON PROJECTIVE BUNDLES

The purpose of this section is to establish the following projective bundle formula for holomorphic Koszul–Brylinski homology.

Theorem 4.1. *Suppose (Z, π) is a compact holomorphic Poisson manifold. Let $\rho : E \rightarrow Z$ be the projective bundle of a holomorphic vector bundle of rank $c \geq 2$ on Z . If Z satisfies the $\partial\bar{\partial}$ -lemma and $\tilde{\pi}$ is a holomorphic Poisson structure on E such that $\rho_*\tilde{\pi} = \pi$, then there is an isomorphism of Koszul–Brylinski homology as \mathbb{C} -vector spaces:*

$$H_{k+1-c}(Z, \pi)^{\oplus c} \cong H_k(E, \tilde{\pi}),$$

for any $k \in \mathbb{Z}$.

To illustrate the basic idea of the proof of the theorem above, we consider the case of $\dim_{\mathbb{C}} Z = 2$ and $c = 3$. Consider the first Chern class of the tautological line bundle over E :

$$\mathbf{h} = c_1(\mathcal{O}_E(1)) \in H_{\bar{\partial}}^{1,1}(E).$$

Set $A_Z^{s,t} := \Gamma(Z, \mathcal{A}_Z^{s,t})$ be the space of differential (s, t) -forms. Observe that $H_k(E, \tilde{\pi})$ is equal to the k -th total cohomology of the double complex $G = (A_E^{\bullet, \bullet}, \partial_{\tilde{\pi}}, \bar{\partial})$, whereas $H_{k-2}(Z, \pi)^{\oplus 3}$ is the k -th total cohomology of the double complex

$$L = \left(\bigoplus_{i=0}^2 A_Z^{\bullet, \bullet}[-2+i, -i], \partial_{\pi}, \bar{\partial} \right).$$

According to the standard spectral sequence theory for double complexes, we have a spectral sequence $\{(\mathcal{G}_r^{\bullet, \bullet}, d_r)\}$ associated to G such that

$$\mathcal{G}_1^{\bullet, \bullet} = H_{\bar{\partial}}^{\bullet, \bullet}(E) \Longrightarrow H_{\bullet}(E, \tilde{\pi}).$$

Similarly, for the double complex L , there exists a spectral sequence $\{(\mathcal{L}_r^{\bullet, \bullet}, \bar{d}_r)\}$ satisfying

$$\mathcal{L}_1^{\bullet, \bullet} = \bigoplus_{i=0}^2 H_{\bar{\partial}}^{\bullet, \bullet}(Z)[-2+i, -i] \Longrightarrow H_{\bullet-2}(Z, \pi)^{\oplus 3}.$$

Note that there exists a well-defined map of bi-graded \mathbb{C} -vector spaces (see the figure below):

$$\Psi := \sum_{i=0}^2 h^i \wedge \rho^*(-) : \bigoplus_{i=0}^2 A_Z^{\bullet, \bullet}[-2+i, -i] \longrightarrow A_E^{\bullet, \bullet}.$$

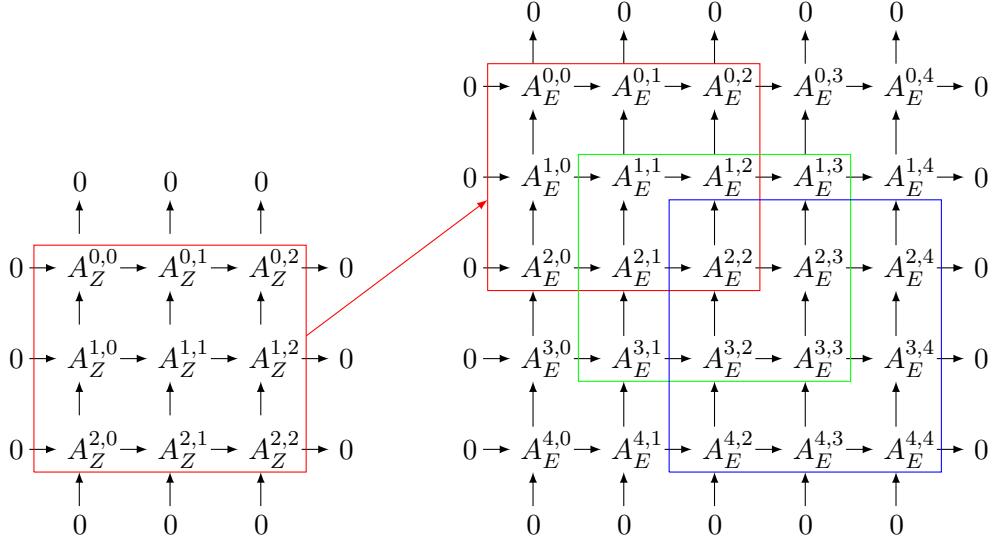
Since $\partial_{\tilde{\pi}}$ is not a derivation, it does not commute with Ψ and therefore Ψ can not give rise to a morphism between the double complexes G and L . Recall that a compact complex manifold X satisfies the $\partial\bar{\partial}$ -lemma, if the equation

$$\ker \partial \cap \ker \bar{\partial} \cap \text{im } d = \text{im } \partial\bar{\partial}$$

holds for the double complex $(A_X^{\bullet, \bullet}, \partial, \bar{\partial})$ (cf. [12]). Under the assumption that Z satisfies the $\partial\bar{\partial}$ -lemma, it is noteworthy that Ψ induces a morphism $\Psi_1 : \mathcal{G}_1^{\bullet, \bullet} \rightarrow \mathcal{L}_1^{\bullet, \bullet}$ which commutes with the differentials d_1 and \bar{d}_1 . Consequently, we get a well-defined morphism of spectral sequences

$$\Psi_r : (\mathcal{G}_r^{\bullet, \bullet}, d_r) \longrightarrow (\mathcal{L}_r^{\bullet, \bullet}, \bar{d}_r).$$

In particular, by the projective bundle formula of Dolbeault cohomology, we conclude that Ψ_1 is an isomorphism, and so is the Ψ_{∞} .



We here state some facts which is necessary for the proof of Theorem 4.1.

Proposition 4.2. *Let (X, π) be a holomorphic Poisson manifold. Then for any d -closed forms α and β on X , the bracket $[\alpha, \beta]_{\partial_\pi}$ is d -exact.*

Proof. Let $d_\pi := \iota_\pi d - d\iota_\pi = \partial_\pi + \bar{\partial}_\pi$. According to a result by Sharygin–Talalaev [36, Lemma 5], the Lie bracket

$$[\alpha, \beta]_{d_\pi} = (-1)^k (d_\pi(\alpha \wedge \beta) - (d_\pi \alpha) \wedge \beta - (-1)^k \alpha \wedge (d_\pi \beta)), \quad \alpha \in A_X^k, \quad \beta \in A_X^l,$$

associated to d_π is d -exact if both α and β are d -closed. Due to the Lemma 2.3, we obtain that $\bar{\partial}_\pi$ is zero, and hence we get $d_\pi = \partial_\pi$. Therefore we in fact have $[\alpha, \beta]_{d_\pi} = [\alpha, \beta]_{\partial_\pi}$, and consequently the Proposition holds. \square

We are now in a position to give the proof of Theorem 4.1.

Proof of Theorem 4.1. Using the same notations as above, the morphism of the first pages $\Psi_1 : (\mathcal{G}_1^{\bullet, \bullet}, d_1 = \partial_\pi) \rightarrow (\mathcal{L}_1^{\bullet, \bullet}, \bar{d}_1 = \partial_{\bar{\pi}})$ is explicitly expressed as:

$$\Psi_1 = \sum_{i=0}^{c-1} h^i \wedge \rho^*(-) : \bigoplus_{i=0}^{c-1} (H_{\bar{\partial}}^{\bullet, \bullet}(Z)[-c+1+i, -i], \partial_\pi) \longrightarrow (H_{\bar{\partial}}^{\bullet, \bullet}(E), \partial_{\bar{\pi}}). \quad (4.1)$$

We claim that Ψ_1 commutes with ∂_π and $\partial_{\bar{\pi}}$. Note that if the $\partial\bar{\partial}$ -lemma holds on Z , it also holds on E (cf. [1, Corollary 12]). Since h is a d -closed real $(1, 1)$ -form on E , it follows from Proposition 4.2 that $[h, h]_{\partial_{\bar{\pi}}}$ is d -exact. On the one hand, since $\partial h = 0$, we get

$$\partial([h, h]_{\partial_{\bar{\pi}}}) = [\partial h, h]_{\partial_{\bar{\pi}}} + [h, \partial h]_{\partial_{\bar{\pi}}} = 0. \quad (4.2)$$

On the other hand, since $[h, h]_{\partial_{\bar{\pi}}}$ is d -exact, it follows from (4.2) that

$$0 = d([h, h]_{\partial_{\bar{\pi}}}) = \bar{\partial}([h, h]_{\partial_{\bar{\pi}}}). \quad (4.3)$$

From the $\partial\bar{\partial}$ -lemma, we obtain $[h, h]_{\partial_{\bar{\pi}}} = \partial\bar{\partial}\beta$ for some β on E . This implies that $[h, h]_{\partial_{\bar{\pi}}}$ represents a zero class in $H_{\bar{\partial}}^{1,2}(E)$ and therefore we get

$$0 = [\partial_{\bar{\pi}} h^i] \in H_{\bar{\partial}}^{i-1, i}(E).$$

For any $[\alpha] \in H_{\bar{\partial}}^{\bullet, \bullet}(Z)$, we have $\partial\alpha \in \ker \bar{\partial} \cap \text{im } \partial$. Since Z satisfies the $\partial\bar{\partial}$ -lemma, there exists ξ on Z such that $\partial\alpha = \partial\bar{\partial}\xi$. Put $\tilde{\alpha} = \alpha - \bar{\partial}\xi$. Then we get $[\alpha] = [\tilde{\alpha}]$ and $\partial\tilde{\alpha} = 0$. In what follows, we always choose the ∂ -closed representatives of the Dolbeault cohomology classes in $H_{\bar{\partial}}^{\bullet, \bullet}(Z)$. Let $[\alpha] \in H_{\bar{\partial}}^{\bullet, \bullet}(Z)$, then we have

$$\begin{aligned}\partial_{\tilde{\pi}}(\Psi_1([\alpha])) &= \partial_{\tilde{\pi}}\left(\sum_{i=0}^{c-1}[h^i \wedge \rho^*(\alpha)]\right) = \sum_{i=0}^{c-1}[\partial_{\tilde{\pi}}(h^i \wedge \rho^*(\alpha))] \\ &= \sum_{i=0}^{c-1}[(\partial_{\tilde{\pi}}h^i) \wedge \rho^*(\alpha) + h^i \wedge (\partial_{\tilde{\pi}} \circ \rho^*(\alpha)) + [h^i, \rho^*(\alpha)]_{\partial_{\tilde{\pi}}}].\end{aligned}$$

Note that $\partial_{\tilde{\pi}}h^i$ is $\bar{\partial}$ -exact and α is $\bar{\partial}$ -closed. We obtain that $[(\partial_{\tilde{\pi}}h^i) \wedge \rho^*(\alpha)] = 0$. Consider $\gamma := [h^i, \rho^*(\alpha)]_{\partial_{\tilde{\pi}}}$. From Proposition 4.2, we know that γ is d -exact. Notice that both h and α are ∂ -closed. This implies $\partial\gamma = [\partial h^i, \rho^*(\alpha)]_{\partial_{\tilde{\pi}}} + [h^i, \partial\rho^*(\alpha)]_{\partial_{\tilde{\pi}}} = 0$. Furthermore, we get that γ is $\bar{\partial}$ -closed. By the $\partial\bar{\partial}$ -lemma on E , we get $\gamma = \bar{\partial}\eta$ for some η on E . This implies $[\gamma] = 0$ in the Dolbeault cohomology group. Consequently, we are led to the conclusion

$$\partial_{\tilde{\pi}}(\Psi_1([\alpha])) = \sum_{i=0}^{c-1}[h^i \wedge \partial_{\tilde{\pi}}(\rho^*(\alpha))] = \sum_{i=0}^{c-1}[h^i \wedge \rho^*(\partial_{\pi}(\alpha))] = \Psi_1(\partial_{\pi}([\alpha])).$$

The morphism Ψ_1 induces the morphism between the second pages of the spectral sequences:

$$\Psi_2 = H(\Psi_1) : \mathcal{G}_2^{\bullet, \bullet} \cong H(\mathcal{G}_1^{\bullet, \bullet}, d_1) \longrightarrow H(\mathcal{L}_1^{\bullet, \bullet}, \bar{d}_1) \cong \mathcal{L}_2^{\bullet, \bullet}.$$

Assume that $\Psi_r : \mathcal{G}_r^{\bullet, \bullet} \longrightarrow \mathcal{L}_r^{\bullet, \bullet}$ commutes with the differentials d_r and \bar{d}_r , where $r \geq 2$. Then we get the induced morphism of the $(r+1)$ -pages:

$$\Psi_{r+1} = H(\Psi_r) : \mathcal{G}_{r+1}^{\bullet, \bullet} \cong H(\mathcal{G}_r^{\bullet, \bullet}, d_r) \longrightarrow H(\mathcal{L}_r^{\bullet, \bullet}, \bar{d}_r) \cong \mathcal{L}_{r+1}^{\bullet, \bullet}.$$

We claim that Ψ_{r+1} commutes with d_{r+1} and \bar{d}_{r+1} . Suppose $\alpha \in A_Z^{p, q}$ represents a class in $\mathcal{G}_{r+1}^{p, q}$; equivalently, α is a cocycle in $\mathcal{G}_j^{p, q}$ for all $1 \leq j \leq r$. Without loss of generality, we assume that α is ∂ -closed. Since α lives to $\mathcal{G}_{r+1}^{p, q}$, by the $\partial\bar{\partial}$ -lemma, it can be extended to a zig-zag of length $r+1$ such that the tail is ∂ -exact. Denote the tail by $\partial\beta$ and then we get $d_{r+1}([\alpha]_{r+1}) = [\partial_{\pi}\partial\beta]_{r+1}$. Observe that the form $\tilde{\alpha} := h^i \wedge \rho^*\alpha$ represents a class in $\mathcal{L}_1^{p, q}$. Using the $\partial\bar{\partial}$ -lemma again, the form $\tilde{\alpha}$ can be extended to a zig-zag of length $(r+1)$ which has the tail $\tilde{\eta} := h^i \wedge \rho^*(\partial\beta) + \partial\tilde{\gamma}$, where $\tilde{\gamma}$ is a form on E . This implies that $\tilde{\alpha}$ lives to $\mathcal{L}_{r+1}^{p, q}$ and hence we get

$$\bar{d}_{r+1}(\Psi_{r+1}([\alpha]_{r+1})) = \bar{d}_{r+1}([\tilde{\alpha}]_{r+1}) = [\partial_{\tilde{\pi}}\tilde{\alpha}]_{r+1}.$$

Via a straightforward computation, we have

$$\begin{aligned}\partial_{\tilde{\pi}}\tilde{\eta} &= (\partial_{\tilde{\pi}}h^i) \wedge \rho^*(\partial\beta) + h^i \wedge \partial_{\tilde{\pi}}\rho^*(\partial\beta) + [h^i, \rho^*(\partial\beta)]_{\partial_{\tilde{\pi}}} + \partial_{\tilde{\pi}}\partial\tilde{\gamma} \\ &= h^i \wedge \rho^*(\partial_{\pi}\partial\beta) - \partial((\partial_{\tilde{\pi}}h^i) \wedge \rho^*\beta) - \partial([h^i, \rho^*\beta]_{\partial_{\tilde{\pi}}}) - \partial\partial_{\tilde{\pi}}\tilde{\gamma} \\ &:= h^i \wedge \rho^*(\partial_{\pi}\partial\beta) + \partial\tilde{\zeta}.\end{aligned}$$

Note that both $\partial_{\tilde{\pi}}\tilde{\eta}$ and $\partial_{\pi}\partial\beta$ are $\bar{\partial}$ -closed, and so is the form $\partial\tilde{\zeta}$. Due to the $\partial\bar{\partial}$ -lemma, we get $\partial\tilde{\zeta} = \partial\bar{\partial}\tilde{\omega}$ for some $\tilde{\omega}$ on E . This implies that $\partial_{\tilde{\pi}}\tilde{\eta}$ and $h^i \wedge \rho^*(\partial_{\pi}\partial\beta)$ represent the same class in \mathcal{L}_1 , i.e., $[\partial_{\tilde{\pi}}\tilde{\eta}]_1 = [h^i \wedge \rho^*(\partial_{\pi}\partial\beta)]_1$ and therefore we get

$$\tilde{d}_{r+1}(\Psi_{r+1}([\alpha]_{r+1})) = [\partial_{\tilde{\pi}}\tilde{\eta}]_{r+1} = [h^i \wedge \rho^*(\partial_{\pi}\partial\beta)]_{r+1} = \Psi_{r+1}(d_{r+1}([\alpha]_{r+1})).$$

Inductively, we obtain a morphism of spectral sequences

$$\Psi_r : (\mathcal{G}_r^{\bullet, \bullet}, d_r) \longrightarrow (\mathcal{L}_r^{\bullet, \bullet}, \bar{d}_r).$$

Thanks to the projective bundle formula for Dolbeault cohomology, we get that Ψ_1 is an isomorphism. By a result on the convergence of spectral sequences [28, Theorem 3.4], for all r , $1 < r \leq \infty$, $\Psi_r : \mathcal{G}_r^{\bullet, \bullet} \rightarrow \mathcal{L}_r^{\bullet, \bullet}$ is an isomorphism, and this completes the proof. \square

Example 4.3. Suppose (X, π_X) and $(\mathbb{P}^n, \pi_{\mathbb{P}^n})$ are two holomorphic Poisson manifolds. Then the product manifold $E = X \times \mathbb{P}^n$ can be thought of as the projectivization of the trivial vector bundle $X \times \mathbb{C}^{n+1} \rightarrow X$. In particular, if we view E as the product of Poisson manifolds X and \mathbb{P}^n , then the associated product Poisson structure $\tilde{\pi}_E$ on E satisfying that the two projections $\rho_1 : E \rightarrow X$ and $\rho_2 : E \rightarrow \mathbb{P}^n$ are Poisson maps with involution property (cf. [26, Proposition 2.5]). Notice that the first Chern class $[h_E]$ of the tautological line bundle over E is the pullback of the first Chern class $[h_{\mathbb{P}^n}]$ of the tautological line bundle over \mathbb{P}^n via ρ_2^* , hence by the involution property of $\tilde{\pi}_E$, for any $\alpha \in \mathcal{A}_X^{\bullet, \bullet}$,

$$[h_E^i, \rho_1^*(\alpha)]_{\partial_{\tilde{\pi}_E}} = [\rho_2^*(h_{\mathbb{P}^n}^i), \rho_1^*(\alpha)]_{\partial_{\tilde{\pi}_E}} = 0.$$

This means in this special case, the map Ψ is a well-defined morphism between the double complexes G and L . With the classical argument, Ψ induces an isomorphism

$$H_k(E, \tilde{\pi}_E) \cong H_{k-n}(X, \pi_X)^{\oplus(n+1)},$$

for any integer $k \geq 0$. Especially, if X is a point, then we have

$$H_k(\mathbb{P}^n, \pi_{\mathbb{P}^n}) = \begin{cases} \mathbb{C}^{n+1}, & k = n, \\ 0, & k \neq n. \end{cases}$$

Assume that $X = \mathbb{P}^1$ and $n = 1$. The moduli space of holomorphic Poisson structures on $E = \mathbb{P}^1 \times \mathbb{P}^1$ is isomorphic to \mathbb{C}^9 (cf. [22, Proposition 2.2]). Consider the projective bundle $\rho : E \rightarrow \mathbb{P}^1$. Let π_E be an arbitrary holomorphic Poisson structure on E . The holomorphic Koszul–Brylinski homology of (E, π_E) has been computed by Stiénon [37, Theorem 7.2], and here we present a new proof of his result by applying the projective bundle formula (Theorem 4.1). For the dimension reason, each holomorphic bivector field on \mathbb{P}^1 is zero. Thus we get $\rho_*(\pi_E) = 0$. As a corollary, we obtain

$$H_k(E, \pi_E) \cong H_{k-1}(\mathbb{P}^1, \pi = 0)^{\oplus 2} \cong \left[\bigoplus_{p-q=2-k} H^q(\mathbb{P}^1, \Omega_{\mathbb{P}^1}^p) \right]^{\oplus 2}.$$

A straightforward computation shows $H_k(E, \pi_E) = 0$ when $k = 0, 1, 3, 4$. When $k = 2$, we have

$$H_2(E, \pi_E) \cong H_1(\mathbb{P}^1, \pi = 0)^{\oplus 2} \cong \mathbb{C}^4,$$

since $H_1(\mathbb{P}^1, \pi = 0) \cong H_{\bar{\partial}}^{0,0}(\mathbb{P}^1) \oplus H_{\bar{\partial}}^{1,1}(\mathbb{P}^1) \cong \mathbb{C}^2$.

5. COMPARISON UNDER POISSON BLOW-UPS

The main purpose of this section is to prove the blow-up formula for holomorphic Koszul–Brylinski homology of compact holomorphic Poisson manifolds.

5.1. Relative Koszul–Brylinski homology. Let X be a compact complex manifold and $\jmath: Z \hookrightarrow X$ a closed complex submanifold of codimension c . Consider the natural morphism

$$\jmath^*: \Omega_X^s \longrightarrow \jmath_* \Omega_Z^s$$

which is defined as follows:

$$\begin{aligned} \jmath^*(V) : \Gamma(V, \Omega_X^s) &\longrightarrow \Gamma(V, \jmath_* \Omega_Z^s) \\ \alpha &\longmapsto (\jmath_{V \cap Z})^* \alpha, \end{aligned}$$

where $V \subset X$ is an open subset and $\jmath_{V \cap Z}: V \cap Z \hookrightarrow V$ is the holomorphic inclusion. We also define the sheaf morphism

$$\jmath^*: \mathcal{A}_X^{p,q} \longrightarrow \jmath_* \mathcal{A}_Z^{p,q}.$$

in a similar way.

Definition 5.1 ([34, 39]). The kernel sheaves

$$\mathcal{K}_{X,Z}^s := \ker(\jmath^*: \Omega_X^s \rightarrow \jmath_* \Omega_Z^s)$$

and

$$\mathcal{K}_{X,Z}^{p,q} := \ker(\jmath^*: \mathcal{A}_X^{p,q} \rightarrow \jmath_* \mathcal{A}_Z^{p,q})$$

are called the s -th relative Dolbeault sheaf and (p, q) -th relative Dolbeault sheaf with respect to Z .

There exist two natural short exact sequences

$$0 \longrightarrow \mathcal{K}_{X,Z}^s \longrightarrow \Omega_X^s \xrightarrow{\jmath^*} \jmath_* \Omega_Z^s \longrightarrow 0 \tag{5.1}$$

and

$$0 \longrightarrow \mathcal{K}_{X,Z}^{p,q} \longrightarrow \mathcal{A}_X^{p,q} \xrightarrow{\jmath^*} \jmath_* \mathcal{A}_Z^{p,q} \longrightarrow 0, \tag{5.2}$$

where $\mathcal{K}_{X,Z}^{p,\bullet}$ is a fine resolution of $\mathcal{K}_{X,Z}^p$. Consider the holomorphic Poisson manifold (X, π) together with the holomorphic Poisson submanifold $(Z, \pi|_Z)$. Then we have

Lemma 5.2. *There exists a short exact sequence of sheaf complexes on X :*

$$0 \longrightarrow (\mathcal{K}_{X,Z}^\bullet, \partial_\pi) \longrightarrow (\Omega_X^\bullet, \partial_\pi) \xrightarrow{\jmath^*} \jmath_*(\Omega_Z^\bullet, \partial_{\pi|_Z})[-c] \longrightarrow 0, \tag{5.3}$$

where $c = \text{codim}_{\mathbb{C}} Z$.

Proof. Since $\jmath^* \circ \iota_\pi = \iota_{\pi|_Z} \circ \jmath^*$ and $\jmath^* \circ \partial = \partial \circ \jmath^*$, we have $\jmath^* \circ \partial_\pi = \partial_{\pi|_Z} \circ \jmath^*$. Hence, for any $k \in \mathbb{Z}$, there is a well-defined induced operator $\partial_\pi: \mathcal{K}_{X,Z}^k \rightarrow \mathcal{K}_{X,Z}^{k-1}$, i.e., $(\mathcal{K}_{X,Z}^\bullet, \partial_\pi)$ is a well-defined sheaf complex. Moreover, by the short exact sequence (5.1), there is a commutative diagram of short exact sequences of sheaves

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{K}_{X,Z}^k & \longrightarrow & \Omega_X^k & \xrightarrow{\jmath^*} & \jmath_* \Omega_Z^k & \longrightarrow 0 \\ & & \partial_\pi \downarrow & & \partial_\pi \downarrow & & \partial_{\pi|_Z} \downarrow & \\ 0 & \longrightarrow & \mathcal{K}_{X,Z}^{k-1} & \longrightarrow & \Omega_X^{k-1} & \xrightarrow{\jmath^*} & \jmath_* \Omega_Z^{k-1} & \longrightarrow 0. \end{array}$$

The lemma follows immediately from the above commutative diagram. \square

Definition 5.3. The sheaf complex $(\mathcal{K}_{X,Z}^{\bullet}, \partial_{\pi})$ is called the *relative Koszul–Brylinski complex* of (X, π) with respect to $(Z, \pi|_Z)$, and its k -th hypercohomology

$$H_k(X, Z; \pi) := \mathbb{H}^k(X, (\mathcal{K}_{X,Z}^{\bullet}, \partial_{\pi}))$$

is called the k -th *relative Koszul–Brylinski homology* of (X, π) with respect to $(Z, \pi|_Z)$.

Taking the hypercohomology of the short exact sequence (5.3), we get a long exact sequence:

$$\cdots \rightarrow H_k(X, Z; \pi) \rightarrow H_k(X, \pi) \rightarrow H_{k-c}(Z, \pi|_Z) \rightarrow H_{k+1}(X, Z; \pi) \rightarrow \cdots$$

Similarly to Proposition 2.5, we have the following:

Proposition 5.4. *The relative Koszul–Brylinski complex $(\mathcal{K}_{X,Z}^{\bullet}, \partial_{\pi})$ is quasi-isomorphic to the total sheaf complex of the double sheaf complex $(\mathcal{K}_{X,Z}^{\bullet, \bullet}, \partial_{\pi}, \bar{\partial})$.*

Proof. As we mentioned above, for each $p \in \mathbb{Z}$, the relative Dolbeault sheaf $\mathcal{K}_{X,Z}^p$ admits a fine resolution $\mathcal{K}_{X,Z}^{p, \bullet}$. Then the proposition follows. \square

5.2. Proof of Theorem 1.1. Given a compact holomorphic Poisson manifold (X, π) with a holomorphic Poisson closed submanifold $\jmath : (Z, \pi|_Z) \hookrightarrow (X, \pi)$ with complex codimension $c \geq 2$. Let $\varphi : \tilde{X} \rightarrow X$ be the blow-up of X along Z with exceptional divisor $E := \varphi^{-1}(Z)$. Suppose that the transverse Poisson structure $\pi_N = 0$. Due to Proposition 3.3, we get a commutative diagram for the blow-up in the holomorphic Poisson category:

$$\begin{array}{ccc} (E, \tilde{\pi}|_E) & \xhookrightarrow{\jmath} & (\tilde{X}, \tilde{\pi}) \\ \rho = \varphi|_E \downarrow & & \downarrow \varphi \\ (Z, \pi|_Z) & \xhookrightarrow{\jmath} & (X, \pi). \end{array} \quad (5.4)$$

The following lemma plays a crucial role in the proof Theorem 1.1.

Lemma 5.5. *The pullback φ^* naturally induces an isomorphism*

$$\varphi^* : H_k(X, Z; \pi) \xrightarrow{\sim} H_k(\tilde{X}, E; \tilde{\pi}).$$

for any $k \in \mathbb{Z}$.

Proof. Note that there exists a natural morphism of bounded double complexes

$$\varphi^* : (\Gamma(X, \mathcal{K}_{X,Z}^{\bullet, \bullet}), \partial_{\pi}, \bar{\partial}) \longrightarrow (\Gamma(\tilde{X}, \mathcal{K}_{\tilde{X}, E}^{\bullet, \bullet}), \partial_{\tilde{\pi}}, \bar{\partial}).$$

Furthermore, we have two spectral sequences:

- $\{E_r, d_r\}$, associated to $(\Gamma(X, \mathcal{K}_{X,Z}^{\bullet, \bullet}), \partial_{\pi}, \bar{\partial})$, converges to the relative Koszul–Brylinski homology $H_{\bullet}(X, Z; \pi)$ with the E_1 -page given by

$$E_1^{p,q} = H^q(X, \mathcal{K}_{X,Z}^{n-p});$$

- $\{\tilde{E}_r, \tilde{d}_r\}$, associated to $(\Gamma(\tilde{X}, \mathcal{K}_{\tilde{X}, E}^{\bullet, \bullet}), \partial_{\tilde{\pi}}, \bar{\partial})$, converges to the relative Koszul–Brylinski homology $H_k(\tilde{X}, E; \tilde{\pi})$ with the \tilde{E}_1 -page given by

$$\tilde{E}_1^{p,q} = H^q(\tilde{X}, \mathcal{K}_{\tilde{X}, E}^{n-p}).$$

For any $r \geq 1$, the morphism φ^* induces a morphism of the spectral sequences

$$\varphi_r^* : E_r \longrightarrow \tilde{E}_r$$

and hence a morphism of relative Koszul–Brylinski homologies

$$\varphi^* : H_k(X, Z; \pi) \longrightarrow H_k(\tilde{X}, E; \tilde{\pi}).$$

Due to [34, Lemma 4.5], the pullback of differential forms induces an isomorphism

$$\varphi^* : H^q(X, \mathcal{K}_{X, Z}^{n-p}) \longrightarrow H^q(\tilde{X}, \mathcal{K}_{\tilde{X}, E}^{n-p}),$$

for any $0 \leq p, q \leq n$. It follows that $\varphi_1^* : E_1 \rightarrow \tilde{E}_1$ is an isomorphism. Consequently, by the standard result in the spectral sequence theory, we get that φ_r^* is isomorphic for any $r > 1$ and therefore the assertion holds. \square

Now we are in the position to prove the blow-up formula of holomorphic Koszul–Brylinski homology.

Proof of Theorem 1.1. For the pairs of compact holomorphic Poisson manifolds (X, Z) and (\tilde{X}, E) , we have two short exact sequences of sheaf complexes

$$0 \longrightarrow (\mathcal{K}_{X, Z}^\bullet, \partial_\pi) \longrightarrow (\Omega_X^\bullet, \partial_\pi) \xrightarrow{j^*} j_*(\Omega_Z^\bullet, \partial_{\pi|_Z})[-c] \longrightarrow 0 \quad (5.5)$$

and

$$0 \longrightarrow (\mathcal{K}_{\tilde{X}, E}^\bullet, \partial_{\tilde{\pi}}) \longrightarrow (\Omega_{\tilde{X}}^\bullet, \partial_{\tilde{\pi}}) \xrightarrow{\tilde{j}^*} \tilde{j}_*(\Omega_E^\bullet, \partial_{\tilde{\pi}|_E})[-1] \longrightarrow 0. \quad (5.6)$$

We next establish a commutative diagram of long exact sequences of holomorphic Koszul–Brylinski homology associated to (5.5) and (5.6). Observe that each complex in (5.5) and (5.6) admits a natural fine resolution. For the pair (X, Z) , by the short exact sequence (5.2), we have the two commutative diagrams of short exact sequences of fine sheaves:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{K}_{X, Z}^{p, q} & \longrightarrow & \mathcal{A}_X^{p, q} & \xrightarrow{j^*} & j_* \mathcal{A}_Z^{p, q} \longrightarrow 0 \\ & & \bar{\partial} \downarrow & & \bar{\partial} \downarrow & & \bar{\partial} \downarrow \\ 0 & \longrightarrow & \mathcal{K}_{X, Z}^{p, q+1} & \longrightarrow & \mathcal{A}_X^{p, q+1} & \xrightarrow{j^*} & j_* \mathcal{A}_Z^{p, q+1} \longrightarrow 0 \end{array}$$

and

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{K}_{X, Z}^{p, q} & \longrightarrow & \mathcal{A}_X^{p, q} & \xrightarrow{j^*} & j_* \mathcal{A}_Z^{p, q} \longrightarrow 0 \\ & & \partial_\pi \downarrow & & \partial_\pi \downarrow & & \partial_{\pi|_Z} \downarrow \\ 0 & \longrightarrow & \mathcal{K}_{X, Z}^{p-1, q} & \longrightarrow & \mathcal{A}_X^{p-1, q} & \xrightarrow{j^*} & j_* \mathcal{A}_Z^{p-1, q} \longrightarrow 0. \end{array}$$

As a consequence, there is a short exact sequence of *double complexes*:

$$0 \longrightarrow (\Gamma(X, \mathcal{K}_{X, Z}^{\bullet, \bullet}), \partial_\pi, \bar{\partial}) \longrightarrow (\Gamma(X, \mathcal{A}_X^{\bullet, \bullet}), \partial_\pi, \bar{\partial}) \xrightarrow{j^*} (\Gamma(Z, \mathcal{A}_Z^{\bullet, \bullet})[-c, 0], \partial_{\pi|_E}, \bar{\partial}) \longrightarrow 0.$$

Similarly, for the pair (\tilde{X}, E) we have a short exact sequence:

$$0 \longrightarrow (\Gamma(\tilde{X}, \mathcal{K}_{\tilde{X}, E}^{\bullet, \bullet}), \partial_{\tilde{\pi}}, \bar{\partial}) \longrightarrow (\Gamma(\tilde{X}, \mathcal{A}_{\tilde{X}}^{\bullet, \bullet}), \partial_{\tilde{\pi}}, \bar{\partial}) \xrightarrow{\tilde{j}^*} (\Gamma(E, \mathcal{A}_E^{\bullet, \bullet})[-1, 0], \partial_{\tilde{\pi}|_E}, \bar{\partial}) \longrightarrow 0.$$

Moreover, the blow-up morphism φ naturally induces a commutative diagram of short exact sequences:

$$\begin{array}{ccccccc}
 0 & \longrightarrow & \Gamma(X, \mathcal{K}_{X,Z}^{p,q}) & \longrightarrow & \Gamma(X, \mathcal{A}_X^{p,q}) & \xrightarrow{\mathcal{J}^*} & \Gamma(Z, \mathcal{A}_Z^{p,q}) \longrightarrow 0 \\
 & & \varphi^* \downarrow \quad \partial_\pi \searrow & & \varphi^* \downarrow \quad \partial_\pi \searrow & & \rho^* \downarrow \quad \partial_\pi \searrow \\
 0 & \longrightarrow & \Gamma(X, \mathcal{K}_{X,Z}^{p-1,q}) & \longrightarrow & \Gamma(X, \mathcal{A}_X^{p-1,q}) & \xrightarrow{\mathcal{J}^*} & \Gamma(Z, \mathcal{A}_Z^{p-1,q}) \longrightarrow 0 \\
 & & \varphi^* \downarrow \quad \partial_\pi \searrow & & \varphi^* \downarrow \quad \partial_\pi \searrow & & \rho^* \downarrow \quad \partial_\pi \searrow \\
 0 & \longrightarrow & \Gamma(\tilde{X}, \mathcal{K}_{\tilde{X},E}^{p,q}) & \dashrightarrow & \Gamma(\tilde{X}, \mathcal{A}_{\tilde{X}}^{p,q}) & \xrightarrow{\tilde{\mathcal{J}}^*} & \Gamma(E, \mathcal{A}_E^{p,q}) \longrightarrow 0 \\
 & & \partial_{\tilde{\pi}} \searrow & & \partial_{\tilde{\pi}} \searrow & & \partial_{\tilde{\pi}} \searrow \\
 0 & \longrightarrow & \Gamma(\tilde{X}, \mathcal{K}_{\tilde{X},E}^{p-1,q}) & \longrightarrow & \Gamma(\tilde{X}, \mathcal{A}_{\tilde{X}}^{p-1,q}) & \xrightarrow{\tilde{\mathcal{J}}^*} & \Gamma(E, \mathcal{A}_E^{p-1,q}) \longrightarrow 0
 \end{array}$$

Therefore, we obtain a commutative diagram of double complexes

$$\begin{aligned}
 0 \rightarrow (\Gamma(X, \mathcal{K}_{X,Z}^{\bullet,\bullet}), \partial_\pi, \bar{\partial}) &\rightarrow (\Gamma(X, \mathcal{A}_X^{\bullet,\bullet}), \partial_\pi, \bar{\partial}) \xrightarrow{\mathcal{J}^*} (\Gamma(Z, \mathcal{A}_Z^{\bullet,\bullet})[-c, 0], \partial_{\pi|_E}, \bar{\partial}) \rightarrow 0 \\
 &\varphi^* \downarrow \quad \varphi^* \downarrow \quad \rho^* \downarrow \\
 0 \rightarrow (\Gamma(\tilde{X}, \mathcal{K}_{\tilde{X},E}^{\bullet,\bullet}), \partial_{\tilde{\pi}}, \bar{\partial}) &\rightarrow (\Gamma(\tilde{X}, \mathcal{A}_{\tilde{X}}^{\bullet,\bullet}), \partial_{\tilde{\pi}}, \bar{\partial}) \xrightarrow{\tilde{\mathcal{J}}^*} (\Gamma(E, \mathcal{A}_E^{\bullet,\bullet})[-1, 0], \partial_{\tilde{\pi}|_E}, \bar{\partial}) \rightarrow 0.
 \end{aligned} \tag{5.7}$$

The commutative diagram (5.7) above yields a commutative diagram of long exact sequences of Koszul–Brylinski homologies:

$$\begin{aligned}
 \cdots &\longrightarrow H_k(X, Z; \pi) \longrightarrow H_k(X, \pi) \longrightarrow H_{k-c}(Z, \pi|_Z) \longrightarrow H_{k+1}(X, Z; \pi) \longrightarrow \cdots \\
 &\varphi^* \downarrow \quad \varphi^* \downarrow \quad \rho^* \downarrow \quad \varphi^* \downarrow \\
 \cdots &\longrightarrow H_k(\tilde{X}, E; \tilde{\pi}) \longrightarrow H_k(\tilde{X}, \tilde{\pi}) \longrightarrow H_{k-1}(E, \tilde{\pi}|_E) \longrightarrow H_{k+1}(\tilde{X}, E; \tilde{\pi}) \longrightarrow \cdots.
 \end{aligned} \tag{5.8}$$

In the above diagram, by Lemma 5.5, for any $l \in \mathbb{Z}$, the morphism

$$\varphi^* : H_l(X, Z; \pi) \xrightarrow{\cong} H_l(\tilde{X}, E; \tilde{\pi})$$

is an isomorphism. Moreover, by Theorem 3.6, both the second and third vertical arrows in (5.8) are injective. Finally, by a standard diagram-chasing in (5.8), we get the following isomorphisms of finite dimensional \mathbb{C} -vector spaces:

$$H_k(\tilde{X}, \tilde{\pi}) \cong H_k(X, \pi) \oplus \left(H_{k-1}(E, \tilde{\pi}|_E) / \rho^* H_{k-c}(Z, \pi|_Z) \right)$$

Furthermore, if Z satisfies the $\partial\bar{\partial}$ -lemma, then by Theorem 4.1,

$$H_k(\tilde{X}, \tilde{\pi}) \cong H_k(X, \pi) \oplus H_{k-c}(Z, \pi|_Z)^{\oplus c-1}.$$

This completes the proof of Theorem 1.1. \square

5.3. Degeneracy of the Dolbeault–Koszul–Brylinski spectral sequence. Let (X, π) be a holomorphic Poisson manifold of complex dimension n . Consider the Koszul–Brylinski double complex $(\Gamma(X, \mathcal{A}_X^{\bullet,\bullet}), \partial_\pi, \bar{\partial})$. Inspired by the Fröhlicher (or Hodge–de Rham) spectral sequence of complex manifolds, we introduce the following:

Definition 5.6. The Fröhlicher-type spectral sequence associated to the double complex $(\Gamma(X, \mathcal{A}_X^{\bullet, \bullet}), \partial_\pi, \bar{\partial})$ satisfying

$$E_1^{s,t} := H_{\bar{\partial}}^{n-s,t}(X) \implies H_{n-s+t}(X, \pi), \quad (5.9)$$

is called the *Dolbeault–Koszul–Brylinski spectral sequence of (X, π)* .

As mentioned before, for a compact complex manifold X with the trivial holomorphic Poisson structure π , the Dolbeault–Koszul–Brylinski spectral sequence degenerates at E_1 -page and we have

$$H_k(X, \pi = 0) \cong \bigoplus_{p-q=n-k} H^q(X, \Omega_X^p) \cong \bigoplus_{p-q=n-k} H_{\bar{\partial}}^{p,q}(X).$$

Analogously to the Hodge–de Rham spectral sequence, in general, the Dolbeault–Koszul–Brylinski spectral sequence (5.9) does not degenerate at E_1 -page (see for example in §§6.3.2). We have the following result.

Lemma 5.7. *Let (X, π) be a compact holomorphic Poisson manifold of complex dimension n . Then its Dolbeault–Koszul–Brylinski spectral sequence degenerates at E_1 -page if and only if*

$$\sum_{p-q=n-k} \dim_{\mathbb{C}} H_{\bar{\partial}}^{p,q}(X) = \dim_{\mathbb{C}} H_k(X, \pi),$$

for any $0 \leq k \leq 2n$.

Proof. Observe that for a compact holomorphic Poisson manifold the holomorphic Koszul–Brylinski homology groups are finite-dimensional; moreover, the following inequality holds

$$\dim_{\mathbb{C}} H_k(X, \pi) \leq \sum_{p-q=n-k} \dim_{\mathbb{C}} H_{\bar{\partial}}^{p,q}(X)$$

for any $0 \leq k \leq 2n$. By definition, the E_1 -degeneracy of the Dolbeault–Koszul–Brylinski spectral sequence is equivalent to the condition

$$\dim_{\mathbb{C}} H_k(X, \pi) = \sum_{p-q=n-k} \dim_{\mathbb{C}} H_{\bar{\partial}}^{p,q}(X),$$

for any $0 \leq k \leq 2n$. □

We are ready to give the proof of Theorem 1.2.

Proof of Theorem 1.2. By the blow-up formula for Dolbeault cohomology [33, Theorem 1.2], we have

$$\sum_{p-q=n-k} \dim_{\mathbb{C}} H_{\bar{\partial}}^{p,q}(\tilde{X}) = \sum_{p-q=n-k} \left[\dim_{\mathbb{C}} H_{\bar{\partial}}^{p,q}(X) + \sum_{i=1}^{c-1} \dim_{\mathbb{C}} H_{\bar{\partial}}^{p-i, q-i}(Z) \right].$$

Consequently, by Theorem 1.1, we get

$$\begin{aligned} & \dim_{\mathbb{C}} H_k(\tilde{X}, \tilde{\pi}) - \sum_{p-q=n-k} \dim_{\mathbb{C}} H_{\bar{\partial}}^{p,q}(\tilde{X}) \\ &= \left[\dim_{\mathbb{C}} H_k(X, \pi) - \sum_{p-q=n-k} \dim_{\mathbb{C}} H_{\bar{\partial}}^{p,q}(X) \right] \\ & \quad + (c-1) \dim_{\mathbb{C}} H_{k-c}(Z, \pi|_Z) - \sum_{p-q=n-k} \left[\sum_{i=1}^{c-1} \dim_{\mathbb{C}} H_{\bar{\partial}}^{p-i, q-i}(Z) \right] \end{aligned}$$

$$\begin{aligned}
&= \left[\dim_{\mathbb{C}} H_k(X, \pi) - \sum_{p-q=n-k} \dim_{\mathbb{C}} H_{\bar{\partial}}^{p,q}(X) \right] \\
&\quad + (c-1) \left[\dim_{\mathbb{C}} H_{k-c}(Z, \pi|_Z) - \sum_{s-t=(n-c)-(k-c)} \dim_{\mathbb{C}} H_{\bar{\partial}}^{s,t}(Z) \right]
\end{aligned}$$

for $0 \leq k \leq 2n$. If the Dolbeault–Koszul–Brylinski spectral sequence degenerates at E_1 -pages for (X, π) and (Z, π_Z) , then it immediately follows that the Dolbeault–Koszul–Brylinski spectral sequence degenerates at E_1 -pages for $(\tilde{X}, \tilde{\pi})$. Conversely, if the Dolbeault–Koszul–Brylinski spectral sequence degenerates at E_1 -pages for $(\tilde{X}, \tilde{\pi})$, then we obtain the following equalities

$$\begin{aligned}
0 &= \underbrace{\dim_{\mathbb{C}} H_k(X, \pi) - \sum_{p-q=n-k} \dim_{\mathbb{C}} H_{\bar{\partial}}^{p,q}(X)}_{\leq 0} \\
&\quad + \underbrace{(c-1) \left[\dim_{\mathbb{C}} H_{k-c}(Z, \pi|_Z) - \sum_{s-t=(n-c)-(k-c)} \dim_{\mathbb{C}} H_{\bar{\partial}}^{s,t}(Z) \right]}_{\leq 0},
\end{aligned}$$

which implies that the E_1 -degeneracy holds for (X, π) and $(Z, \pi|_Z)$. \square

6. EXAMPLES

In this section, as applications of the main theorems, we compute the Koszul–Brylinski homology for some special holomorphic Poisson manifolds, such as del Pezzo surfaces and Iwasawa manifolds.

6.1. del Pezzo surfaces. Recall that a *del Pezzo surface* is a smooth Fano surface which is exactly one of the following: $\mathbb{P}^1 \times \mathbb{P}^1$, \mathbb{P}^2 and blow-up of \mathbb{P}^2 at r ($1 \leq r \leq 8$) generic points (denoted by M_r). The holomorphic Koszul–Brylinski homology of $\mathbb{P}^1 \times \mathbb{P}^1$ has been computed in Example 4.3; see also [37, Theorem 7.2]. We now consider the rest cases. Define the space

$$V_r^2 = \{\text{holomorphic bi-vector fields on } \mathbb{P}^2 \text{ vanishing at the blow-up points of } M_r\}.$$

By a result of Kodaira [23, page 225], the blow-up transformation $\varphi : M_r \rightarrow \mathbb{P}^2$ induces an isomorphism from the space of holomorphic bi-vector fields on M_r to the space V_r^2 . Equivalently, the holomorphic Poisson structures π on \mathbb{P}^2 vanishing at the blow-up points of M_r are one-one corresponding to the holomorphic Poisson structures $\tilde{\pi}$ on M_r such that φ is a Poisson morphism.

In general, given a holomorphic Poisson structure π on \mathbb{P}^n , the E_1 -page of the Dolbeault–Koszul–Brylinski spectral sequence of (\mathbb{P}^n, π) is

$$E_1^{s,t} = H^t(\mathbb{P}^n, \Omega^{n-s}) = \begin{cases} \mathbb{C}, & s+t=n, \\ 0, & \text{otherwise.} \end{cases}$$

Via a direct checking we get $d_r \equiv 0$ for any $r \geq 1$. This implies that the Dolbeault–Koszul–Brylinski spectral sequence of (\mathbb{P}^n, π) degenerates at E_1 -page, and therefore we obtain

$$H_k(\mathbb{P}^n, \pi) = \begin{cases} \mathbb{C}^{n+1}, & k=n, \\ 0, & k \neq n. \end{cases}$$

Consider the Poisson blow-up $\varphi : (M_r, \tilde{\pi}) \rightarrow (\mathbb{P}^2, \pi)$. From the blow-up formula in Theorem 1.1, we get

$$H_k(M_r, \tilde{\pi}) = \begin{cases} \mathbb{C}^{r+3}, & k = 2, \\ 0, & k \neq 2. \end{cases}$$

6.2. Iwasawa manifolds. To begin with, let us recall some basic facts on complex nilmanifolds. Let G be a complex nilpotent Lie group with Lie algebra \mathfrak{g} whose complexification is $\mathfrak{g}_{\mathbb{C}} := \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$, and let H be a discrete subgroup of G . Suppose $M = G/H$ is the associated nilmanifold endowed with a left-invariant complex structure J and a left-invariant holomorphic Poisson bi-vector field π . Then there exists a natural inclusion of complexes

$$i : (\wedge^{p,\bullet} \mathfrak{g}_{\mathbb{C}}^*, \bar{\partial}) \hookrightarrow (\Gamma(M, \mathcal{A}_M^{p,\bullet}), \bar{\partial}), \quad (6.1)$$

for any $p \geq 0$. Set $n := \dim_{\mathbb{C}} M$.

Lemma 6.1. *If the map (6.1) is a quasi-isomorphism, then the total cohomology of the double complex $(\wedge^{\bullet,\bullet} \mathfrak{g}_{\mathbb{C}}^*, \partial_{\pi}, \bar{\partial})$ is isomorphic to $H_{\bullet}(M, \pi)$.*

Proof. Observe that (6.1) induces a morphism of double complexes

$$i : (\wedge^{\bullet,\bullet} \mathfrak{g}_{\mathbb{C}}^*, \partial_{\pi}, \bar{\partial}) \longrightarrow (\Gamma(M, \mathcal{A}_M^{\bullet,\bullet}), \partial_{\pi}, \bar{\partial}). \quad (6.2)$$

On the one hand, we know that $(\wedge^{\bullet,\bullet} \mathfrak{g}_{\mathbb{C}}^*, \partial_{\pi}, \bar{\partial})$ admits a spectral sequence $\{\tilde{E}_r, \tilde{d}_r\}$ converging to the corresponding total cohomology such that the \tilde{E}_1 -page states as

$$\tilde{E}_1^{p,q} = H^q(\wedge^{n-p,\bullet} \mathfrak{g}_{\mathbb{C}}^*, \bar{\partial}).$$

On the other hand, the Dolbeault–Koszul–Brylinski spectral sequence $\{E_r, d_r\}$ converges to the holomorphic Koszul–Brylinski homology $H_{\bullet}(M, \pi)$ and has the \tilde{E}_1 -page

$$E_1^{p,q} = H^q(M, \Omega_M^{n-p}).$$

For any $r \geq 1$, the inclusion (6.2) induces a morphism of the spectral sequences

$$i_r^* : \tilde{E}_r \longrightarrow E_r.$$

Since (6.1) is a quasi-isomorphism, i.e., $i_1^* : \tilde{E}_1 \rightarrow E_1$ is an isomorphism, by the standard result in the spectral sequence theory, i_r^* is an isomorphism for any $r \geq 2$. This implies that the total cohomology of double complex $(\wedge^{\bullet,\bullet} \mathfrak{g}_{\mathbb{C}}^*, \partial_{\pi}, \bar{\partial})$ is isomorphic to the holomorphic Koszul–Brylinski homology $H_{\bullet}(M, \pi)$. \square

Remark 6.2. A result of Sakane [35, Theorem 1] states that if a complex nilmanifold is complex parallelisable (i.e., the holomorphic tangent bundle is holomorphically trivial), then the inclusion (6.1) is a quasi-isomorphism.

Next we consider a concrete example. Let $H(3; \mathbb{C})$ be the Heisenberg Lie group:

$$H(3; \mathbb{C}) = \left\{ \begin{pmatrix} 1 & z_1 & z_2 \\ 0 & 1 & z_3 \\ 0 & 0 & 1 \end{pmatrix} \middle| z_1, z_2, z_3 \in \mathbb{C} \right\} \subset \mathrm{GL}(3; \mathbb{C}).$$

As a complex manifold, $H(3; \mathbb{C})$ is isomorphic to \mathbb{C}^3 . Consider the discrete group $G_3 := \mathrm{GL}(3; \mathbb{Z}[\sqrt{-1}]) \cap H(3; \mathbb{C})$, where $\mathbb{Z}[\sqrt{-1}] = \{a + b\sqrt{-1} \mid a, b \in \mathbb{Z}\}$ is the Gaussian integers. The

left multiplication gives rise to a natural G_3 -action on $H(3; \mathbb{C})$, and the corresponding faithful G_3 -action on \mathbb{C}^3 is given by

$$(a_1, a_2, a_3) \cdot (z_1, z_2, z_3) := (z_1 + a_1, z_2 + a_1 z_3 + a_2, z_3 + a_3),$$

where $a_1, a_2, a_3 \in \mathbb{Z}[\sqrt{-1}]$. Such a G_3 -action yields a monomorphism $f : G_3 \rightarrow \text{Aff}(\mathbb{C}^3)$. Here $\text{Aff}(\mathbb{C}^3)$ is the affine transformation group of \mathbb{C}^3 . Therefore, such a G_3 -action is properly discontinuous. Furthermore, the G_3 -quotient space

$$\mathbb{I}_3 := \mathbb{C}^3/G_3$$

is a compact complex Calabi–Yau threefold, called the *Iwasawa manifold*, which is non-Kähler, non-formal, and complex parallelisable.

Denote by $(\mathfrak{g}_\mathbb{C}^*)^{1,0}$ the space of left-invariant holomorphic differential forms on $H(3; \mathbb{C})$. Then $(\mathfrak{g}_\mathbb{C}^*)^{1,0}$ has a basis:

$$w^1 = dz_1, \quad w^2 = dz_2 - z_1 dz_3, \quad w^3 = dz_3,$$

satisfying the structure equations:

$$\begin{cases} w^1 = 0, \\ dw^3 = 0, \\ dw^2 = -w^1 \wedge w^3. \end{cases}$$

The dual basis of Lie algebra of left-invariant holomorphic vector fields on $H(3; \mathbb{C})$, denoted by $\mathfrak{g}_\mathbb{C}^{1,0}$, is

$$X_1 = \frac{\partial}{\partial z_1}, \quad X_2 = \frac{\partial}{\partial z_2}, \quad X_3 = \frac{\partial}{\partial z_3} + z_1 \frac{\partial}{\partial z_2}$$

with the structure equations $[X_1, X_2] = [X_2, X_3] = 0, [X_1, X_3] = X_2$.

Note that each left-invariant holomorphic bi-vector field π on \mathbb{I}_3 is of the form $\pi = c_1 X_1 \wedge X_2 + c_2 X_1 \wedge X_3 + c_3 X_2 \wedge X_3$, where c_1, c_2 and c_3 are constants. In particular, a direct checking shows that $[\pi, \pi] = 0$ holds if and only if $c_2 = 0$. Since π is left-invariant and \mathbb{I}_3 is complex parallelisable, by Lemma 6.1, the holomorphic Koszul–Brylinski homology of (\mathbb{I}_3, π) can be computed in terms of the total cohomology of the double complex $(\wedge^{\bullet, \bullet} \mathfrak{g}_\mathbb{C}^*, \partial_\pi, \bar{\partial})$. Observe that π is the linear combination of two compatible Poisson bi-vector fields $\pi_{12} = X_1 \wedge X_2$ and $\pi_{23} = X_2 \wedge X_3$. Since $\partial_{\pi_{12}} = \partial_{\pi_{23}} = 0$ we get $\partial_\pi = 0$. It follows that the Dolbeault–Koszul–Brylinski spectral sequence of (\mathbb{I}_3, π) degenerates at E_1 -page and therefore the Koszul–Brylinski homology $H_\bullet(\mathbb{I}_3, \pi)$ can be read off from the Hodge diamond of \mathbb{I}_3 (see figure below).

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & & 3 & 2 & & \\ & & 3 & 6 & 2 & & \\ 1 & 6 & 6 & & 1 & & \\ 2 & 6 & 3 & & & & \\ 2 & & 3 & & & & \\ & & & 1 & & & \end{array}$$

(Hodge diamond of \mathbb{I}_3)

As a result, we have the following table which records the holomorphic Koszul–Brylinski homology of (\mathbb{I}_3, π) .

| k | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|--------------------------|--------------|----------------|-------------------|-------------------|-------------------|----------------|--------------|
| $H_k(\mathbb{I}_3, \pi)$ | \mathbb{C} | \mathbb{C}^5 | \mathbb{C}^{11} | \mathbb{C}^{14} | \mathbb{C}^{11} | \mathbb{C}^5 | \mathbb{C} |

Note that (\mathbb{I}_3, π) is a unimodular holomorphic Poisson manifold. Due to Proposition 2.8 and the isomorphism (2.4), the holomorphic Koszul–Brylinski homology of (\mathbb{I}_3, π) is isomorphic to its Lichnerowicz–Poisson cohomology $H^\bullet(\mathbb{I}_3, \pi)$. So we obtain the following table.

| k | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|--------------------------|--------------|----------------|-------------------|-------------------|-------------------|----------------|--------------|
| $H^k(\mathbb{I}_3, \pi)$ | \mathbb{C} | \mathbb{C}^5 | \mathbb{C}^{11} | \mathbb{C}^{14} | \mathbb{C}^{11} | \mathbb{C}^5 | \mathbb{C} |

6.3. A six-dimensional complex nilmanifold. Motivated by the construction of the Iwasawa manifold, we consider the nilpotent Lie group

$$G = \left\{ A = \begin{pmatrix} 1 & z_1 & z_2 & z_3 \\ 0 & 1 & z_4 & z_5 \\ 0 & 0 & 1 & z_6 \\ 0 & 0 & 0 & 1 \end{pmatrix} \middle| z_1, z_2, \dots, z_6 \in \mathbb{C} \right\} \subset \mathrm{GL}(4; \mathbb{C}).$$

As a complex manifold, G is isomorphic to the complex vector space \mathbb{C}^6 . Consider the discrete subgroup $H := \mathrm{GL}(4; \mathbb{Z}[\sqrt{-1}]) \cap G$. Analogously, the left multiplication defines a natural H -action on G and the corresponding faithful H -action on \mathbb{C}^6 is given by

$$\begin{aligned} (a_1, a_2, a_3, a_4, a_5, a_6) \cdot (z_1, z_2, z_3, z_4, z_5, z_6) \\ = (z_1 + a_1, z_2 + a_1 z_4 + a_2, z_3 + a_1 z_5 + a_2 z_6 + a_3, z_4 + a_4, z_5 + a_4 z_6 + a_5, z_6 + a_6). \end{aligned}$$

Therefore, this H -action is properly discontinuous, and the associated H -quotient space

$$\mathbb{I}_6 := \mathbb{C}^6 / H$$

is a compact complex manifold with complex dimension 6. Let $(\mathfrak{g}_\mathbb{C}^*)^{1,0}$ be the space of left-invariant holomorphic differential forms on G . Then a basis of $(\mathfrak{g}_\mathbb{C}^*)^{1,0}$ is given by

$$\begin{aligned} w_1 &= dz_1, \quad w_2 = dz_2 - z_1 dz_4, \quad w_3 = dz_3 - z_1 dz_5 + (z_1 z_4 - z_2) dz_6, \\ w_4 &= dz_4, \quad w_5 = dz_5 - z_4 dz_6, \quad w_6 = dz_6. \end{aligned}$$

The structure equations are

$$\begin{cases} dw_1 = dw_4 = dw_6 = 0, \\ dw_2 = -w_1 \wedge w_4, \\ dw_3 = -w_1 \wedge w_5 - w_2 \wedge w_6, \\ dw_5 = -w_4 \wedge w_6. \end{cases}$$

Dually, Lie algebra of left-invariant holomorphic vector fields of G , denoted by $\mathfrak{g}_\mathbb{C}^{1,0}$, has a basis:

$$\begin{aligned} X_1 &= \frac{\partial}{\partial z_1}, \quad X_2 = \frac{\partial}{\partial z_2}, \quad X_3 = \frac{\partial}{\partial z_3}, \\ X_4 &= \frac{\partial}{\partial z_4} + z_1 \frac{\partial}{\partial z_2}, \quad X_5 = \frac{\partial}{\partial z_5} + z_1 \frac{\partial}{\partial z_3}, \quad X_6 = \frac{\partial}{\partial z_6} + z_2 \frac{\partial}{\partial z_3} + z_4 \frac{\partial}{\partial z_5}. \end{aligned}$$

The only non-trivial relations of the dual basis are

$$[X_1, X_4] = X_2, \quad [X_1, X_5] = X_3 = [X_2, X_6], \quad [X_4, X_6] = X_5.$$

It follows that \mathbb{I}_6 is a complex parallelisable, non-Kähler, Calabi–Yau manifold with dimension 6.

Now we consider some special holomorphic Poisson structures on \mathbb{I}_6 given by left-invariant holomorphic bi-vector fields. Akin to the Iwasawa manifold, the holomorphic Koszul–Brylinski homology of \mathbb{I}_6 can be computed in terms of the total cohomology of the double complex $(\wedge^{\bullet, \bullet} \mathfrak{g}_{\mathbb{C}}^*, \partial_{\pi}, \bar{\partial})$. For the simplicity, we write $w^{i_1 \cdots i_p \bar{j}_1 \cdots \bar{j}_q} = w^{i_1} \wedge \cdots \wedge w^{i_p} \wedge w^{\bar{j}_1} \wedge \cdots \wedge w^{\bar{j}_q}$, for any $1 \leq p, q \leq 6$. We study the holomorphic Koszul–Brylinski homology of \mathbb{I}_6 with respect to the following three holomorphic Poisson bi-vector fields:

$$\pi_1 = X_2 \wedge X_3, \quad \pi_2 = X_1 \wedge X_6, \quad \text{and} \quad \pi_3 = X_1 \wedge X_3.$$

6.3.1. Computation of $H_{\bullet}(\mathbb{I}_6, \pi_1)$. We claim that the Dolbeault–Koszul–Brylinski spectral sequence of (\mathbb{I}_6, π_1) degenerates at E_1 -page. Observe that the only possible elements which are ∂_{π_1} -closed are of the form $w^{23i_1 \cdots i_{p-2} \bar{j}_1 \cdots \bar{j}_q}$. However, a straightforward computation shows

$$\begin{aligned} \partial_{\pi_1} w^{23i_1 \cdots i_{p-2} \bar{j}_1 \cdots \bar{j}_q} &= (\iota_{\pi_1} \circ \partial - \partial \circ \iota_{\pi_1}) w^{23i_1 \cdots i_{p-2} \bar{j}_1 \cdots \bar{j}_q} \\ &= \iota_{\pi_1} (w^{23} \wedge \partial w^{i_1 \cdots i_{p-2} \bar{j}_1 \cdots \bar{j}_q}) - \partial w^{i_1 \cdots i_{p-2} \bar{j}_1 \cdots \bar{j}_q} \\ &= \partial w^{i_1 \cdots i_{p-2} \bar{j}_1 \cdots \bar{j}_q} - \partial w^{i_1 \cdots i_{p-2} \bar{j}_1 \cdots \bar{j}_q} \\ &= 0. \end{aligned}$$

This implies that the holomorphic volume form ω^{123456} is ∂_{π_1} -closed, which means (\mathbb{I}_6, π_1) is unimodular, and the Dolbeault–Koszul–Brylinski spectral sequence of (\mathbb{I}_6, π_1) degenerates at E_1 -page. Consequently, we get

$$H_k(\mathbb{I}_6, \pi_1) = \bigoplus_{6-(p-q)=k} H_{\bar{\partial}}^{p,q}(\mathbb{I}_6). \quad (6.3)$$

From the isomorphism (2.4), we have

$$H_k(\mathbb{I}_6, \pi_1) \cong H_{12-k}(\mathbb{I}_6, \pi_1).$$

From the Hodge diamond of \mathbb{I}_6 (see Appendix A) and Proposition 2.8, we get the following table recording the holomorphic Koszul–Brylinski homology of (\mathbb{I}_6, π_1) up to degree 6 (the rest are obtained by the holomorphic Evens–Lu–Weinstein duality).

| k | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---------------------------------|--------------|----------------|-------------------|--------------------|--------------------|--------------------|--------------------|
| $H_k(\mathbb{I}_6, \pi_1)$ | \mathbb{C} | \mathbb{C}^9 | \mathbb{C}^{38} | \mathbb{C}^{101} | \mathbb{C}^{191} | \mathbb{C}^{274} | \mathbb{C}^{308} |
| $H^{12-k}(\mathbb{I}_6, \pi_1)$ | \mathbb{C} | \mathbb{C}^9 | \mathbb{C}^{38} | \mathbb{C}^{101} | \mathbb{C}^{191} | \mathbb{C}^{274} | \mathbb{C}^{308} |

Remark 6.3. If the Dolbeault–Koszul–Brylinski spectral sequence for a holomorphic Poisson manifold degenerates at the E_1 -page, then we can read off its holomorphic Koszul–Brylinski homology from the Hodge diamond using the same method as in the computation of $H_{\bullet}(\mathbb{I}_6, \pi_1)$. However, the E_1 -degeneracy of the Dolbeault–Koszul–Brylinski spectral sequence is not a necessary condition for a holomorphic Poisson manifold.

6.3.2. *E*₁-non-degeneracy for (\mathbb{I}_6, π_2) . Consider the holomorphic Poisson manifold (\mathbb{I}_6, π_2) . Observe that $(\mathfrak{g}_\mathbb{C}^*)^{6,0} = \langle w^{123456} \rangle$ and $\partial_{\pi_2} \omega^{123456} = 0$; we obtain

$$H_0(\mathbb{I}_6, \pi_2) = \langle [w^{123456}] \rangle \cong \mathbb{C}.$$

On the one hand, note that $(\mathfrak{g}_\mathbb{C}^*)^{5,0} = \langle w^{23456}, w^{13456}, w^{12456}, w^{12356}, w^{12346}, w^{12345} \rangle$, and we have

$$\begin{aligned} \partial_{\pi_2} w^{23456} &= \partial_{\pi_2} w^{12456} = \partial_{\pi_2} w^{12345} = 0, \\ \partial_{\pi_2} w^{13456} &= -w^{2456}, \\ \partial_{\pi_2} w^{12356} &= w^{1345} - w^{2346}, \\ \partial_{\pi_2} w^{12346} &= -w^{1245}. \end{aligned}$$

On the other hand, since

$$(\mathfrak{g}_\mathbb{C}^*)^{6,1} = \langle w^{123456\bar{1}}, w^{123456\bar{2}}, w^{123456\bar{3}}, w^{123456\bar{4}}, w^{123456\bar{5}}, w^{123456\bar{6}} \rangle,$$

the following equalities hold:

$$\partial_{\pi_2}|(\mathfrak{g}_\mathbb{C}^*)^{6,1} = 0 \text{ and } \ker \bar{\partial} \cap (\mathfrak{g}_\mathbb{C}^*)^{6,1} = \langle w^{123456\bar{1}}, w^{123456\bar{4}}, w^{123456\bar{6}} \rangle.$$

Consequently, we get

$$H_1(\mathbb{I}_6, \pi_2) = \langle [w^{23456}], [w^{12456}], [w^{12345}], [w^{123456\bar{1}}], [w^{123456\bar{4}}], [w^{123456\bar{6}}] \rangle \cong \mathbb{C}^6.$$

Assuming that the Dolbeault–Koszul–Brylinski spectral sequence of (\mathbb{I}_6, π_2) degenerates at the E_1 page, we get

$$H_1(\mathbb{I}_6, \pi_2) = H_{\bar{\partial}}^{5,0}(\mathbb{I}_6) \oplus H_{\bar{\partial}}^{6,1}(\mathbb{I}_6). \quad (6.4)$$

Notice that $H_{\bar{\partial}}^{5,0}(\mathbb{I}_6) \cong \mathbb{C}^6$ and $H_{\bar{\partial}}^{6,1}(\mathbb{I}_6) \cong \mathbb{C}^3$ (see Appendix A). This leads to a contradiction to the equality (6.4), and therefore the Dolbeault–Koszul–Brylinski spectral sequence of (\mathbb{I}_6, π_2) does not degenerate at the E_1 -page.

6.3.3. *Computation of $H_\bullet(\mathbb{I}_6, \pi_3)$.* A direct computation shows that the non-trivial ∂_{π_3} -closed monomials are given by:

- (1) On $(\mathfrak{g}_\mathbb{C}^*)^{5,q}$, $\partial_{\pi_3} w^{12356\bar{j}_1 \cdots \bar{j}_q} = -w^{1456\bar{j}_1 \cdots \bar{j}_q}$;
- (2) On $(\mathfrak{g}_\mathbb{C}^*)^{4,q}$,

$$\begin{aligned} \partial_{\pi_3} w^{1235\bar{j}_1 \cdots \bar{j}_q} &= -w^{145\bar{j}_1 \cdots \bar{j}_q}, \\ \partial_{\pi_3} w^{1236\bar{j}_1 \cdots \bar{j}_q} &= -w^{146\bar{j}_1 \cdots \bar{j}_q}, \\ \partial_{\pi_3} w^{2356\bar{j}_1 \cdots \bar{j}_q} &= w^{456\bar{j}_1 \cdots \bar{j}_q}; \end{aligned}$$

- (3) On $(\mathfrak{g}_\mathbb{C}^*)^{3,q}$,

$$\begin{aligned} \partial_{\pi_3} w^{123\bar{j}_1 \cdots \bar{j}_q} &= -w^{14\bar{j}_1 \cdots \bar{j}_q}, \\ \partial_{\pi_3} w^{235\bar{j}_1 \cdots \bar{j}_q} &= w^{45\bar{j}_1 \cdots \bar{j}_q}, \\ \partial_{\pi_3} w^{236\bar{j}_1 \cdots \bar{j}_q} &= w^{46\bar{j}_1 \cdots \bar{j}_q}; \end{aligned}$$

- (4) On $(\mathfrak{g}_\mathbb{C}^*)^{2,q}$, $\partial_{\pi_3} w^{23\bar{j}_1 \cdots \bar{j}_q} = w^{4\bar{j}_1 \cdots \bar{j}_q}$.

It follows that (\mathbb{I}_6, π_3) is unimodular, and the Dolbeault–Koszul–Brylinski spectral sequence of (\mathbb{I}_6, π_3) does not degenerate at the E_1 -page. By Lemma 6.1 and Proposition 2.8, we have the following table.

| k | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---------------------------------|--------------|----------------|-------------------|-------------------|--------------------|--------------------|--------------------|
| $H_k(\mathbb{I}_6, \pi_3)$ | \mathbb{C} | \mathbb{C}^8 | \mathbb{C}^{31} | \mathbb{C}^{78} | \mathbb{C}^{143} | \mathbb{C}^{202} | \mathbb{C}^{226} |
| $H^{12-k}(\mathbb{I}_6, \pi_3)$ | \mathbb{C} | \mathbb{C}^8 | \mathbb{C}^{31} | \mathbb{C}^{78} | \mathbb{C}^{143} | \mathbb{C}^{202} | \mathbb{C}^{226} |

6.3.4. *Poisson blow-up of (\mathbb{I}_6, π_3) .* Take

$$\Gamma_2 = \left\{ A = \begin{pmatrix} 1 & z_1 & z_2 & z_3 \\ 0 & 1 & a_{23} & a_{24} \\ 0 & 0 & 1 & a_{34} \\ 0 & 0 & 0 & 1 \end{pmatrix} \middle| z_1, z_2, z_3 \in \mathbb{C}, a_{23}, a_{24}, a_{34} \in \mathbb{Z}[\sqrt{-1}] \right\}.$$

Then $Y_2 := \Gamma_2/H$ is a 3-dimensional Kählerian nilmanifold. Furthermore, $(Y_2, \pi_3|_{Y_2} = X_1 \wedge X_3)$ is a closed holomorphic Poisson submanifold of (\mathbb{I}_6, π_3) whose transverse Poisson structure vanishes. One can check that $\partial_{\pi_3|_{Y_2}} = 0$ and thus the Dolbeault–Koszul–Brylinski spectral sequence of $(Y_2, \pi_3|_{Y_2})$ degenerates at the E_1 -page. Note that the Hodge diamond of Y_2 is

$$\begin{array}{ccccccc} & & & & 1 & & \\ & & & & 3 & 3 & \\ & & & & 3 & 9 & 3 \\ & & & & 1 & 9 & 9 & 1 \\ & & & & 3 & 9 & 3 & \\ & & & & 3 & 3 & & \\ & & & & & & 1 & \\ & & & & & & & \end{array}$$

As a corollary, we get the holomorphic Koszul–Brylinski homology of $(Y_2, \pi_3|_{Y_2})$ as follows:

| k | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|--------------------------|--------------|----------------|-------------------|-------------------|-------------------|----------------|----------------|
| $H_k(Y_2, \pi_3 _{Y_2})$ | \mathbb{C} | \mathbb{C}^6 | \mathbb{C}^{15} | \mathbb{C}^{20} | \mathbb{C}^{15} | \mathbb{C}^6 | \mathbb{C}^1 |

Let $\varphi : \text{Bl}_{Y_2}\mathbb{I}_6 \rightarrow \mathbb{I}_6$ be the blow-up of \mathbb{I}_6 along Y_2 . By Proposition 3.3, the holomorphic Poisson structure π_3 can be lifted to a unique holomorphic Poisson structure $\tilde{\pi}_3$ on $\text{Bl}_{Y_2}\mathbb{I}_6$. By Theorem 1.1, we get the following table recording the holomorphic Koszul–Brylinski homology of $(\text{Bl}_{Y_2}\mathbb{I}_6, \tilde{\pi}_3)$.

| k | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|--------------|----------------|-------------------|-------------------|--------------------|--------------------|--------------------|
| $H_k(\text{Bl}_{Y_2}\mathbb{I}_6, \tilde{\pi}_3)$ | \mathbb{C} | \mathbb{C}^8 | \mathbb{C}^{31} | \mathbb{C}^{80} | \mathbb{C}^{155} | \mathbb{C}^{232} | \mathbb{C}^{266} |

APPENDIX A. HODGE DIAMOND OF \mathbb{I}_6

Note that \mathbb{I}_6 is complex parallelisable. As mentioned in the main text, the Dolbeault cohomology of \mathbb{I}_6 can be computed by means of left-invariant forms ([35, Theorem 1]). Consider the associated double complex $(\wedge^{\bullet, \bullet} \mathfrak{g}_\mathbb{C}^*, \partial, \bar{\partial})$. By Leibniz rule, we have

$$\bar{\partial}w^{i_1 \cdots i_p \bar{j}_1 \cdots \bar{j}_q} = (-1)^p w^{i_1 \cdots i_p} \wedge \bar{\partial}w^{\bar{j}_1 \cdots \bar{j}_q}.$$

In particular, we get $h^{i,j} = \binom{6}{i} \cdot h^{0,j}$, where $h^{i,j} := \dim_{\mathbb{C}} H^j(\wedge^{i,0} \mathfrak{g}_\mathbb{C}^*, \bar{\partial})$ is the Lie algebra Hodge number. For this reason, to compute the Hodge diamond of \mathbb{I}_6 , we only need to compute $h^{0,0}, h^{0,1}, \dots, h^{0,6}$. Since \mathbb{I}_6 is a compact complex manifold we have $H_\partial^{0,0}(\mathbb{I}_6) \cong \mathbb{C}$. The monomials in $(\mathfrak{g}_\mathbb{C}^*)^{0,j}$ which are *not* $\bar{\partial}$ -closed are stated as follows:

(1) On $(\mathfrak{g}_{\mathbb{C}}^*)^{0,1}$, since

$$\bar{\partial}w^{\bar{2}} = -w^{\bar{1}\bar{4}}, \quad \bar{\partial}w^{\bar{3}} = -w^{\bar{1}\bar{5}} - w^{\bar{2}\bar{6}}, \quad \bar{\partial}w^{\bar{5}} = -w^{\bar{4}\bar{6}},$$

we get

$$H_{\bar{\partial}}^{0,1}(\mathbb{I}_6) = \langle [w^{\bar{1}}], [w^{\bar{4}}], [w^{\bar{6}}] \rangle \cong \mathbb{C}^3.$$

(2) On $(\mathfrak{g}_{\mathbb{C}}^*)^{0,2}$, since

$$\begin{aligned} \bar{\partial}w^{\bar{1}\bar{3}} &= w^{\bar{1}\bar{2}\bar{6}}, & \bar{\partial}w^{\bar{1}\bar{5}} &= w^{\bar{1}\bar{4}\bar{6}}, & \bar{\partial}w^{\bar{2}\bar{3}} &= w^{\bar{1}\bar{3}\bar{4}} - w^{\bar{1}\bar{2}\bar{5}}, \\ \bar{\partial}w^{\bar{2}\bar{6}} &= -w^{\bar{1}\bar{4}\bar{6}}, & \bar{\partial}w^{\bar{3}\bar{6}} &= -w^{\bar{1}\bar{5}\bar{6}}, & \bar{\partial}w^{\bar{3}\bar{4}} &= w^{\bar{1}\bar{4}\bar{5}} + w^{\bar{2}\bar{4}\bar{6}}, \\ \bar{\partial}w^{\bar{2}\bar{5}} &= -w^{\bar{1}\bar{4}\bar{5}} + w^{\bar{2}\bar{4}\bar{6}}, & \bar{\partial}w^{\bar{3}\bar{5}} &= w^{\bar{2}\bar{5}\bar{6}} + w^{\bar{3}\bar{4}\bar{6}}, \end{aligned}$$

we get

$$H_{\bar{\partial}}^{0,2}(\mathbb{I}_6) = \langle [w^{\bar{1}\bar{2}}], [w^{\bar{1}\bar{6}}], [w^{\bar{2}\bar{4}}], [w^{\bar{4}\bar{5}}], [w^{\bar{5}\bar{6}}] \rangle \cong \mathbb{C}^5.$$

(3) On $(\mathfrak{g}_{\mathbb{C}}^*)^{0,3}$, since

$$\begin{aligned} \bar{\partial}w^{\bar{1}\bar{2}\bar{5}} &= -w^{\bar{1}\bar{2}\bar{4}\bar{6}}, & \bar{\partial}w^{\bar{1}\bar{3}\bar{4}} &= -w^{\bar{1}\bar{2}\bar{4}\bar{6}}, & \bar{\partial}w^{\bar{2}\bar{3}\bar{4}} &= w^{\bar{1}\bar{2}\bar{4}\bar{5}}, \\ \bar{\partial}w^{\bar{2}\bar{5}\bar{6}} &= -w^{\bar{1}\bar{4}\bar{5}\bar{6}}, & \bar{\partial}w^{\bar{3}\bar{4}\bar{5}} &= -w^{\bar{2}\bar{4}\bar{5}\bar{6}}, & \bar{\partial}w^{\bar{3}\bar{4}\bar{6}} &= w^{\bar{1}\bar{4}\bar{5}\bar{6}}, \\ \bar{\partial}w^{\bar{1}\bar{3}\bar{5}} &= -w^{\bar{1}\bar{2}\bar{5}\bar{6}} - w^{\bar{1}\bar{3}\bar{4}\bar{6}}, & \bar{\partial}w^{\bar{2}\bar{3}\bar{5}} &= w^{\bar{1}\bar{3}\bar{4}\bar{5}} - w^{\bar{2}\bar{3}\bar{4}\bar{6}}, & \bar{\partial}w^{\bar{2}\bar{3}\bar{6}} &= w^{\bar{1}\bar{3}\bar{4}\bar{6}} - w^{\bar{1}\bar{2}\bar{5}\bar{6}}, \end{aligned}$$

we get

$$H_{\bar{\partial}}^{0,3}(\mathbb{I}_6) = \langle [w^{\bar{1}\bar{2}\bar{4}}], [w^{\bar{1}\bar{3}\bar{4}}], [w^{\bar{1}\bar{3}\bar{6}}], [w^{\bar{2}\bar{4}\bar{5}}], [w^{\bar{3}\bar{5}\bar{6}}], [w^{\bar{4}\bar{5}\bar{6}}] \rangle \cong \mathbb{C}^6.$$

(4) On $(\mathfrak{g}_{\mathbb{C}}^*)^{0,4}$, since

$$\bar{\partial}w^{\bar{1}\bar{2}\bar{3}\bar{5}} = w^{\bar{1}\bar{2}\bar{3}\bar{4}\bar{6}}, \quad \bar{\partial}w^{\bar{1}\bar{3}\bar{4}\bar{5}} = w^{i_1 \dots i_p \bar{1}\bar{2}\bar{4}\bar{5}\bar{6}}, \quad \bar{\partial}w^{\bar{2}\bar{3}\bar{4}\bar{6}} = w^{\bar{1}\bar{2}\bar{4}\bar{5}\bar{6}}, \quad \bar{\partial}w^{\bar{2}\bar{3}\bar{5}\bar{6}} = w^{\bar{1}\bar{3}\bar{4}\bar{5}\bar{6}},$$

we get

$$H_{\bar{\partial}}^{0,4}(\mathbb{I}_6) = \langle [w^{\bar{1}\bar{2}\bar{3}\bar{4}}], [w^{\bar{1}\bar{2}\bar{3}\bar{6}}], [w^{\bar{1}\bar{3}\bar{5}\bar{6}}], [w^{\bar{2}\bar{3}\bar{4}\bar{5}}], [w^{\bar{3}\bar{4}\bar{5}\bar{6}}] \rangle \cong \mathbb{C}^5.$$

(5) Since $\bar{\partial}|_{(\mathfrak{g}_{\mathbb{C}}^*)^{0,5}} = \bar{\partial}|_{(\mathfrak{g}_{\mathbb{C}}^*)^{0,6}} = 0$, we have

$$H_{\bar{\partial}}^{0,5}(\mathbb{I}_6) = \langle [w^{\bar{1}\bar{2}\bar{3}\bar{4}\bar{5}}], [w^{\bar{1}\bar{2}\bar{3}\bar{5}\bar{6}}][w^{\bar{2}\bar{3}\bar{4}\bar{5}\bar{6}}] \rangle \cong \mathbb{C}^3 \quad \text{and} \quad H_{\bar{\partial}}^{0,6}(\mathbb{I}_6) = \langle [w^{\bar{1}\bar{2}\bar{3}\bar{4}\bar{5}\bar{6}}] \rangle \cong \mathbb{C}.$$

By the discussion in the above, we obtain the Hodge diamond of \mathbb{I}_6 as follows:

$$\begin{array}{cccccccc} & & & & 1 & & & \\ & & & & 6 & & 3 & \\ & & & & 15 & & 18 & & 5 \\ & & & & 20 & & 45 & & 30 & & 6 \\ & & & & 15 & & 60 & & 75 & & 36 & & 5 \\ & & & & 6 & & 45 & & 100 & & 90 & & 30 & & 3 \\ & & & & 1 & & 18 & & 75 & & 120 & & 75 & & 18 & & 1 \\ & & & & 3 & & 30 & & 90 & & 100 & & 45 & & 6 \\ & & & & 5 & & 36 & & 75 & & 60 & & 15 & & & \\ & & & & 6 & & 30 & & 45 & & 20 & & & & \\ & & & & 5 & & 18 & & 15 & & & & & \\ & & & & 3 & & 6 & & & & & & & \\ & & & & & & 1 & & & & & & & \end{array}$$

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